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SERIES REPRESENTATION OF POWER FUNCTION

KOLOSOV PETRO

ABSTRACT. In this paper we discuss a problem of generalization of binomial distributed triangle, that is sequence A287326 in OEIS. The main property of A287326 that it returns a perfect cube n as sum of n -th row terms over k , $0 \leq k \leq n-1$ or $1 \leq k \leq n$, by means of its symmetry. In this paper we have derived a similar triangles in order to receive powers $m = 5, 7$ as row items sum and generalized obtained results in order to receive every odd-powered monomial n^{2m+1} , $m \geq 0$ as sum of row terms of corresponding triangle.

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1. STRUCTURE OF THE MANUSCRIPT

The problem of finding expansions of monomials, binomials, trinomials, etc. is classical and a lot of theorems have been found, the most prominent examples are Binomial Theorem [2], Multinomial theorem, Wozpitsky Identity [30], Stirling numbers of second kind identity, etc. In this paper we try to solve the classical problem of finding expansions of monomials. We start from binomial distributed triangle A287326 [11] in OEIS. The main property of A287326 that it returns a perfect cube n as n -th row sum, starting from $0, \dots, n-1$ or from $1, \dots, n$ by means of its symmetry. Therefore, the following question stated:

- Can we find similar to A287326 triangles in order to receive monomial n^t , $t > 3$ as sum of row terms? In other words, can A287326 be generalized in order to receive monomial n^t , $t > 3$ as sum of row terms?

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Finding an analogs for $t = 5, 7$ in section 3, we answer to above questions positively. Could this process be continued for each $t = 1, 3, 5, 7, \dots$ similarly? Positive answer to this question is given by theorem (3.29).

2. INTRODUCTION

Let describe the derivation of the sequence A287326 in OEIS. Sequence A287326 returns the perfect cube n as row sum over k , $0 \leq k \leq n-1$, as well as sum over $1 \leq k \leq n$, by means of its symmetry. First, consider a difference table of perfect cubes ([4], eq. 7)

(2.1)

n	$\Delta^0(n^3)$	$\Delta^1(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	6
6	216	127	42	6
7	343	169	48	6
8	512	217	54	
9	729	271		
10	1000			

Table 1: Difference table of perfect cubes n , $0 \leq n \leq 10$ up to 3rd order.

Reviewing above table, we have noticed that

$$\begin{aligned}
 (2.2) \quad \Delta(0^3) &= 1 + 6 \cdot 0 = 6\binom{1}{2} + \binom{1}{0} \\
 \Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 = 6\binom{2}{2} + \binom{2}{0} \\
 \Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 = 6\binom{3}{2} + \binom{3}{0} \\
 \Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 = 6\binom{4}{2} + \binom{4}{0} \\
 &\vdots \\
 \Delta(n^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \dots + 6 \cdot n = 6\binom{n+1}{2} + \binom{n+1}{0}
 \end{aligned}$$

Above difference identity is closely related to Faulhaber's sum of cubes, where $n^3 = 6\binom{n+1}{3} + \binom{n+1}{1}$, see ([21], p. 9). Note that $\Delta^2(n^3)$ could be found similarly using above identity $\Delta^2(n^3) = 6\binom{n+1}{3-2} + \binom{n+1}{1-2}$.

Property 2.3. (Generalized finite difference of power using Faulhaber's formula). Consider the identities, ([21], p. 9).

$$\begin{cases}
 n^1 = \binom{n}{1} \\
 n^3 = 6\binom{n+1}{3} + \binom{n}{1} \\
 n^5 = 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1}
 \end{cases}$$

We can find the first order finite difference of odd power as decreasing the variable of corresponding binomial coefficients by 1, for example

$$\begin{cases} \Delta n^1 = \binom{n}{0} \\ \Delta n^3 = 6\binom{n+1}{2} + \binom{n}{0} \\ \Delta n^5 = 120\binom{n+2}{4} + 30\binom{n+1}{2} + \binom{n}{0} \end{cases}$$

Continue similarly, we can express each difference of order $t \geq 1$. The coefficients $\{1, 6, 1, 120, 30, 1\}$ in above identities are generated by

$$(2.4) \quad V_{n,k} = \frac{1}{r} \sum_{j=0}^r (-1)^j \binom{2r}{j} (r-j)^{2n},$$

where $r = n - k + 1$, this formula was provided by Peter Luschny in [27]. Therefore, for every odd $t > 0$ and $m \geq 0$, we have

$$\Delta^t n^{2m+1} = \sum_{\substack{0 \leq k \leq m \\ l \leq 2(m-k)+1-t \\ l \text{ is even}}} V_{m,k} \binom{n+m-k}{l}, \text{ if } t > 0 \text{ and odd}$$

Let be $m \geq 0$, $t > 1$ and even, then

$$\Delta^t n^{2m+1} = \sum_{\substack{0 \leq k \leq m \\ l \leq 2(m-k)+1-t \\ l \text{ is odd}}} V_{m,k} \binom{n+m-k}{l}, \text{ if } t > 1 \text{ and even}$$

Let show finite differences, set $m \geq 1$, $t > 1$, then we have finite difference identity

$$\Delta^t n^{2m} = \sum_{\substack{0 \leq k \leq m \\ l \leq 2(m-k)+1-t \\ l \text{ is even}}} \frac{1}{n} V_{m,k} \binom{n+m-k}{l}, \text{ if } t > 0 \text{ and odd}$$

And

$$\Delta^t n^{2m} = \sum_{\substack{0 \leq k \leq m \\ l \leq 2(m-k)+1-t \\ l \text{ is odd}}} \frac{1}{n} V_{m,k} \binom{n+m-k}{l}, \text{ if } t > 1 \text{ and even}$$

By the identity $\sum_{k=0}^{n-1} \Delta n^m = n^m$, we have right to represent perfect cube n as

$$(2.5) \quad n^3 = 6\binom{1}{2} + \binom{1}{0} + 6\binom{2}{2} + \binom{2}{0} + 6\binom{3}{2} + \binom{3}{0} + \cdots + 6\binom{n+1}{2} + \binom{n+1}{0}$$

Let rewrite it again and display every binomial coefficient as summation $\binom{n+1}{2} = 1 + 2 + \cdots + n$, then

$$n^3 = (1 + 6 \cdot 0) + (1 + 6 \cdot 0 + 6 \cdot 1) + \cdots + (1 + 6 \cdot 0 + \cdots + 6 \cdot (n-1))$$

Particularizing above expression, we get

$$(2.6) \quad n^3 = n + (n-0) \cdot 6 \cdot 0 + (n-1) \cdot 6 \cdot 1 + \cdots + (n-(n-1)) \cdot 6 \cdot (n-1)$$

Provided that n is natural. Now we apply a compact sigma notation on (2.6), thus

$$(2.7) \quad n^3 = n + \sum_{1 \leq k \leq n} 6k(n-k)$$

As sum $\sum_{1 \leq k \leq n} 6k(n-k)$ consists of n terms, we have right to move n in (2.7) under sigma notation, we get

$$(2.8) \quad n^3 = \sum_{1 \leq k \leq n} 6k(n-k) + 1$$

Property 2.9. (*Proof of symmetry*). Let be a sets $A(n) := \{1, 2, \dots, n\}$, $B(n) := \{0, 1, \dots, n\}$, $C(n) := \{0, 1, \dots, n-1\}$, let be expression (2.8) defined as

$$M(n, C(n)) \stackrel{\text{def}}{=} \sum_{k \in C(n)} 6k(n-k) + 1$$

where x is natural-valued variable and $C(n)$ is iteration set of (2.8), then we have equality

$$(2.10) \quad M(n, A(n)) = M(n, C(n))$$

Let review and define expression (2.6) as

$$U(n, C(n)) \stackrel{\text{def}}{=} n + 6 \cdot \sum_{k \in C(n)} k(n-k)$$

then

$$(2.11) \quad U(n, A(n)) = U(n, B(n)) = U(n, C(n))$$

Other words, changing of iteration sets of (2.6) and (2.8) by $A(n)$, $B(n)$, $C(n)$ and $A(n)$, $C(n)$, respectively, doesn't change resulting value for each natural x .

Proof. Let be a plot $y(n, k) = 6k(n-k) + 1$, $k \in \mathbb{R}$, $0 \leq k \leq 10$, given $n = 10$

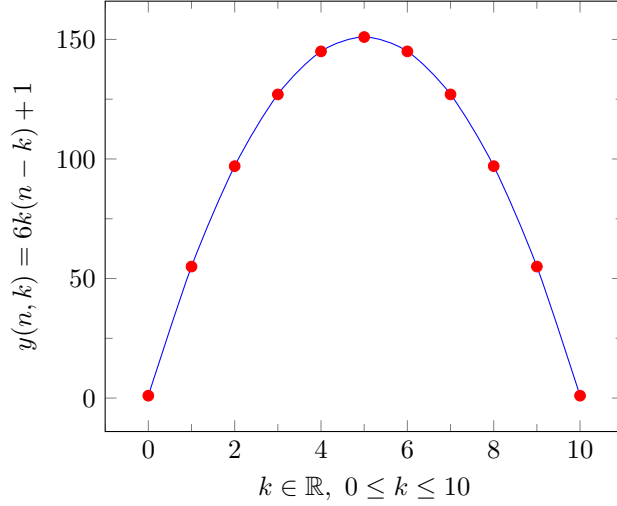


Figure 2. Plot of $6k(n-k) + 1$, $k \in \mathbb{R}$, $0 \leq k \leq n$, where $n = 10$.

Obviously, being a parabolic function, it's symmetrical over $\frac{n}{2}$, hence equivalent $M(n, A(n)) = M(n, C(n))$ follows. Reviewing (2.6) and denote $u(n, k) = kn - k^2$, we can conclude, that $u(n, 0) = u(n, n) = 0$, then equality of $U(n, A(n)) = U(n, B(n)) = U(n, C(n))$ immediately follows. This completes the proof. \square

Review above property (2.9). Let be an example of triangle built using

Definition 2.12. For every $n \geq 0$

$$(2.13) \quad L_1(n, k) \stackrel{\text{def}}{=} 6k(n - k) + 1, \quad 0 \leq k \leq n$$

over n from 0 to $n = 4$, where n denotes corresponding row and k shows the item of row n .

$$(2.14) \quad \begin{array}{rcccccc} \text{Row 0:} & & & & & 1 \\ \text{Row 1:} & & & & 1 & 1 \\ \text{Row 2:} & & & 1 & 7 & 1 \\ \text{Row 3:} & & 1 & 13 & 13 & 1 \\ \text{Row 4:} & 1 & 19 & 25 & 19 & 1 \end{array}$$

Figure 3. Triangle generated by $L_1(n, k)$ from 0 to $n = 4$, sequence A287326 in OEIS, [11].

Note that n -th row sum of Triangle (2.14) over $0 \leq k \leq n - 1$ returns perfect cube n . We can see that each row with respect to variable $n = 0, 1, 2, 3, 4, \dots$, has Binomial distribution of row terms. One could compare Triangle (2.14) with Pascal's triangle [1], [12]

$$\begin{array}{rcccccc} \text{Row 0:} & & & & & 1 \\ \text{Row 1:} & & & & 1 & 1 \\ \text{Row 2:} & & & 1 & 2 & 1 \\ \text{Row 3:} & & 1 & 3 & 3 & 1 \\ \text{Row 4:} & 1 & 4 & 6 & 4 & 1 \end{array}$$

Figure 4. Pascal's triangle read by rows, sequence A007318 in OEIS, [1].

Let us approach to show a few properties of triangle (2.14) and $L_1(n, k)$.

Properties 2.15. *Properties of triangle (2.14).*

- (1) *Summation of items $L_1(n, k)$ of n -th row of triangle (2.14) over k from 0 to $n - 1$ returns perfect cube $n \geq 0$ as follows*

$$(2.16) \quad \sum_{1 \leq k \leq n} L_1(n, k) = n^3$$

- (2) *Relation between $\alpha_{0,n}$ and $\alpha_{1,n}$*

$$\alpha_{0,n+1} = \alpha_{1,n}, \quad n \geq 1$$

- (3) *First item of each row's number corresponding to central polygonal numbers sequence $a(n) = \frac{n^2+n+2}{2}$ (sequence A000124 in OEIS, [13]) returns finite difference of consequent perfect cubes. For example, let be a k -th row of triangle (2.14), such that $k = \frac{n^2+n+2}{2}$, $n = 0, 1, 2, \dots$, then item*

$$(2.17) \quad L_1\left(\frac{n^2 + n + 2}{2}, 1\right) = (n + 1)^3 - n^3$$

- (4) *Items of (2.14) have Binomial distribution over rows.*

(5) *Linear recurrence, for every k and $n > 0$*

$$(2.18) \quad 2L_1(n, k) = L_1(n + 1, k) + L_1(n - 1, k)$$

This linear recurrence is direct result of second order binomial transform of $L_1(n, k)$ over n .

(6) *Linear recurrence, for each $n > k$*

$$(2.19) \quad 2L_1(n, k) = L_1(2n - k, k) + L_1(2n - k, 0)$$

(7) *From (1.24) for every $n \geq 0$ follows*

$$(2.20) \quad \sum_{1 \leq k \leq n} L_1(n, k) = \sum_{1 \leq k \leq n} L_1\left(\frac{n^2 + n + 2}{2}, 1\right) = n^3$$

(8) *Triangle (2.14) is symmetric, i.e*

$$(2.21) \quad L_1(n, k) = L_1(n, n - k)$$

Property 2.22. *(Generalized binomial series by means of identity (2.16). Let review identity (2.16) in sense of*

$$\sum_{1 \leq k \leq t} L_1(n, k) = \alpha_{0,t}n - \beta_{0,t}$$

By property (2.9) we rewrite above expression as

$$\sum_{0 \leq k \leq t} L_1(n, k) = \alpha_{1,t}n - \beta_{1,t}$$

where subscripts $0, t$ and $1, t$ denote the ranges of summation, respectively. Running over $t > 0$ above identities produce sets of coefficients $\{\alpha_{0,t}\}_t$, $\{\beta_{0,t}\}_t$, $\{\alpha_{1,t}\}_t$ and $\{\beta_{1,t}\}_t$. Below table shows initial terms of these sequences

t	$\alpha_{0,t}$	$\beta_{0,t}$	$\alpha_{1,t}$	$\beta_{1,t}$
1	1	0	6	5
2	6	4	18	28
3	18	27	36	81
4	36	80	60	176
5	60	175	90	325
6	90	324	126	540
7	126	539	168	833
8	168	832	216	1216
9	216	1215	270	1701
10	270	1700	330	2300

Table 5. Array of coefficients $\alpha_{\overline{0,1},n}$, $\beta_{\overline{0,1},n}$ given $n = 1, \dots, 10$.

Therefore, perfect cube n could be rewritten as binomials of the form

$$n^3 = \begin{cases} \alpha_{0,n-1}n - \beta_{0,n-1}, & \text{if } t = n - 1; \\ \alpha_{1,n}n - \beta_{1,n}, & \text{if } t = n \end{cases}$$

By the main power property, for every $m \in \mathbb{N}$

$$n^m = \begin{cases} \alpha_{0,n-1}n^{m-2} - \beta_{0,n-1}n^{m-3} \\ \alpha_{1,n}n^{m-2} - \beta_{1,n}n^{m-3} \end{cases}$$

We denote above equation as

$$n^m = \alpha_{\overline{0,1,n-1,n}} n^{m-2} - \beta_{\overline{0,1,n-1,n}} n^{m-3}$$

Let rewrite the right part of above expression regarding to itself as recursion

$$\begin{aligned} n^m &= \alpha_{\overline{0,1,n-1,n}} (\alpha_{\overline{0,1,n-1,n}} n^{m-4} - \beta_{\overline{0,1,n-1,n}} n^{m-5}) \\ &\quad - \beta_{\overline{0,1,n-1,n}} (\alpha_{\overline{0,1,n-1,n}} n^{m-5} - \beta_{\overline{0,1,n-1,n}} n^{m-6}) \\ &= \alpha_{\overline{0,1,n-1,n}}^2 n^{m-4} - 2\alpha_{\overline{0,1,n-1,n}} \beta_{\overline{0,1,n-1,n}} n^{m-5} + \beta_{\overline{0,1,n-1,n}}^2 n^{m-6} \end{aligned}$$

We can observe corresponding binomial coefficient present before each $\alpha_{\overline{0,1,n-1,n}}$ times $\beta_{\overline{0,1,n-1,n}}$. Continuous j -times recursion gives

$$n^m = \sum_{k \geq 0}^{\infty} (-1)^k \binom{j}{k} \alpha_{\overline{0,1,n-1,n}}^{j-k} \beta_{\overline{0,1,n-1,n}} n^{m-2j-k}, \quad j \geq 0$$

Sequences $\alpha_{1,t}$, $\alpha_{0,t>1}$ are generated by $3n^2 + 3n$, sequence A028896 in OEIS, [23]. Sequence $\beta_{1,t}$ is generated by $2n^3 + 3n^2$, sequence A275709 in OEIS, [20].

In this section we have reached binomial distributed triangle (2.14), such that perfect cube n could be found as sum of n -th row terms of (2.14). Therefore, the follow question is stated

Question 2.23. Can we find similar to A287326 triangles in order to receive monomial n^t , $t > 3$ as sum of row terms? Is it exist $L_v(n, k)$, $v \neq 1$, such that

$$n^t \equiv \sum_{1 \leq k \leq n} L_v(n, k), \quad v \neq t ?$$

3. GENERALIZATION OF SEQUENCE A287326

In order to get analogs of Triangle (2.14) one should solve a system of equations, where unknowns are coefficients of polynomial and variable of polynomial is $k(n-k)$. Let show a triangle generated by $L_2(n, k)$, such that sum of n -th row terms returns n^5 .

Example 3.1. We suspect that n -th row of triangle is generated by

$$(3.2) \quad L_2(n, k) = A_{2,2}(n-k)^2 k^2 + A_{2,1}(n-k)k + A_{2,0}$$

where $A_{2,2}, A_{2,1}, A_{2,0}$ are unknown coefficients and $n \geq 0$, $0 \leq k \leq n$. Assume that for every $n \geq 0$, $m \geq 0$ holds

$$(3.3) \quad \sum_{1 \leq k \leq n} L_2(n, k) \equiv n^5$$

In more explicit view

$$\begin{aligned}
(3.4) \quad & A_{2,2} \sum_{1 \leq k \leq n} k^2(n-k)^2 + A_{2,1} \sum_{1 \leq k \leq n} k(n-k) + A_{2,0}n \\
&= A_{2,2} \sum_{1 \leq k \leq n} k^2(n^2 - 2nk + k^2) + A_{2,1} \sum_{1 \leq k \leq n} kn - k^2 + A_{2,0}n \\
&= A_{2,2} \sum_{1 \leq k \leq n} k^2n^2 - 2nk^3 + k^4 + A_{2,1} \sum_{1 \leq k \leq n} kn - k^2 + A_{2,0}n \\
&= A_{2,2}n^2 \sum_{1 \leq k \leq n} k^2 - 2A_{2,2}n \sum_{1 \leq k \leq n} k^3 + A_{2,2} \sum_{1 \leq k \leq n} k^4 + A_{2,1}n \sum_{1 \leq k \leq n} k \\
&\quad - A_{2,1} \sum_{1 \leq k \leq n} k^2 + A_{2,0}n
\end{aligned}$$

Thus, we have received expression containing sums of powers of successive natural numbers, where powers are $\{1, 2, 3, 4\}$. By the Faulhaber's formula [7], the following identities hold

$$(3.5) \quad \sum_{1 \leq k \leq n} k = \frac{n^2 + n}{2},$$

$$(3.6) \quad \sum_{1 \leq k \leq n} k^2 = \frac{2n^3 + 3n^2 + n}{6},$$

$$(3.7) \quad \sum_{1 \leq k \leq n} k^3 = \frac{n^4 + 2n^3 + n^2}{4},$$

$$(3.8) \quad \sum_{1 \leq k \leq n} k^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}.$$

Now we substitute above identities to (3.4), respectively, we get

$$\begin{aligned}
& A_{2,2}n^2 \frac{2n^3 + 3n^2 + n}{6} - 2A_{2,2}n \frac{n^4 + 2n^3 + n^2}{4} + A_{2,2} \frac{6n^5 + 15n^4 + 10n^3 - n}{30} \\
& + A_{2,1}n \frac{n^2 + n}{2} - A_{2,1} \frac{2n^3 + 3n^2 + n}{6} + A_{2,0}n
\end{aligned}$$

Particularizing the elements of above expression and moving them under the common divisor, we get

$$(3.9) \quad \frac{A_{2,2}n^5 - A_{2,2}n + 30A_{2,0}}{30} + A_{2,1} \left(\frac{n^3 - n}{6} \right)$$

We have to remember that expression (3.9) is the left side of the input equation (2.2). Therefore,

$$(3.10) \quad \frac{A_{2,2}n^5 - A_{2,2}n + 30A_{2,0}}{30} + A_{2,1} \left(\frac{n^3 - n}{6} \right) = n^5, \quad n \geq 0$$

In order to satisfy (3.10) for each natural n , coefficients $A_{2,0}, A_{2,1}, A_{2,2}$ should be a solutions of following system of equations

$$\begin{cases} \frac{1}{30}A_{2,2} & = 1 \\ A_{2,1} & = 1 \\ 30A_{2,0} - A_{2,2} & = 0 \end{cases}$$

The only solution of above system is $A_{2,2} = 30$, $A_{2,1} = 0$, $A_{2,0} = 1$. Hereby, $L_2(n, k)$ takes the form

$$(3.11) \quad L_2(n, k) = 30k^2(n - k)^2 + 1$$

And for each natural n holds

$$(3.12) \quad \sum_{1 \leq k \leq n} 30k^2(n - k)^2 + 1 = n^5$$

Let show initial rows of triangle built by $L_2(n, k)$

$$(3.13) \quad \begin{array}{ccccccc} & & & & 1 & & \\ & & & & & & 1 \\ & & & 1 & & & 1 \\ & & & & 31 & & 1 \\ & & 1 & & 121 & & 121 & & 1 \\ & & & 1 & & 271 & & 481 & & 271 & & 1 \\ & & & & 1 & & 480 & & 1081 & & 1081 & & 481 & & 1 \\ & & & & & & & & & & & & & & & \dots \end{array}$$

Figure 6. Triangle generated by $L_2(n, k)$, $0 \leq k \leq n$, sequence A300656 in OEIS, [15].

Similarly, finding the coefficients $A_{3,0}, A_{3,1}, A_{3,2}, A_{3,3}$ in

$$(3.14) \quad L_3(n, k) = A_{3,3}k^3(n - k)^3 + A_{3,2}k^2(n - k)^2 + A_{3,1}k(n - k) + A_{3,0}$$

we get $A_{3,3} = 140$, $A_{3,2} = -14$, $A_{3,1} = 0$, $A_{3,0} = 1$, therefore, for each $n \geq 0$ holds

$$(3.15) \quad \sum_{1 \leq k \leq n} 140k^3(n - k)^3 - 14k^2(n - k)^2 + 1 = n^7$$

Below we show a few initial rows of triangle built by $L_3(n, k)$

$$(3.16) \quad \begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & & & 1 & & & & 1 \\ & & & & & & & & & & 127 & & & & 1 \\ & & & & 1 & & 1093 & & 1093 & & & & 1 \\ & & & & & 1 & & 3793 & & 8905 & & 3793 & & 1 \\ & & & & & & 1 & & 8905 & & 30157 & & 30157 & & 8905 & & 1 \\ & & & & & & & & & & & & & & & & \dots \end{array}$$

Figure 7. Triangle generated by $L_3(n, k)$, $0 \leq k \leq n$, sequence A300785 in OEIS, [16].

We assume now that generalization of A287326 holds for odd powers only. To generalize our sequences A287326, A300656, A300785 for every odd power $2m+1$, $m = 0, 1, 2, \dots$ we have to review the generating functions of corresponding sequences, that is

$$(3.17) \quad \sum_{1 \leq k \leq n} \sum_{0 \leq j \leq m} A_{m,j} k^j (n - k)^j = n^{2m+1}, \quad m = 1, 2, 3$$

Where $A_{m,j}$ are unknown coefficients of polynomials (2.1) and (2.13).

Definition 3.18. Let define the part of (2.1) as

$$\sum_{0 \leq j \leq m} A_{m,j} k^j (n-k)^j \stackrel{\text{def}}{=} L_m(n, k) \stackrel{\text{def}}{=} \sum_{0 \leq j \leq m} A_{m,j} T^j(n, k)$$

where

$$T(n, k) \stackrel{\text{def}}{=} k(n-k).$$

Note that $L_m(n, k)$ is generalization of definitions (2.12) for $m = 1$ and (3.11) for $m = 2$, respectively.

For example, generating functions of sequences A287326, A300656, A300785 are

$$\begin{cases} L_1(n, k) = 1 + 6k(n-k), & \text{for A287326} \\ L_2(n, k) = 1 - 0k(n-k) + 30k^2(n-k)^2, & \text{for A300656} \\ L_3(n, k) = 1 - 14k(n-k) - 0k^2(n-k)^2 + 140k^3(n-k)^3, & \text{for A300785} \end{cases}$$

Where coefficients $A_{m,j}$, for $m = 1, 2, 3$ are $\{A_{1,j}\}_{j=0}^1 = \{1, 6\}$, $\{A_{2,j}\}_{j=0}^2 = \{1, 0, 30\}$, $\{A_{3,j}\}_{j=0}^3 = \{1, -14, 0, 140\}$ in definitions of generating functions of A287326, A300656, A300785, respectively. To generalize above result in order to receive monomial n^{2m+1} as $\sum_{1 \leq k \leq n} L_m(n, k) = n^{2m+1}$, $m = 0, 1, 2, \dots$ one has to solve the system of equations. Complete set of coefficients $\{A_{m,0}, \dots, A_{m,m}\}$ such that $\sum_{1 \leq k \leq n} L_m(n, k) = n^{2m+1}$, $m \geq 0$ holds can be found solving follow system of equations

$$(3.19) \quad \begin{cases} L_m(1, 0) = 1^{2m+1} \\ L_m(2, 0) + L_m(2, 1) = 2^{2m+1} \\ L_m(3, 0) + L_m(3, 1) + L_m(3, 2) = 3^{2m+1} \\ \vdots \\ L_m(r, 0) + L_m(r, 1) + \dots + L_m(r, r-1) = r^{2m+1}, \quad r \geq m \end{cases}$$

List of solutions¹ of system (2.4) is split and assigned to OEIS under the numbers A302971 (numerators of $A_{m,j}$) and A304042 (denominators of $A_{m,j}$). To reach recurrent formula of $A_{m,j}$, first let fix the unused values $A_{m,j} = 0$, for $j < 0$ or $j > m$, so we don't need to care about the summation range for j , then by expanding $(n-k)^j$ and using Faulhaber's formula [7], we get

$$(3.20) \quad \begin{aligned} \sum_{k=0}^{n-1} (n-k)^j k^j &= \sum_{k=0}^{n-1} \sum_i^{\infty} \binom{j}{i} n^{j-i} (-1)^i k^{i+j} \\ &= \sum_i^{\infty} \binom{j}{i} n^{j-i} \frac{(-1)^i}{i+j+1} \left[\sum_t^{\infty} \binom{i+j+1}{t} B_t n^{i+j+1-t} - B_{i+j+1} \right] \\ &= \underbrace{\sum_{i,t}^{\infty} \binom{j}{i} \frac{(-1)^i}{i+j+1} \binom{i+j+1}{t} B_t n^{2j+1-t}}_{(\star)} - \underbrace{\sum_i^{\infty} \binom{j}{i} \frac{(-1)^i}{i+j+1} B_{i+j+1} n^{j-i}}_{(\diamond)} \end{aligned}$$

¹One can produce a list of solutions of system (2.4) up to $t = 11$ using Mathematica code solutions_system_2.4.txt, [24].

where B_t are Bernoulli numbers [14]. Now, we notice that

$$(3.21) \quad \sum_i^{\infty} \binom{j}{i} \frac{(-1)^i}{i+j+1} \binom{i+j+1}{t} = \begin{cases} \frac{1}{(2j+1)\binom{2j}{j}}, & \text{if } t = 0; \\ \frac{(-1)^j}{t} \binom{j}{2j-t+1}, & \text{if } t > 0 \end{cases}$$

In particular, the last sum is zero for $0 < t \leq j$. Now we substitute the terms from right part of (3.25) into (\star) , thus

$$\begin{aligned} \sum_{i,t}^{\infty} \binom{j}{i} \frac{(-1)^i}{i+j+1} \binom{i+j+1}{t} B_t n^{2j+1-t} &= \frac{1}{(2j+1)\binom{2j}{j}} \\ &+ \sum_{t>0} \frac{(-1)^j}{t} \binom{j}{2j-t+1} B_t n^{2j+1-t} \end{aligned}$$

Therefore, (3.24) takes the form

$$\begin{aligned} (*) \quad \sum_{k=0}^{n-1} (n-k)^j k^j &= \underbrace{\frac{1}{(2j+1)\binom{2j}{j}} + \sum_{t>0} \frac{(-1)^j}{t} \binom{j}{2j-t+1} B_t n^{2j+1-t}}_{(*)} \\ &- \underbrace{\sum_i^{\infty} \binom{j}{i} \frac{(-1)^i}{i+j+1} B_{i+j+1} n^{j-i}}_{(\diamond)} \end{aligned}$$

Now, we keep our attention to $(*)$ and we have to remember that if the sum over some variable i contains $\binom{j}{i}$, then instead of limiting its summation range to $i = 0, \dots, j$, we can let $i = -\infty, \dots, +\infty$ since $\binom{j}{i} = 0$ for i outside the range $i = 0, \dots, j$ (i.e., when $i < 0$ or $i > j$). It's much easier to review such sum as summing from $-\infty$ to $+\infty$ (unless specified otherwise), where only a finite number of terms are nonzero, this fact is discussed in [28] as well. To combine or cancel identical terms across the two sums in $(*)$ more easily, we introduce $\ell = 2j + 1 - t$ to (\star) and $\ell = j - i$ to (\diamond) , we get

$$\begin{aligned} (3.22) \quad \sum_{k=0}^{n-1} (n-k)^j k^j &= \frac{1}{(2j+1)\binom{2j}{j}} n^{2j+1} + \sum_{\ell=-\infty}^{\infty} \frac{(-1)^j}{2j+1-\ell} \binom{j}{\ell} B_{2j+1-\ell} n^{\ell} \\ &- \sum_{\ell=-\infty}^{\infty} \binom{j}{\ell} \frac{(-1)^{j-\ell}}{2j+1-\ell} B_{2j+1-\ell} n^{\ell} \\ &= \frac{1}{(2j+1)\binom{2j}{j}} n^{2j+1} + 2 \sum_{\text{odd } \ell}^{\infty} \frac{(-1)^j}{2j+1-\ell} \binom{j}{\ell} B_{2j+1-\ell} n^{\ell}. \end{aligned}$$

Now, using the definition of $A_{m,j}$, we obtain the following identity for polynomials in n

$$\begin{aligned} (3.23) \quad \sum_j^{\infty} A_{m,j} \frac{1}{(2j+1)\binom{2j}{j}} n^{2j+1} &+ 2 \sum_{j, \text{ odd } \ell}^{\infty} A_{m,j} \binom{j}{\ell} \frac{(-1)^j}{2j+1-\ell} B_{2j+1-\ell} n^{\ell} \\ &\equiv n^{2m+1}. \end{aligned}$$

Taking the coefficient of n^{2m+1} in above expression, we get $A_{m,m} = (2m+1)\binom{2m}{m}$, and taking the coefficient of x^{2d+1} for an integer d in the range $m/2 \leq d < m$ we

get $A_{m,d} = 0$. Taking the coefficient of n^{2d+1} in (2.8) for $m/4 \leq d < m/2$, we get

$$(3.24) \quad A_{m,d} \frac{1}{(2d+1) \binom{2d}{d}} + 2(2m+1) \binom{2m}{m} \binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0,$$

i.e

$$(3.25) \quad A_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}.$$

Continue similarly, we can express $A_{m,j}$ for each integer j in range $m/2^{s+1} \leq j < m/2^s$ (iterating consecutively $s = 1, 2, \dots$) via previously determined values of $A_{m,d}$, $d < j$ as follows

$$(3.26) \quad A_{m,j} = (2j+1) \binom{2j}{j} \sum_{d=2j+1}^m A_{m,d} \binom{d}{2j+1} \frac{(-1)^{d-1}}{d-j} B_{2d-2j}.$$

The same formula holds also for $m = 0$. Note that in above sum m have to be $m \geq 2j+1$ to return nonzero term $A_{m,j}$.

Definition 3.27. We define here a generalized sequence of coefficients $A_{m,j}$, such that $\sum_{k=0}^{n-1} \sum_{j=0}^m A_{m,j} (n-k)^j k^j = n^{2m+1}$, $n \geq 0$, $m = 0, 1, 2, \dots$

$$A_{m,j} := \begin{cases} 0, & \text{if } j < 0 \text{ or } j > m \\ (2j+1) \binom{2j}{j} \sum_{d=2j+1}^m A_{m,d} \binom{d}{2j+1} \frac{(-1)^{d-1}}{d-j} B_{2d-2j}, & \text{if } 0 \leq j < m \\ (2j+1) \binom{2j}{j}, & \text{if } j = m \end{cases}$$

Five initial rows of triangle generated by $A_{m,j}$ are

$$(3.28) \quad \begin{array}{cccccc} & & & & & 1 \\ & & & & & & 1 & 6 \\ & & & & & 1 & 0 & 30 \\ & & & & 1 & -14 & 0 & 140 \\ & & 1 & -120 & 0 & 0 & 630 \\ & 1 & -1386 & 660 & 0 & 0 & 2772 \\ & \dots & & & & & & \end{array}$$

Figure 8. Triangle generated by $A_{m,j}$, $0 \leq j \leq m$, sequences A302971 (numerators of $A_{m,j}$) and A304042 (denominators of $A_{m,j}$).

Note that starting from row $m \geq 11$ the terms of Triangle (3.28) consist fractional numbers, for example, $A_{11,1} = 800361655623,6$. One can find complete list of the numerators and denominators of $A_{m,j}$ in OEIS under the identifiers A302971 and A304042, respectively, see [17],[18]. To verify the terms that definition (3.27) produces one should refer to Mathematica code². Hereby, let be theorem

Theorem 3.29. For every positive integers n and m holds

$$\sum_{1 \leq k \leq n} \sum_j A_{m,j} k^j (n-k)^j = n^{2m+1}$$

²def.2.12.txt, [25]

One can verify results concerning above theorem via Mathematica code³. Therefore, theorem (3.29) answers to the question question (2.23) positively, since for every $m \geq 0$ exists a triangle, generated by $\sum_j A_{m,j} k^j (n-k)^j = n^{2m+1}$, such that odd power n^{2m+1} can be reached as sum of n -th row of corresponding triangle over k and A287326 is partial case for $m = 1$.

3.1. Properties of $L_m(n, k)$ and $A_{m,j}$. Here we show a few properties of definition $L_m(n, k)$, some of them correlates with properties of partial case $L_1(n, k)$ in 2.15.

(1) Sum of $A_{m,j}$, $m \geq 0$ gives

$$\sum_{j \geq 0} A_{m,j} = 2^{2m+1} - 1$$

(2) Similarly to particular property (1.28), items of $\{L_m(n, k)\}_{k=0}^n$, $m \geq 0$ is symmetric, i.e

$$L_m(n, k) = L_m(n, n - k), \quad n \geq 0, \quad 0 \leq k \leq n$$

(3) From (2) for every $n \geq 0$, $m \geq 0$ immediately follows

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n, k) = \sum_{0 \leq k \leq n-1} \sum_{j \geq 0} A_{m,j} T^j(n, k)$$

(4) $A_{m,m}$, $m = 0, 1, 2, \dots$ are terms of A002457.

(5) For every $m \geq 0$

$$A_{m,0} = 1$$

(6) For each $m \geq 0$

$$\sum_{j \geq 0} A_{m,j} = \sum_{j \geq 0} \binom{2m+1}{j} - 1$$

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n, k) = n + \sum_{2 \leq k \leq n} \sum_{j \geq 1} A_{m,j} T^j(n, k)$$

(7) For each even power $2m$, $m \geq 0$ and $n \in \mathbb{Z}$ we have

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} \frac{1}{n} A_{m,j} T^j(n, k) = n^{2m}$$

(8) Forward and inverse summation identity

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n, k) = \sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,m-j} T^{m-j}(n, k)$$

³expression.2.1.txt, [26].

3.2. **Example of use.** Recall existing pattern

$$(3.30) \quad \begin{array}{cccccc} & & & & & 1 \\ & & & & & 1 & 6 \\ & & & & 1 & 0 & 30 \\ & & 1 & & -14 & 0 & 140 \\ & 1 & & -120 & 0 & 0 & 630 \\ 1 & & -1386 & & 660 & 0 & 0 & 2772 \\ & \dots & & & & & & \end{array}$$

Figure 9. Triangle generated by $A_{m,j}$, $0 \leq j \leq m$.

By received formula $\sum_{k=0}^{n-1} \sum_{j \geq 0} A_{m,j} T^j(n, k) = n^{2m+1}$ each line of above triangle being multiplied by $T^j(n, k)$ and summed up to n or $n - 1$ over k from 0 or 1, respectively, will result odd power of n , depending on which row of $A_{m,j}$, $0 \leq j \leq m$ is applied. Consider the case $n = 3$, $m = 2$, we introduce triangle built using $T(n, k)$, $1 \leq k \leq n$,

$$(3.31) \quad \begin{array}{cccc} & & & 0 \\ & & & 1 & 0 \\ & & 2 & 2 & 0 \\ & 3 & 4 & 3 & 0 \end{array}$$

Figure 10. Triangle generated by $T(n, k)$, $1 \leq k \leq n$, sequence A094053, [29] in OEIS.

Then,

$$\begin{aligned} 3^{2 \cdot 2 + 1} &= 1 + 0 \cdot 2^1 + 30 \cdot 2^2 \\ &+ 1 + 0 \cdot 2^1 + 30 \cdot 2^2 \\ &+ 1 + 0 \cdot 0^1 + 30 \cdot 0^2 \\ &= 121 + 121 + 1 = 243 \end{aligned}$$

We've highlighted the terms of $A_{2,j}$ and $T(3, k)$ with different colors to be more easily to see regularity. Result we received are terms of the third row of triangle A300656.

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5. CONCLUSION

In this paper particular pattern, that is binomial distributed triangle A287326 in OEIS, which shows perfect cube n as sum of row terms over $0 \leq k \leq n-1$ or $1 \leq k \leq n$ is generalized. Firstly, we discussed analogs of A287326 for powers $2m+1 = 5, 7$, sequences A300656, A300785, respectively, then we derived coefficients $A_{m,j}$, such that for every $n \geq 0$ and $m \geq 0$ holds

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n, k) = n^{2m+1}$$

where $A_{m,j}$ is defined by definition (3.27). Therefore, question question (2.23) is answered positively. Section 3 is totally dedicated to complete and extended derivation of identity $\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n, k) = n^{2m+1}$. Properties of triangle (2.14) and $L_m(n, k)$ are shown in properties 2.15 and subsection 3.1, respectively. Relation between Faulhaber's sum $\sum n^m$ and finite differences of power are shown in 2.3.

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