Higher integral moments of automorphic L-functions in short intervals

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HIGHER INTEGRAL MOMENTS OF AUTOMORPHIC L-FUNCTIONS IN SHORT INTERVALS

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Abstract. We consider the higher integral moments for automorphic L-functions in short intervals and give a proof for the conjecture of Conrey et al. under Generalized Riemann Hypothesis for automorphic L-function.

Keywords. L-functions, automorphic forms, Moment

1. Introduction

The central values of L-functions is one of the most important problems in number theory. A typical example is the Lindelöf problem. In many arithmetic applications, we only ask for its mean values. The simplest example is the $2r$th moment of the Riemann ζ-function :

$$M_r(T;\zeta)=\int_T^{2T}|\zeta(\tfrac{1}{2}+\mathrm{i}\tau)|^{2r}\mathrm{d}\tau \tag{1.1}$$

for $r>0$ and $T\geqslant 1$. Hardy and Littlewood [8] proved that

$$M_1(T;\zeta)\sim T\log T \qquad (T\to\infty).$$

Ingham [12] showed that

$$M_2(T;\zeta)\sim\frac{1}{2\pi^2}T(\log T)^4.$$

Titchmarsh [31, Theorem 7.19] showed that for all integer $r>0$

$$\int_0^{\infty}|\zeta(\tfrac{1}{2}+\mathrm{i}\tau)|^{2r}\mathrm{e}^{-t/T}\mathrm{d}\tau\gg T(\log T)^{r^2}.$$

Ramachandra [23] strengthened the result to the lower bound

$$M_r(T;\zeta)\gg_r T(\log T)^{r^2}, \tag{1.2}$$

when $2r$ is a positive integer. Ramachandra [22] also showed that

$$M_r(T;\zeta)\gg T(\log T)^{r^2}(\log\log T)^{-\theta_r}$$

for real $r\geqslant\frac{1}{2}$, θ_r being a constant depending, possibly, on r, and moreover that, under the Riemann Hypothesis (RH), (1.2) holds for all real $r>0$. In other direction he proved [24] the upper bound

$$M_r(T;\zeta)\ll T(\log T)^{r^2} \tag{1.3}$$

unconditionally for $r=\frac{1}{2}$ and under the RH for $0<r<2$. Heath-Brown [10] proved that (1.2) holds for all rational number $r>0$. He also proved that (1.3) holds for $r=1/n$ with

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integer $n \geqslant 1$ and for all $0 < r \leqslant 2$ under RH. Radziwiłł in [20] improved this conditional upper bound for all $0 < r \leqslant 2.181$. In [28], Soundararajan proved that for all $r > 0$ and $\varepsilon > 0$ RH implies

$$M_r(T;\zeta) \ll_{r,\varepsilon} T(\log T)^{r^2+\varepsilon} \quad (T \geqslant 2).$$

Radziwiłł and Soundararajan [21] proved the lower bound (1.2) for all $r > 1$ unconditionally by extending the idea of Rudnick and Soundararajan ([26] and [27]) (who developed a new method of obtaining lower bounds for rational moments of L-functions varying in certain families) to obtain the result in the irrational case.

Very recently Harper [9] succeeded in removing ε by further refining Soundararajan's method: under RH, one has

$$M_r(T;\zeta) \ll_r T(\log T)^{r^2}. \tag{1.4}$$

It is conjectured that there is a positive constant C_r such that

$$M_r(T;\zeta) \sim C_r T(\log T)^{r^2} \qquad (T \to \infty). \tag{1.5}$$

A precise value for C_r was conjectured by Keating and Snaith [15] based on considerations from random matrix theory. Subsequently, Diaconu, Goldfeld and Hoffstein [3] gave an alternative approach based on multiple Dirichlet series and produced the same conjecture. Recently Conrey et al. [2] gave a more precise conjecture, identifying lower order terms in an asymptotic expansion for $M_r(T;\zeta)$.

It is natural to consider the analogue of $M_r(T;\zeta)$ for automorphic L-functions. For positive even integer κ, denote by $\mathcal{H}_\kappa^*$ the set of all normalized Hecke primitive cuspforms of weight κ for the modular group $SL_2(\mathbb{Z})$. The Fourier series expansion of $f \in \mathcal{H}_\kappa^*$ at the cusp ∞ is

$$f(z) = \sum_{n\geqslant 1} \lambda_f(n) n^{(\kappa-1)/2} \mathrm{e}^{2\pi \mathrm{i} n z} \qquad (\Im m\, z > 0),$$

where $\lambda_f(n)$ is the nth normalized Fourier coefficient of f with $\lambda_f(1) = 1$. The automorphic L-function attached to f is defined as

$$L(s,f) := \sum_{n\geqslant 1} \lambda_f(n) n^{-s} \qquad (\sigma > 1).$$

Here and in the sequel, we write implicitly $s = \sigma + \mathrm{i}\tau$. It is well known that $L(s,f)$ can be analytically prolonged to $\mathbb{C}$ and satisfies the functional equation over $\mathbb{C}$:

$$(2\pi)^{-s}\Gamma\big(s + \tfrac{1}{2}(\kappa-1)\big)L(s,f) = \mathrm{i}^\kappa (2\pi)^{-(1-s)}\Gamma\big(1-s+\tfrac{1}{2}(\kappa-1)\big)L(1-s,f). \tag{1.6}$$

Similarly to (1.1), for $f \in \mathcal{H}_\kappa^*$ and $r > 0$, we define

$$M_r(T;f) := \int_T^{2T} |L(\tfrac{1}{2} + \mathrm{i}\tau, f)|^{2r} \mathrm{d}\tau. \tag{1.7}$$

The study of mean squares for L-functions over $GL(2)$ was mainly pioneered by Good [7, 6], who showed that

$$M_1(T;f) \sim C_f T \log T \quad \text{as} \quad T \to \infty$$

for some positive constant C_f depending on f. In [29], Sun and Lü considered fractional power moments and proved that

$$M_r(T;f) \gg T(\log T)^{r^2},$$

for all $r = \frac{p}{q} \leqslant \frac{1}{2}$ with positive integers p and q, and also that

$$M_r(T; f) \ll T(\log T)^{r^2}$$

for positive even integer q under GRH for $L(s, f)$.

Pi considers in [19] the moments of automorphic L-functions on $GL(m)$, under Generalized Ramanujan Conjecture (GRC), he proved the lower bound for any non-negative rational number $r > 0$ (for all non-negative real numbers under GRC and GRH). He also shows that under GRC and GRH the upper bound holds for $0 \leqslant r \leqslant 2/m - \varepsilon$ for arbitrary $\varepsilon > 0$. The lower bound is also considered by Akbary and Fodden [1]. They proved the lower bound under GRH and a weaker conjecture instead of GRC for the local parameters at unramified primes. The moments of products of automorphic L-functions are considered by Milinovich and Turnage-Butterbaugh [17] with the method of frequency of large values introduced by Soundararajan.

Conrey et al. (see [2, Conjecture 2.5.4]) provided us the conjecture about $M_r(T; f)$ by considering the attached shifted moments, from what we can deduce the following one without considering the exact coefficients.

Conjecture 1. *For $f \in \mathcal{H}_\kappa^*$ and $r > 0$, we have*

$$M_r(T; f) \ll_{f,r} T(\log T)^{r^2},$$

where the implied constant depends on f and r.

In this paper, we consider a more general problem, i.e., the higher moment of $L(\frac{1}{2} + \mathrm{i}\tau, f)$ in short intervals :

$$M_r(T, H; f) := \int_T^{T+H} |L(\tfrac{1}{2} + \mathrm{i}\tau, f)|^{2r} \mathrm{d}\tau. \tag{1.8}$$

Thus $M_r(T; f) = M_r(T, T; f)$.

Our result is as follows.

Theorem 1.1. *Let $f \in \mathcal{H}_\kappa^*$, $r > 0$ and $0 < \varepsilon \leqslant 1$. Assuming GRH for $L(s, f)$, we have*

$$H(\log T)^{r^2} \ll M_r(T, H; f) \ll H(\log T)^{r^2} \tag{1.9}$$

uniformly for $T \geqslant T_0(f, r, \varepsilon)$ and $T^\varepsilon \leqslant H \leqslant T$, where the constant $T_0(f, r, \varepsilon)$ and the implied constants depend on f, r and ε only.

Particularly, the lower bound above holds for rational number r unconditionally.

The particular case of Theorem 1.1 shows that the analogues of Harper's upper bound result [9] and Heath-Brown's lower bound result [10] on $M_r(T; \zeta)$ also hold for $M_r(T; f)$. Our result improves the upper bound of the particular case of Milinovich and Turnage-Butterbaugh [17] and extends the range of validity of Pi's result [19] in the case of automorphic L-functions on $SL_2(\mathbb{Z})$. For the completeness of the result, the lower bound in short interval is also considered in the present article, although it seems to be a trivial generalization of the result of Pi and Akbary and Fodden [1].

Our approach is an adaptation of [28, 9] for upper bound part and of [10] for lower bound part. In [28], Soundararajan built on Selberg's work on the distribution of $\log \zeta(\frac{1}{2} + \mathrm{i}\tau)$. He removed the effect of zeros very near $\frac{1}{2} + \mathrm{i}\tau$ by finding an inequality and gave an upper bound for $\log \zeta(\frac{1}{2} + \mathrm{i}\tau)$ (see [28, Proposition]). By choosing a suitable length of the Dirichlet

polynomial, he investigated the frequency with which large values of $\log|\zeta(\frac{1}{2}+\mathrm{i}\tau)|$ can occur and then deduced an estimation for the moment of zeta function.

Harper in [9] improves the method. His work is based on that of Soundararajan. He divides the Dirichlet polynomial into small intervals, chooses a longer length for the polynomial and splits the integral into pieces according to the large values of these polynomials over small intervals. With these delicate decoupage, we can take more advantage of the information about large values. Moreover, he doesn't estimate the frequency of large values but works throughout with moment-type objects. Furthermore, he uses a similar lemma of Radziwiłł (see Lemma 2.5 below) to deal with the integral of a product over primes of terms $\cos(\tau\log p)$. With these 'almost-equations', we can save more for the contributions from these Dirichlet polynomials.

In this article, we succeed in extending the method of Soundararajan and Harper to the moments of automorphic L-functions in short intervals . The difference is that: if just follow their methods, we have to assume RH for $\zeta(s)$ and GRH for $L(s,\mathrm{sym}^2 f)$ additionally. Our argument does not truncate the second summation over prime squares by $\log T$ as Harper has done, but considers it together with the first summation directly. The contribution from this part is also negligible with our method. These allow us to avoid RH for $\zeta(s)$ and GRH for $L(s,\mathrm{sym}^2 f)$ and to give some simplification for the Soundararajan-Harper method. Moreover, in the last part, we use the method of Heath-Brown and apply the Rankin-Selberg L-function to prove the exact lower bound.

The result can be generalised to modular forms on congruence subgroups of $SL_2(\mathbb{Z})$. Besides, the method can be used to improve the result of Milinovich and Turnage-Butterbaugh [17] and the work of Milinovich and Ng [16], but the Generalised Ramanujan Conjecture is needed for the case $GL(m)$.

2. Automorphic L-functions and some preliminary lemmas

Let $f\in\mathcal{H}_\kappa^*$. According to Deligne, for any prime number p there are complex numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\begin{cases}|\alpha_f(p)|=\alpha_f(p)\beta_f(p)=1,\\ \lambda_f(p^\nu)=\sum_{0\leqslant j\leqslant\nu}\alpha_f(p)^{\nu-j}\beta_f(p)^j \quad (\nu\geqslant 1).\end{cases}\tag{2.1}$$

Hence, $\lambda_f(n)$ is real and satisfies the Hecke relation

$$\lambda_f(m)\lambda_f(n)=\sum_{d|(m,n)}\lambda_f\left(\frac{mn}{d^2}\right)\tag{2.2}$$

for all $m\geqslant 1$ and $n\geqslant 1$, and the Deligne's inequality

$$|\lambda_f(n)|\leqslant d(n)\qquad (n\geqslant 1),\tag{2.3}$$

where $d(n)$ is the divisor function. $L(s,f)$ admits the Euler product :

$$L(s,f)=\prod_p(1-\alpha_f(p)p^{-s})^{-1}(1-\beta_f(p)p^{-s})^{-1}\qquad(\sigma>1).$$

It is well known that (see e.g. [25], see also [4])

$$\sum_{p\leqslant x}\frac{\lambda_f(p)^2}{p}=\log_2 x+O_f(1)\quad(x\geqslant 3),\tag{2.4}$$

where we denote by $\log_j$ the j-fold iterated logarithm.

The main aim of this section is to prove Proposition 2.1 below. Let us begin by citing some properties on the logarithmic derivative of $L(s,f)$ which will be needed in the proof of Proposition 2.1. First we define $b_f(n)$ by the formula :

$$-\frac{L'}{L}(s,f)=\sum_{n\geqslant 1}\frac{\Lambda(n)b_f(n)}{n^s}\qquad(\sigma>1),\tag{2.5}$$

where $\Lambda(n)$ is the von Mangodt function and $b_f(n)$ is supported on prime powers such that

$$b_f(p^\nu)=\alpha_f(p)^\nu+\beta_f(p)^\nu\tag{2.6}$$

for prime number p and integer $\nu\geqslant 1$. Particularly, $b_f(p)=\lambda_f(p)$.

• In view of the functional equation (1.6), the logarithmic derivative $\frac{L'}{L}(s,f)$ have "trivial poles" at the points $s=-n-\frac{1}{2}(\kappa-1)$ for $n=0,1,2,\dots$.

• According to [14, Proposition 5.7], for any s in the vertical strip $-\frac{1}{2}<\sigma<2$ we have

$$\frac{L'}{L}(s,f)\ll_f \log(|\tau|+3)+\sum_{|s-\rho|<1}\frac{1}{s-\rho},$$

where ρ denotes the non-trivial zero of $L(s,f)$. If we write $s=\sigma+\mathrm{i}v$, then the number of zeros ρ with $|v-\Im m\,\rho|<1$ is $\ll\log(|v|+3)$. Hence by varying v by a bounded amount, we can ensure that

$$|v-\Im m\,\rho|\gg\log^{-1}(|v|+3).\tag{2.7}$$

With the present choice of v, we have for $-\frac{1}{2}<\sigma<2$

$$\frac{L'}{L}(\sigma+\mathrm{i}v,f)\ll_f\log(|v|+3)+\sum_{|v-\Im m\,\rho|<1}\log(|v|+3)\ll_f\log^2(|v|+3).\tag{2.8}$$

• By the functional equation (1.6) and Stirling's formula [30, 4.42] we have for $\sigma<-\frac{1}{2}$

$$-\frac{L'}{L}(s,f)=\frac{\Gamma'}{\Gamma}\left(1-s+\frac{\kappa-1}{2}\right)+\frac{\Gamma'}{\Gamma}\left(s+\frac{\kappa-1}{2}\right)+O_f(1)\ll_f\log(|s|+3).\tag{2.9}$$

.

The main result of this section is the following proposition, which will consists in the starting point of the proof of upper bound part in Theorem 1.1.

Proposition 2.1. *Let $f\in\mathcal{H}_\kappa^*$ and let $\varpi_0=0.4912...$ denote the unique positive real number satisfying $\mathrm{e}^{-\varpi_0}=\varpi_0+\frac{1}{2}\varpi_0^2$. Assuming GRH for $L(s,f)$, we have*

$$\log|L(\tfrac{1}{2}+\mathrm{i}\tau,f)|\leqslant\Re e\sum_{n\leqslant x}\frac{\Lambda(n)b_f(n)\log(x/n)}{n^{1/2+\varpi/\log x+\mathrm{i}\tau}(\log n)\log x}+(\varpi+1)\frac{\log T}{\log x}+O\left(\frac{1}{\log x}\right)\tag{2.10}$$

$$\log|L(\tfrac{1}{2}+\mathrm{i}\tau,f)|\leqslant\Re e\sum_{p\leqslant x}\frac{b_f(p)\log(x/p)}{p^{1/2+\varpi/\log x+\mathrm{i}\tau}\log x}+\frac{1}{2}\Re e\sum_{p\leqslant\sqrt{x}}\frac{b_f(p^2)\log(x/p^2)}{p^{1+2\varpi/\log x+\mathrm{i}2\tau}\log x}+(\varpi+1)\frac{\log T}{\log x}+O(1)\tag{2.11}$$

for $T\geqslant 2$, $T<\tau\leqslant 2T$, $2\leqslant x\leqslant T^2$ and $\varpi\geqslant\varpi_0$, where the implied constants depend on f.

Proof. Let $c := \max(1, 2-\sigma)$. Under GRH, we can denote $\rho = \frac{1}{2} + \mathrm{i}\gamma$ to be the non-trivial zeros of $L(s,f)$. By the Perron formula, we can write

$$\sum_{n\leqslant x} \frac{\Lambda(n)b_f(n)}{n^s} \log\Big(\frac{x}{n}\Big) = -\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}v}^{c+\mathrm{i}v} \frac{L'}{L}(s+w,f)\frac{x^w}{w^2}\mathrm{d}w + O\Big(\frac{x^c\log^2 x}{v^2}\Big),$$

where $s = \sigma + \mathrm{i}\tau$ with $\sigma > \frac{1}{2}$ and $T < \tau \leqslant 2T$ and $v \geqslant 3$ is a parameter tending to infinity but satisfying (2.7). Shift the segment of integration to the path $\mathcal{C}$ consisting of straight lines joining $c-\mathrm{i}v$, $-v-\mathrm{i}v$, $-v+\mathrm{i}v$ and $c+\mathrm{i}v$. By the residue theorem, one gets that

$$\begin{aligned}\sum_{n\leqslant x} \frac{\Lambda(n)b_f(n)}{n^s} \log\Big(\frac{x}{n}\Big) &= -\frac{L'}{L}(s,f)\log x - \Big(\frac{L'}{L}(s,f)\Big)' - \sum_{|\gamma-\tau|<v} \frac{x^{\rho-s}}{(\rho-s)^2} \\ &- \sum_{n+(\kappa-1)/2+\sigma<v} \frac{x^{-n-(\kappa-1)/2-s}}{(n+(\kappa-1)/2+s)^2} - \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}} \frac{L'}{L}(s+w,f)\frac{x^w}{w^2}\mathrm{d}w + O\Big(\frac{x^c\log^2 x}{v}\Big).\end{aligned} \tag{2.12}$$

We apply (2.9) and (2.8) to estimate the integrals over horizontal segments from $-v$ to $-\frac{1}{2}$ and from $-\frac{1}{2}$ to c, respectively. Then it is

$$\ll \frac{\log(|s|+v)}{v^2\sqrt{x}\log x} + \frac{x^c\log^2(T+v)}{v^2}.$$

The integral over the vertical segment is $\ll x^{-v}v^{-1}\log(|s|+v)$. Inserting these estimates into (2.12) and making $v\to\infty$, it follows that

$$\begin{aligned}-\frac{L'}{L}(s,f) &= \sum_{n\leqslant x} \frac{\Lambda(n)b_f(n)}{n^s}\frac{\log(x/n)}{\log x} + \frac{1}{\log x}\left(\frac{L'}{L}(s,f)\right)' \\ &\quad + \frac{1}{\log x}\sum_{\rho}\frac{x^{\rho-s}}{(\rho-s)^2} + \frac{1}{\log x}\sum_{n\geqslant 0}\frac{x^{-n-(\kappa-1)/2-s}}{(n+(\kappa-1)/2+s)^2}.\end{aligned} \tag{2.13}$$

Taking real parts of both sides and integrating with respect to σ over $\sigma_0 > \frac{1}{2}$ to ∞, we get

$$\begin{aligned}\log|L(s_0,f)| = \Re e\Big\{ &\sum_{n\leqslant x}\frac{\Lambda(n)b_f(n)}{n^{s_0}\log n}\frac{\log(x/n)}{\log x} - \frac{1}{\log x}\frac{L'}{L}(s_0,f) \\ &+ \frac{1}{\log x}\sum_{\rho}\int_{\sigma_0}^{\infty}\frac{x^{\rho-s}}{(\rho-s)^2}\mathrm{d}\sigma\Big\} + O\big(x^{-\kappa/2}\big),\end{aligned} \tag{2.14}$$

where $s_0 = \sigma_0 + \mathrm{i}\tau$ and the implied constant is absolute.

For $s = \sigma + \mathrm{i}\tau$ with $\sigma > \frac{1}{2}$ and $T \leqslant \tau \leqslant 2T$, define

$$F(s) := \sum_{\rho} \Re e\,\frac{1}{s-\rho} = \sum_{\rho}\frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(\tau-\gamma)^2} > 0. \tag{2.15}$$

According to [14, Theorem 5.6], we have

$$-\Re e\,\frac{L'}{L}(s,f) = -\log(2\pi) + \Re e\,\frac{\Gamma'}{\Gamma}\Big(s+\frac{\kappa-1}{2}\Big) - \Re e\,B - \Re e\sum_{\rho}\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big),$$

where $\Re e\, B = -\sum_\rho \Re e\,(\frac{1}{\rho})$. Thus for $s=\sigma+\mathrm{i}\tau$ with $\sigma>\frac12$ and $T\leqslant\tau\leqslant 2T$, we have

$$-\Re e\,\frac{L'}{L}(s,f) = \log T - F(s) + O_\kappa(1), \tag{2.16}$$

where we have evaluated $\frac{\Gamma'}{\Gamma}(s+\frac{\kappa-1}{2}) = \log T + O_\kappa(1)$ by the Stirling formula. Observing

$$\sum_\rho \left|\int_{\sigma_0}^\infty \frac{x^{\rho-s}}{(\rho-s)^2}\mathrm{d}\sigma\right| \leqslant \sum_\rho \int_{\sigma_0}^\infty \frac{x^{1/2-\sigma}}{|\rho-s_0|^2}\mathrm{d}\sigma = \sum_\rho \frac{x^{1/2-\sigma_0}}{|\rho-s_0|^2\log x} = \frac{x^{1/2-\sigma_0}F(s_0)}{(\sigma_0-\frac12)\log x},$$

together with (2.14) and (2.16), we deduce that

$$\log|L(s_0,f)| \leqslant \Re e \sum_{n\leqslant x}\frac{\Lambda(n)b_f(n)}{n^{s_0}\log n}\frac{\log(x/n)}{\log x} + \frac{\log T - F(s_0)+O(1)}{\log x} + \frac{x^{1/2-\sigma_0}F(s_0)}{(\sigma_0-\frac12)(\log x)^2}.$$

Integrating (2.16) as σ varies from $\frac12$ to σ_0 ($>\frac12$), we obtain

$$\log|L(\tfrac12+\mathrm{i}\tau,f)| - \log|L(s_0,f)| = \{\log T + O(1)\}(\sigma_0-\tfrac12) - \int_{1/2}^{\sigma_0} F(\sigma+\mathrm{i}\tau)\mathrm{d}\sigma.$$

According to the definition of $F(\sigma+\mathrm{i}\tau)$, we have

$$\int_{1/2}^{\sigma_0} F(\sigma+\mathrm{i}\tau)\mathrm{d}\sigma = \frac12\sum_\rho \log\left(1+\frac{(\sigma_0-1/2)^2}{(\tau-\gamma)^2}\right) \geqslant \frac12 F(s_0),$$

since we have $\log(1+x^2)\geqslant x^2/(1+x^2)$. Then it follows that

$$\log|L(\tfrac12+\mathrm{i}\tau,f)| - \log|L(s_0,f)| \leqslant \{\log T - \tfrac12 F(s_0) + O(1)\}(\sigma_0-\tfrac12).$$

Together with the precedent inequality, it follows that

$$\begin{aligned}\log|L(\tfrac12+\mathrm{i}\tau,f)| \leqslant \Re e \sum_{n\leqslant x}\frac{\Lambda(n)b_f(n)}{n^{s_0}\log n}\frac{\log(x/n)}{\log x} + \big((\sigma_0-\tfrac12)\log x+1\big)\frac{\log T}{\log x}\\ + F(s_0)(\sigma_0-\tfrac12)^{-1}(\log x)^{-2}G\big((\sigma_0-\tfrac12)\log x\big) + O\big((\log x)^{-1}+\sigma_0-\tfrac12\big)\end{aligned}$$

where $G(\varpi) := \mathrm{e}^{-\varpi}-\varpi-\frac12\varpi^2$. We take $\sigma_0 = \frac12+\frac{\varpi}{\log x}$ with $\varpi\geqslant\varpi_0$. It is easy to see that $G(\varpi)$ is decreasing and $G(\varpi_0)=0$. Since $F(s_0)>0$, we have $F(s_0)G(\varpi)\leqslant 0$ for $\varpi\geqslant\varpi_0$ and therefore this term can be omitted. Then (2.10) follows.

The inequality (2.11) is a simple consequence of (2.10) since the contribution of p^ν with $\nu\geqslant 3$ to the sum on the right-hand side of (2.10) is

$$\sum_{p^\nu\leqslant x,\,\nu\geqslant 3}\frac{b_f(p^\nu)}{p^{\nu/2+\nu\mathrm{i}\tau+\nu\varpi_0/\log x}\nu}\frac{\log(x/p^\nu)}{\log x} \ll \sum_{p^\nu\leqslant x,\,\nu\geqslant 3}\frac{1}{p^{\nu/2}} \ll 1.$$

This completes the proof. □

Finally we shall cite some mean value theorems and an elementary lemma, which will be useful later. The first one is a slight variant of [28, Lemma 3]. The proof is more or less the same.

Lemma 2.2. *Let $0<\varepsilon\leqslant 1$. For any complex numbers $a(p)$, we have*

$$\int_T^{T+H}\left|\sum_{p\leqslant x}\frac{a(p)}{p^{1/2+\mathrm{i}\tau}}\right|^{2r}\mathrm{d}\tau \ll r!H\left(\sum_{p\leqslant x}\frac{|a(p)|^2}{p}\right)^r$$

uniformly for $r \in \mathbb{N}$, $T \geqslant 2$, $T^\varepsilon \leqslant H \leqslant T$ and $2 \leqslant x \leqslant (H/\log H)^{1/r}$, where the implied constant depends on ε at most.

The second lemma is [18, Corollary 3]).

Lemma 2.3. *For any complex sequence $\{a_n\}_{n\geqslant 1}$ verifying $\sum_{n\geqslant 1} n|a_n|^2 < \infty$, we have*

$$\int_0^T \Big|\sum_{n\geqslant 1} a_n n^{-\mathrm{i}\tau}\Big|^2 \mathrm{d}\tau = \sum_{n\geqslant 1}^{\infty} |a_n|^2\{T + O(n)\}$$

uniformly for $T \geqslant 2$, where the implied constant is absolute.

The following lemma is a corollary of [5, Theorem 2].

Lemma 2.4. *Let $F(s)$ be regular in the vertical strip $\alpha < \sigma < \beta$ and continuous for $\alpha \leqslant \sigma \leqslant \beta$. Suppose $F(s) \to 0$ as $|\tau| \to \infty$ uniformly for $\alpha \leqslant \sigma \leqslant \beta$. Then for $\alpha \leqslant \gamma \leqslant \beta$ and any $q > 0$ we have*

$$\int_{\mathbb{R}} |F(\gamma + \mathrm{i}\tau)|^q \mathrm{d}\tau \leqslant \left(\int_{\mathbb{R}} |F(\alpha + \mathrm{i}\tau)|^q \mathrm{d}\tau\right)^{(\beta-\gamma)/(\beta-\alpha)} \left(\int_{\mathbb{R}} |F(\beta + \mathrm{i}\tau)|^q \mathrm{d}\tau\right)^{(\gamma-\alpha)/(\beta-\alpha)}.$$

The following lemma is a simple generalization of [9, Proposition 2].

Lemma 2.5. *Let $M = p_1^{\mu_1}\cdots p_r^{\mu_r}$ and $N = q_1^{\nu_1}\cdots q_t^{\nu_t}$, where the p_j and q_k are all distinct primes of one another, the μ_j, ν_k are positive integers and the r, t are non negative integers. We have*

$$\int_T^{T+H} \prod_{1\leqslant j\leqslant r} (\cos(2\tau\log p_j))^{\mu_j} \prod_{1\leqslant k\leqslant t} (\cos(\tau\log q_k))^{\nu_k} \mathrm{d}\tau = H\Theta(MN) + O(M^2N), \tag{2.17}$$

uniformly for $2 \leqslant H \leqslant T$, where

$$\Theta(MN) := \prod_{1\leqslant j\leqslant r} \frac{1}{2^{\mu_j}}\binom{\mu_j}{\mu_j/2} \prod_{1\leqslant k\leqslant t} \frac{1}{2^{\nu_k}}\binom{\nu_k}{\nu_k/2}.$$

Here by convention, $\binom{\nu}{\nu/2} = \frac{\nu!}{((\nu/2)!)^2}$ if ν is even and $\binom{\nu}{\nu/2} = 0$ if ν is odd.

Proof. Notice that

$$(\cos(a\tau))^\nu = \frac{(\mathrm{e}^{\mathrm{i}a\tau} + \mathrm{e}^{-\mathrm{i}a\tau})^\nu}{2^\nu} = \frac{1}{2^\nu}\binom{\nu}{\nu/2} + \sum_{0\leqslant\ell\leqslant\nu,\, \ell\neq\nu/2} \frac{1}{2^\nu}\binom{\nu}{\ell}\mathrm{e}^{\mathrm{i}(\nu-2\ell)a\tau}.$$

Therefore the integral on the left-hand side of (2.17) is equal to $H\Theta(MN) + R$ with

$$R := \int_T^{T+H} \sum_{(\ell_{11},\dots,\ell_{1r},\ell_{21},\dots,\ell_{2t})} \prod_{1\leqslant j\leqslant r} \prod_{1\leqslant k\leqslant t} \frac{1}{2^{\mu_j+\nu_k}}\binom{\mu_j}{\ell_{1j}}\binom{\nu_j}{\ell_{2k}} \mathrm{e}^{\mathrm{i}\tau\log\left(p_1^{2c_1}\cdots p_r^{2c_r} q_1^{d_1}\cdots q_t^{d_t}\right)} \mathrm{d}\tau$$

with $c_j := \mu_j - 2\ell_{1j}$ and $d_k := \nu_k - 2\ell_{2k}$. Here $0 \leqslant \ell_{1j} \leqslant \mu_j$ and $0 \leqslant \ell_{2k} \leqslant \nu_k$ such that $\sum_j c_j^2 + \sum_k d_k^2 \neq 0$. Since the p_j, q_k are distinct and $|c_j| \leqslant \mu_j$ and $|d_k| \leqslant \nu_k$, clearly we have

$|p_1^{2c_1}\cdots p_r^{2c_r}q_1^{d_1}\cdots q_t^{d_t}-1|\geqslant M^{-2}N^{-1}$. Thus $\big|\log\big(p_1^{2c_1}\cdots p_r^{2c_r}q_1^{d_1}\cdots q_t^{d_t}\big)\big|\gg M^{-2}N^{-1}$ and

$$\begin{aligned}R&\ll M^2N\sum_{(\ell_{11},\dots,\ell_{1r},\ell_{21},\dots,\ell_{2t})}\prod_{1\leqslant j\leqslant r}\prod_{1\leqslant k\leqslant t}\frac{1}{2^{\mu_j+\nu_k}}\binom{\mu_j}{\ell_{1j}}\binom{\nu_j}{\ell_{2k}}\\&\ll M^2N\prod_{1\leqslant j\leqslant r}\prod_{1\leqslant k\leqslant t}\sum_{\ell_{1j}}\frac{1}{2^{\mu_j}}\binom{\mu_j}{\ell_{1j}}\sum_{\ell_{2k}}\frac{1}{2^{\nu_k}}\binom{\nu_k}{\ell_{2k}}\\&\ll M^2N.\end{aligned}$$

Then the result follows. □

3. Harper's refinement

Harper's method is a refinement of that of Soundararajan. Its starting point is (2.11). Harper proposed a delicate decoupage on the first sum on the right-hand side and considered the contribution of the second sum. We do the same decoupage for the first sum and our treatment for the second sum is a little different: we do not truncate this sum by $\log T$ (this asks for additional assumptions of RH for $\zeta(s)$ and GRH for $L(s,\mathrm{sym}^2 f)$), but consider it directly with the first sum by choosing proper values for parameters.

By the prime number theorem, we have trivially

$$\Big|\sum_{p\leqslant\sqrt{x}}\frac{b_f(p^2)\log p^2}{p^{1+2/\log x+2\mathrm{i}\tau}\log x}\Big|\leqslant\sum_{p\leqslant\sqrt{x}}\frac{4\log p}{p\log x}\ll 1.$$

Combining this with (2.11), we find, for $(\log T)^{10}\leqslant x\leqslant T^2$,

$$\log|L(\tfrac12+\mathrm{i}\tau,f)|\leqslant\Re e\sum_{p\leqslant x}\frac{b_f(p)\log(x/p)}{p^{1/2+1/\log x+\mathrm{i}\tau}\log x}+\frac12\Re e\sum_{p\leqslant\sqrt{x}}\frac{b_f(p^2)}{p^{1+\mathrm{i}2\tau}}+2\frac{\log T}{\log x}+O_f(1).\tag{3.1}$$

For $0<\varepsilon\leqslant 1$, $r>0$, $T\geqslant 100$ and large positive constant $c(\varepsilon)$ depending on ε, we define the real sequence $\{\psi_i\}_{i\geqslant 0}$ and the integer I by

$$\psi_0:=0,\qquad \psi_i:=\frac{20^{i-1}}{(\log_2 T)^2}\quad(i\geqslant 1),\tag{3.2}$$

$$I=I_{\varepsilon,r,T}:=1+\max\{i:\psi_i\leqslant \mathrm{e}^{-c(\varepsilon)r}\}\leqslant(2/\log 20)\log_3 T.\tag{3.3}$$

Then define the set $\mathscr{T}=\mathscr{T}_{r,T,H}$ by

$$\mathscr{T}:=\big\{\tau\in[T,\,T+H]:|F_i(\tau)|\leqslant\psi_i^{-3/4}\ \ (1\leqslant i\leqslant I)\big\},\tag{3.4}$$

where $T^\varepsilon\leqslant H\leqslant T$ and

$$F_i(\tau):=\Re e\sum_{T^{\psi_{i-1}}<p\leqslant T^{\psi_i}}\frac{\lambda_f(p)\log(T^{\psi_I}/p)}{p^{1/2+1/(\psi_I\log T)+\mathrm{i}\tau}\log T^{\psi_I}}.\tag{3.5}$$

Lemma 3.1. *Let $f\in\mathcal{H}_\kappa^*$, $r>0$ and $0<\varepsilon\leqslant 1$. Under the previous notation, we have*

$$\int_{\mathscr{T}}\exp\bigg(2r\Re e\sum_{p\leqslant T^{\psi_I}}\frac{\lambda_f(p)\log(T^{\psi_I}/p)}{p^{1/2+1/(\psi_I\log T)+\mathrm{i}\tau}\log T^{\psi_I}}\bigg)\mathrm{d}\tau\ll H(\log T)^{r^2},\tag{3.6}$$

uniformly for $T\geqslant 100$ and $T^\varepsilon\leqslant H\leqslant T$, where the implied constant depends on f, r and ε.

Proof. Denoting by $\mathfrak{I}$ the integral on the left-hand side of (3.6), we can write

$$\mathfrak{I} = \int_{\tau\in\mathscr{T}} \prod_{1\leqslant i\leqslant I} \big(\exp\{rF_i(\tau)\}\big)^2 \mathrm{d}\tau.$$

On the other hand, we have

$$\mathrm{e}^t = \sum_{0\leqslant j\leqslant J} \frac{t^j}{j!} + O(\mathrm{e}^{-J}) \tag{3.7}$$

uniformly for $J \geqslant 0$ and $|t| \leqslant \frac{1}{9}J$, where we have used the Stirling formula to write

$$\mathrm{e}^J \sum_{j>J} \frac{t^j}{j!} \ll \sum_{j>J} \frac{(\mathrm{e}t)^j}{(j/\mathrm{e})^j\sqrt{j}} \ll \sum_{j>J} \frac{(\mathrm{e}^2/9)^j}{\sqrt{j}} \ll 1.$$

By the definition of $\mathscr{T}$, we have $|F_i(\tau)| \leqslant \psi_i^{-3/4}$ for $\tau\in\mathscr{T}$ and $1\leqslant i\leqslant I$. Applying (3.7) of the form $\mathrm{e}^t = \{1+O(\mathrm{e}^{-J})\}\sum_{0\leqslant j\leqslant J} t^j/j!$ with $J = [100r\psi_i^{-3/4}]$ and $t = rF_i(\tau)$, we get

$$\begin{aligned}\mathfrak{I} &= \int_{\mathscr{T}} \prod_{1\leqslant i\leqslant I} \Big\{1+O\Big(\mathrm{e}^{-100r\psi_i^{-3/4}}\Big)\Big\}\Big(\sum_{0\leqslant j\leqslant 100r\psi_i^{-3/4}} \frac{(rF_i(\tau))^j}{j!}\Big)^2 \mathrm{d}\tau \\ &\ll \int_T^{T+H} \prod_{1\leqslant i\leqslant I} \Big(\sum_{0\leqslant j\leqslant 100r\psi_i^{-3/4}} \frac{(rF_i(\tau))^j}{j!}\Big)^2 \mathrm{d}\tau,\end{aligned} \tag{3.8}$$

where we have used the following estimates :

$$\begin{aligned}\sum_{1\leqslant i\leqslant I} \mathrm{e}^{-100r\psi_i^{-3/4}} &= \sum_{1\leqslant i\leqslant I} \mathrm{e}^{-ab^{-i}} \qquad (a = 20^{3/4}100r(\log_2 T)^{3/2},\ b = 20^{3/4}) \\ &\leqslant \int_1^{I+1} \mathrm{e}^{-ab^{-t}}\mathrm{d}t \qquad (u = ab^{-t},\ \mathrm{d}t = -\mathrm{d}u/(u\log b)) \\ &= \frac{1}{\log b}\int_{100b^{-1}r\psi_I^{-3/4}}^{a/b} \frac{\mathrm{e}^{-u}}{u}\mathrm{d}u \ll 1.\end{aligned} \tag{3.9}$$

Developing the square and then $(rF_i(\tau))^j$, we can deduce that

$$\mathfrak{I} \ll_r \sum_{\mathbf{j},\mathbf{k},\mathbf{p},\mathbf{q}} C_{\mathbf{j},\mathbf{k},\mathbf{p},\mathbf{q}} \int_T^{T+H} \prod_{1\leqslant i\leqslant I} \Big(\prod_{\substack{1\leqslant m\leqslant j_i\\ 1\leqslant n\leqslant k_i}} \cos\big(\tau\log p_i(m)\big)\cos\big(\tau\log q_i(n)\big)\Big)\mathrm{d}\tau, \tag{3.10}$$

where $\mathbf{j} := (j_1, j_2, \dots, j_I)$, $\mathbf{k} := (k_1, k_2, \dots, k_I)$ with $0\leqslant j_i,\ k_i \leqslant 100r\psi_i^{-3/4}$, and

$$\begin{aligned}\mathbf{p} &:= \big(p_1(1),\dots,p_1(j_1); p_2(1),\dots,p_2(j_2);\dots; p_I(1),\dots,p_I(j_I)\big) \\ \mathbf{q} &:= \big(q_1(1),\dots,q_1(k_1); q_2(1),\dots,q_2(k_2);\dots; q_I(1),\dots,q_I(k_I)\big)\end{aligned}$$

with primes $p_i(m)$ and $q_i(n)$ satisfying

$$T^{\psi_{i-1}} < p_i(1),\dots,p_i(j_i); q_i(1),\dots,q_i(k_i) \leqslant T^{\psi_i} \qquad (1\leqslant i\leqslant I),$$

and

$$C_{\mathbf{j},\mathbf{k},\mathbf{p},\mathbf{q}} := \prod_{1\leqslant i\leqslant I} \frac{r^{j_i+k_i}}{j_i!k_i!}\Big(\prod_{\substack{1\leqslant m\leqslant j_i\\ 1\leqslant n\leqslant k_i}} \frac{\lambda_f(p_i(m))\log(T^{\psi_I}/p_i(m))}{p_i(m)^{1/2+1/(\psi_I\log T)}\log T^{\psi_I}} \frac{\lambda_f(q_i(n))\log(T^{\psi_I}/q_i(n))}{q_i(n)^{1/2+1/(\psi_I\log T)}\log T^{\psi_I}}\Big).$$

Using Lemma 2.5 with $M=1$, we have

$$\mathfrak{I} \ll_r H\mathfrak{I}_1 + \mathfrak{I}_2, \tag{3.11}$$

where

$$\mathfrak{I}_1 := \sum_{\mathbf{j},\mathbf{k},\mathbf{p},\mathbf{q}} D_{\mathbf{j},\mathbf{k},\mathbf{p},\mathbf{q}}\Theta\Big(\prod_{1\leqslant i\leqslant I}\prod_{\substack{1\leqslant m\leqslant j_i\\1\leqslant n\leqslant k_i}} p_i(m)q_i(n)\Big),$$
$$\mathfrak{I}_2 := \sum_{\mathbf{j},\mathbf{k},\mathbf{p},\mathbf{q}} D_{\mathbf{j},\mathbf{k},\mathbf{p},\mathbf{q}}\prod_{1\leqslant i\leqslant I}\prod_{\substack{1\leqslant m\leqslant j_i\\1\leqslant n\leqslant k_i}} p_i(m)q_i(n),$$

with

$$D_{\mathbf{j},\mathbf{k},\mathbf{p},\mathbf{q}} := \prod_{1\leqslant i\leqslant I}\frac{r^{j_i+k_i}}{j_i!k_i!}\Big(\prod_{\substack{1\leqslant m\leqslant j_i\\1\leqslant n\leqslant k_i}}\frac{|\lambda_f(p_i(m))\lambda_f(q_i(n))|}{\sqrt{p_i(m)q_i(n)}}\Big).$$

Since

$$\prod_{1\leqslant i\leqslant I}\prod_{\substack{1\leqslant m\leqslant j_i\\1\leqslant n\leqslant k_i}} p_i(m)q_i(n) \leqslant \prod_{1\leqslant i\leqslant I} T^{\psi_i(j_i+k_i)} \leqslant \prod_{1\leqslant i\leqslant I} T^{200r\psi_i^{1/4}} \leqslant T^{\varepsilon/10},$$

we have

$$\begin{aligned}\mathfrak{I}_2 &\ll T^{\varepsilon/10}\prod_{1\leqslant i\leqslant I}\Big\{\sum_{0\leqslant j\leqslant 100r\psi_i^{-3/4}}\frac{r^j}{j!}\Big(\sum_{T^{\psi_{i-1}}<p\leqslant T^{\psi_i}}\frac{|\lambda_f(p)|}{\sqrt{p}}\Big)^j\Big\}^2\\ &\ll T^{\varepsilon/10}\prod_{1\leqslant i\leqslant I}T^{200r\psi_i^{1/4}}\Big(\sum_{0\leqslant j\leqslant 100r\psi_i^{-3/4}}\frac{r^j}{j!}\Big)^2\\ &\ll T^{2\varepsilon/10}\mathrm{e}^{2rI} \ll_{\varepsilon,r} T^{3\varepsilon/10}.\end{aligned} \tag{3.12}$$

For the first term on the right-hand side of (3.11), we have

$$\begin{aligned}\mathfrak{I}_1 &\leqslant \prod_{1\leqslant i\leqslant I}\sum_{0\leqslant m\leqslant 200r\psi_i^{-3/4}}\sum_{\substack{j+k=m\\j,\,k\geqslant 0}}\frac{r^m}{j!k!}\sum_{T^{\psi_{i-1}}<p_1,\dots,p_m\leqslant T^{\psi_i}}\frac{\Theta(p_1\cdots p_m)|\lambda_f(p_1)\cdots\lambda_f(p_m)|}{\sqrt{p_1\cdots p_m}}\\ &\leqslant \prod_{1\leqslant i\leqslant I}\sum_{0\leqslant m\leqslant 200r\psi_i^{-3/4}}\frac{r^m2^m}{m!}\sum_{T^{\psi_{i-1}}<p_1,\,\dots,\,p_m\leqslant T^{\psi_i}}\frac{\Theta(p_1\cdots p_m)|\lambda_f(p_1)\cdots\lambda_f(p_m)|}{\sqrt{p_1\cdots p_m}}.\end{aligned}$$

According to the definition of $\Theta(\cdot)$ in Lemma 2.5, we can assume $m=2n$. Thus

$$\mathfrak{I}_1 \leqslant \prod_{1\leqslant i\leqslant I}\sum_{0\leqslant n\leqslant 100r\psi_i^{-3/4}}\frac{(2r)^{2n}}{(2n)!}\sum_{T^{\psi_{i-1}}<p_1,\,\dots,\,p_n\leqslant T^{\psi_i}}\frac{|\lambda_f(p_1)\cdots\lambda_f(p_n)|^2}{p_1\cdots p_n}\Phi(p_1,\,\dots,\,p_n), \tag{3.13}$$

where

$$\Phi(p_1,\,\dots,\,p_n) := \Theta(p_1^2\cdots p_n^2)\frac{|\{(q_1,\,\dots,\,q_{2n}) : q_1\cdots q_{2n} = p_1^2\cdots p_n^2\}|}{|\{(q_1,\,\dots,\,q_n) : q_1\cdots q_n = p_1\cdots p_n\}|} \tag{3.14}$$

with primes $T^{\psi_{i-1}} < q_1, \dots, q_{2n} \leqslant T^{\psi_i}$. Write $p_1^2 \cdots p_n^2 = p_{n_1}^{2\nu_1} \cdots p_{n_\ell}^{2\nu_\ell}$ with $(p_{n_i}, p_{n_j}) = 1$ for $i \neq j$. By direct calculation, we have

$$\Theta(p_1^2 \cdots p_n^2) = \frac{1}{2^{2n}} \prod_{1 \leqslant j \leqslant \ell} \frac{(2\nu_j)!}{(\nu_j!)^2},$$

$$\big|\big\{(q_1, \dots, q_{2n}) : q_1 \cdots q_{2n} = (p_1 \cdots p_n)^2\big\}\big| = \frac{(2n)!}{\prod_{1 \leqslant j \leqslant \ell} (2\nu_j)!},$$

$$|\{(q_1, \dots, q_n) : q_1 \cdots q_n = p_1 \cdots p_n\}| = \frac{n!}{\prod_{1 \leqslant j \leqslant \ell} \nu_j!}.$$

These imply that

$$\Phi(p_1, \dots, p_n) = \frac{(2n)!}{2^{2n} n! \prod_{1 \leqslant j \leqslant \ell} \nu_j!} \leqslant \frac{(2n)!}{2^{2n} n!}.$$

Inserting this into (3.13) and using (2.4), we find that

$$\begin{aligned} \mathfrak{I}_1 &\leqslant \prod_{1 \leqslant i \leqslant I} \sum_{0 \leqslant n \leqslant 100 r \psi_i^{-3/4}} \frac{1}{n!} \bigg(r^2 \sum_{T^{\psi_{i-1}} < p \leqslant T^{\psi_i}} \frac{|\lambda_f(p)|^2}{p} \bigg)^n \\ &\leqslant \exp\bigg(r^2 \sum_{p \leqslant T^{\psi_I}} \frac{|\lambda_f(p)|^2}{p} \bigg) \ll_{\varepsilon, r} (\log T)^{r^2}. \end{aligned} \tag{3.15}$$

Now the required bound follows from (3.11), (3.12) and (3.15). □

Next we shall consider further the integral on $\mathscr{T}$ together with the second sum inside.

Proposition 3.2. *Let $f \in \mathcal{H}_\kappa^*$, $r > 0$ and $0 < \varepsilon \leqslant 1$. Under the previous notation, we have*

$$\int_{\mathscr{T}} \exp\bigg\{ 2r \Re e \bigg(\sum_{p \leqslant T^{\psi_I}} \frac{\lambda_f(p) \log(T^{\psi_I}/p)}{p^{1/2 + 1/(\psi_I \log T) + \mathrm{i}\tau} \log T^{\psi_I}} + \frac{1}{2} \sum_{p \leqslant T^{\psi_I/2}} \frac{b_f(p^2)}{p^{1+2\mathrm{i}\tau}} \bigg) \bigg\} \mathrm{d}\tau \ll H (\log T)^{r^2},$$

uniformly for $T \geqslant 100$ and $T^\varepsilon \leqslant H \leqslant T$, where the implied constant depends on f, r and ε.

Proof. For $0 \leqslant m \leqslant M_T := [\psi_I \log T / \log 4]$ ($2^m \leqslant T^{\psi_I/2}$), define

$$P_m(\tau) := \Re e \bigg(\frac{1}{2} \sum_{2^m < p \leqslant 2^{m+1}} \frac{b_f(p^2)}{p^{1+2\mathrm{i}\tau}} \bigg),$$

$$\mathscr{P}(m) := \Big\{ \tau \in \mathscr{T} : |P_m(\tau)| > 2^{-m/10} \text{ but } |P_j(\tau)| \leqslant 2^{-j/10} \ (m+1 \leqslant j \leqslant M_T) \Big\}.$$

If τ belongs to none of these sets, then $|P_j(\tau)| \leqslant 2^{-j/10}$ for all $j \leqslant M_T$ and

$$\Re e \bigg(\frac{1}{2} \sum_{p \leqslant T^{\psi_I/2}} \frac{b_f(p^2)}{p^{1+2\mathrm{i}\tau}} \bigg) \ll 1.$$

If we denote by $\mathfrak{J}$ the integral to estimate, the contribution of such part of τ to $\mathfrak{J}$ is $\ll H(\log T)^{r^2}$ thanks to Lemma 3.1. Let $\mathfrak{J}_m$ be the integral of the same integrand over $\mathscr{P}(m)$. Then

$$\mathfrak{J} \ll H(\log T)^{r^2} + \mathfrak{J}_0 + \cdots + \mathfrak{J}_{M_T}. \tag{3.16}$$

Using the fact that $|P_m(\tau)| \geqslant 2^{-m/10}$ for $\tau \in \mathscr{P}(m)$ and Lemma 2.5 with $N = 1$, we have

$$\begin{aligned}|\mathscr{P}(m)| &= \int_{\mathscr{P}(m)} \mathrm{d}\tau \leqslant \int_T^{T+H} \big(2^{m/10} P_m(\tau)\big)^{2n} \mathrm{d}\tau \\ &= 2^{mn/5} \sum_{2^m < p_1, \dots, p_{2n} \leqslant 2^{m+1}} \Big(\prod_{1 \leqslant i \leqslant 2n} \frac{b_f(p_i^2)}{2p_i} \Big) \int_T^{T+H} \prod_{1 \leqslant i \leqslant 2n} \cos(2\tau \log p_i) \mathrm{d}\tau \\ &= 2^{mn/5} \sum_{2^m < p_1, \dots, p_{2n} \leqslant 2^{m+1}} \Big(\prod_{1 \leqslant i \leqslant 2n} \frac{b_f(p_i^2)}{2p_i} \Big) \big\{ H\Theta(p_1 \cdots p_{2n}) + O\big((p_1 \cdots p_{2n})^2\big) \big\},\end{aligned}$$

where

$$n = n(m) := \begin{cases} [2^{3m/4}] & \text{if } 2^m \leqslant \log T, \\ c(\varepsilon, r) & \text{if } 2^m > \log T. \end{cases} \tag{3.17}$$

Here $c(\varepsilon, r)$ is a positive constant large enough and $c(\varepsilon, r) \in [6r^2 + 2r, \mathrm{e}^{c(\varepsilon)r}\varepsilon/700]$. With such choice of n, it is easy to see that the contribution of the error term $O\big((p_1 \cdots p_{2n})^2\big)$ to $|\mathscr{P}(m)|$ is

$$\ll 2^{mn/5} 2^{5mn} \ll T^{\varepsilon/10},$$

since $|b_f(p^2)| \leqslant 2$. For the main term, according to the definition of $\Theta(\cdot)$, we can assume that $p_1 \cdots p_{2n} = q_1^2 \cdots q_n^2$. We can bound the contribution of the main term to $\mathscr{P}(m)$ as before to write, with the notation (3.14),

$$\begin{aligned}|\mathscr{P}(m)| &\ll H 2^{mn/5} \sum_{2^m < q_1, \dots, q_n \leqslant 2^{m+1}} \Big(\prod_{1 \leqslant i \leqslant n} \frac{b_f(q_i^2)^2}{4q_i^2} \Big) \Phi(q_1, \dots, q_n) + H^{1/10} \\ &\leqslant H 2^{mn/5} \Big(\sum_{2^m < q \leqslant 2^{m+1}} \frac{1}{q^2} \Big)^n \frac{(2n)!}{2^{2n} n!} + H^{1/10} \\ &\ll H 2^{-(4/5)mn} n^n + H^{1/10}.\end{aligned} \tag{3.18}$$

By Cauchy's inequality, Proposition 3.1 and (3.18) allow us to deduce that

$$\begin{aligned}\mathfrak{J}_m &\ll \Big\{ \int_{\mathscr{T}} \exp\Big(4r \Re e \sum_{p \leqslant T^{\psi_I}} \frac{\lambda_f(p) \log(T^{\psi_I}/p)}{p^{1/2 + 1/(\psi_I \log T) + \mathrm{i}\tau} \log T^{\psi_I}} \Big) \mathrm{d}\tau \int_{\mathscr{P}(m)} (\log T)^{2r} \mathrm{d}\tau \Big\}^{1/2} \\ &\ll \Big(H (\log T)^{4r^2} \times \big(H 2^{-(4/5)mn} n^n + H^{1/10} \big) (\log T)^{2r} \Big)^{1/2} \\ &\ll H (\log T)^{2r^2 + r} 2^{-(2/5)mn} n^{n/2} + H^{3/5}.\end{aligned} \tag{3.19}$$

This and (3.17) imply that

$$\begin{aligned}&\sum_{(\log_2 T)^2 \leqslant 2^m \leqslant T^{\psi_I/2}} \mathfrak{J}_m \\ &\ll H (\log T)^{2r^2 + r} \Big(\sum_{(\log_2 T)^2 \leqslant 2^m \leqslant \log T} 2^{-(1/40)mn} + \sum_{\log T \leqslant 2^m \leqslant T^{\psi_I/2}} 2^{-(2/5)mn} \Big) + H^{9/10} \\ &\ll H (\log T)^{2r^2 + r} \big(\mathrm{e}^{-(\log_2 T)^{3/2}/30} + (\log T)^{-c(\varepsilon, r)/10} \big) + H^{9/10} \\ &\ll H (\log T)^{r^2}.\end{aligned} \tag{3.20}$$

Now we bound $\mathfrak{J}_m$ when $2^m \leqslant (\log_2 T)^2$ following the argument of Harper. By the definition of $\mathscr{P}(m)$, we have

$$\Re e\bigg(\sum_{p\leqslant 2^{m+1}}\frac{\lambda_f(p)\log(T^{\psi_I}/p)}{p^{1/2+1/(\psi_I\log T)+\mathrm{i}\tau}\log T^{\psi_I}}+\frac{1}{2}\sum_{p\leqslant T^{\psi_I/2}}\frac{b_f(p^2)}{p^{1+2\mathrm{i}\tau}}\bigg)\ll\sum_{p\leqslant 2^{m+1}}\bigg(\frac{1}{\sqrt{p}}+\frac{1}{p}\bigg)+1\leqslant c2^{m/2}$$

for $\tau\in\mathscr{P}(m)$, where $c>0$ is an absolute positive constant. Introduce the notation

$$F(\tau):=\Re e\sum_{2^{m+1}<p\leqslant T^{\psi_I}}\frac{\lambda_f(p)\log(T^{\psi_I}/p)}{p^{1/2+1/(\psi_I\log T)+\mathrm{i}\tau}\log T^{\psi_I}}.$$

According to the definition of $\mathscr{T}$ and $\mathscr{T}(m)$, and similarly to (3.8), we have

$$\begin{aligned}\mathfrak{J}_m&\ll \mathrm{e}^{cr2^{m/2}}\int_{\mathscr{P}(m)}\exp\{2rF(\tau)\}\mathrm{d}\tau\\&\ll \mathrm{e}^{cr2^{m/2}}2^{mn/5}\int_{\mathscr{T}}P_m(\tau)^{2n}\prod_i\bigg(\sum_{0\leqslant j\leqslant 100r\psi_i^{-3/4}}\frac{(rF_i(\tau))^j}{j!}\bigg)^2\mathrm{d}\tau\end{aligned}$$

where $F_i(\tau)$ is defined by (3.5) and i in the product satisfies $T^{\psi_{i-1}}>2^{m+1}$ and $i<I$. Since the primes p in $P_m(\tau)$ are different from those in $F(\tau)$, Lemma 2.5 is applicable with M (product of primes from $P_m(\tau)$) and N (product of primes from $F(\tau)$). A similar argument for proving (3.13) and (3.15) allows us to deduce, for $2^m\leqslant(\log_2 T)^2$ and $n=[2^{3m/4}]$,

$$\begin{aligned}\mathfrak{J}_m&\ll H\mathrm{e}^{cr2^{m/2}}2^{mn/5}\exp\bigg(r^2\sum_{2^{m+1}<p\leqslant T^{\psi_I}}\frac{|\lambda_f(p)|^2}{p}\bigg)\sum_{2^m<p_1,\dots,p_{2n}\leqslant 2^{m+1}}\frac{\Theta(p_1\cdots p_{2n})}{p_1\cdots p_{2n}}+H^{3/10}\\&\ll H(\log T)^{r^2}\mathrm{e}^{cr2^{m/2}}2^{mn/5}\frac{(2n)!}{2^{2n}n!}\bigg(\sum_{2^m<p\leqslant 2^{m+1}}\frac{1}{p^2}\bigg)^n+H^{3/10}\\&\ll H(\log T)^{r^2}\mathrm{e}^{cr2^{m/2}-2^{3m/4}}.\end{aligned}\tag{3.21}$$

This implies that

$$\sum_{2^m\leqslant(\log_2 T)^2}\mathfrak{J}_m\ll H(\log T)^{r^2}\sum_{2^m\leqslant(\log_2 T)^2}\mathrm{e}^{cr2^{m/2}-2^{3m/4}}\ll H(\log T)^{r^2}.\tag{3.22}$$

Inserting (3.20) and (3.22) into (3.16), we get the required inequality. □

For $1\leqslant i\leqslant j\leqslant I$, define

$$G_{i,j}(\tau):=\Re e\bigg(\sum_{T^{\psi_{i-1}}<p\leqslant T^{\psi_i}}\frac{\lambda_f(p)\log(T^{\psi_j}/p)}{p^{1/2+1/(\psi_j\log T)+\mathrm{i}\tau}\log T^{\psi_j}}\bigg).$$

And for $0\leqslant j\leqslant I-1$, define

$$\begin{aligned}\mathscr{S}_j:=\big\{\tau\in[T,T+H]:|G_{i,\ell}(\tau)|\leqslant\psi_i^{-3/4}\ (1\leqslant i\leqslant j\ \text{and}\ i\leqslant\ell\leqslant I)\\ \text{but}\ |G_{j+1,\ell}(\tau)|>\psi_{j+1}^{-3/4}\ \text{for some}\ j+1\leqslant\ell\leqslant I\big\}.\end{aligned}\tag{3.23}$$

Similarly to Lemma 3.1, we have the following lemma.

Lemma 3.3. *Let $f \in \mathcal{H}_\kappa^*$, $r > 0$ and $0 < \varepsilon \leqslant 1$. Then we have*

$$\int_{\mathscr{S}_j} \exp\bigg(2r\Re e \sum_{p\leqslant T^{\psi_j}} \frac{\lambda_f(p)\log(T^{\psi_j}/p)}{p^{1/2+1/(\psi_j\log T)+\mathrm{i}\tau}\log T^{\psi_j}}\bigg)\mathrm{d}\tau \ll H(\log T)^{r^2}\exp\big(-\tfrac{\varepsilon}{51}\psi_{j+1}^{-1}\log\psi_{j+1}^{-1}\big)$$

uniformly for $T \geqslant 100$, $T^\varepsilon \leqslant H \leqslant T$ and $1 \leqslant j \leqslant I-1$, where the implied constant depends on f, r and ε. What's more, we have

$$|\mathscr{S}_0| \ll H\mathrm{e}^{-(\log_2 T)^2/10}.$$

Proof. Since the proof is similar to that of Proposition 3.1, we just sketch the proof here. For $1 \leqslant j < k$, define

$$\mathscr{S}_{j,k} := \big\{\tau \in [T, T+H] : |G_{i,j}(\tau)| \leqslant \psi_i^{-3/4}\ (1\leqslant i\leqslant j), \text{but}\, |G_{j+1,k}(\tau)| > \psi_{j+1}^{-3/4}\big\}.$$

Denote by $\mathfrak{K}_j$ the integral to bound and by $\mathfrak{K}_{j,k}$ the corresponding integral over $\mathscr{S}_{j,k}$. Then

$$\mathfrak{K}_j \leqslant \mathfrak{K}_{j,j+1} + \mathfrak{K}_{j,j+2} + \cdots + \mathfrak{K}_{j,I} \tag{3.24}$$

and with the notation $\ell_\varepsilon := [\varepsilon/(10\psi_{j+1})]$

$$\begin{aligned}
\mathfrak{K}_{j,k} &= \int_{\mathscr{S}_{j,k}} \prod_{1\leqslant i\leqslant j} \big(\exp\{rG_{i,j}(\tau)\}\big)^2\mathrm{d}\tau \\
&\ll \int_{\substack{|G_{i,j}(\tau)|\leqslant\psi_i^{-3/4}\\(1\leqslant i\leqslant j)}} \prod_{1\leqslant i\leqslant j} \big(\exp\{rG_{i,j}(\tau)\}\big)^2\big(\psi_{j+1}^{3/4}G_{j+1,k}(\tau)\big)^{2\ell_\varepsilon}\mathrm{d}\tau \\
&\ll \psi_{j+1}^{(3/2)\ell_\varepsilon}\int_T^{T+H} \prod_{1\leqslant i\leqslant j}\bigg(\sum_{0\leqslant n\leqslant 100r\psi_i^{-3/4}} \frac{(rG_{i,j}(\tau))^n}{n!}\bigg)^2 G_{j+1,k}(\tau)^{2\ell_\varepsilon}\mathrm{d}\tau.
\end{aligned}$$

In the last inequality, we have a similar argument to establish the first inequality of (3.10). Expand the square and the power n and $2\ell_\varepsilon$, and proceed as the proof in Lemma 3.1 and Proposition 3.2 for estimating the last integral. We can obtain

$$\mathfrak{K}_{j,k} \ll \psi_{j+1}^{(3/2)\ell_\varepsilon}\bigg\{H\exp\bigg(r^2\sum_{p\leqslant T^{\psi_j}}\frac{|\lambda_f(p)|^2}{p}\bigg)\bigg(\frac{\varepsilon\psi_{j+1}^{-1}}{20}\sum_{T^{\psi_j}<p\leqslant T^{\psi_{j+1}}}\frac{|\lambda_f(p)|^2}{p}\bigg)^{\ell_\varepsilon} + H^{4/5}\bigg\}$$

for $j+1 \leqslant k \leqslant I$. Inserting it into (3.24), we get

$$\begin{aligned}
\mathfrak{K}_j &\ll H(I-j)\psi_{j+1}^{(3/2)\ell_\varepsilon}\times \\
&\quad\times\bigg\{\exp\bigg(r^2\sum_{p\leqslant T^{\psi_j}}\frac{|\lambda_f(p)|^2}{p}\bigg)\bigg(\frac{\varepsilon\psi_{j+1}^{-1}}{20}\sum_{T^{\psi_j}<p\leqslant T^{\psi_{j+1}}}\frac{|\lambda_f(p)|^2}{p}\bigg)^{\ell_\varepsilon} + H^{-1/5}\bigg\}.
\end{aligned} \tag{3.25}$$

For $j = 0$, the left-hand side of (3.25) is $|\mathscr{S}_0|$ which is

$$\ll HI\psi_1^{(3/2)[\varepsilon/(10\psi_1)]}\big\{(\frac{\psi_1}{20})^{-[\varepsilon/(10\psi_1)]} + H^{-1/5}\big\} \ll H\mathrm{e}^{-\varepsilon(\log_2 T)^2/10},$$

with the help of (2.4) and the fact that $I \ll \log_3 T$ and $\psi_1 = (\log_2 T)^{-2}$.

For $1 \leqslant j \leqslant I-1$, the contribution of $H^{-1/5}$ on the right-hand side of (3.25) to $\mathfrak{K}_j$ is

$$\ll (I-j)\psi_{j+1}^{(3/2)[\varepsilon/(10\psi_{j+1})]}H^{4/5} \ll H\exp\big(-\tfrac{\varepsilon}{51}\psi_{j+1}^{-1}\log\psi_{j+1}^{-1}\big)$$

since $I \leqslant \log_3 T$. Observing that $I - j = \log(\psi_I/\psi_j)/\log 20 \leqslant \frac{\log \psi_{j+1}^{-1}}{\log 20}$, and

$$\sum_{T^{\psi_j} < p \leqslant T^{\psi_{j+1}}} \frac{|\lambda_f(p)|^2}{p} \leqslant 4 \sum_{T^{\psi_j} < p \leqslant T^{\psi_{j+1}}} \frac{1}{p} \leqslant 40,$$

the contribution of the first term on the right-hand side of (3.25) to $\mathfrak{K}_j$ is

$$\ll H(\log T)^{r^2} \exp\left(-\tfrac{\varepsilon}{51}\psi_{j+1}^{-1}\log\psi_{j+1}^{-1}\right),$$

according to (2.4) again. □

Similarly to Proposition 3.2, we have the following proposition. The proof is very similar. The only difference is to apply Lemma 3.3 in place of Lemma 3.1.

Proposition 3.4. *Let $f \in \mathcal{H}_\kappa^*$, $r > 0$ and $0 < \varepsilon \leqslant 1$. Then we have*

$$\int_{\mathscr{I}_j} \exp\left\{2r\Re e\left(\sum_{p \leqslant T^{\psi_j}} \frac{\lambda_f(p)\log(T^{\psi_j}/p)}{p^{1/2+1/(\psi_j \log T)+\mathrm{i}\tau}\log T^{\psi_j}} + \frac{1}{2}\sum_{p \leqslant T^{\psi_j/2}} \frac{b_f(p^2)}{p^{1+2\mathrm{i}\tau}}\right)\right\}\mathrm{d}\tau$$
$$\ll H(\log T)^{r^2}\exp\left(-\tfrac{\varepsilon}{51}\psi_{j+1}^{-1}\log\psi_{j+1}^{-1}\right)$$

uniformly for $T \geqslant 100$, $T^\varepsilon \leqslant H \leqslant T$ and $1 \leqslant j \leqslant I-1$, where the implied constant depends on f, r and ε.

Next we follow the method of Soundararajan to give a weaker estimate for $M_r(T, H; f)$, which will consist in the first step of the iteration presented in Section 4.

Proposition 3.5. *Let $f \in \mathcal{H}_\kappa^*$, $r > 0$ and $0 < \varepsilon \leqslant 1$. Assuming GRH for $L(s, f)$, there is a constant $c_0(r, \varepsilon)$ such that*

$$M_r(T, H; f) \ll_{f,r,\varepsilon} H(\log T)^{c_0(r,\varepsilon)}$$

holds uniformly for $T \geqslant 2$ and $T^\varepsilon \leqslant H \leqslant T$.

Proof. For $T \geqslant 2$, $T^\varepsilon \leqslant H \leqslant T$ and $v \in \mathbb{R}$, define

$$\mathscr{S}_{T,H}(v) := \big|\{\tau \in [T, T+H] : \log|L(\tfrac{1}{2}+\mathrm{i}\tau, f)| \geqslant v\}\big|.$$

We can write

$$M_r(T, H; f) = -\int_{\mathbb{R}} \mathrm{e}^{2rv}\mathrm{d}\mathscr{S}_{T,H}(v) = 2r\int_{\mathbb{R}} \mathrm{e}^{2rv}\mathscr{S}_{T,H}(v)\mathrm{d}v. \tag{3.26}$$

Define $x := T^{4/v}$ and $z := T^{4/(v\log_2 T)}$. By bounding the second sum on the right-hand side of (2.11) of Proposition 2.1 trivially and by taking $\varpi = \frac{1}{2}$, we have

$$\log|L(\tfrac{1}{2}+\mathrm{i}\tau, f)| \leqslant S_1(\tau) + S_2(\tau) + \tfrac{1}{2}v$$

for $v \geqslant 10\log_2 T$ and $T \geqslant T_0(f, r, \varepsilon)$, where

$$S_1(\tau) := \left|\sum_{p \leqslant z} \frac{b_f(p)}{p^{1/2+1/2\log x+\mathrm{i}\tau}}\frac{\log(x/p)}{\log x}\right|,$$
$$S_2(\tau) := \left|\sum_{z < p \leqslant x} \frac{b_f(p)}{p^{1/2+1/2\log x+\mathrm{i}\tau}}\frac{\log(x/p)}{\log x}\right|.$$

When $\log|L(\frac12+\mathrm{i}\tau,f)|\geqslant v$, we have

$$S_1(\tau)\geqslant \tfrac38 v=:v_1 \qquad \text{or} \qquad S_2(\tau)\geqslant \tfrac18 v=:v_2.$$

With the help of Lemma 2.2, (2.4) and the Stirling formula $n!\sim\sqrt{2\pi n}(n/\mathrm{e})^n$ $(n\to\infty)$, we can deduce, for any positive integer $\ell\leqslant \log(H/\log H)/\log z$,

$$\big|\{\tau\in[T,T+H]: S_1(\tau)\geqslant v_1\}\big|\leqslant \int_T^{T+H}\Big(\frac{S_1(\tau)}{v_1}\Big)^{2\ell}\mathrm{d}\tau \ll H\sqrt{\ell}\Big(\frac{\ell\log_2 T}{\mathrm{e}v_1^2}\Big)^{\ell}.$$

Taking $\ell=[v_1^2/\log_2 T]$ if $v\leqslant\frac{\varepsilon}{2}(\log_2 T)^2$ and $\ell=[10v]$ if $v>\frac{\varepsilon}{2}(\log_2 T)^2$, then we get

$$|\{\tau\in[T,T+H]: S_1(\tau)\geqslant v_1\}|\ll\begin{cases}\dfrac{Hv}{\sqrt{\log_2 T}}\mathrm{e}^{-9v^2/(64\log_2 T)} & \text{if } v\leqslant\frac{\varepsilon}{2}(\log_2 T)^2,\\ H\mathrm{e}^{-4v\log v} & \text{if } v>\frac{\varepsilon}{2}(\log_2 T)^2.\end{cases}\tag{3.27}$$

Similarly we have

$$\big|\{\tau\in[T,T+H]: S_2(\tau)\geqslant v_2\}\big|\leqslant\int_T^{T+H}\Big|\frac{8S_2(\tau)}{v}\Big|^{2\ell}\mathrm{d}\tau.$$

Taking $\ell=[v\varepsilon/4-1]$ and using Lemma 2.2 again, we obtain

$$\big|\{\tau\in[T,T+H]: S_2(\tau)\geqslant \tfrac18 v\}\big|\ll H\mathrm{e}^{-\frac{\varepsilon}{8}v\log v}.$$

This and (3.27) imply that

$$\mathscr{S}_{T,H}(v)\ll\begin{cases}\dfrac{Hv}{\sqrt{\log_2 T}}\mathrm{e}^{-9v^2/(64\log_2 T)} & \text{for } 10\log_2 T\leqslant v\leqslant\frac12(\log_2 T)\log_3 T,\\ H\mathrm{e}^{-\frac{\varepsilon}{33}v\log v} & \text{for } v>\frac12(\log_2 T)\log_3 T.\end{cases}$$

Together with (3.26) and the trivial bound $\mathscr{S}_{T,H}(v)\leqslant H$ for $v\leqslant 10\log_2 T$, we can obtain the required inequality. □

4. Proof of the upper bound for the higher moments

We have been ready for the proof of the upper bound of (1.9). Let $f\in\mathcal{H}_\kappa^*$, $r>0$ and $0<\varepsilon\leqslant 1$. Let I, $\mathscr{T}$ and $\mathscr{S}_j$ be defined as in (3.3), (3.4), and (3.23). Then

$$[T,T+H]=\mathscr{T}\cup\big(\bigcup_{0\leqslant j\leqslant I-1}\mathscr{S}_j\big).$$

Thus we can write

$$M_r(T,H;f)\leqslant\mathfrak{L}+\mathfrak{L}_0+\mathfrak{L}_1+\cdots+\mathfrak{L}_{I-1},\tag{4.1}$$

where

$$\mathfrak{L}:=\int_{\mathscr{T}}|L(\tfrac12+\mathrm{i}\tau,f)|^{2r}\mathrm{d}\tau,\qquad \mathfrak{L}_j:=\int_{\mathscr{S}_j}|L(\tfrac12+\mathrm{i}\tau,f)|^{2r}\mathrm{d}\tau.$$

Assume GRH for $L(s,f)$.

First we can apply (3.1) with $x=T^{\psi_I}$ and Proposition 3.2 to deduce immediately that

$$\mathfrak{L}\ll H(\log T)^{r^2}.\tag{4.2}$$

Secondly the Cauchy-Schwarz inequality, Lemma 3.3 and Propositions 3.5 imply

$$\mathfrak{L}_0 \leqslant \Big(|\mathscr{S}_0|\int_{\mathscr{S}_0}|L(\tfrac12+\mathrm{i}\tau,f)|^{4r}\mathrm{d}\tau\Big)^{1/2} \ll \Big(H\mathrm{e}^{-(\log_2 T)^2/10}H(\log T)^{c_0(\varepsilon,r)}\Big)^{1/2} \ll H. \tag{4.3}$$

Finally, for $1\leqslant j\leqslant I-1$ inequality (3.1) with $x=T^{\psi_j}$ implies that

$$\log|L(\tfrac12+\mathrm{i}\tau,f)| \leqslant \Re e\Big(\sum_{p\leqslant T^{\psi_j}}\frac{\lambda_f(p)\log(T^{\psi_j}/p)}{p^{1/2+1/(\psi_j\log T)+\mathrm{i}\tau}\log T^{\psi_j}}+\frac12\sum_{p\leqslant T^{\psi_j/2}}\frac{b_f(p^2)}{p^{1+2\mathrm{i}\tau}}\Big)+\frac{2}{\psi_j}+O_f(1).$$

By Proposition 3.4, it follows that

$$\mathfrak{L}_j \ll H(\log T)^{r^2}\exp\big(-\tfrac{\varepsilon}{51}\psi_{j+1}^{-1}\log\psi_{j+1}^{-1}+4r\psi_j^{-1}\big).$$

Summing over $1\leqslant j\leqslant I-1$ and using a similar argument to (3.9), we can deduce that

$$\mathfrak{L}_1+\cdots+\mathfrak{L}_{I-1}\ll H(\log T)^{r^2}. \tag{4.4}$$

Together with (4.2), (4.3) and (4.4), we get the upper bound in Theorem 1.1.

5. Proof of the lower bound

In this section, we adapt Heath-Brown's method [11] to prove the lower bound part in Theorem 1.1. As indicated in the introduction, in order to obtain the correct order of $M_r(T,H;f)$, we need to apply the Rankin-Selberg theory.

Let $f\in\mathcal{H}_\kappa^*$ and $r>0$. When $L(s,f)\neq0$, we define $L(s,f)^r$ by

$$L(s,f)^r=\exp(r\log L(s,f)),$$

where $\log L(s,f)=\log|L(s,f)|+\arg L(s,f)$ with $-\pi<\arg L(s,f)\leqslant\pi$. We also define $\lambda_{f,r}(n)$ by the formula

$$L(s,f)^r=\sum_{n\geqslant1}\lambda_{f,r}(n)n^{-s}\qquad(\sigma>1). \tag{5.1}$$

Clearly $\lambda_{f,r}(n)$ is multiplicative, and for all positive integers j and n we have

$$\lambda_{f,rj}(n)=\sum_{n=n_1n_2\cdots n_j}\lambda_{f,r}(n_1)\lambda_{f,r}(n_2)\cdots\lambda_{f,r}(n_j). \tag{5.2}$$

Specially, we have

$$\lambda_{f,r}(p)=r\lambda_f(p)\qquad\text{and}\qquad\forall\varepsilon>0:\lambda_{f,r}(n)\ll_{r,\varepsilon}n^{\varepsilon} \tag{5.3}$$

for all prime numbers p and integers n.

In the sequel we write $r=u/v$. Here $v=1$ and u is any positive real number when GRH holds for $L(s,f)$ (to ensure that $g_{f,r}(s,N)$ defined in (5.6) is a regular function); if not, u and v are positive co-prime integers.

5.1. Some convexity estimates.

For $\frac12\leqslant\sigma\leqslant\frac34$ and $2\leqslant H\leqslant T$, define

$$J_{T,H}(\sigma):=\int_{\mathbb{R}}|L(\sigma+\mathrm{i}\tau,f)|^{2r}w_{T,H}(\tau)\mathrm{d}\tau\quad\text{with}\quad w_{T,H}(\tau):=\int_{\Delta_1}^{\Delta_2}\mathrm{e}^{-2r(\tau-t)^2}\mathrm{d}t,$$

where $\Delta_1=T+H^{1/4}$ and $\Delta_2=T+H-H^{1/4}$.

With the help of Lemma 2.4, we can prove the following convexity estimate.

Lemma 5.1. *Let $f \in \mathcal{H}_\kappa^*$, $r > 0$ and $0 < \varepsilon \leqslant 1$. Then we have*

$$J_{T,H}(\tfrac{1}{2}) \ll T^{r(2\sigma-1)} J_{T,H}(\sigma) + \mathrm{e}^{-rT^2/20} \tag{5.4}$$

$$J_{T,H}(\sigma) \ll H^{(\sigma-1/2)} J_{T,H}(\tfrac{1}{2})^{3/2-\sigma} \tag{5.5}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$, $T \geqslant 2$ and $T^\varepsilon \leqslant H \leqslant T$, where the implied constants depend on f, r and ε.

Proof. With the help of the functional equation (1.6) and the Stirling formula, we have

$$|L(1-\sigma+\mathrm{i}\tau, f)| \ll_f |L(\sigma+\mathrm{i}\tau, f)|(1+|\tau|)^{2\sigma-1}.$$

Define $F(s) = L(s,f)\mathrm{e}^{(s-\mathrm{i}t)^2}$ for $t \geqslant 2$. Then it follows that

$$\int_{\mathbb{R}} |F(1-\sigma+\mathrm{i}\tau)|^{2r}\mathrm{d}\tau \ll \int_{\mathbb{R}} |L(\sigma+\mathrm{i}\tau, f)|^{2r}(1+|\tau|)^{2r(2\sigma-1)}\mathrm{e}^{-2r(\tau-t)^2}\mathrm{d}\tau.$$

In view of the convexity bound for $L(\sigma+\mathrm{i}\tau, f)$, the contribution of the lines $(-\infty, t/2] \cup [3t/2, \infty)$ to the last integral is

$$\ll \bigg(\int_{-\infty}^{t/2} + \int_{3t/2}^{\infty}\bigg)(1+|\tau|)^{2r}\mathrm{e}^{-2r(\tau-t)^2}\mathrm{d}\tau \ll t^{2r}\mathrm{e}^{-rt^2/2} \ll \mathrm{e}^{-rt^2/3}.$$

Therefore

$$\int_{\mathbb{R}} |F(1-\sigma+\mathrm{i}\tau)|^{2r}\mathrm{d}\tau \ll \mathrm{e}^{-rt^2/3} + t^{2r(2\sigma-1)}\int_{\mathbb{R}} |L(\sigma+\mathrm{i}\tau, f)|^{2r}\mathrm{e}^{-2r(\tau-t)^2}\mathrm{d}\tau.$$

Applying Lemma 2.4 to $F(s)$ with $(\alpha, \gamma, \beta) = (1-\sigma, \frac{1}{2}, \sigma)$ and $q = 2r$ and using the preceding inequality, we can deduce

$$\int_{\mathbb{R}} |L(\tfrac{1}{2}+\mathrm{i}\tau, f)|^{2r}\mathrm{e}^{-2r(\tau-t)^2}\mathrm{d}\tau \ll \mathrm{e}^{-rt^2/8} + t^{r(2\sigma-1)}\int_{\mathbb{R}} |L(\sigma+\mathrm{i}\tau, f)|^{2r}\mathrm{e}^{-2r(\tau-t)^2}\mathrm{d}\tau.$$

Then we can get (5.4) by integrating for $\Delta_1 \leqslant t \leqslant \Delta_2$.

Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}} |F(\sigma+\mathrm{i}\tau)|^{2r}\mathrm{d}\tau &\leqslant \bigg(\int_{\mathbb{R}} |F(\tfrac{1}{2}+\mathrm{i}\tau)|^{2r}\mathrm{d}\tau\bigg)^{3/2-\sigma}\bigg(\int_{\mathbb{R}} |F(\tfrac{3}{2}+\mathrm{i}\tau)|^{2r}\mathrm{d}\tau\bigg)^{\sigma-1/2} \\ &\ll \bigg(\int_{\mathbb{R}} |L(\tfrac{1}{2}+\mathrm{i}\tau, f)|^{2r}\mathrm{e}^{-2r(\tau-t)^2}\mathrm{d}\tau\bigg)^{3/2-\sigma},\end{aligned}$$

since $\int_{\mathbb{R}} |F(\frac{3}{2}+\mathrm{i}\tau)|^{2r}\mathrm{d}\tau \ll 1$. Finally, integrating for $\Delta_1 \leqslant t \leqslant \Delta_2$ and using the Hölder inequality, we obtain (5.5). □

For $N \geqslant 2$, $r = u/v$ and $\sigma \geqslant \frac{1}{2}$, define

$$S_{f,r}(s;N) := \sum_{n\leqslant N} \lambda_{f,r}(n)n^{-s}, \qquad g_{f,r}(s;N) := L(s,f)^u - S_{f,r}(s;N)^v, \tag{5.6}$$

where $\lambda_{f,r}(n)$ is defined as in (5.1). Define

$$K_{T,H,N}(\sigma) := \int_{\mathbb{R}} |g_{f,r}(\sigma+\mathrm{i}\tau; N)|^{2/v} w_{T,H}(\tau)\mathrm{d}\tau.$$

Lemma 5.2. *Let $f \in \mathcal{H}_\kappa^*$, $r > 0$ and $0 < \varepsilon \leqslant 1$. Then we have*

$$K_{T,H,N}(\sigma) \ll_{f,r,\varepsilon} K_{T,H,N}(\tfrac{1}{2})^{3/2-\sigma}(HN^{-2/v+\varepsilon})^{\sigma-1/2}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$, $T \geqslant 2$, $T^\varepsilon \leqslant H \leqslant T$ and $T^{\varepsilon/2} \leqslant N \leqslant T^\varepsilon$.

Proof. Applying Lemma 2.4 to $F(s) = g_{f,r}(s;N)\mathrm{e}^{u(s-\mathrm{i}t)^2}$ with $(\alpha,\gamma,\beta) = (\frac{1}{2},\sigma,\frac{3}{2})$ and $q = 2/v$ where $t \geqslant 2$ is a parameter, it follows that for $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$,

$$\int_{\mathbb{R}} |F(\sigma+\mathrm{i}\tau)|^{2/v}\mathrm{d}\tau \leqslant \bigg(\int_{\mathbb{R}} |F(\tfrac{1}{2}+\mathrm{i}\tau)|^{2/v}\mathrm{d}\tau\bigg)^{3/2-\sigma}\bigg(\int_{\mathbb{R}} |F(\tfrac{3}{2}+\mathrm{i}\tau)|^{2/v}\mathrm{d}\tau\bigg)^{\sigma-1/2}.$$

Integrating for $\Delta_1 \leqslant t \leqslant \Delta_2$ and using the Hölder inequality, we have

$$K_{T,H,N}(\sigma) \leqslant K_{T,H,N}(\tfrac{1}{2})^{3/2-\sigma}K_{T,H,N}(\tfrac{3}{2})^{\sigma-1/2}. \tag{5.7}$$

Observing that $S_{f,r}(s;N) \ll N \ll T^\varepsilon$ for $\frac{1}{2} \leqslant \sigma \leqslant 2$, $\tau \in \mathbb{R}$ and $L(\frac{3}{2}+\mathrm{i}\tau,f) \ll 1$ $(\tau \in \mathbb{R})$, we conclude that $g_{f,r}(\frac{3}{2}+\mathrm{i}\tau;N) \ll T^{v\varepsilon}$ $(\tau \in \mathbb{R})$. Therefore

$$\begin{aligned} K_{T,H,N}(\tfrac{3}{2}) &\ll \int_{\Delta_1}^{\Delta_2}\int_{t-t^\varepsilon}^{T+H} \frac{|g_{f,r}(\frac{3}{2}+\mathrm{i}\tau;N)|^{2/v}}{\mathrm{e}^{2r(\tau-t)^2}}\mathrm{d}\tau\mathrm{d}t + \int_{\Delta_1}^{\Delta_2}\bigg(\int_{-\infty}^{t-t^\varepsilon}+\int_{T+H}^{\infty}\bigg)\frac{T^{2\varepsilon}\mathrm{d}\tau\mathrm{d}t}{\mathrm{e}^{2r(\tau-t)^2}} \\ &\ll \int_{\Delta_1}^{\Delta_2}\int_{t-t^\varepsilon}^{T+H} |g_{f,r}(\tfrac{3}{2}+\mathrm{i}\tau;N)|^{2/v}\mathrm{e}^{-2r(\tau-t)^2}\mathrm{d}\tau\mathrm{d}t + \mathrm{e}^{-rT^{2\varepsilon}} \\ &\ll \int_{T-3H}^{T+3H} |g_{f,r}(\tfrac{3}{2}+\mathrm{i}\tau;N)|^{2/v}w_{T,H}(\tau)\mathrm{d}\tau + \mathrm{e}^{-rT^{2\varepsilon}}. \end{aligned} \tag{5.8}$$

Since v is always a positive integer, in view of (5.2), we can write

$$g_{f,r}(s;N) = L(s,f)^{rv} - S_{f,r}(s;N)^v = \sum_{n>N} a_n n^{-s} \qquad (\sigma > 1),$$

where $a_n \ll n^\varepsilon$ $(n \geqslant 1)$ for any $\varepsilon > 0$ thanks to (5.3). So Lemma 2.3 implies that

$$\int_{T-3H}^{T+3H} |g_{f,r}(\tfrac{3}{2}+\mathrm{i}\tau;N)|^2\mathrm{d}\tau \ll H\sum_{n\geqslant N}|a_n|^2n^{-3} + \sum_{n\geqslant N}|a_n|^2n^{-2} \ll HN^{-2+\varepsilon},$$

since $N \leqslant H$. So we have

$$\begin{aligned} \int_{T-3H}^{T+3H} |g_{f,r}(\tfrac{3}{2}+\mathrm{i}\tau;N)|^{2/v}\mathrm{d}\tau &\ll \bigg(\int_{T-3H}^{T+3H} |g_{f,r}(\tfrac{3}{2}+\mathrm{i}\tau;N)|^2\mathrm{d}\tau\bigg)^{1/v}H^{1-1/v} \\ &\ll (HN^{-2+\varepsilon})^{1/v}H^{1-1/v} \ll HN^{-2/v+\varepsilon}. \end{aligned}$$

Putting it back to (5.8) and then to (5.7) and noticing that the term $\mathrm{e}^{-rT^{2\varepsilon}}$ can be absorbed by $HN^{-2/v+\varepsilon}$, we obtain the required inequality. □

5.2. **Companion to $J_{T,H}(\sigma)$ and $K_{T,H,N}(\sigma)$.** In this subsection we shall apply the Rankin-Selberg theory to prove Lemma 5.3 below, which will consist in the main tool in this section. For $f \in \mathcal{H}_\kappa^*$, the Rankin-Selberg L-function is defined by

$$L(s,f\times f) := \prod_p \big(1-\alpha_f(p)^2p^{-s}\big)^{-1}\big(1-\beta_f(p)^2p^{-s}\big)^{-1}\big(1-p^{-s}\big)^{-2} \qquad (\sigma > 1), \tag{5.9}$$

where $\alpha_f(p)$ and $\beta_f(p)$ are the local parameters of f. According to the Rankin-Selberg theory, it is well know that $L(s, f\times f)$ has a simple pole at $s=1$ (see e.g. [13]). Thus there are two positive constants A_f and B_f such that

$$A_f \leqslant (s-1)L(s,f\times f)\leqslant B_f \qquad (1<s\leqslant 2). \tag{5.10}$$

Lemma 5.3. *Let $f\in\mathcal{H}_\kappa^*$ and $r>0$. There are positive constants $C_{f,r}^*$ and $N_{f,r}$ such that*

$$S_{f,r}^*(2\sigma;N):=\sum_{n\leqslant N}\lambda_{f,r}(n)^2n^{-2\sigma}\asymp(\sigma-\tfrac12)^{-r^2} \tag{5.11}$$

uniformly for $N\geqslant N_{f,r}$ and $\frac12+\frac{C_{f,r}^*}{\log N}\leqslant\sigma\leqslant 1$. Moreover for $N\geqslant N_{f,r}$ we have

$$S_{f,r}^*(1;N)\asymp(\log N)^{r^2}. \tag{5.12}$$

Here the implied constants depend on f and r.

Proof. We write $\sigma=\frac12+\delta$ with $\delta>0$. Denote by $\mu(n)$ the Möbius function. Then

$$\begin{aligned}S_{f,r}^*(2\sigma;N)&\geqslant\sum_{n\geqslant1}\lambda_{f,r}(n)^2\mu(n)^2n^{-1-2\delta}\{1-(n/N)^\delta\}\\&=G_f(1+2\delta)-N^{-\delta}G_f(1+\delta),\end{aligned}$$

where

$$G_f(s):=\sum_{n\geqslant1}\lambda_{f,r}(n)^2\mu(n)^2n^{-s}=\prod_p\big(1+r^2\lambda_f(p)^2p^{-s}\big) \tag{5.13}$$

for real $s>1$ by (5.3). Write $H_f(s):=L(s,f\times f)^{-r^2}G_f(s)$. Since $H_f(s)>0$ for $s>1$, we can define $h_f(s):=\log H_f(s)$ for these s. Using (5.9) and (5.13), for $s>1$ we have

$$h_f(s)=\sum_p\Big(\log\big(1+r^2\lambda_f(p)^2p^{-s}\big)-r^2\log\big\{\big(1-\alpha_f(p)^2p^{-s}\big)^{-1}\big(1-\beta_f(p)^2p^{-s}\big)^{-1}\big(1-p^{-s}\big)^{-2}\big\}\Big).$$

Since $\alpha_f(p)^2+\beta_f(p)^2+2=\lambda_f(p)^2$, the series on the right-hand side is absolutely convergent for $s>\frac12$. Thus $H_f(s)\neq0$ for $\frac12<s\leqslant2$. Combining this with (5.10) and in view of continuity of $H_f(s)$ on $(\frac12,2]$, there are two positive constants $C_{f,r}$ and $D_{f,r}$ such that

$$C_{f,r}\leqslant H_f(s)\leqslant D_{f,r}\qquad(1\leqslant s\leqslant2). \tag{5.14}$$

Then

$$S_{f,r}^*(2\sigma;N)\geqslant(A_f/2\delta)^{r^2}C_{f,r}-N^{-\delta}(B_f/\delta)^{r^2}D_{f,r}\geqslant\tfrac12(A_f/2)^{r^2}C_{f,r}\delta^{-r^2},$$

provided $C_{f,r}^*$ is so large that $N^\delta\geqslant\mathrm{e}^{C_{f,r}^*}\geqslant2(2B_f/A_f)^{r^2}D_{f,r}/C_{f,r}$. This proves the lower bound part of (5.11).

Since $\lambda_{f,r}(n)$ is multiplicative and $\lambda_{f,r}(n)\ll n^\varepsilon$, we have

$$\begin{aligned}S_{f,r}^*(2\sigma;N)&\leqslant\prod_p\big\{1+r^2\lambda_f(p)^2p^{-2\sigma}+O(p^{-4\sigma+\varepsilon})\big\}\\&=L(2\sigma,f\times f)^{r^2}H_f(2\sigma)\prod_p\big\{1+O(p^{-4\sigma+\varepsilon})\big\}.\end{aligned}$$

Now the upper bound of (5.11) follows from (5.10) and (5.14). We take $\sigma=\frac12+\frac{C_{f,r}^*}{\log N}$ in (5.11) and we have $n^{-2\sigma}\asymp n^{-1}$ for $1\leqslant n\leqslant N$. Then (5.12) follows immediately from (5.11). □

Let $S_{f,r}(s;N)$ be defined in (5.6) and $0<\varepsilon\leqslant\frac{1}{2}$. For $\frac{1}{2}\leqslant\sigma\leqslant\frac{3}{4}$, $T\geqslant 2$, $T^{\varepsilon}\leqslant H\leqslant T$ and $T^{\varepsilon/2}\leqslant N\leqslant H^{1-\varepsilon}$, define

$$L_{T,H,N}(\sigma):=\int_{\mathbb{R}}|S_{f,r}(\sigma+\mathrm{i}\tau;N)|^2w_{T,H}(\tau)\mathrm{d}\tau.$$

Since $w_{T,H}(\tau)\gg 1$ for $\Delta_1+\frac{1}{2}H^{1/4}\leqslant\tau\leqslant\Delta_2-\frac{1}{2}H^{1/4}$, we can apply Lemma 2.3 to write

$$\begin{aligned}L_{T,H,N}(\sigma)&\gg\int_{\Delta_1+\frac{1}{2}H^{1/4}}^{\Delta_2-\frac{1}{2}H^{1/4}}|S_{f,r}(\sigma+\mathrm{i}\tau;N)|^2\mathrm{d}\tau\\&\gg\sum_{n\leqslant N}\lambda_{f,r}(n)^2n^{-2\sigma}\{H+O(n)\}\\&\gg H\sum_{n\leqslant N}\lambda_{f,r}(n)^2n^{-2\sigma}+O(N^{1+\varepsilon}).\end{aligned}$$

On the other hand, we have $w_{T,H}(\tau)\ll 1$ for all τ and $w_{T,H}(\tau)\ll\exp\{-r(H^{1/2}+\tau^2)/19\}$ for $\tau\leqslant\Delta_1-\frac{1}{2}H^{1/4}$ or $\tau\geqslant\Delta_2+\frac{1}{2}H^{1/4}$. It implies that

$$\begin{aligned}L_{T,H,N}(\sigma)&\ll\int_{\Delta_1-\frac{1}{2}H^{1/4}}^{\Delta_2+\frac{1}{2}H^{1/4}}|S_{f,r}(\sigma+\mathrm{i}\tau;N)|^2\mathrm{d}\tau+O(1)\\&\ll\sum_{n\leqslant N}\lambda_{f,r}(n)^2n^{-2\sigma}\{H+O(n)\}+O(1)\\&\ll H\sum_{n\leqslant N}\lambda_{f,r}(n)^2n^{-2\sigma}+O(N^{1+\varepsilon}).\end{aligned}$$

Then consequently by (5.11) of Lemma 5.3

$$L_{T,H,N}(\sigma)\asymp_{f,r,\varepsilon}H(\sigma-\tfrac{1}{2})^{-r^2}\tag{5.15}$$

for $T\geqslant T_0(f,r,\varepsilon)$, $T^{\varepsilon}\leqslant H\leqslant T$, $T^{\varepsilon/2}\leqslant N\leqslant H^{1-\varepsilon}$ and $\frac{1}{2}+\frac{C^*_{f,r}}{\log N}\leqslant\sigma\leqslant\frac{3}{4}$; and by (5.12) of Lemma 5.3

$$L_{T,H,N}(\tfrac{1}{2})\asymp_{f,r,\varepsilon}H(\log T)^{r^2}.\tag{5.16}$$

for $T\geqslant T_0(f,r,\varepsilon)$, $T^{\varepsilon}\leqslant H\leqslant T$ and $T^{\varepsilon/2}\leqslant N\leqslant H^{1-\varepsilon}$.

5.3. **End the proof of the lower bound.** Trivially we have

$$|S_{f,r}(s;N)^v|^{2/v}=|L(s,f)^u-g_{f,r}(s;N)|^{2/v}\ll|L(s,f)|^{2r}+|g_{f,r}(s;N)|^{2/v}.$$

Then it follows that

$$L_{T,H,N}(\sigma)\ll J_{T,H}(\sigma)+K_{T,H,N}(\sigma)\tag{5.17}$$

$$J_{T,H}(\sigma)\ll L_{T,H,N}(\sigma)+K_{T,H,N}(\sigma)\tag{5.18}$$

$$K_{T,H,N}(\tfrac{1}{2})\ll L_{T,H,N}(\tfrac{1}{2})+J_{T,H}(\tfrac{1}{2})\tag{5.19}$$

for $\frac{1}{2}\leqslant\sigma\leqslant\frac{3}{4}$, $T\geqslant T_0(f,r,\varepsilon)$, $T^{\varepsilon}\leqslant H\leqslant T$ and $T^{\varepsilon/2}\leqslant N\leqslant T^{\varepsilon}$, where $0<\varepsilon\leqslant\frac{1}{2}$ and $T_0(f,r,\varepsilon)$ is a constant depending on f,r,ε.

If $K_{T,H,N}(\frac{1}{2})\leqslant H$, (5.16) and (5.17) imply that

$$J_{T,H}(\tfrac{1}{2})\gg H(\log T)^{r^2}.\tag{5.20}$$

If $K_{T,H,N}(\frac{1}{2}) \geqslant H$, Lemma 5.2 follows that

$$K_{T,H,N}(\sigma) \ll K_{T,H,N}(\tfrac{1}{2})N^{-(2/v-\varepsilon)(\sigma-1/2)}. \tag{5.21}$$

We take $N = T^{\varepsilon/2}$ and $\sigma = \sigma_0 := \frac{1}{2} + \frac{2C^*_{f,r,\varepsilon}}{\log T}$, where $C^*_{f,r,\varepsilon}$ is a large constant depending on f, r and ε. Then (5.17), (5.21) and (5.19) yield

$$\begin{aligned} L_{T,H,N}(\sigma_0) &\ll_r J_{T,H}(\sigma_0) + \big(L_{T,H,N}(\tfrac{1}{2}) + J_{T,H}(\tfrac{1}{2})\big)T^{-\varepsilon(1/v-\varepsilon/2)(\sigma_0-1/2)} \\ &\ll_r J_{T,H}(\sigma_0) + J_{T,H}(\tfrac{1}{2})\mathrm{e}^{-2C^*_{f,r,\varepsilon}(1/v-\varepsilon/2)\varepsilon} + L_{T,H,N}(\tfrac{1}{2})\mathrm{e}^{-2C^*_{f,r,\varepsilon}(1/v-\varepsilon/2)\varepsilon}. \end{aligned}$$

On the other hand, by (5.16) and (5.15), there is a positive constant $C_0(f,r,\varepsilon)$ such that

$$\begin{aligned} L_{T,H,N}(\tfrac{1}{2})\mathrm{e}^{-2C^*_{f,r,\varepsilon}(1/v-\varepsilon/2)\varepsilon} &\leqslant C_0(f,r,\varepsilon)\mathrm{e}^{-2C^*_{f,r,\varepsilon}(1/v-\varepsilon/2)\varepsilon}H(\log T)^{r^2}, \\ L_{T,H,N}(\sigma_0) &\geqslant C_0(f,r,\varepsilon)(2C^*_{f,r,\varepsilon})^{-r^2}H(\log T)^{r^2}. \end{aligned}$$

Combining these with the precedent inequality, we can deduce that

$$L_{T,H,N}(\sigma_0) \ll J_{T,H}(\sigma_0) + J_{T,H}(\tfrac{1}{2})\mathrm{e}^{-2C^*_{f,r,\varepsilon}(1/v-\varepsilon)\varepsilon}. \tag{5.22}$$

Together with Lemma 5.1 and (5.15) again, we have

$$H(\log T)^{r^2} \ll L_{T,H,N}(\sigma_0) \ll J_{T,H}(\tfrac{1}{2})^{3/2-\sigma_0} + J_{T,H}(\tfrac{1}{2}) \ll J_{T,H}(\tfrac{1}{2}),$$

where the implied constants depend on f, r and ε. So we can conclude that (5.20) holds no matter $K_{T,H,N}(\frac{1}{2}) \leqslant H$ or not. Since $w_{T,H}(\tau) \ll 1$ for all τ, we have

$$J_{T,H}(\tfrac{1}{2}) \ll M_r(T,H;f) + \left(\int_{-\infty}^{T} + \int_{T+H}^{\infty}\right) w_{T,H}(\tau)\mathrm{d}\tau \ll M_r(T,H;f) + O(1).$$

Then the lower bound follows from (5.20).

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References

[1] A. Akbary and B. Fodden. Lower bounds for power moments of L-functions. *Acta Arith.*, 151(1):11–38, 2012.

[2] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith. Integral moments of L-functions. *Proc. London Math. Soc. (3)*, 91(1):33–104, 2005.

[3] A. Diaconu, D. Goldfeld, and J. Hoffstein. Multiple Dirichlet series and moments of zeta and L-functions. *Compositio Math.*, 139(3):297–360, 2003.

[4] O. M. Fomenko. Fourier coefficients of parabolic forms, and automorphic L-functions. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 237(Anal. Teor. Chisel i Teor. Funkts. 14):194–226, 231, 1997.

[5] R. M. Gabriel. Some Results Concerning the Integrals of Moduli of Regular Functions Along Certain Curves. *J. London Math. Soc.*, S1-2(2):112.

[6] A. Good. The square mean of Dirichlet series associated with cusp forms. *Mathematika*, 29(2):278–295 (1983), 1982.

[7] A. Good. The convolution method for Dirichlet series. In *The Selberg trace formula and related topics (Brunswick, Maine, 1984)*, volume 53 of *Contemp. Math.*, pages 207–214. Amer. Math. Soc., Providence, RI, 1986.

[8] G. H. Hardy and J. E. Littlewood. Contributions to the theory of the riemann zeta-function and the theory of the distribution of primes. *Acta Math.*, 41(1):119–196, 1916.
[9] A. J. Harper. Sharp conditional bounds for moments of the Riemann zeta function. *Preprint available on-line at http://arxiv.org/abs/1305.4618.*, 2013.
[10] D. R. Heath-Brown. The fourth power moment of the Riemann zeta function. *Proc. London Math. Soc. (3)*, 38(3):385–422, 1979.
[11] D. R. Heath-Brown. Fractional moments of the Riemann zeta function. *J. London Math. Soc. (2)*, 24(1):65–78, 1981.
[12] A. E. Ingham. Mean-Value Theorems in the Theory of the Riemann Zeta-Function. *Proc. London Math. Soc.*, S2-27(1):273.
[13] H. Iwaniec. *Topics in classical automorphic forms*, volume 17 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.
[14] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.
[15] J. P. Keating and N. C. Snaith. Random matrix theory and $\zeta(1/2+it)$. *Comm. Math. Phys.*, 214(1):57–89, 2000.
[16] M. B. Milinovich and N. Ng. Simple zeros of modular l-functions. *Proc. Lond. Math. Soc.*, 109(6):1465–1506, 2014.
[17] M. B. Milinovich and C. L. Turnage-Butterbaugh. Moments of products of automorphic L-functions. *J. Number Theory*, 139:175–204, 2014.
[18] H. L. Montgomery and R. C. Vaughan. Hilbert's inequality. *J. London Math. Soc. (2)*, 8:73–82, 1974.
[19] Q. Pi. Fractional moments of automorphic L-functions on $\mathrm{GL}(m)$. *Chin. Ann. Math. Ser. B*, 32(4):631–642, 2011.
[20] M. Radziwiłł. The 4.36th moment of the Riemann zeta-function. *Int. Math. Res. Not. IMRN*, (18):4245–4259, 2012.
[21] M. Radziwiłł and K. Soundararajan. Continuous lower bounds for moments of zeta and L-functions. *Mathematika*, 59(1):119–128, 2013.
[22] K. Ramachandra. Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. I. *Hardy-Ramanujan J.*, 1:15, 1978.
[23] K. Ramachandra. Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. II. *Hardy-Ramanujan J.*, 3:1–24, 1980.
[24] K. Ramachandra. Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. III. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 5(1):145–158, 1980.
[25] R. A. Rankin. Sums of powers of cusp form coefficients. *Math. Ann.*, 263(2):227–236, 1983.
[26] Z. Rudnick and K. Soundararajan. Lower bounds for moments of L-functions. *Proc. Natl. Acad. Sci. USA*, 102(19):6837–6838, 2005.
[27] Z. Rudnick and K. Soundararajan. Lower bounds for moments of L-functions: symplectic and orthogonal examples. In *Multiple Dirichlet series, automorphic forms, and analytic number theory*, volume 75 of *Proc. Sympos. Pure Math.*, pages 293–303. Amer. Math. Soc., Providence, RI, 2006.
[28] K. Soundararajan. Moments of the Riemann zeta function. *Ann. of Math. (2)*, 170(2):981–993, 2009.
[29] H. Sun and G. Lü. On fractional power moments of L-functions associated with certain cusp forms. *Acta Appl. Math.*, 109(2):653–667, 2010.
[30] E. C. Titchmarsh. *Theory of Functions*. Oxford University Press, second edition, 1976.
[31] E. C. Titchmarsh. *The theory of the Riemann zeta-function*. The Clarendon Press, Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.

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