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# Series Representation of Power Function

Kolosov Petro

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## Abstract

This paper presents the way to make expansion for the next form function:  $y = x^n$ ,  $\forall(x, n) \in \mathbb{N}$  to the numerical series. The most widely used methods to solve this problem are Newton's Binomial Theorem and Fundamental Theorem of Calculus (that is, derivative and integral are inverse operators). The paper provides the other kind of solution, based on induction from particular to general case, except above described theorems.

**Keywords.** power, power function, power series, third power, series, finite difference, divided difference, high order finite difference, derivative, binomial coefficient, binomial theorem, Newton's binomial theorem, binomial expansion, n-th difference of n-th power, number theory, cubic number, cube, Euler number, exponential function, Pascal triangle, Pascal's triangle

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## 1 Introduction

Let basically describe Newton's Binomial Theorem and Fundamental Theorem of Calculus and some their properties. In elementary algebra, the binomial theorem (or binomial expansion) describes the algebraic expansion of powers of a binomial. The theorem describes expanding of the power of  $(x + y)^n$  into a sum involving terms of the form  $ax^by^c$  where the exponents  $b$  and  $c$  are nonnegative integers with  $b + c = n$ , and the coefficient  $a$  of each term is a specific positive integer depending on  $n$  and  $b$ . The coefficient  $a$  in the term of  $ax^by^c$  is known as the binomial coefficient. The main properties of the binomial theorem are next:

- I. the powers of  $x$  go down until it reaches  $x_0 = 1$  starting value is  $n$  (the  $n$  in  $(x + y)^n$ )
- II. the powers of  $y$  go up from 0 ( $y^0 = 1$ ) until it reaches  $n$  (also  $n$  in  $(x + y)^n$ )
- III. the  $n$ -th row of the Pascal's Triangle (see [1]) will be the coefficients of the expanded binomial.

IV. for each line, the number of products (i.e. the sum of the coefficients) is equal to  $x + 1$

V. for each line, the number of product groups is equal to  $2^n$

According to the theorem, it is possible to expand any power of  $x + y$  into a sum of the form (see [2]):

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \quad (1)$$

By using binomial theorem for our case we obtain next form function:

$$x^n = \sum_{k=0}^{x-1} \left( nk^{n-1} + \binom{n}{2} k^{n-2} + \dots + \binom{n}{n-1} k + 1 \right) \quad (2)$$

We can reach the same result by using Fundamental Theorem of Calculus, we have, respectively:

$$x^n = \int_0^x nt^{n-1} dt$$

by means of addition of integrals

$$= \sum_{k=0}^{x-1} \int_k^{k+1} nt^{n-1} dt = \sum_{k=0}^{x-1} (k+1)^n - k^n \quad (3)$$

For presented in this paper method the properties of binomial theorem are not corresponded and the function, which we use summation operator on, has the **recursive** structure depending on  $x$ , basic view is the next:  $x^n = \sum_k \theta(x, k)$ . Below is represented theoretical algorithm deducing such a function, which, when be substituted to the summation operator, with some  $k$  number of iterations, returns the correct value of the number  $x_i \in \mathbb{R}$  to power  $n = 3$ . The main idea that the law of basic elements distribution of the value  $x_i$  to third power seen in finding the 3-rd order finite difference (note that  $n$ -order difference is written as  $\Delta^n(x_i^k)$ ) between nearest two items  $\Delta(x_i^3) = x_{i+1}^3 - x_i^3$ ,  $x_i = i\Delta x$ ,  $i \in \mathbb{N}$ ,  $\Delta x = x_{i+1} - x_i$ ,  $\Delta x \in \mathbb{R}$ . For example, let be a set of numbers, which  $x_i$  has constant difference between numbers, constant difference is a key of this method (example for  $\Delta x = 1$ , i.e  $x_i = i$ ):

$i$	$x_i^3$	$\Delta(x_i^3)$	$\Delta^2(x_i^3)$	$\Delta^3(x_i^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

(4)

Figure 1: Difference table of third power

Where differences  $\Delta^k(x_i^3)$  of orders  $k \in [1, 3]$ , for  $x_i = i \cdot \Delta x$  distribution are defined as:

$$\begin{aligned}\Delta(x_i^3) &\stackrel{\text{def}}{=} x_{i+1}^3 - x_i^3 = (i+1)^3 \cdot \Delta x^3 - i^3 \cdot \Delta x^3 \\ \Delta^2(x_i^3) &= \Delta(x_{i+1}^3) - \Delta(x_i^3) \\ \Delta^3(x_i^3) &= \Delta^2(x_{i+1}^3) - \Delta^2(x_i^3)\end{aligned}$$

More generally, the  $n$ -th difference of sequence  $\{f(x_i)\}_{i=1}^\infty$  could be written as (see [10]):

$$\Delta^n f(x_i) = \Delta^{n-1} f(x_{i+1}) - \Delta^{n-1} f(x_i) = \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot f(x_{i+n-k})$$

Basically, the  $n$ -order difference of the function  $f(x_i) = x_i^m$  has the follow view:

$$\Delta^n(x_i^m) = \begin{cases} m! \cdot \Delta x^m, & \text{if } n = m \\ \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot (i+n-k)^m \cdot \Delta x^m, & \text{if } n < m \\ (i+1)^m \cdot \Delta x^m - i^m \cdot \Delta x^m, & \text{if } n = 1 \end{cases} \quad (5)$$

According to Figure 1, in case of  $f(x_i) = x_i^3$ ,  $\Delta x = 1$ , the first order finite difference has next regularity (sequence A008458 in OEIS):

$$\begin{aligned}\Delta(x_0^3) &= x_1^3 - x_0^3 = 1 + 3! \cdot 0 \\ \Delta(x_1^3) &= x_2^3 - x_1^3 = 1 + 3! \cdot 0 + 3! \cdot 1 \\ \Delta(x_2^3) &= x_3^3 - x_2^3 = 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 \\ &\vdots \\ \Delta(x_m^3) &= x_{m+1}^3 - x_m^3 = 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + \cdots + 3! \cdot m\end{aligned}$$

**Theorem 1.** For each function  $f(x_i) = x_i^n$ ,  $x_i = i\Delta x$ ,  $\Delta x = \text{const}$ ,  $\Delta x \in \mathbb{R}$ ,  $\forall(i, n) \in \mathbb{N}$ ,  $x_i \neq x$ , holds the following equality:  $\frac{d^n(x^n)}{dx^n} = \frac{\Delta^n(x_i^n)}{(\Delta x)^n} = n!$ , i.e for any power function with natural number as exponent holds the equality between finite difference and derivative with order respectively to exponent and equals to exponent under factorial sign.

*Proof.* Since, the  $(x^n)'_x = nx^{n-1}$  (see [3]), we can say:  $\left. \frac{d^k}{dx^k} \cdot f(x) \right|_{f(x)=x^n} = n \cdot (n-1) \times \cdots \times (n-k+1) \cdot x^{n-k}$ ,  $n \in \mathbb{N}$ . Using limit notation, we have:

$$\underbrace{\lim_{m \rightarrow n^-} \left( \frac{d^m(\vartheta(x))}{dx^m} \right) \bigg|_{\vartheta(x)=x^n}}_{\phi(m)} = \frac{d^{n-0}(\vartheta(x))}{dx^{n-0}} = n!$$

By means of expression (5), in case of  $m < n$ ,  $n \in \mathbb{N}$ , we have the limit:

$$\begin{aligned} \lim_{m \rightarrow n^-} \left( \frac{\Delta^m(x_i^n)}{(\Delta x)^m} \right) &= \lim_{m \rightarrow n^-} \underbrace{\left( \sum_{k=0}^m \binom{m}{k} (-1)^k \cdot (i + m - k)^m \right)}_{\omega(m)} \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k \cdot (i + n - k)^{n-0} = n! \end{aligned}$$

Obviously, if  $\phi(m) = n!$ ,  $\omega(m) = n! \longrightarrow \phi(m) = \omega(m)$ , so we have right to equal

$$\frac{d^n(x^n)}{dx^n} = \frac{\Delta^n(x_i^n)}{(\Delta x)^n} = n!$$

This completes the proof.  $\square$

As we can see, according to Figure 1, the values of third rank differences of  $x_i^3$ , when  $\Delta x = 1$  are equal to  $3!$  (see [6], [7]) and constant for each index  $i$ . According to the values from Figure 1, the difference functions are corresponding to next recursively defined expressions:

$$\Delta^3(x_i^3) = j \cdot (\Delta x)^3, \quad \Delta^2(x_i^3) = \sum_{m=0}^{i-1} \Delta^3(x_i^3) \cdot m$$

$$\Delta(x_i^3) = (\Delta x)^3 + \sum_{k=0}^{i-1} \left( \Delta^2(x_k^3) \right)$$

Where  $j$ , according to theorem 1 and ([6], [7]), equals to  $3!$ . Hence, we have follow expansion for natural cube numbers (also the sum of sequences A008458 over  $k$  from 0 to  $x-1$ ):

$$\begin{aligned} x^3 &= (1 + 3! \cdot 0) + (1 + 3! \cdot 0 + 3! \cdot 1) + (1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2) + \dots \\ &\quad \dots + (1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + \dots + 3! \cdot (x-1)) \\ x^3 &= x + (x-0) \cdot 3! \cdot 0 + (x-1) \cdot 3! \cdot 1 + (x-2) \cdot 3! \cdot 2 + \dots \\ &\quad \dots + (x-(x-1)) \cdot 3! \cdot (x-1) \end{aligned} \tag{6}$$

Note that step  $\Delta x$  from equation (6) is set as  $\Delta x = 1$  and not displayed. Using summation notation over  $k$  from 0 to  $x-1$  on (6), we obtain:

$$x^3 = x + j \sum_{m=0}^{x-1} mx - m^2 \tag{7}$$

Or

$$x^3 = \sum_{m=0}^{x-1} \left( 1 + j \sum_{u=0}^m u \right) \equiv j \sum_{m=1}^{x-1} \left( mx - m^2 + \frac{x}{j(x-1)} \right) \Big|_{x>1}, \quad x \in \mathbb{N} \tag{8}$$

**Property 1.** Let be expression (7) written as  $\psi(\Omega(x)) = x + \sum_{m \in \Omega(x)} \chi(m)$ ,  $\Omega(x) \subset \mathbb{N}_0$ . Let be set  $\Omega(x)$  defined as:  $\Omega(x) := \{\{0, \underbrace{\{1, 2, 3, 4, \dots, x-1\}}_{\Lambda(x)}, x\}\}$  Since the  $\chi(0) = \chi(x) = 0$ , we can say that  $\forall(7) : \psi(\Omega(x)) \equiv \psi(\Lambda(x)) \equiv \psi(\Xi(x))$ ,  $(\Lambda(x), \Xi(x)) \subset \Omega(x)$ ,  $\Lambda(x) \not\subset \Xi(x)$ . It shows that changing iteration sets of expression (7) the function value is not changed.

Now we have successful expression, which disperses any natural number  $x^3$  to the numerical series (this example shows only expansion for any number  $x \in \mathbb{N}$  to power  $n = 3$ , but this method works for floats numbers also, it depends of start set of numbers, function's form also depends of the chosen set, step  $\Delta x$  between numbers should be constant all the time). Going from it, in the next section we introduce changing over the function (7) to power  $n > 3$ ,  $n \in \mathbb{N}$ .

## 2 Change over to higher powers expression

In this section are reviewed the ways to change obtained in previous annex expression (7) to higher powers i.e  $n > 3$ . Examples are shown for case  $\Delta x = 1$ ,  $i = x$ ,  $x \in \mathbb{N}$ . By means of Fundamental Theorem of Calculus, we know next (3):

$$x^n = \int_0^x nt^{n-1} dt = \sum_{k=0}^{x-1} \int_k^{k+1} nt^{n-1} dt = \sum_{k=0}^{x-1} (k+1)^n - k^n$$

According to property 1, expression (7) could be written as:

$$x^3 = x + j \sum_{m \in (\Omega(x) \equiv \Xi(x) \equiv \Lambda(x))} mx - m^2$$

Next, let derive the relationship between sums of forward and backward differences:

$$x^n = \sum_{k=0}^{x-1} (k+1)^n - k^n \equiv \sum_{k=1}^x k^n - (k-1)^n \quad (9)$$

As we can see, iteration sets of (9) are  $\Xi(x) = \{0, 1, \dots, x-1\}$  and  $\Lambda(x) = \{1, 2, \dots, x\}$ , respectively. Consequently, going from forward difference and by means of property 1, we have right to make the transition to the expansion of the form:

$$\begin{aligned} x^3 &= \sum_{k=0}^{x-1} (k+1)^3 - k^3 = \sum_{k \in \Xi(x)} \left( (k+1)^3 - \left\{ k - j \sum_{m \in \Omega(k)} mk - m^2 \right\} \right) \\ &\equiv \sum_{k \in \Xi(x)} \left( \left\{ (k+1) + j \sum_{m \in \Lambda(k)} (m(k+1) - m^2) \right\} - k^3 \right) \end{aligned} \quad (10)$$

Otherwise, let derive the series representation of  $x^3$  by means of backward finite difference (9), that is:

$$\begin{aligned} x^3 &= \sum_{k \in \Lambda(x)} k^3 - (k-1)^3 = \sum_{k \in \Lambda(x)} \left( \left\{ k + j \sum_{m \in \Omega(k)} (mk - m^2) \right\} - (k-1)^3 \right) \\ &\equiv \sum_{k \in \Lambda(x)} \left( k^3 - \left\{ k + 1 + j \sum_{m \in \Xi(k)} (m \cdot k + m - m^2) \right\} \right) \end{aligned} \quad (11)$$

By means of (10), in case of  $\phi(x) = x^n$ ,  $\forall(x, n) \in \mathbb{N}$  follows:

$$\begin{aligned} x^n &= \sum_{k \in \Xi(x)} \left( \left\{ (k+1)^{n-2} + j \sum_{m \in \Omega(k)} (m(k+1)^{n-2} - m^2(k+1)^{n-3}) \right\} - k^n \right) \\ &\equiv \sum_{k \in \Xi(x)} \left( (k+1)^n - \left\{ k^{n-2} + j \sum_{m \in \Omega(k)} m \cdot k^{n-2} - m^2 \cdot k^{n-3} \right\} \right) \end{aligned}$$

By means of main property of the power function  $x^n = x^k \cdot x^{n-k}$ , from equation (11) for  $\phi(x) = x^n$  we receive:

$$\begin{aligned} x^n &= \sum_{k \in \Lambda(x)} \left( \left\{ k^{n-2} + j \sum_{m \in \Xi(k)} (m \cdot k^{n-2} - m^2 \cdot k^{n-3}) \right\} - (k-1)^n \right) \\ &\equiv \sum_{k \in \Lambda(x)} \left( k^n - \left\{ (k+1)^{n-2} + j \sum_{m \in \Xi(k)} (m(k+1)^{n-2} - m^2(k+1)^{n-3}) \right\} \right) \end{aligned} \quad (12)$$

Since the  $x^n = x^k \cdot x^{n-k}$ , from the expression (8) for  $\phi(x) = x^n$ , we derive:

$$x^n = j \sum_{k \in \Xi(x)} \left( k \cdot x^{n-2} - k^2 \cdot x^{n-3} + \frac{x^{n-2}}{j(x-1)} \right), \quad 1 < x \in \mathbb{N} \quad (13)$$

In case of  $\phi(x) = x^n$ ,  $x \geq 0$ ,  $x \in \mathbb{N}$  expression (7) could be written as follows:

$$x^n = x^{n-2} + j \sum_{k \in \Xi(x)} k \cdot x^{n-2} - k^2 \cdot x^{n-3} \quad (14)$$

Continuum case:

$$\forall(u_1 \pm u_2 \pm \dots \pm u_k) \geq 1$$

holds

$$\begin{aligned} (u_1 \pm u_2 \pm \dots \pm u_k)^3 &= \sum_{g \leq k} u_g + j \sum_{m=0}^{u_1 \pm u_2 \pm \dots \pm u_k - 1} \left( m \cdot \sum_{g \leq k} u_g - m^2 \right) \\ (u_1 \pm u_2 \pm \dots \pm u_k)^n &= \left( \sum_{g \leq k} u_g \right)^{n-2} + j \sum_{m=0}^{u_1 \pm u_2 \pm \dots \pm u_k - 1} \left( m \cdot \left( \sum_{g \leq k} u_g \right)^{n-2} - m^2 \cdot \left( \sum_{g \leq k} u_g \right)^{n-3} \right) \end{aligned}$$

### 3 Binomial Theorem Representation

By means of Binomial theorem (1) expansion of  $(x+1)^n$  will be next (see [4]):

$$(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad (15)$$

According expressions (1) and (3), we have the next corresponding:

$$x^n = \sum_{k=0}^{x-1} (k+1)^n - k^n = \sum_{k=0}^{x-1} \left( \sum_{m=0}^n \binom{n}{m} k^m - k^n \right) \quad (16)$$

Let, going from expression (7), change the binomial expansion of  $\phi(x) = x^3$ :

$$x^3 = x + j \sum_{m \in \Xi(x)} mx - m^2$$

Consequently, for  $(x+1)^3$  we obtain:

$$(x+1)^3 = (x+1) + j \sum_{m \in \Omega(x)} m(x+1) - m^2 \quad (17)$$

So, for  $\phi(x) = x^3$ , by means of expressions (17) and (15), binomial expansion is next:

$$x^3 = \sum_{k=0}^{x-1} \left( - \sum_{i=0}^{k-1} \left( \sum_{t=0}^3 \binom{3}{t} i^t - i^3 \right) + \left[ (k+1) + j \sum_{m=0}^k m(k+1) - m^2 \right] \right) \quad (18)$$

Going from (18), by means of power function property  $x^n = x^k \cdot x^{n-k}$ , we can only to multiply by  $x$  every product of the series, by this way, for  $\phi(x) = x^n$ , from expression (16), we have next changes in binomial expansion:

$$\begin{aligned} x^n &= \sum_{k=0}^{x-1} \left( \sum_{m=0}^n \binom{n}{m} k^m - k^n \right) \\ &= \sum_{k=0}^{x-1} \left( \left\{ - \sum_{i=0}^{k-1} \left( \sum_{t=0}^n \binom{n}{t} i^t - i^n \right) \right\} + \left[ (k+1)^{n-2} + j \sum_{m=0}^k m(k+1)^{n-2} - m^2 k^{n-3} \right] \right) \\ &= \sum_{k=0}^{x-1} \left( \left\{ - \sum_{i=0}^{k-1} \left( \sum_{t=0}^n \binom{n}{t} i^t - i^n \right) \right\} + \left[ (k+1)^{n-2} + j \sum_{m=1}^{k+1} m(k+1)^{n-2} - m^2 k^{n-3} \right] \right) \end{aligned}$$

### 4 $e^x$ Representation

According above method, we have right to present function  $y = e^x$  by the follow view (as the exponential function is the infinite sum of powers of  $x$  divided by value of factorial according to iteration step, see [5]):

$$e^x = \sum_{m=0}^{\infty} \left( j \sum_{k \in \Lambda(x)} \left( \frac{kx^{m-2} - k^2 x^{m-3} + \frac{x^{m-3}}{j}}{m!} \right) \right) \quad (19)$$



## 5 Difference from Binomial Theorem

To show changes from binomial theorem let use other algorithm to reach expansion of  $f(\mu) = \mu^3$ ,  $\mu \in \mathbb{N}$ , which finally returns the binomial expansion. The algorithm is also based on  $\Delta$ -functions from (5). By means of theorem 1:

$$\chi^{(n)}(\mu) \Big|_{\chi(\mu)=\mu^n} = \Delta^n(x_i^n) \Big|_{\Delta x=1} = n!$$

We have right to integrate the  $f^{(3)}(\mu)$  to  $f'(\mu)$  and represent the  $\mu^3$  as:

$$\mu^3 = \sum_{\tau \in \Xi(\mu)} f'(\tau)$$

For third-order derivative we have next equality:

$$d^3 f(\mu) = \Delta^3(x_i^3) \cdot d\mu^3$$

Let derive the  $f''(x)$ :

$$\int d(f''(\mu)) = \Delta^3(x_i^3) \cdot \int d\mu = \Delta^3(x_i^3) \cdot \mu + C_1$$

Let be  $C_1 = \Delta^2(x_0^3)$ , so we have:

$$f''(\mu) = \Delta^3(x_i^3) \cdot \mu + \Delta^2(x_0^3)$$

First derivative is next:

$$\int d(f'(\mu)) = \Delta^3(x_i^3) \cdot \int \mu d\mu + \Delta^2(x_0^3) \cdot \int d\mu + C_2(i)$$

Let calculate the  $C_2(i)$  function, basic formula is next:

$$C_k(i) = \Delta^{n-k}(x_0^n) - \left( \Delta^{n-k}(x_0^n) - (\Delta^{n-k}(x_1^n) - f^{(n-k)}(1)) \right) i$$

And for  $f(\mu) = \mu^3$  equals to:

$$C_2(i) = 1 - (1 - (7 - f'(1)))i = 1 - 3i$$

In case of  $i = \mu$  we obtain the first derivative:

$$f'(\mu) = 3\mu^2 + 3\mu + 1$$

So,

$$\mu^3 = \sum_{\tau \in \Xi(\mu)} 3\tau^2 + 3\tau + 1$$

and corresponds to binomial expansion. Also, the difference is lay in the iteration limits of the expression (7), the binomial type expansion has the the summation limit  $[0, x - 1]$  and it's necessary to provide the  $x$ -products summation to calculate the number to power, while equation (7), according to property 1, has the lesser products iteration set  $[1, x - 1]$ , which shows, that for such type function is necessary lesser by 1 number of iteration steps, than for binomial expansion.

## 6 Conclusion

In this paper was reviewed a method of power function  $f(x_i) = x_i^n$ ,  $\forall(i, n) \in \mathbb{N}$  expanding to the numerical series. The disadvantages of this method are complicated form of expression and the complexity of calculating the value of these expressions of the some variables. Advantage of this method is the possibility of the successful application of this method in the solution of some problems in number theory, the theory of series, due to the differences from the common theory, displayed the difference from binomial expansion, presented example for exponential function representation by means of method from section 2. The paper doesn't consist all combinations of binomial theorem representation (by means of the property 1 and transformation (9)). In the Application 1 (Section 7) are shown Microsoft Visual Basic 6.0 program codes for the most important expressions (by authors' opinion). Future research in this direction could take as result the success polynomial kind expansion.

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## 7 Application 1. Visual Basic 6.0 Program codes

Expression (7):

```
j = 6
x = Val(Text1.Text)
r = 0
For k = 1 To x
r = r + j * (k * x - k2)
Next k
r = r + x
```

Expression (8) part 1:

```
j = 6
x = Val(Text1.Text)
r = 0
For k = 0 To x - 1 Step 1
For m = 0 To k Step 1
r = r + j * m
Next m
r = r + 1
Next k
```

Expression (8) part 2:

```
j = 6
x = Val(Text1.Text)
r = 0
For k = 1 To x - 1
r = r + j * (k * x - k2 + x / (j * (x - 1)))
Next k
```

Expression (11), part 1:

```
j = 6
x = Val(Text1.Text)
r = 0
For k = 1 To x Step 1
For m = 0 To k - 1 Step 1
r = r + j * (m * k - m2)
Next m
r = r + k - (k - 1)3
Next k
```

Expression (11), part 2:

```
j = 6
x = Val(Text1.Text)
r = 0
For k = 1 To x Step 1
```

```

For m = 0 To k - 1 Step 1
r = r - j * (m * (k - 1) - m^2)
Next m
r = r + k^3 - (k - 1)
Next k

```

Expression (12), part 1:

```

j = 6
x = Val(Text1.Text)
n = Val(Text2.Text)
r = 0
For k = 1 To x Step 1
For m = 0 To k - 1 Step 1
r = r + j * (m * k^{n-2} - m^2 * k^{n-3})
Next m
r = r + k^{n-2} - (k - 1)^n
Next k

```

Expression (12), part 2:

```

j = 6
x = Val(Text1.Text)
n = Val(Text2.Text)
r = 0
t = 0
For k = 1 To x Step 1
For m = 0 To k - 1 Step 1
r = r + j * (m * (k - 1)^{n-2} - m^2 * (k - 1)^{n-3})
Next m
t = t + k^n - (k - 1)^{n-2} - r
Next k

```

Expression (13):

```

j = 6
x = Val(Text1.Text)
n = Val(Text2.Text)
r = 0
For k = 1 To x - 1
r = r + j * (k * x^{n-2} - k^2 * x^{n-3} + x^{n-2} / (j * (x - 1)))
Next k

```

Expression (14):

```

j = 6
x = Val(Text1.Text)
n = Val(Text2.Text)
r = 0
For k = 1 To x

```

```

 $r = r + j * (k * x^{n-2} - k^2 * x^{n-3})$ 
Next  $k$ 
 $r = r + x^{n-2}$ 

```

$e^x$  Representation (19):

```

 $j = 6$ 
 $e = 0$ 
 $x = \text{Val}(\text{Text1.Text})$ 
 $r = \text{Val}(\text{Text2.Text})$ 
 $f = 0$ 
For  $m = 0$  To  $r$  Step 1
If  $m = 0$  Then
 $f = 1$ 
Else
 $f = f * m$ 
End If
For  $k = 1$  To  $x$ 
 $e = e + ((j * (k * x^{m-2} - k^2 * x^{m-3})) + x^{m-3}) / f$ 
Next  $k$ 
Next  $m$ 

```

Templates of all the programs available online at [this link](#).