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# Minimal transitive factorizations of a permutation of type $(p, q)$

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**Abstract.** We give a combinatorial proof of Goulden and Jackson’s formula for the number of minimal transitive factorizations of a permutation when the permutation has two cycles. We use the recent result of Goulden, Nica, and Oancea on the number of maximal chains of annular noncrossing partitions of type  $B$ .

**Résumé.** Nous donnons une preuve combinatoire de formule de Goulden et Jackson pour le nombre de factorisations transitives minimales d’une permutation lorsque la permutation a deux cycles. Nous utilisons le résultat récent de Goulden, Nica, et Oancea sur le nombre de chaînes maximales des partitions non-croisées annulaires de type  $B$ .

**Keywords:** minimal transitive factorizations, annular noncrossing partitions, bijective proof

## 1 Introduction

Given an integer partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$  of  $n$ , denote by  $\alpha_\lambda$  the permutation

$$(1 \dots \lambda_1)(\lambda_1 + 1 \dots \lambda_1 + \lambda_2) \dots (n - \lambda_\ell + 1 \dots n)$$

of the set  $\{1, 2, \dots, n\}$  in the cycle notation. Let  $\mathcal{F}_\lambda$  be the set of all  $(n + \ell - 2)$ -tuples  $(\eta_1, \dots, \eta_{n+\ell-2})$  of transpositions such that

- (1)  $\eta_1 \cdots \eta_{n+\ell-2} = \alpha_\lambda$  and
- (2)  $\{\eta_1, \dots, \eta_{n+\ell-2}\}$  generates the symmetric group  $\mathcal{S}_n$ .

Such tuples are called *minimal transitive factorizations* of the permutation  $\alpha_\lambda$  of type  $\lambda$ , which are related to the branched covers of the sphere suggested by Hurwitz [Hur91, Str96].

In 1997, using algebraic methods Goulden and Jackson [GJ97] proved that

$$|\mathcal{F}_\lambda| = (n + \ell - 2)! n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}. \quad (1)$$

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Bousquet-Mélou and Schaeffer [BMS00] proved a more general formula than (1) and obtained (1) using the principle of inclusion and exclusion. Irving [Irv09] studied the enumeration of minimal transitive factorizations into cycles instead of transpositions.

If  $\lambda = (n)$ , the formula (1) yields

$$|\mathcal{F}_{(n)}| = n^{n-2}, \tag{2}$$

and there are several combinatorial proofs of (2) [Bia02, GY02, Mos89].

If  $\lambda = (p, q)$ , the formula (1) yields

$$|\mathcal{F}_{(p,q)}| = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q. \tag{3}$$

A few special cases of (3) have bijective proofs: by Kim and Seo [KS03] for the case  $(p, q) = (1, n - 1)$ , and by Rattan [Rat06] for the cases  $(p, q) = (2, n - 2)$  and  $(p, q) = (3, n - 3)$ . There are no simple combinatorial proofs for other  $(p, q)$ .

Recently, Goulden et al. [GNO11] showed that the number of maximal chains in the poset  $NC^{(B)}(p, q)$  of annular noncrossing partitions of type  $B$  is

$$\binom{p+q}{q} p^p q^q + \sum_{c \geq 1} 2c \binom{p+q}{p-c} p^{p-c} q^{q+c}. \tag{4}$$

Interestingly it turns out that half the sum in (4) is equal to the number in (3):

$$\sum_{c \geq 1} c \binom{p+q}{p-c} p^{p-c} q^{q+c} = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q.$$

In this paper we will give a combinatorial proof of (3) using the results in [GNO11]. The rest of this paper is organized as follows. In Section 2 we recall the poset  $\mathcal{S}_{nc}^B(p, q)$  of annular noncrossing permutations of type  $B$  which is isomorphic to the poset  $NC^{(B)}(p, q)$  of annular noncrossing partitions of type  $B$ , and show that the number of connected maximal chains in  $\mathcal{S}_{nc}^B(p, q)$  is equal to  $\frac{2pq}{p+q} \binom{p+q}{q} p^p q^q$ . In Section 3 we prove that there is a 2-1 map from the set of connected maximal chains in  $\mathcal{S}_{nc}^B(p, q)$  to  $\mathcal{F}_{(p,q)}$ , thus completing a combinatorial proof of (3).

We note that the present paper is part of [KSS12]. In the full version [KSS12] we give another combinatorial proof of (3) by introducing marked annular noncrossing permutations of type  $A$ .

## 2 Connected maximal chains

A *signed permutation* is a permutation  $\sigma$  on  $\{\pm 1, \dots, \pm n\}$  satisfying  $\sigma(-i) = -\sigma(i)$  for all  $i \in \{1, \dots, n\}$ . We denote by  $B_n$  the set of signed permutations on  $\{\pm 1, \dots, \pm n\}$ .

We will use the two notations

$$\begin{aligned} [a_1 a_2 \dots a_k] &= (a_1 a_2 \dots a_k - a_1 - a_2 \dots - a_k), \\ ((a_1 a_2 \dots a_k)) &= (a_1 a_2 \dots a_k)(-a_1 - a_2 \dots - a_k), \end{aligned}$$

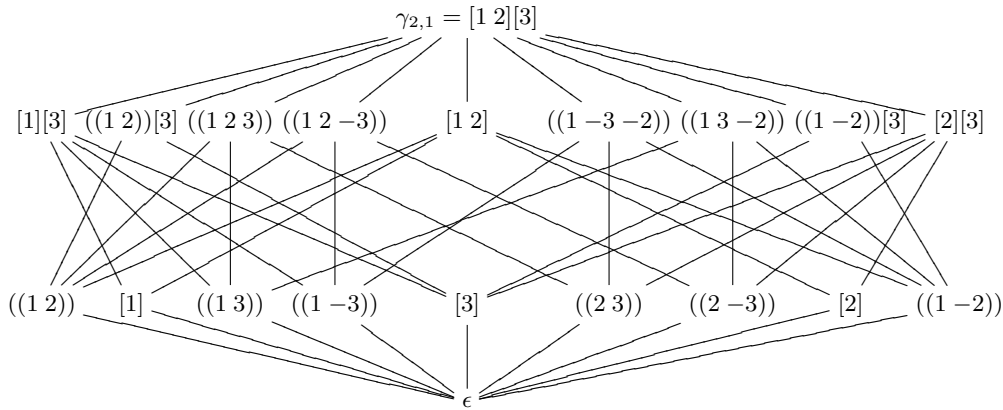


Fig. 1: The Hasse diagram for  $S_{nc}^B(2, 1)$ .

and call  $[a_1 a_2 \dots a_k]$  a *zero cycle* and  $((a_1 a_2 \dots a_k))$  a *paired nonzero cycle*. We also call the cycles  $\epsilon_i := [i] = (i - i)$  and  $((i j))$  *type B transpositions*, or simply *transpositions* if there is no possibility of confusion.

For  $\pi \in B_n$ , the *absolute length*  $\ell(\pi)$  is defined to be the smallest integer  $k$  such that  $\pi$  can be written as a product of  $k$  type  $B$  transpositions. The *absolute order* on  $B_n$  is defined by

$$\pi \leq \sigma \iff \ell(\sigma) = \ell(\pi) + \ell(\pi^{-1}\sigma).$$

From now, we fix positive integers  $p$  and  $q$ . The poset  $S_{nc}^B(p, q)$  of *annular noncrossing permutations of type B* is defined by

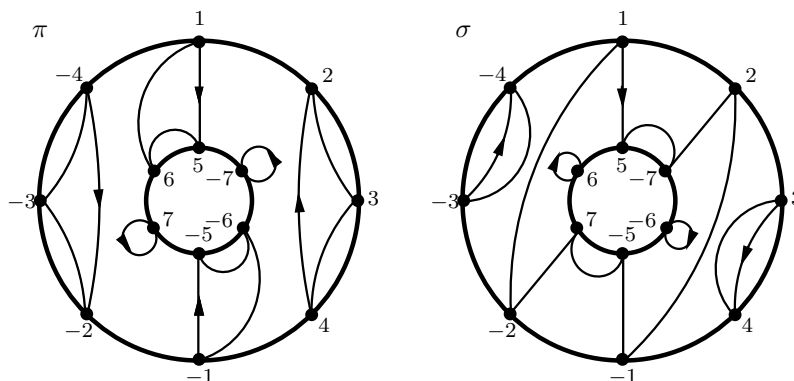
$$S_{nc}^B(p, q) := [\epsilon, \gamma_{p,q}] = \{\sigma \in B_{p+q} : \epsilon \leq \sigma \leq \gamma_{p,q}\} \subseteq B_{p+q},$$

where  $\epsilon$  is the identity in  $B_{p+q}$  and  $\gamma_{p,q} = [1 \dots p][p+1 \dots p+q]$ . Figure 1 shows the Hasse diagram for  $S_{nc}^B(2, 1)$ . Then  $S_{nc}^B(p, q)$  is a graded poset with rank function

$$\text{rank}(\sigma) = (p + q) - (\# \text{ of paired nonzero cycles of } \sigma). \tag{5}$$

Nica and Oancea [NO09] showed that  $\sigma \in S_{nc}^B(p, q)$  if and only if  $\sigma$  can be drawn without crossing arrows inside an annulus in which the outer circle has integers  $1, 2, \dots, p, -1, -2, \dots, -p$  in clockwise order and the inner circle has integers  $p+1, p+2, \dots, p+q, -p-1, -p-2, \dots, -p-q$  in counterclockwise order, see Figure 2. They also showed that  $S_{nc}^B(p, q)$  is isomorphic to the poset  $NC^{(B)}(p, q)$  of annular noncrossing partitions of type  $B$ .

A paired nonzero cycle  $((a_1 a_2 \dots a_k))$  is called *connected* if the set  $\{a_1, \dots, a_k\}$  intersects with both  $\{\pm 1, \dots, \pm p\}$  and  $\{\pm(p+1), \dots, \pm(p+q)\}$ , and *disconnected* otherwise. A zero cycle is always considered to be disconnected. For  $\sigma \in S_{nc}^B(p, q)$ , the *connectivity* of  $\sigma$  is the number of connected paired nonzero cycles of  $\sigma$ .



**Fig. 2:**  $\pi = ((1\ 5\ 6))((2\ 3\ 4))$  and  $\sigma = [1\ 5\ -7\ 2]((3\ 4))$  in  $\mathcal{S}_{nc}^B(4, 3)$

We say that a maximal chain  $C = \{\epsilon = \pi_0 < \pi_1 < \dots < \pi_{p+q} = \gamma_{p,q}\}$  of  $\mathcal{S}_{nc}^B(p, q)$  is *disconnected* if the connectivity of each  $\pi_i$  is zero. Otherwise,  $C$  is called *connected*. Denote by  $\mathcal{CM}(\mathcal{S}_{nc}^B(p, q))$  the set of connected maximal chains of  $\mathcal{S}_{nc}^B(p, q)$ .

For a maximal chain  $C = \{\pi_0 < \pi_1 < \dots < \pi_n\}$  of the interval  $[\pi_0, \pi_n]$ , we define  $\varphi(C) = (\tau_1, \tau_2, \dots, \tau_n)$ , where  $\tau_i = \pi_i^{-1}\pi_{i+1}$ . Note that each  $\tau_i$  is a type  $B$  transposition and  $\pi_i = \tau_1\tau_2 \dots \tau_i$  for all  $i = 1, 2, \dots, n$ .

**Lemma 1** *If  $C$  is a connected maximal chain of  $\mathcal{S}_{nc}^B(p, q)$ , then  $\varphi(C)$  has no transpositions of the form  $\epsilon_i = [i]$  and has at least one connected transposition. If  $C$  is a disconnected maximal chain of  $\mathcal{S}_{nc}^B(p, q)$ , then  $\varphi(C)$  has only disconnected transpositions.*

**Proof:** By (5),  $\sigma$  covers  $\pi$  in  $\mathcal{S}_{nc}^B(p, q)$  if and only if one of the following conditions holds, see [NO09, Proposition 2.2]:

- (a)  $\pi^{-1}\sigma = \epsilon_i$  and the cycle containing  $i$  in  $\pi$  is nonzero, i.e.,  $\pi$  has  $((i \dots))$  and  $\sigma$  has  $[i \dots]$ .
- (b)  $\pi^{-1}\sigma = ((i\ j))$  and no two of  $i, -i, j, -j$  belong to the same cycle in  $\pi$  with  $|i| \neq |j|$ , i.e.,  $\pi$  has  $((i \dots))((j \dots))$  and  $\sigma$  has  $((i \dots j \dots))$ .
- (c)  $\pi^{-1}\sigma = ((i\ j))$  and the cycle containing  $i$  in  $\pi$  is nonzero and the cycle containing  $j$  in  $\pi$  is zero with  $|i| \neq |j|$ , i.e.,  $\pi$  has  $((i \dots))[j \dots]$  and  $\sigma$  has  $[i \dots j \dots]$ .
- (d)  $\pi^{-1}\sigma = ((i\ j))$  and  $i$  and  $-j$  belong to the same nonzero cycle in  $\pi$  with  $|i| \neq |j|$ , i.e.,  $\pi$  has  $((i \dots -j \dots))$  and  $\sigma$  has  $[i \dots][-j \dots]$ .

If  $\sigma$  covers  $\pi$  in  $\mathcal{S}_{nc}^B(p, q)$ , we have  $zc(\sigma) \geq zc(\pi)$ , where  $zc(\sigma)$  is the the number of zero cycles in  $\sigma$ . More precisely we have

$$zc(\sigma) - zc(\pi) = \begin{cases} 0 & \text{if type (b) or (c),} \\ 1 & \text{if type (a),} \\ 2 & \text{if type (d).} \end{cases}$$

Since  $\gamma_{p,q}$  has two zero cycles, each  $\pi \in \mathcal{S}_{nc}^B(p, q)$  has at most two zero cycles. Moreover, if  $\pi$  has two zero cycles, then one of them belongs to  $\{\pm 1, \dots, \pm p\}$  and the other belongs to  $\{\pm(p+1), \dots, \pm(p+q)\}$ . Consider a maximal chain  $C$  in  $\mathcal{S}_{nc}^B(p, q)$ .

- If  $C$  has a permutation  $\pi$  with  $zc(\pi) = 1$ , there are two cover relations of type (a) and no cover relations of type (d) in  $C$ . For each cover relation  $\pi < \sigma$  of type (a), (b), or (c),  $\sigma$  is obtained by merging cycles in  $\pi$ . Since  $\gamma_{p,q}$  has only disconnected cycles, all permutations in  $C$  are disconnected, which implies that  $C$  is disconnected.
- Otherwise, there is a cover relation  $\pi < \sigma$  of type (d) in  $C$ . Then  $\sigma$  has two zero cycles  $[i \dots]$  and  $[-j \dots]$ , one of which is contained in  $\{\pm 1, \dots, \pm p\}$  and the other is contained in  $\{\pm(p+1), \dots, \pm(p+q)\}$ . Thus  $\pi$  has a connected nonzero cycle  $((i \dots -j \dots))$ , and  $C$  is connected. Since  $C$  has no cover relations of type (a),  $\varphi(C)$  has no transposition of the form  $\epsilon_i$ .

Therefore, if  $C$  is a disconnected maximal chain of  $\mathcal{S}_{nc}^B(p, q)$ , then  $\varphi(C)$  has two transpositions of the form  $\epsilon_i$ . So all transpositions of  $\varphi(C)$  are disconnected. Also, if  $C$  is a connected maximal chain of  $\mathcal{S}_{nc}^B(p, q)$ , then  $\varphi(C)$  has no transposition of the form  $\epsilon_i$  and has at least one connected transposition.  $\square$

The following proposition is a refinement of (4).

**Proposition 2** *The number of disconnected maximal chains of  $\mathcal{S}_{nc}^B(p, q)$  is equal to*

$$\binom{p+q}{q} p^p q^q \tag{6}$$

and the number of connected maximal chains of  $\mathcal{S}_{nc}^B(p, q)$  is equal to

$$\sum_{c \geq 1} 2c \binom{p+q}{p-c} p^{p-c} q^{q+c}. \tag{7}$$

We now prove the following identity that appears in the introduction. The proof is due to Krattenthaler [Kra].

**Lemma 3** *We have*

$$\sum_{c \geq 1} c \binom{p+q}{p-c} p^{p-c} q^{q+c} = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q. \tag{8}$$

**Proof:** Since  $c = p \cdot \frac{q+c}{p+q} - q \cdot \frac{p-c}{p+q}$ , we have

$$\begin{aligned} \sum_{c=0}^p c \binom{p+q}{p-c} p^{p-c} q^{q+c} &= \sum_{c=0}^p \left( p \cdot \frac{q+c}{p+q} - q \cdot \frac{p-c}{p+q} \right) \binom{p+q}{p-c} p^{p-c} q^{q+c} \\ &= \sum_{c=0}^p \left( \binom{p+q-1}{p-c} p^{p-c+1} q^{q+c} - \binom{p+q-1}{p-c-1} p^{p-c} q^{q+c+1} \right) \\ &= \binom{p+q-1}{p} p^{p+1} q^q = \frac{pq}{p+q} \binom{p+q}{p} p^p q^q. \end{aligned}$$

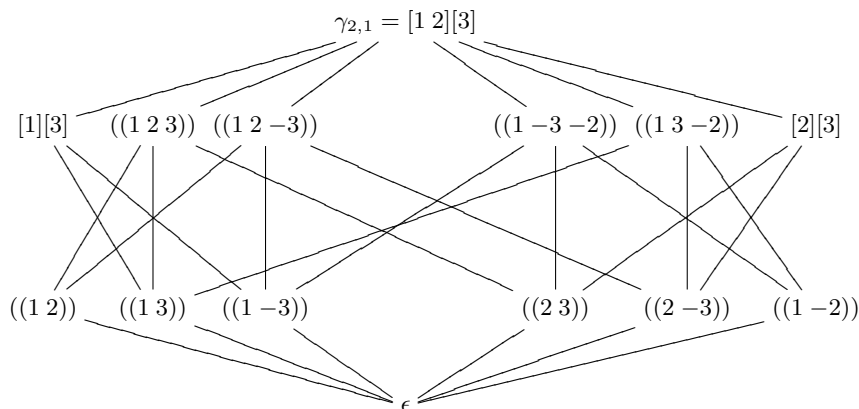


Fig. 3: Connected maximal chains in  $\mathcal{S}_{nc}^B(2, 1)$ .

□

By Proposition 2 and Lemma 3, we get the following.

**Corollary 4** *The number of connected maximal chains of  $\mathcal{S}_{nc}^B(p, q)$  is equal to*

$$\frac{2pq}{p+q} \binom{p+q}{q} p^p q^q. \tag{9}$$

For example, Figure 3 illustrates  $16 = \frac{4}{3} \binom{3}{1} 2^2$  connected maximal chains of  $\mathcal{S}_{nc}^B(2, 1)$ .

By Corollary 4, in order to prove (3) combinatorially it is sufficient to find a 2-1 map from  $\mathcal{CM}(\mathcal{S}_{nc}^B(p, q))$  to  $\mathcal{F}_{(p,q)}$ . We will find such a map in the next section.

*Remark 1.* One can check that the factorizations  $\varphi(C)$  coming from connected maximal chains  $C$  in  $\mathcal{S}_{nc}^B(p, q)$  are precisely the minimal factorizations of  $\gamma_{p,q}$  in the Weyl group  $D_{p+q}$ . Thus Corollary 4 can be restated as follows: the number of minimal factorizations of  $\gamma_{p,q}$  in  $D_{p+q}$  is equal to  $\frac{2pq}{p+q} \binom{p+q}{q} p^p q^q$ . Goupil [Gou95, Theorem 3.1] also proved this result by finding a recurrence relation.

*Remark 2.* Since the proof of Lemma 3 is a simple manipulation, it is easy and straightforward to construct a combinatorial proof for the identity in Lemma 3. Together with the result in Section 3 we get a combinatorial proof of (9). It would be interesting to find a direct bijective proof of (9) without using Lemma 3.

### 3 A 2-1 map from $\mathcal{CM}(\mathcal{S}_{nc}^B(p, q))$ to $\mathcal{F}_{(p,q)}$

Recall that a minimal transitive factorization of  $\alpha_{p,q} = (1 \dots p)(p + 1 \dots p + q)$  is a sequence  $(\eta_1, \dots, \eta_{p+q})$  of transpositions in  $\mathcal{S}_{p+q}$  such that

(1)  $\eta_1 \cdots \eta_{p+q} = \alpha_{p,q}$  and

(2)  $\{\eta_1, \dots, \eta_{p+q}\}$  generates  $\mathcal{S}_{p+q}$ ,

and  $\mathcal{F}_{(p,q)}$  is the set of minimal transitive factorizations of  $\alpha_{p,q}$ .

In this section we will prove the following theorem.

**Theorem 5** *There is a 2-1 map from the set of connected maximal chains in  $\mathcal{S}_{\text{nc}}^B(p, q)$  to the set  $\mathcal{F}_{(p,q)}$  of minimal transitive factorizations of  $\alpha_{p,q}$ .*

In order to prove Theorem 5 we need some definitions.

**Definition 6 (Two maps  $(\cdot)^+$  and  $|\cdot|$ )** *We introduce the following two maps.*

(1) *The map  $(\cdot)^+ : B_n \rightarrow B_n$  is defined by*

$$\sigma^+(i) = \begin{cases} |\sigma(i)| & \text{if } i > 0, \\ -|\sigma(i)| & \text{if } i < 0. \end{cases}$$

(2) *The map  $|\cdot| : B_n \rightarrow \mathcal{S}_n$  is defined by  $|\sigma|(i) = |\sigma(i)|$  for all  $i \in \{1, \dots, n\}$ .*

**Definition 7** *A  $(p+q)$ -tuple  $(\tau_1, \dots, \tau_{p+q})$  of transpositions in  $B_{p+q}$  is called a minimal transitive factorization of type  $B$  of  $\gamma_{p,q} = [1 \dots p][p+1 \dots p+q]$  if it satisfies*

(1)  $\tau_1 \dots \tau_{p+q} = \gamma_{p,q}$

(2)  $\{|\tau_1|, \dots, |\tau_{p+q}|\}$  generates  $\mathcal{S}_{p+q}$ .

*Denote by  $\mathcal{F}_{(p,q)}^{(B)}$  the set of minimal transitive factorizations of type  $B$  of  $\gamma_{p,q}$ .*

**Definition 8** *A  $(p+q)$ -tuple  $(\sigma_1, \dots, \sigma_{p+q})$  of transpositions in  $B_{p+q}$  is called a positive minimal transitive factorization of type  $B$  of  $\beta_{p,q} = ((1 \dots p))(p+1 \dots p+q)$  if it satisfies*

(1)  $\sigma_1 \dots \sigma_{p+q} = \beta_{p,q}$ ,

(2)  $\{|\sigma_1|, \dots, |\sigma_{p+q}|\}$  generates  $\mathcal{S}_{p+q}$ ,

(3)  $\sigma_i = \sigma_i^+$  for all  $i = 1, \dots, p+q$ .

*Denote by  $\mathcal{F}_{(p,q)}^+$  the set of positive minimal transitive factorizations of type  $B$  of  $\beta_{p,q}$ .*

For the rest of this section we will prove the following:

1. The map  $\varphi : \mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q)) \rightarrow \mathcal{F}_{(p,q)}^{(B)}$  is a bijection. (Lemma 9)
2. There is a 2-1 map  $(\cdot)^+ : \mathcal{F}_{(p,q)}^{(B)} \rightarrow \mathcal{F}_{(p,q)}^+$ . (Lemma 11)
3. There is a bijection  $|\cdot| : \mathcal{F}_{(p,q)}^+ \rightarrow \mathcal{F}_{(p,q)}$ . (Lemma 10)

By the above three statements the composition  $|\varphi^+| := |\cdot| \circ (\cdot)^+ \circ \varphi$  is a 2-1 map from  $\mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q))$  to  $\mathcal{F}_{(p,q)}$ , which completes the proof of Theorem 5. Since the proofs of the first and the third statements are simpler, we will present these first.



**Lemma 9** *The map  $\varphi : \mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q)) \rightarrow \mathcal{F}_{(p, q)}^{(B)}$  is a bijection.*

**Proof:** Given a connected maximal chain  $C = \{\epsilon = \pi_0 < \pi_1 < \dots < \pi_{p+q} = \gamma_{p, q}\}$  in  $\mathcal{S}_{\text{nc}}^B(p, q)$ , the elements in the sequence  $\varphi(C) = (\tau_1, \dots, \tau_{p+q})$  are transpositions with  $\tau_1 \cdots \tau_{p+q} = \gamma_{p, q}$ . By Lemma 1, at least one of  $\tau_i$ 's is connected. Thus  $\{|\tau_1|, \dots, |\tau_{p+q}|\}$  generates  $\mathcal{S}_{p+q}$ , and  $\varphi(C) \in \mathcal{F}_{(p, q)}^{(B)}$ . Conversely, if  $\tau = (\tau_1, \dots, \tau_{p+q}) \in \mathcal{F}_{(p, q)}^{(B)}$ , then  $\varphi^{-1}(\tau) = \{\epsilon = \pi_0 < \pi_1 < \dots < \pi_{p+q} = \gamma_{p, q}\}$ , where  $\pi_i = \tau_1 \cdots \tau_i$ , is a connected maximal chain in  $\mathcal{S}_{\text{nc}}^B(p, q)$  because  $\{|\tau_1|, \dots, |\tau_{p+q}|\}$  generates  $\mathcal{S}_{p+q}$ .  $\square$

**Lemma 10** *There is a bijection  $|\cdot| : \mathcal{F}_{(p, q)}^+ \rightarrow \mathcal{F}_{(p, q)}$ .*

**Proof:** Let  $(\sigma_1, \dots, \sigma_{p+q}) \in \mathcal{F}_{(p, q)}^+$ . Each  $\sigma_i$  can be written as  $\sigma_i = ((j \ k))$  for some positive integers  $j$  and  $k$ . In this case we let  $\eta_i = |\sigma_i| = (j \ k) \in \mathcal{S}_{p+q}$ . Then the map  $|\cdot| : \mathcal{F}_{(p, q)}^+ \rightarrow \mathcal{F}_{(p, q)}$  sending  $(\sigma_1, \dots, \sigma_{p+q})$  to  $(\eta_1, \dots, \eta_{p+q})$  is a bijection.  $\square$

Recall  $\epsilon_i = [i] = (i \ -i)$ . We write  $\overline{((i \ j))} := ((i \ -j))$ . It is easy to see that for  $i, j \in \{\pm 1, \dots, \pm(p+q)\}$ , we have

$$[i \ j] = \epsilon_i((i \ j)) = ((i \ j))\epsilon_j = \overline{((i \ j))}\epsilon_i = \epsilon_j\overline{((i \ j))}. \quad (10)$$

**Lemma 11** *There is a 2-1 map  $(\cdot)^+ : \mathcal{F}_{(p, q)}^{(B)} \rightarrow \mathcal{F}_{(p, q)}^+$ .*

Here we only describe the map  $(\cdot)^+$ : For  $(\tau_1, \tau_2, \dots, \tau_{p+q}) \in \mathcal{F}_{(p, q)}^{(B)}$ , we define  $(\tau_1, \tau_2, \dots, \tau_{p+q})^+ = (\tau_1^+, \tau_2^+, \dots, \tau_{p+q}^+)$ . Since  $\tau_1^+ \cdots \tau_{p+q}^+ = \gamma_{p, q}^+ = \beta_{p, q}$ , we have  $(\tau_1, \tau_2, \dots, \tau_{p+q})^+ \in \mathcal{F}_{(p, q)}^+$ . Let us fix  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{p+q}) \in \mathcal{F}_{(p, q)}^+$ . If  $\tau = (\tau_1, \dots, \tau_{p+q}) \in \mathcal{F}_{(p, q)}^{(B)}$  satisfies  $\tau^+ = \sigma$ , then  $\tau' = (\tau'_1 \cdots \tau'_{p+q}) \in \mathcal{F}_{(p, q)}^{(B)}$  defined by

$$\tau'_i = \begin{cases} \tau_i & \text{if } \tau_i \text{ is disconnected} \\ \overline{\tau_i} & \text{if } \tau_i \text{ is connected,} \end{cases} \quad (11)$$

also satisfies  $(\tau')^+ = \sigma$ . Thus the map  $(\cdot)^+$  is two-to-one. For the detailed proof, see [KSS12].

For example, let  $\sigma = ((1 \ 2)), ((2 \ 5)), ((2 \ 3)), ((4 \ 5)), ((3 \ 4)) \in \mathcal{F}_{(3, 2)}^+$  be the following factorization

$$\beta_{3, 2} = ((1 \ 2 \ 3))((4 \ 5)) = ((1 \ 2))((2 \ 5))((2 \ 3))((4 \ 5))((3 \ 4)).$$

Since  $\gamma_{3, 2} = \epsilon_4 \epsilon_1 \beta_{3, 2}$ , we can obtain a factorization of  $\gamma_{3, 2}$  from  $\sigma$  as follows:

$$\begin{aligned} \gamma_{3, 2} &= [1 \ 2 \ 3][4 \ 5] = \epsilon_4 \epsilon_1 ((1 \ 2))((2 \ 5))((2 \ 3))((4 \ 5))((3 \ 4)) \\ &= \epsilon_4 \epsilon_2 \overline{((1 \ 2))}((2 \ 5))((2 \ 3))((4 \ 5))((3 \ 4)) \\ &= \epsilon_4 \epsilon_3 ((1 \ 2))\overline{((2 \ 5))}\overline{((2 \ 3))}((4 \ 5))((3 \ 4)) \\ &= \epsilon_4 \epsilon_4 ((1 \ 2))\overline{((2 \ 5))}\overline{((2 \ 3))}\overline{((4 \ 5))}\overline{((3 \ 4))} \\ &= ((1 \ 2))\overline{((2 \ 5))}\overline{((2 \ 3))}\overline{((4 \ 5))}\overline{((3 \ 4))}. \end{aligned}$$

Thus  $\tau = \left( ((1\ 2)), \overline{((2\ 5))}, ((2\ 3)), \overline{((4\ 5))}, \overline{((3\ 4))} \right) \in \mathcal{F}_{(3,2)}^{(B)}$  satisfies  $\tau^+ = \sigma$ . The factorization  $\tau' = \left( ((1\ 2)), ((2\ 5)), ((2\ 3)), \overline{((4\ 5))}, ((3\ 4)) \right)$  obtained by toggling the connected transpositions of  $\tau$  also satisfies  $(\tau')^+ = \sigma$ .

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