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# *An algorithm which generates linear extensions for a non-simply-laced $d$ -complete poset with uniform probability*

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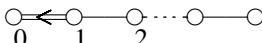
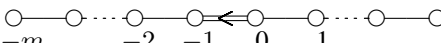
**Abstract.** The purpose of this paper is to present an algorithm which generates linear extensions for a non-simply-laced  $d$ -complete poset with uniform probability.

**Résumé.** Le but de ce papier est présenter un algorithme qui produit des extensions linéaires pour une non-simply-laced  $d$ -complete poset avec probabilité constante.

**Keywords:**  $d$ -complete posets, algorithm, linear extension, uniform generation

## 1 Introduction

In [7](Theorem 4.2), J. Stembridge classified irreducible minuscule elements of Kac-Moody Weyl group over a root system  $\Phi$  into three classes below:

- $\Phi$  is simply-laced,
- $\Phi$  has the form  (namely, of type  $B$ ), or
- $\Phi$  has the form  (we say *type  $F_m$* , for simplicity).

In [5][6], the author and S. Okamura constructed an algorithm which generates reduced decompositions for a given minuscule element of simply-laced Weyl group with uniform probability. The algorithm in [6] is described in terms of graphs. Simply-laced minuscule elements are described as certain simple acyclic di-graphs. The transitive-closure of the graph is called a  $d$ -complete poset. Then, the reduced decompositions are identified with linear extensions of the graph. This algorithm gives a proof of the hook formula [1] for the number of reduced decompositions of a minuscule element in simply-laced case.

In this paper, we present an algorithm (algorithm A) in terms of graphs (See Section 2 for details). This algorithm is a generalization of an algorithm in [5][6]. We define a certain acyclic multi-di-graph corresponding to a minuscule element of type  $B$  (resp. type  $F_m$ ) in Section 3 (resp. Section 4). Our

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main result (Theorem 5.1) is that the algorithm A generates linear extensions for a minuscule element of type  $B$  and  $F_m$  with uniform probability. More precisely, the probability the algorithm A generates linear extension  $L$  of a graph  $S$  is given by:

$$\frac{\prod_{v \in S} (1 + \#H_S(v)^+)}{\#S!}, \quad (1.1)$$

where  $H_S(v)^+$  is a certain subset of  $S$  (See Section 2 for detail). This (1.1) is independent from the choice of  $L$ . Hence, we get the hook formula for the number of linear extensions of a given shape  $S$  of type  $B$  and  $F_m$ . Namely, the number of linear extensions of a shape  $S$  is given by:

$$\frac{\#S!}{\prod_{v \in S} (1 + \#H_S(v)^+)}.$$

In section 6, we give a Lie theoretical description of shape of type  $B$  and  $F_m$ .

## 2 An algorithm for a graph $\Gamma$

Let  $\Gamma = (\Gamma; A, o, i)$  be a finite acyclic multi-di-graph, where  $A$  denotes the set of arrows of  $\Gamma$ ,  $i(a)$  the sink of  $a \in A$ , and  $o(a)$  the source of  $a \in A$ .

**Definition 2.1** Put  $d := \#\Gamma$ . A bijection  $L : \{1, \dots, d\} \rightarrow \Gamma$  is said to be a linear extension of  $\Gamma$  if:

$$L(k) = o(a) \text{ and } i(a) = L(l) \text{ implies } k > l, \quad k, l \in \{1, \dots, d\}, \quad a \in A.$$

The set of linear extensions of  $\Gamma$  is denoted by  $\mathcal{L}(\Gamma)$ .

For a given  $v \in \Gamma$ , we define a set  $H_\Gamma(v)^+$  by:

$$H_\Gamma(v)^+ := \{a \in A(\Gamma) \mid v = o(a)\}.$$

For a given  $\Gamma$ , we call the following algorithm the *algorithm A* for  $\Gamma$ :

GNW1. Set  $i := 0$  and set  $\Gamma_0 := \Gamma$ .

GNW2. (Now  $\Gamma_i$  has  $d - i$  nodes.) Set  $j := 1$  and pick a node  $v_1 \in \Gamma_i$  with the probability  $1/(d - i)$ .

GNW3. If  $\#H_{\Gamma_i}(v_j)^+ \neq 0$ , pick an arrow  $a_{j+1} \in H_{\Gamma_i}(v_j)^+$  with the probability  $1/\#H_{\Gamma_i}(v_j)^+$ . If not, go to GNW5.

GNW4. Set  $v_{j+1} := i(a_j)$ . Set  $j := j + 1$  and return to GNW3.

GNW5. (Now  $\#H_{\Gamma_i}(v_j)^+ = 0$ .) Set  $L(i + 1) := v_j$  and set  $\Gamma_{i+1} := \Gamma_i \setminus v_j$  (the graph deleted  $v_j$  from  $\Gamma_i$ ).

GNW6. Set  $i := i + 1$ . If  $i < d$ , return to GNW2; if  $i = d$ , terminate.

We note that the algorithm A stops in finite time since  $\Gamma$  is acyclic. By the definition of the algorithm A for  $\Gamma$ , the map  $L : i \mapsto L(i)$  generated above is a linear extension of  $\Gamma$ . We denote by  $\text{Prob}_\Gamma(L)$  the probability we get  $L \in \mathcal{L}(\Gamma)$  by the algorithm A.

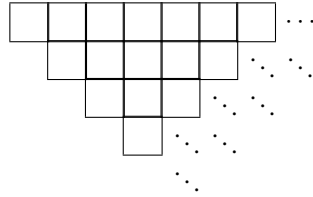
### 3 Shapes of type $B$

We denote by  $\mathbb{N}$  the set of non-negative integers. We define a set  $\mathbb{B}$  by:

$$\mathbb{B} := \{ (i, j) \in \mathbb{N} \times \mathbb{N} \mid i \leq j \}.$$

The set  $\mathbb{B}$  is depicted in FIGURE 3.1. We equip the  $\mathbb{B}$  with the partial order:

$$(i, j) \leq (i', j') \iff i \geq i' \text{ and } j \geq j'.$$



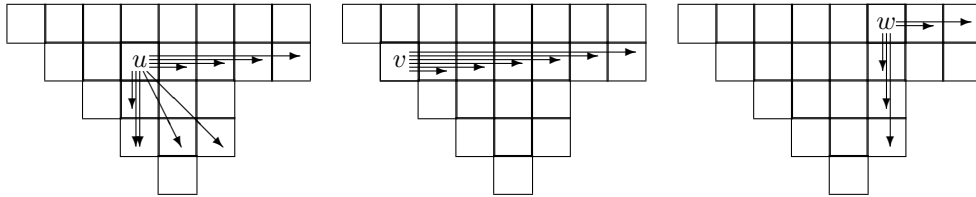
**Fig. 3.1:** The set  $\mathbb{B}$

**Definition 3.1** Let  $S$  be a finite order filter of  $\mathbb{B}$ . We induce to  $S$  a graph structure by:

$$(i, j) \rightarrow (i', j') \text{ if and only if } \begin{cases} "i = j \text{ and } i' = i, j' > j", \\ "i < j \text{ and } i' = i, j' > j", \\ "i < j \text{ and } i' > i, j' = j", \\ \text{or } "i < j \text{ and } i' = j, j' > i", \end{cases}$$

$$(i, j) \rightrightarrows (i', j') \text{ if and only if } "i < j \text{ and } i' = j' = j",$$

and there exists no other adjacency relation. Here,  $v \rightarrow v'$  means there exists exactly one arrow from  $v$  to  $v'$ , and  $v \rightrightarrows v'$  there exists exactly two arrows from  $v$  to  $v'$ . A graph  $S$  is called a shape of type  $B$ . See FIGURE 3.2 for examples of  $H_S(v)^+$ .



**Fig. 3.2:**  $H_S(u)^+$ ,  $H_S(v)^+$ , and  $H_S(w)^+$ .

**Remark 3.2** A shape of type  $B$  as poset is order-isomorphic to a shifted shape. Shifted shapes are also realized as  $d$ -complete posets over a root system of type  $D$ . The graph-structure of shapes of type  $D$  is described in [6] and compatible with notion of hooks (or called bars) of shifted shapes. The algorithm  $A$  depends not only on poset-structure but on graph-structure. Hence, we do not consider shapes of type  $B$  as shifted shapes.

### 4 Shapes of type $F_m$ ( $m \geq 2$ ).

We denote by  $\mathbb{Z}$  the set of integers. Let  $m$  be an integer greater than or equal to 2. We define a set  $\mathbb{F}_m$  by:

$$\mathbb{F}_m := \left\{ (i, j) \in \mathbb{N} \times \mathbb{Z} \left| \begin{array}{l} i = 0 \text{ and } j \geq -m, \\ i = 1 \text{ and } j \geq 0, \text{ or} \\ 2 \leq i \leq m \text{ and } j = 0 \end{array} \right. \right\}$$

For example, the set  $\mathbb{F}_3$  is depicted in FIGURE 4.1. We equip the  $\mathbb{F}_m$  with the partial order:

$$(i, j) \leq (i', j') \iff i \geq i' \text{ and } j \geq j'.$$

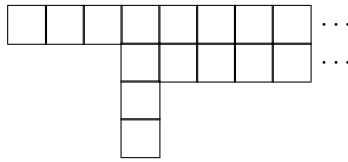


Fig. 4.1: The set  $\mathbb{F}_3$

**Definition 4.1** Let  $S$  be a finite order filter of  $\mathbb{F}_m$ . We induce to  $S$  a graph structure by:

$$(i, j) \rightarrow (i', j') \text{ if and only if } \begin{cases} "i = 0, j \leq -1 \text{ and } i' \neq -j, j' > j", \\ "i = 0, j = 0, \text{ and } j' > 0", \\ "i = 1, j = 0, \text{ and } i' = 1, j' > 0", \\ "i = 1, j = 0, \text{ and } i' > 1, j' = 0", \\ "i \geq 2, j = 0, \text{ and } i' > i, j' = 0", \\ "j \geq 1 \text{ and } i' = i, j' > j", \\ \text{or } "j \geq 1 \text{ and } i' > i, j' = j", \end{cases}$$

$$(i, j) \Rightarrow (i', j') \text{ if and only if } "i = 0, j = 0, \text{ and } 0 < i', j' = 0",$$

and there exists no other adjacency relation. A graph  $S$  is called a shape of type  $F_m$ . See FIGURE 4.2 for examples of  $H_S(v)^+$ .

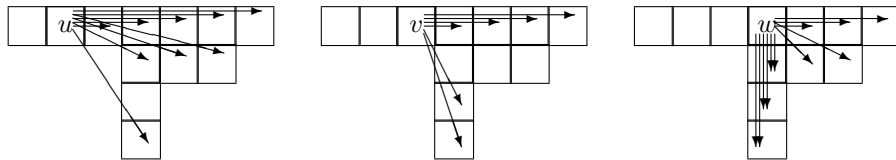


Fig. 4.2:  $H_S(u)^+$ ,  $H_S(v)^+$ , and  $H_S(w)^+$ .

## 5 Main result

Now, we can state the main theorem:

**Theorem 5.1** *Let  $S$  be a shape of type  $B$  or type  $F_m$  for some  $m \geq 2$ . Let  $L \in \mathcal{L}(S)$ . Then the algorithm  $A$  for  $S$  generates  $L$  with the probability*

$$\text{Prob}_S(L) = \frac{\prod_{v \in S} (1 + \#H_S(v)^+)}{\#S!}. \quad (5.1)$$

Since the right hand side of (5.1) is independent from the choice of  $L \in \mathcal{L}(S)$ , we have:

**Corollary 5.2** *Let  $S$  be a shape of type  $B$  or type  $F_m$  for some  $m \geq 2$ . Then we have:*

$$\#\mathcal{L}(S) = \frac{\#S!}{\prod_{v \in S} (1 + \#H_S(v)^+)}.$$

## 6 Lie theoretical description of main result and Remarks

In this section, we fix a (not necessary simply-laced) Kac-Moody Lie algebra  $\mathfrak{g}$  with a simple root system  $\Pi = \{\alpha_i \mid i \in I\}$ . For all undefined terminology in this section, we refer the reader to [2] [3].

**Definition 6.1** *An integral weight  $\lambda$  is said to be pre-dominant if:*

$$\langle \lambda, \beta^\vee \rangle \geq -1 \quad \text{for each } \beta^\vee \in \Phi_+^\vee,$$

where  $\Phi_+^\vee$  denotes the set of positive real coroots. The set of pre-dominant integral weights is denoted by  $P_{\geq -1}$ . For  $\lambda \in P_{\geq -1}$ , we define the set  $D(\lambda)^\vee$  by:

$$D(\lambda)^\vee := \{ \beta^\vee \in \Phi_+^\vee \mid \langle \lambda, \beta^\vee \rangle = -1 \}.$$

The set  $D(\lambda)^\vee$  is called the shape of  $\lambda$ . If  $\#D(\lambda)^\vee < \infty$ , then  $\lambda$  is called finite.

**Proposition 6.2 (see [4])** *Let  $\lambda \in P_{\geq -1}$  be finite and  $\beta^\vee, \gamma^\vee \in D(\lambda)^\vee$  satisfy  $\beta^\vee > \gamma^\vee$  in the ordinary order of coroots. Then we have:*

$$\langle \beta, \gamma^\vee \rangle = 0, 1, \text{ or } 2.$$

By proposition 6.2, we introduce graph-structure into  $D(\lambda)^\vee$  by:

$$\begin{aligned} \beta^\vee \rightarrow \gamma^\vee &\Leftrightarrow \beta^\vee > \gamma^\vee \text{ and } \langle \beta, \gamma^\vee \rangle = 1. \\ \beta^\vee \rightrightarrows \gamma^\vee &\Leftrightarrow \beta^\vee > \gamma^\vee \text{ and } \langle \beta, \gamma^\vee \rangle = 2. \end{aligned}$$

If  $\beta^\vee \not> \gamma^\vee$ , or  $\beta^\vee > \gamma^\vee$  and  $\langle \beta, \gamma^\vee \rangle = 0$ , then no arrows from  $\beta^\vee$  to  $\gamma^\vee$  exist.

Thus, we get a finite acyclic multi-di-graph  $D(\lambda)^\vee$  for a finite  $\lambda \in P_{\geq -1}$ .

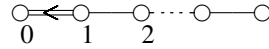
**Remark 6.3** *The finite pre-dominant integral weights  $\lambda$  are identified with the minuscule elements  $w$  [4]. And, we have  $D(\lambda)^\vee = \{ \beta^\vee \in \Phi_+^\vee \mid w^{-1}(\beta^\vee) < 0 \}$ . Furthermore, the linear extensions of  $D(\lambda)^\vee$  are identified with the reduced decompositions of  $w$  [4] by the following one-to-one correspondence:*

$$\text{Red}(w) \ni (s_{i_1}, s_{i_2}, \dots, s_{i_d}) \longleftrightarrow L \in \mathcal{L}(D(\lambda)^\vee), \quad L(k) = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})^\vee \in D(\lambda)^\vee \quad (k = 1, \dots, d),$$

where  $\text{Red}(w)$  denotes the set of reduced decompositions of  $w$ ,  $d = \ell(w)$  the length of  $w$ .

### 6.1 Case of type B

Suppose that the underlying Dynkin diagram is of type B:



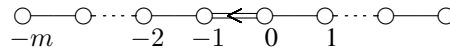
Let  $W = \langle s_0, s_1, s_2, \dots \rangle$  be the Weyl group. Let  $\Lambda_0$  be the 0-th fundamental weight. Then each  $\lambda \in W\Lambda_0$  is a finite pre-dominant integral weight. And,  $D(\lambda)^\vee$  is graph-isomorphic with some shape of type B defined in section 3.

**Remark 6.4** Let  $W_0 := \langle s_1, s_2, \dots \rangle$  be a maximal parabolic subgroup of  $W$ , which is the Weyl group of type A. Then a minimal coset representative  $w$  in  $W/W_0$  is called a Lagrangian Grassmannian element.

Let  $\lambda \in W\Lambda_0$ . Then the corresponding minuscule element  $w$  in remark 6.3 is a Lagrangian Grassmannian element. Our result gives the number of reduced decompositions of Lagrangian Grassmannian element  $w$ .

### 6.2 Case of type $F_m$ ( $m \geq 2$ )

Let  $m \in \mathbb{Z}$  be greater than or equal to 2. Suppose that the underlying Dynkin diagram is of type  $F_m$ :



Let  $W = \langle s_{-m}, \dots, s_{-2}, s_{-1}, s_0, s_1, \dots \rangle$  be the Weyl group. Let  $\Lambda_{-m}$  be the  $(-m)$ -th fundamental weight. Then each  $\lambda \in P_{\geq -1} \cap W\Lambda_{-m}$  is a finite pre-dominant integral weight. And,  $D(\lambda)^\vee$  is graph-isomorphic with some shape of type  $F_m$  defined in section 4.

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