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Series Representation of Power Function

Kolosov Petro

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Abstract

This paper presents the way to make expansion for the next form function: $y = x^n$, $\forall(x, n) \in \mathbb{N}$ to the numerical series. The most widely used methods to solve this problem are Newton's Binomial Theorem and Fundamental Theorem of Calculus (that is, derivative and integral are inverse operators). The paper provides the other kind of solution, except above described theorems.

1 Introduction

Let basically describe Newton's Binomial Theorem and Fundamental Theorem of Calculus and some their properties. In elementary algebra, the binomial theorem (or binomial expansion) describes the algebraic expansion of powers of a binomial. The theorem describes expanding of the power of $(x + y)^n$ into a sum involving terms of the form $ax^b y^c$ where the exponents b and c are nonnegative integers with $b + c = n$, and the coefficient a of each term is a specific positive integer depending on n and b . The coefficient a in the term of $ax^b y^c$ is known as the binomial coefficient. The main properties of the binomial theorem are next:

- I. the powers of x go down until it reaches $x_0 = 1$ starting value is n (the n in $(x + y)^n$)
- II. the powers of y go up from 0 ($y^0 = 1$) until it reaches n (also n in $(x + y)^n$)
- III. the n -th row of the Pascal's Triangle (see [1]) will be the coefficients of the expanded binomial.
- IV. for each line, the number of products (i.e. the sum of the coefficients) is equal to $x + 1$
- V. for each line, the number of product groups is equal to 2^n

According to the theorem, it is possible to expand any power of $x + y$ into a sum of the form (see [2]):

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \quad (1)$$

By using binomial theorem for our case we obtain next form function:

$$x^n = \sum_{k=0}^{x-1} \left(n k^{n-1} + \binom{n}{2} k^{n-2} + \dots + \binom{n}{n-1} k + 1 \right) \quad (2)$$

We can reach the same result by using Fundamental Theorem of Calculus, accordingly we have:

$$x^n = \int_0^x nt^{n-1} dt =$$

by means of the addition of integrals

$$= \sum_{k=0}^{x-1} \int_k^{k+1} nt^{n-1} dt = \sum_{k=0}^{x-1} (k+1)^n - k^n \quad (3)$$

For presented in this paper method the properties of binomial theorem are not corresponded and prime function (i.e function, which we use with sum operator) has the recursion structure for x basic view is the next: $x^n = \sum f(x, k)$. Below is represented theoretical algorithm deducing such a function, which, when substituted to the sum operator, with some k number of iterations, returns the correct value of a number $x \in \mathbb{N}$ to power $n = 3$. The main idea that the law for basic elements distribution of the value to third powers seen in finding the n -rank difference (n -rank difference is written as Δ^n) between nearest two items $\Delta_i^1 = x_i^3 - x_{i-1}^3$, $x_i = i\Delta x$, $i \in \mathbb{N}$, $\Delta x \in \mathbb{R}$. For example, let there be a set of numbers, which x has constant difference between numbers, constant difference is a key of this method (example for $\Delta x = 1 \rightarrow i = x_i$):

i	x_i^3	Δ_i^1	Δ_i^2	Δ^3
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1: Numbers according third power

Where “deltas” for $x_i = i \cdot \Delta x$ distribution are equal to:

$$\Delta_i^1 := x_{i+1}^n - x_i^n$$

...

$$\Delta^n = \Delta_i^{n-1} - \Delta_{i-1}^{n-1} = n!(\Delta x)^n$$

Basically, the delta function has view:

$$\Delta^n(x_i, m) = \begin{cases} m! \cdot \Delta x^m, & n = m \\ \sum_{k=0}^n \binom{n}{k} \frac{(i + (n-k))^m}{(-1)^{k-n}} \cdot \Delta x^m, & n < m \\ x_{i+1}^m - x_i^m, & n = 1 \end{cases} \quad (5)$$

where x_i and m are argument and exponent from the function $y = x_i^m$, $\forall(i, m) \in \mathbb{N}$. According (4) (for case $y = x^3$), the first rank delta has next regularity:

$$\Delta_1^1 = 1 - 0 = 1 + 3! \cdot 0$$

$$\Delta_2^1 = 8 - 1 = 1 + 3! \cdot 0 + 3! \cdot 1$$

$$\Delta_3^1 = 27 - 8 = 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2$$

$$\Delta_4^1 = 64 - 27 = 1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + 3! \cdot 3$$

Theorem 1. For each function $f(x) = x_i^n$, $x_i = i \cdot \Delta x$, $\Delta x = \text{const}$, $\Delta x \in \mathbb{R}$, $i \in \mathbb{N}$, $x_i \neq x$, holds the following equality: $\frac{d^n(x^n)}{dx^n} = \frac{\Delta^n(x_i^n)}{(\Delta x)^n} = n!$

Proof. Since, the $(x^n)' = nx^{n-1}$ (see [3]), we can say: $\frac{d^k}{dx^k} \cdot f(x) \Big|_{f(x)=x^n} = n \cdot (n-1) \times \dots \times (n-k+1) \cdot x^{n-k}$, $n \in \mathbb{N}$. Using limit notation, we have:

$$\underbrace{\lim_{m \rightarrow n^-} \left(\frac{d^m(\vartheta(x))}{dx^m} \Big|_{\vartheta(x)=x^n} \right)}_{\phi(m)} = \frac{d^{n-0}(\vartheta(x))}{dx^{n-0}} = n!$$

By means of expression (5), for $m < n$, $n \in \mathbb{N}$ case, we have the limit:

$$\lim_{m \rightarrow n^-} \left(\frac{\Delta^m(x_i^n)}{(\Delta x)^m} \right) = \lim_{m \rightarrow n^-} \underbrace{\left(\sum_{k=0}^n \binom{n}{k} \frac{(i + (n-k))^m}{(-1)^{k-n}} \right)}_{\omega(m)} = \sum_{k=0}^n \binom{n}{k} \frac{(i + (n-k))^{n-0}}{(-1)^{k-n}} = n!$$

Obviously, if $\phi(m) = n!$, $\omega(m) = n! \rightarrow \phi(m) = \omega(m)$, so we have right to equal

$$\frac{d^n(x^n)}{dx^n} = \frac{\Delta^n(x_i^n)}{(\Delta x)^n} = n!$$

This completes the proof. □

Note, that superscript shows the rank of the difference and does not mean power sign. As we can see, according (4), the values of third rank difference are equal to $3!$ and constant for each i . Going from it, we can to make conclusion of the next power function's $f(x) = x_i^n$ property: $\Delta^n = n!(\Delta x)^n$, where $\Delta x = x_i - x_{i-1}$ and constant for each i . According the values from (4), the "delta" functions are corresponding to next expressions:

$$\Delta^3 = 3!, \Delta_i^2 = j \sum_{m=0}^{i-1} m, \Delta_i^1 = j \sum_{k=0}^{i-1} m + (ji - j)^{-1}$$

Where $j = 3!$, hence, we have:

$$x^3 = (1 + 3! \cdot 0) + (1 + 3! \cdot 0 + 3! \cdot 1) + (1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2) + \dots$$

$$\begin{aligned} & \dots + (1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + \dots + 3! \cdot (x - 1)) \\ x^3 = & x + (x - 0) \cdot 3! \cdot 0 + (x - 1) \cdot 3! \cdot 1 + (x - 2) \cdot 3! \cdot 2 + \dots \\ & \dots + (x - (x - 1)) \cdot 3! \cdot (x - 1) \end{aligned} \quad (6)$$

Let use summation notation for expression (6), we obtain:

$$x^3 = x + j \sum_{k=0}^{x-1} kx - k^2 \quad (7)$$

Or

$$x^3 = \sum_{k=0}^{x-1} \left(1 + j \sum_{m=0}^k m \right) = j \sum_{k=1}^{x-1} \left(kx - k^2 + \frac{x}{j(x-1)} \right) \Bigg|_{x>1}, \quad x \in \mathbb{N} \quad (8)$$

Now we have successful formula, which disperses any natural number x^3 to the numerical series (this example shows only expansion for any number $x \in \mathbb{N}$ to third powers, but this method works for floats numbers also, it depends of start set of numbers, function's form also depends of the chosen set, step Δx between numbers should be constant every time). Going from it, the next section shows change over the function to the range $x \in \mathbb{N}$, $3 < n \in \mathbb{N}$.

2 Change over to higher powers expression

In this section are reviewed the ways to change obtained in previous annex expression (7) to higher powers i.e $n > 3$. Examples are shown for case $\Delta x = 1$, $i = x$, $x \in \mathbb{N}$. By means of Fundamental Theorem of Calculus, we know next (3):

$$x^n = \int_0^x nt^{n-1} dt = \sum_{k=0}^{x-1} \int_k^{k+1} nt^{n-1} dt = \sum_{k=0}^{x-1} (k+1)^n - k^n$$

Expression (7) has the follow view:

$$x^3 = x + j \sum_{m=0}^{x-1} mx - m^2$$

As we can see, iteration limits for (3) and (7) are: $\{0, x - 1\}$, for each expression. Consequently, going from (7), to make the transition to the functions of the form:

$$x^3 = \sum_{k=0}^{x-1} \left((k+1)^3 - j \sum_{m=0}^{k-1} \omega(m) \right)$$

is not possible. Let change the formula (3) by the next way:

$$x^n = \sum_{k=0}^{x-1} (k+1)^n - k^n = \sum_{k=1}^x k^n - (k-1)^n \quad (9)$$

Next, give the formula (9) follow changes:

$$x^3 = \sum_{k=1}^x k^3 - (k-1)^3 \rightarrow x^3 = \sum_{k=1}^x \left(j \sum_{m=0}^{k-1} \omega(m) - (k-1)^3 \right)$$

Going from expression (9), in case of $f(x) = x^3$, we obtain:

$$\begin{aligned} x^3 &= \sum_{k=1}^x k^3 - (k-1)^3 = \sum_{k=1}^x \left(k + j \sum_{m=0}^{k-1} (mk - m^2) - (k-1)^3 \right) \\ &= \sum_{k=1}^x \left(j \sum_{m=0}^{k-1} (mk - m^2) - (k-1)^3 + k \right) \\ &= \sum_{k=0}^{x-1} \left(\left[j \sum_{m=0}^k (m(k+1) - m^2) + (k+1) \right] - k^3 \right) \end{aligned} \quad (10)$$

By means of main property of the powers function $x^n = x^k \cdot x^{n-k}$, from formula (10) for $y = x^n$ we receive:

$$\begin{aligned} x^n &= \sum_{k=1}^x \left(j \sum_{m=0}^{k-1} (mk^{n-2} - m^2 k^{n-3}) - (k-1)^n + k^{n-2} \right) \\ &= \sum_{k=0}^{x-1} \left(\left[j \sum_{m=0}^k (m(k+1)^{n-2} - m^2 k^{n-3}) + (k+1)^{n-2} \right] - k^n \right), \quad \forall (x, n) \in \mathbb{N} \end{aligned} \quad (11)$$

According to the above property, from the expression (8) for $y = x^n$, we derive:

$$x^k \cdot x^{n-k} = j \sum_{k=0}^{x-1} \left(\frac{x^{n-2}}{j(x-1)} + kx^{n-2} - k^2 x^{n-3} \right), \quad 1 < x \in \mathbb{N} \quad (12)$$

For $y = x^n$, $x \geq 0$, $x \in \mathbb{N}$ expression (7) takes the next form:

$$x^k \cdot x^{n-k} = j \sum_{k=0}^{x-1} (kx^{n-2} - k^2 x^{n-3}) + x^{n-2}, \quad x \geq 0 \quad (13)$$

Expression (11) has the next property, as well right for (7, 10, 12, 13):

$$\begin{aligned} x^n &= \sum_{k=1}^x \left(j \sum_{m=0}^{k-1} (mk^{n-2} - m^2 k^{n-3}) - (k-1)^n + k^{n-2} \right) \\ &= \sum_{k=1}^x \left(j \sum_{m=1}^k (mk^{n-2} - m^2 k^{n-3}) - (k-1)^n + k^{n-2} \right) \end{aligned}$$

3 Binomial Theorem Representation

By means of Binomial theorem (1) for $(x + 1)^n$, we have expression (see [4]):

$$(x + 1)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad (14)$$

According expressions (1) and (3), we have the next corresponding:

$$x^n = \sum_{k=0}^{x-1} (k + 1)^n - k^n = \sum_{k=0}^{x-1} \left(\sum_{m=0}^n \binom{n}{m} k^m - k^n \right) \quad (15)$$

Let, going from expression (7), change the binomial expansion for $f(x) = x^3$:

$$\begin{aligned} x^3 &= x + j \sum_{k=0}^{x-1} kx - k^2 = x + j \sum_{k=1}^x kx - k^2 \rightarrow \\ &\rightarrow (x + 1)^3 = (x + 1) + j \sum_{k=0}^x k(x + 1) - k^2 \end{aligned} \quad (16)$$

So, for $f(x) = x^3$, by means of expressions (13) and (14), binomial expansion is next:

$$x^3 = \sum_{k=0}^{x-1} \left(- \sum_{i=0}^{k-1} \left(\sum_{t=0}^3 \left(\binom{3}{t} i^t - i^3 \right) + \left[(k + 1) + j \sum_{m=0}^k m(k + 1) - m^2 \right] \right) \right) \quad (17)$$

Going from it, by means of power function property $x^n = x^k \cdot x^{n-k}$, we can only to multiply by x every product of the series, by this way, for $y = x^n$, from expression (15), we have next changes in binomial expansion:

$$\begin{aligned} x^n &= \sum_{k=0}^{x-1} \left(\sum_{m=0}^n \binom{n}{m} k^m - k^n \right) \\ &= \sum_{k=0}^{x-1} \left(\left\{ - \sum_{i=0}^{k-1} \left(\sum_{t=0}^n \binom{n}{t} i^t - i^n \right) \right\} + \left[(k + 1)^{n-2} + j \sum_{m=0}^k m(k + 1)^{n-2} - m^2 k^{n-3} \right] \right) \\ &= \sum_{k=0}^{x-1} \left(\left\{ - \sum_{i=0}^{k-1} \left(\sum_{t=0}^n \binom{n}{t} i^t - i^n \right) \right\} + \left[(k + 1)^{n-2} + j \sum_{m=1}^{k+1} m(k + 1)^{n-2} - m^2 k^{n-3} \right] \right) \end{aligned}$$

4 e^x Representation

According above method, we have right to present function $y = e^x$ by the follow view (as the exponential function is the infinite sum of powers of x divided by value of factorial according to iteration step, see [5]):

$$e^x = \sum_{m=0}^{\infty} \left(j \sum_{k=1}^x \left(\frac{kx^{m-2} - k^2 x^{m-3} + \frac{x^{m-3}}{j}}{m!} \right) \right) \quad (18)$$

By means of general $y = e^x$ determination (see [5]), we have right to represent, also, next way:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{f^{(n)}(x)} \sum_{i=0}^n \int f^{(n-i)}(x) dx_i \right) \Big|_{f(x)=x^n} \quad (19)$$

5 Difference from Binomial Theorem

To show changes from binomial theorem let use other algorithm for $f(x) = x^3$, $x \in \mathbb{N}$, which finally returns the binomial expansion. The algorithm is also based on Δ -functions from (5). By means of theorem 1, we have:

$$\frac{d^n(x^n)}{dx^n} = \frac{\Delta^n(x_i^n)}{(\Delta x)^n} = n!$$

We have right to integrate the $f^{(3)}(x)$ to $f'(x)$ and represent the x^3 as:

$$x^3 = \sum_{t=0}^{x-1} f'(t)$$

For third derivative we have next equation:

$$d^3 y = \Delta^3 dx^3 \rightarrow f^{(3)}(x) = \Delta^3$$

Let derive the $f''(x)$:

$$\int d(f''(x)) = \Delta^3 \cdot \int dx \rightarrow f''(x) = \Delta^3 x + C_1$$

Let be $C_1 = \Delta_0^2$, so we have:

$$f''(x) = \Delta^3 x + \Delta_0^2$$

First derivative is next:

$$\int d(f'(x)) = \int (\Delta^3 x + \Delta_0^2) dx + C_2(i) = \Delta^3 \frac{x^2}{2} + \Delta_0^2 x + C_2(i)$$

Let calculate the $C_2(i)$ function, basic formula is next:

$$C_k(i) = \Delta_0^{n-k} - \left(\Delta_0^{n-k} - (\Delta_1^{n-k} - f^{(n-k)}(1)) \right) i$$

And for $y = x^3$ equals to:

$$C_2(i) = 1 - (1 - (7 - f'(1)))i = 1 - 3i$$

In case of $i = x$ we obtain the first derivative:

$$f'(x) = 3x^2 + 3x + 1$$

So,

$$x^3 = \sum_{t=0}^{x-1} 3t^2 + 3t + 1$$

and corresponds to binomial expansion. Also, the difference is lay in the iteration limits of the expression (8), the binomial type expansion has the the summation limits $\{0, x - 1\}$ and it's necessary to provide the x -products summation to calculate the number to power, while equation (8) has the limits in range $\{1, x - 1\}$, which shows, that for such type function is necessary lesser by 1 number of iteration steps, then for binomial expansion.

6 Conclusion

In the paper presented a method of expanding the function of the form $y = x^n$, $x \in \mathbb{N}$, $n \in \mathbb{N}$ to the numerical series. The disadvantages of this method are sophisticated form of expression and the complexity of calculating the value of these expressions of the some variables. Advantage of this method is the possibility of the successful application of this method in the solution of some problems in number theory, the theory of series, due to the differences from the common theory, displayed the difference from binomial expansion, presented example for exponential function representation by means of method from section 2. The paper doesn't consist the all combinations of power function representation (by means of the function (11) property and transformation (9)). In the Application 1 (Section 7) are shown Microsoft Visual Basic 6.0 program codes for the most important expressions (by authors' opinion). Future research in this direction could take as result the success polynomial kind expansion.

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7 Application 1. Visual Basic 6.0 Program codes

Expression (7) :

```
j = 6
x = Val(Text1.Text)
n = Val(Text2.Text)
r = 0
For k = 1 To x
r = r + j * (k * x - k2)
Next k
r = r + x
```

Expression (8) :

```
j = 6
x = Val(Text1.Text)
n = Val(Text2.Text)
r = 0
For k = 0 To x - 1 Step 1
For m = 0 To k Step 1
r = r + j * m
Next m
r = r + 1
Next k
```

Expression (10) :

```
j = 6
x = Val(Text1.Text)
n = Val(Text2.Text)
r = 0
For k = 1 To x Step 1
For m = 0 To k - 1 Step 1
r = r + j * (m * k - m2)
Next m
r = r + k - (k - 1)3
Next k
```

Expression (11) :

```
j = 6
x = Val(Text1.Text)
n = Val(Text2.Text)
r = 0
For k = 1 To x Step 1
For m = 0 To k - 1 Step 1
r = r + j * (m * kn-2 - m2 * kn-3)
Next m
r = r + kn-2 - (k - 1)n
```

Next k

e^x Representation (19) :

j = 6

e = 0

x = Val(Text1.Text)

r = Val(Text2.Text)

f = 0

For m = 0 To r Step 1

If m = 0 Then

f = 1

Else

f = f * m

End If

For k = 1 To x

$e = e + ((j * (k * x^{m-2} - k^2 * x^{m-3})) + x^{m-3})/f$

Next k

Next m