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► **To cite this version:**

Kento Nakada. An algorithm which generates linear extensions for a non-simply-laced d -complete poset with uniform probability. 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012), 2012, Nagoya, Japan. pp.655-660, 10.46298/dmtcs.3071 . hal-01283096

HAL Id: hal-01283096

<https://hal.science/hal-01283096>

Submitted on 5 Mar 2016

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An algorithm which generates linear extensions for a non-simply-laced d -complete poset with uniform probability

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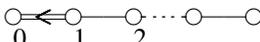
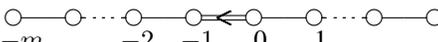
Abstract. The purpose of this paper is to present an algorithm which generates linear extensions for a non-simply-laced d -complete poset with uniform probability.

Résumé. Le but de ce papier est présenter un algorithme qui produit des extensions linéaires pour une non-simply-laced d -complete poset avec probabilité constante.

Keywords: d -complete posets, algorithm, linear extension, uniform generation

1 Introduction

In [7](Theorem 4.2), J. Stembridge classified irreducible minuscule elements of Kac-Moody Weyl group over a root system Φ into three classes below:

- Φ is simply-laced,
- Φ has the form  (namely, of type B), or
- Φ has the form  (we say *type F_m* , for simplicity).

In [5][6], the author and S. Okamura constructed an algorithm which generates reduced decompositions for a given minuscule element of simply-laced Weyl group with uniform probability. The algorithm in [6] is described in terms of graphs. Simply-laced minuscule elements are described as certain simple acyclic di-graphs. The transitive-closure of the graph is called a d -complete poset. Then, the reduced decompositions are identified with linear extensions of the graph. This algorithm gives a proof of the hook formula [1] for the number of reduced decompositions of a minuscule element in simply-laced case.

In this paper, we present an algorithm (algorithm A) in terms of graphs (See Section 2 for details). This algorithm is a generalization of an algorithm in [5][6]. We define a certain acyclic multi-di-graph corresponding to a minuscule element of type B (resp. type F_m) in Section 3 (resp. Section 4). Our

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main result (Theorem 5.1) is that the algorithm A generates linear extensions for a minuscule element of type B and F_m with uniform probability. More precisely, the probability the algorithm A generates linear extension L of a graph S is given by:

$$\frac{\prod_{v \in S} (1 + \#H_S(v)^+)}{\#S!}, \quad (1.1)$$

where $H_S(v)^+$ is a certain subset of S (See Section 2 for detail). This (1.1) is independent from the choice of L . Hence, we get the hook formula for the number of linear extensions of a given shape S of type B and F_m . Namely, the number of linear extensions of a shape S is given by:

$$\frac{\#S!}{\prod_{v \in S} (1 + \#H_S(v)^+)}.$$

In section 6, we give a Lie theoretical description of shape of type B and F_m .

2 An algorithm for a graph Γ

Let $\Gamma = (\Gamma; A, o, i)$ be a finite acyclic multi-di-graph, where A denotes the set of arrows of Γ , $i(a)$ the sink of $a \in A$, and $o(a)$ the source of $a \in A$.

Definition 2.1 Put $d := \#\Gamma$. A bijection $L : \{1, \dots, d\} \rightarrow \Gamma$ is said to be a linear extension of Γ if:

$$L(k) = o(a) \text{ and } i(a) = L(l) \text{ implies } k > l, \quad k, l \in \{1, \dots, d\}, \quad a \in A.$$

The set of linear extensions of Γ is denoted by $\mathcal{L}(\Gamma)$.

For a given $v \in \Gamma$, we define a set $H_\Gamma(v)^+$ by:

$$H_\Gamma(v)^+ := \{a \in A(\Gamma) \mid v = o(a)\}.$$

For a given Γ , we call the following algorithm the *algorithm A* for Γ :

- GNW1. Set $i := 0$ and set $\Gamma_0 := \Gamma$.
- GNW2. (Now Γ_i has $d - i$ nodes.) Set $j := 1$ and pick a node $v_1 \in \Gamma_i$ with the probability $1/(d - i)$.
- GNW3. If $\#H_{\Gamma_i}(v_j)^+ \neq 0$, pick an arrow $a_{j+1} \in H_{\Gamma_i}(v_j)^+$ with the probability $1/\#H_{\Gamma_i}(v_j)^+$. If not, go to GNW5.
- GNW4. Set $v_{j+1} := i(a_j)$. Set $j := j + 1$ and return to GNW3.
- GNW5. (Now $\#H_{\Gamma_i}(v_j)^+ = 0$.) Set $L(i + 1) := v_j$ and set $\Gamma_{i+1} := \Gamma_i \setminus v_j$ (the graph deleted v_j from Γ_i).
- GNW6. Set $i := i + 1$. If $i < d$, return to GNW2; if $i = d$, terminate.

We note that the algorithm A stops in finite time since Γ is acyclic. By the definition of the algorithm A for Γ , the map $L : i \mapsto L(i)$ generated above is a linear extension of Γ . We denote by $\text{Prob}_\Gamma(L)$ the probability we get $L \in \mathcal{L}(\Gamma)$ by the algorithm A.

3 Shapes of type B

We denote by \mathbb{N} the set of non-negative integers. We define a set \mathbb{B} by:

$$\mathbb{B} := \{ (i, j) \in \mathbb{N} \times \mathbb{N} \mid i \leq j \}.$$

The set \mathbb{B} is depicted in FIGURE 3.1. We equip the \mathbb{B} with the partial order:

$$(i, j) \leq (i', j') \iff i \geq i' \text{ and } j \geq j'.$$

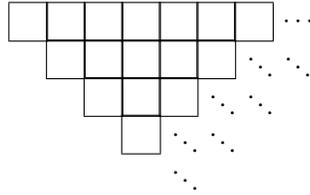


Fig. 3.1: The set \mathbb{B}

Definition 3.1 Let S be a finite order filter of \mathbb{B} . We induce to S a graph structure by:

$$(i, j) \rightarrow (i', j') \text{ if and only if } \begin{cases} "i = j \text{ and } i' = i, j' > j", \\ "i < j \text{ and } i' = i, j' > j", \\ "i < j \text{ and } i' > i, j' = j", \\ \text{or } "i < j \text{ and } i' = j, j' > i", \end{cases}$$

$$(i, j) \rightrightarrows (i', j') \text{ if and only if } "i < j \text{ and } i' = j' = j",$$

and there exists no other adjacency relation. Here, $v \rightarrow v'$ means there exists exactly one arrow from v to v' , and $v \rightrightarrows v'$ there exists exactly two arrows from v to v' . A graph S is called a shape of type B . See FIGURE 3.2 for examples of $H_S(v)^+$.

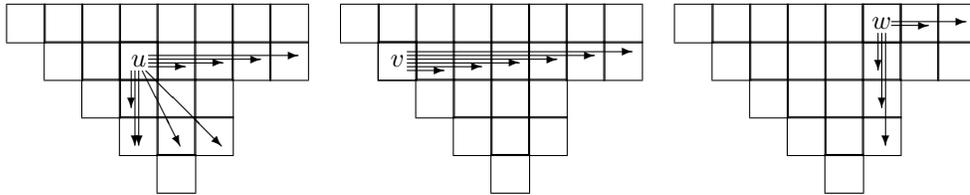


Fig. 3.2: $H_S(u)^+$, $H_S(v)^+$, and $H_S(w)^+$.

Remark 3.2 A shape of type B as poset is order-isomorphic to a shifted shape. Shifted shapes are also realized as d -complete posets over a root system of type D . The graph-structure of shapes of type D is described in [6] and compatible with notion of hooks (or called bars) of shifted shapes. The algorithm A depends not only on poset-structure but on graph-structure. Hence, we do not consider shapes of type B as shifted shapes.

4 Shapes of type F_m ($m \geq 2$).

We denote by \mathbb{Z} the set of integers. Let m be an integer greater than or equal to 2. We define a set \mathbb{F}_m by:

$$\mathbb{F}_m := \left\{ (i, j) \in \mathbb{N} \times \mathbb{Z} \left| \begin{array}{l} i = 0 \text{ and } j \geq -m, \\ i = 1 \text{ and } j \geq 0, \text{ or} \\ 2 \leq i \leq m \text{ and } j = 0 \end{array} \right. \right\}$$

For example, the set \mathbb{F}_3 is depicted in FIGURE 4.1. We equip the \mathbb{F}_m with the partial order:

$$(i, j) \leq (i', j') \iff i \geq i' \text{ and } j \geq j'.$$

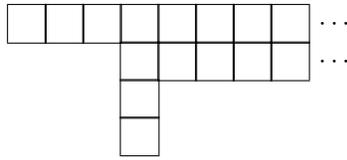


Fig. 4.1: The set \mathbb{F}_3

Definition 4.1 Let S be a finite order filter of \mathbb{F}_m . We induce to S a graph structure by:

$$(i, j) \rightarrow (i', j') \text{ if and only if } \begin{cases} "i = 0, j \leq -1 \text{ and } i' \neq -j, j' > j", \\ "i = 0, j = 0, \text{ and } j' > 0", \\ "i = 1, j = 0, \text{ and } i' = 1, j' > 0", \\ "i = 1, j = 0, \text{ and } i' > 1, j' = 0", \\ "i \geq 2, j = 0, \text{ and } i' > i, j' = 0", \\ "j \geq 1 \text{ and } i' = i, j' > j", \\ \text{or } "j \geq 1 \text{ and } i' > i, j' = j", \end{cases}$$

$$(i, j) \rightrightarrows (i', j') \text{ if and only if } "i = 0, j = 0, \text{ and } 0 < i', j' = 0",$$

and there exists no other adjacency relation. A graph S is called a shape of type F_m . See FIGURE 4.2 for examples of $H_S(v)^+$.

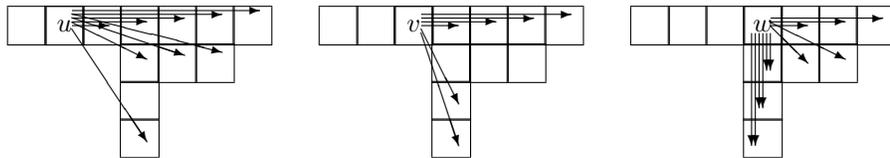


Fig. 4.2: $H_S(u)^+$, $H_S(v)^+$, and $H_S(w)^+$.

5 Main result

Now, we can state the main theorem:

Theorem 5.1 *Let S be a shape of type B or type F_m for some $m \geq 2$. Let $L \in \mathcal{L}(S)$. Then the algorithm A for S generates L with the probability*

$$\text{Prob}_S(L) = \frac{\prod_{v \in S} (1 + \#H_S(v)^+)}{\#S!}. \quad (5.1)$$

Since the right hand side of (5.1) is independent from the choice of $L \in \mathcal{L}(S)$, we have:

Corollary 5.2 *Let S be a shape of type B or type F_m for some $m \geq 2$. Then we have:*

$$\#\mathcal{L}(S) = \frac{\#S!}{\prod_{v \in S} (1 + \#H_S(v)^+)}.$$

6 Lie theoretical description of main result and Remarks

In this section, we fix a (not necessary simply-laced) Kac-Moody Lie algebra \mathfrak{g} with a simple root system $\Pi = \{\alpha_i \mid i \in I\}$. For all undefined terminology in this section, we refer the reader to [2] [3].

Definition 6.1 *An integral weight λ is said to be pre-dominant if:*

$$\langle \lambda, \beta^\vee \rangle \geq -1 \quad \text{for each } \beta^\vee \in \Phi_+^\vee,$$

where Φ_+^\vee denotes the set of positive real coroots. The set of pre-dominant integral weights is denoted by $P_{\geq -1}$. For $\lambda \in P_{\geq -1}$, we define the set $D(\lambda)^\vee$ by:

$$D(\lambda)^\vee := \{ \beta^\vee \in \Phi_+^\vee \mid \langle \lambda, \beta^\vee \rangle = -1 \}.$$

The set $D(\lambda)^\vee$ is called the shape of λ . If $\#D(\lambda)^\vee < \infty$, then λ is called finite.

Proposition 6.2 (see [4]) *Let $\lambda \in P_{\geq -1}$ be finite and $\beta^\vee, \gamma^\vee \in D(\lambda)^\vee$ satisfy $\beta^\vee > \gamma^\vee$ in the ordinary order of coroots. Then we have:*

$$\langle \beta, \gamma^\vee \rangle = 0, 1, \text{ or } 2.$$

By proposition 6.2, we introduce graph-structure into $D(\lambda)^\vee$ by:

$$\begin{aligned} \beta^\vee \rightarrow \gamma^\vee &\Leftrightarrow \beta^\vee > \gamma^\vee \text{ and } \langle \beta, \gamma^\vee \rangle = 1. \\ \beta^\vee \rightrightarrows \gamma^\vee &\Leftrightarrow \beta^\vee > \gamma^\vee \text{ and } \langle \beta, \gamma^\vee \rangle = 2. \end{aligned}$$

If $\beta^\vee \not> \gamma^\vee$, or $\beta^\vee > \gamma^\vee$ and $\langle \beta, \gamma^\vee \rangle = 0$, then no arrows from β^\vee to γ^\vee exist.

Thus, we get a finite acyclic multi-di-graph $D(\lambda)^\vee$ for a finite $\lambda \in P_{\geq -1}$.

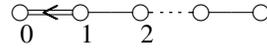
Remark 6.3 *The finite pre-dominant integral weights λ are identified with the minuscule elements w [4]. And, we have $D(\lambda)^\vee = \{ \beta^\vee \in \Phi_+^\vee \mid w^{-1}(\beta^\vee) < 0 \}$. Furthermore, the linear extensions of $D(\lambda)^\vee$ are identified with the reduced decompositions of w [4] by the following one-to-one correspondence:*

$$\text{Red}(w) \ni (s_{i_1}, s_{i_2}, \dots, s_{i_d}) \longleftrightarrow L \in \mathcal{L}(D(\lambda)^\vee), \quad L(k) = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})^\vee \in D(\lambda)^\vee \quad (k = 1, \dots, d),$$

where $\text{Red}(w)$ denotes the set of reduced decompositions of w , $d = \ell(w)$ the length of w .

6.1 Case of type B

Suppose that the underlying Dynkin diagram is of type B:



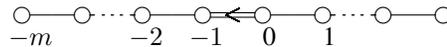
Let $W = \langle s_0, s_1, s_2, \dots \rangle$ be the Weyl group. Let Λ_0 be the 0-th fundamental weight. Then each $\lambda \in W\Lambda_0$ is a finite pre-dominant integral weight. And, $D(\lambda)^\vee$ is graph-isomorphic with some shape of type B defined in section 3.

Remark 6.4 Let $W_0 := \langle s_1, s_2, \dots \rangle$ be a maximal parabolic subgroup of W , which is the Weyl group of type A. Then a minimal coset representative w in W/W_0 is called a Lagrangian Grassmannian element.

Let $\lambda \in W\Lambda_0$. Then the corresponding minuscule element w in remark 6.3 is a Lagrangian Grassmannian element. Our result gives the number of reduced decompositions of Lagrangian Grassmannian element w .

6.2 Case of type F_m ($m \geq 2$)

Let $m \in \mathbb{Z}$ be greater than or equal to 2. Suppose that the underlying Dynkin diagram is of type F_m :



Let $W = \langle s_{-m}, \dots, s_{-2}, s_{-1}, s_0, s_1, \dots \rangle$ be the Weyl group. Let Λ_{-m} be the $(-m)$ -th fundamental weight. Then each $\lambda \in P_{\geq -1} \cap W\Lambda_{-m}$ is a finite pre-dominant integral weight. And, $D(\lambda)^\vee$ is graph-isomorphic with some shape of type F_m defined in section 4.

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