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Samuele Giraudo

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# PLURIASOCIATIVE ALGEBRAS I: THE PLURIASOCIATIVE OPERAD

SAMUELE GIRAUDO

**ABSTRACT.** Diassociative algebras form a category of algebras recently introduced by Loday. A diassociative algebra is a vector space endowed with two associative binary operations satisfying some very natural relations. Any diassociative algebra is an algebra over the diassociative operad, and, among its most notable properties, this operad is the Koszul dual of the dendriform operad. We introduce here, by adopting the point of view and the tools offered by the theory of operads, a generalization on a nonnegative integer parameter  $\gamma$  of diassociative algebras, called  $\gamma$ -pluriassociative algebras, so that 1-pluriassociative algebras are diassociative algebras. Pluriassociative algebras are vector spaces endowed with  $2\gamma$  associative binary operations satisfying some relations. We provide a complete study of the  $\gamma$ -pluriassociative operads, the underlying operads of the category of  $\gamma$ -pluriassociative algebras. We exhibit a realization of these operads, establish several presentations by generators and relations, compute their Hilbert series, show that they are Koszul, and construct the free objects in the corresponding categories. We also study several notions of units in  $\gamma$ -pluriassociative algebras and propose a general way to construct such algebras. This paper ends with the introduction of an analogous generalization of the triassociative operad of Loday and Ronco.

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## INTRODUCTION

In the recent years, several algebraic structures on vector spaces based on various sets of combinatorial objects and endowed with more or less complicated operations on these have been considered by algebraic combinatorists. As most famous examples, we can cite free pre-Lie algebras [CL01], which are vector spaces of rooted trees endowed with a grafting product, and free dendriform algebras [Lod01], which are vector spaces of binary trees endowed with two products operating by shuffling binary trees. Other well-known examples include free Zinbiel algebras [Lod95, Lod01] endowing the space of all permutations with a shuffle product, nonassociative permutative algebras [MY91, Liv06] endowing the space of all rooted trees with a grafting product at the root, and duplicial algebras [Lod08] endowing the space of all binary with two grafting operations.

Instead of studying all these algebraic structures separately, it is possible to ask and treat some general questions about these under a uniform point of view. The theory of operads is an efficient tool to regard different categories of algebraic structures in a unified manner. This theory (see [LV12] for a complete exposition and also [Cha08] for an exposition highlighting the combinatorial aspects of the theory) has been introduced in the context of algebraic topology [May72, BV73]. Roughly speaking, an operad is a space of abstract operators consisting in several inputs and one output that can be composed to form bigger ones. The point is that any operad encodes a category of algebras and working with an operad amounts to work with the algebras all together of this category. Moreover, the use of the theory of operads leads to the discovery of connections between different sorts of algebras by terms of morphisms of operads. As a simple example, the well-known fact that any associative algebra gives rise to a Lie algebra by considering its associator as a Lie bracket comes from the fact that there is a morphism from the underlying operad of the category of Lie algebras to the underlying operad of the category of associative algebras.

The present work is concerned with the definition of a coherent generalization of dialgebras, algebraic structures introduced by Loday in [Lod01]. A dialgebra is a vector space endowed with two associative binary operations  $\dashv$  and  $\vdash$  satisfying some relations. From a combinatorial point of view, the bases of the free dialgebra over one generator are indexed by ordered pairs  $(n, k)$  of integers, denoted by  $\epsilon_{n,k}$ , and satisfying  $1 \leq k \leq n$ . The operations  $\dashv$  and  $\vdash$  admit simple set-theoretic descriptions over this basis [Cha05]. In a previous work [Gir12, Gir15], we introduced a new construction for the operad  $\text{Dias}$ , the underlying operad of the category of diassociative algebras, and we raised the question whether this construction can be extended to obtain operads generalizing  $\text{Dias}$  and hence, to obtain generalizations of dialgebras.

Let us give some explanations about our construction of  $\text{Dias}$ . In [Gir12, Gir15], we defined a general functorial construction  $\mathbb{T}$  producing an operad from any monoid. This construction  $\mathbb{T}$  sends a monoid  $M$  to the operad  $\mathbb{T}M$  of all words on  $M$ , where  $M$  is seen as an alphabet. The arity of a word is its length and the operadic partial composition  $u \circ_i v$  of two words  $u$  and  $v$  of  $\mathbb{T}M$  consists in replacing the  $i$ th letter  $u_i$  of  $u$  by a version of  $v$  obtained by multiplying to the left all its letters by  $u_i$ . The operad  $\text{Dias}$  is the suboperad of  $\mathbb{T}M$ , where  $M$  is the multiplicative monoid on  $\{0, 1\}$ , generated by the two words 01 and 10 of arity two. In the

Operad	Objects	Dimensions
$\text{Dias}_\gamma$	Words on $\{0, 1, \dots, \gamma\}$ with exactly one 0	$n\gamma^{n-1}$
$\text{As}_\gamma$	$\gamma$ -corollas	$\gamma$
$\text{Trias}_\gamma$	Words on $\{0, 1, \dots, \gamma\}$ with at least one 0	$(\gamma + 1)^n - \gamma^n$

TABLE 1. The main operads defined in this paper. All these operads depend on a nonnegative integer parameter  $\gamma$ . The shown dimensions are the ones of the homogeneous components of arities  $n \geq 2$  of the operads.

present paper, we rely on  $\mathbb{T}$  to construct a generalization on a nonnegative integer parameter  $\gamma$  of  $\text{Dias}$ , denoted by  $\text{Dias}_\gamma$ , in such a way that  $\text{Dias}_1 = \text{Dias}$  and  $\text{Dias}_\gamma$  is a suboperad of  $\text{Dias}_{\gamma+1}$  for any  $\gamma \geq 0$ . The operads  $\text{Dias}_\gamma$ , called  $\gamma$ -pluriassociative operads, are set-operads involving words on the alphabet  $\{0, 1, \dots, \gamma\}$  with exactly one occurrence of 0. Besides, this work naturally leads to the consideration and the definition of several new operads. Table 1 summarizes some information about these. We provide for instance a generalization on a nonnegative integer parameter  $\gamma$  of the triassociative operad  $\text{Trias}$  [LR04], denoted by  $\text{Trias}_\gamma$ .

The main rationale for this work is to establish the necessary foundations to propose a generalization on a nonnegative integer parameter  $\gamma$  of dendriform algebras [Lod01]. Since  $\text{Dias}$  is the Koszul dual [GK94] of the operad  $\text{Dendr}$ , the underlying operad of the category of dendriform algebras, our objective is to propose the definition of the operads  $\text{Dendr}_\gamma$ , defined each as the Koszul dual of  $\text{Dias}_\gamma$ . Moreover, since  $\text{Dias}$  admits a description far simpler than  $\text{Dendr}$ , starting by constructing a generalization of  $\text{Dias}$  to obtain a generalization of  $\text{Dendr}$  by Koszul duality is a convenient path to explore. This strategy is developed in the continuation of this work [Gir16], where the operads  $\text{Dendr}_\gamma$  are studied. This lead to new sorts of algebras, providing analogs of dendriform algebras and different from already existing ones (see for instance [LR04, AL04, Ler04, Ler07, Nov14]).

This paper is organized as follows. Section 1 contains a conspectus of the tools used in this paper. We recall here the definition of the construction  $\mathbb{T}$  [Gir12, Gir15] and provide a reformulation of results of Hoffbeck [Hof10] and Dotsenko and Khoroshkin [DK10] to prove that an operad is Koszul by using convergent rewrite rules. Besides, this part provides self-contained definitions about nonsymmetric operads, algebras over operads, free operads, and rewrite rules on trees. This section ends by some recalls about the diassociative operad and diassociative algebras.

Section 2 is devoted to the introduction and the study of the operad  $\text{Dias}_\gamma$ . We begin by detailing the construction of  $\text{Dias}_\gamma$  as a suboperad of the operad obtained by the construction  $\mathbb{T}$  applied on the monoid  $\mathcal{M}_\gamma$  with  $\{0, 1, \dots, \gamma\}$  as underlying set and with the operation  $\max$  as product. More precisely,  $\text{Dias}_\gamma$  is defined as the suboperad of  $\mathbb{T}\mathcal{M}_\gamma$  generated by the words  $0a$  and  $a0$  for all  $a \in \{1, \dots, \gamma\}$ . We then provide a presentation by generators and relations of

$\text{Dias}_\gamma$  (Theorem 2.2.6), and show that it is a Koszul operad (Theorem 2.3.1). We also establish some more properties of this operad: we compute its group of symmetries (Proposition 2.3.2), show that it is a basic operad in the sense of [Val07] (Proposition 2.3.3), and show that it is a rooted operad in the sense of [Cha14] (Proposition 2.3.3). We end this section by introducing an alternating basis of  $\text{Dias}_\gamma$ , the K-basis, defined through a partial ordering relation over the words indexing the bases of  $\text{Dias}_\gamma$ . After describing how the partial composition of  $\text{Dias}_\gamma$  expresses over the K-basis (Theorem 2.3.7), we provide a presentation of  $\text{Dias}_\gamma$  over this basis (Proposition 2.3.8). Despite the fact that this alternative presentation is more complex than the original one of  $\text{Dias}_\gamma$  provided by Theorem 2.2.6, the computation of the Koszul dual  $\text{Dendr}_\gamma$  of  $\text{Dias}_\gamma$  from this second presentation leads to a surprisingly plain presentation of  $\text{Dendr}_\gamma$  considered later in [Gir16].

In Section 3, algebras over  $\text{Dias}_\gamma$ , called  $\gamma$ -pluriassociative algebras, are studied. The free  $\gamma$ -pluriassociative algebra over one generator is described as a vector space of words on the alphabet  $\{0, 1, \dots, \gamma\}$  with exactly one occurrence of 0, endowed with  $2\gamma$  binary operations (Proposition 3.1.1). We next study two different notions of units in  $\gamma$ -pluriassociative algebras, the bar-units and the wire-units, that are generalizations of definitions of Loday introduced into the context of diassociative algebras [Lod01]. We show that the presence of a wire-unit in a  $\gamma$ -pluriassociative algebra leads to many consequences on its structure (Proposition 3.2.1). Besides, we describe a general construction  $M$  to obtain  $\gamma$ -pluriassociative algebras by starting from  $\gamma$ -multiprojection algebras, that are algebraic structures with  $\gamma$  associative products and endowed with  $\gamma$  endomorphisms with extra relations (Theorem 3.3.2). The main interest of the construction  $M$  is that  $\gamma$ -multiprojection algebras are simpler algebraic structures than  $\gamma$ -pluriassociative algebras. The bar-units and wire-units of the  $\gamma$ -pluriassociative algebras obtained by this construction are then studied (Proposition 3.3.3). We end this section by listing five examples of  $\gamma$ -pluriassociative algebras constructed from  $\gamma$ -multiprojection algebras, including the free  $\gamma$ -pluriassociative algebra over one generator considered in Section 3.1.3.

Finally, by using almost the same tools as the one used in Section 2, we propose in Section 4 a generalization on a nonnegative integer parameter  $\gamma$  of the triassociative operad  $\text{Trias}$  of Loday and Ronco [LR04], denoted by  $\text{Trias}_\gamma$ . This follows a very simple idea: like  $\text{Dias}_\gamma$ ,  $\text{Trias}_\gamma$  is defined as a suboperad of  $\mathcal{TM}_\gamma$  generated by the same generators as those of  $\text{Dias}_\gamma$ , plus the word  $00$ . In a previous work [Gir12, Gir15], we showed that  $\text{Trias}_1$  is the triassociative operad. We provide here an expression for the Hilbert series of  $\text{Trias}_\gamma$  obtained from the description of its elements (Proposition 4.1.1) and a presentation (Theorem 4.2.1).

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*Notations and general conventions.* All the algebraic structures of this article have a field of characteristic zero  $\mathbb{K}$  as ground field. If  $S$  is a set,  $\text{Vect}(S)$  denotes the linear span of the elements of  $S$ . For any integers  $a$  and  $c$ ,  $[a, c]$  denotes the set  $\{b \in \mathbb{N} : a \leq b \leq c\}$  and  $[n]$ , the set  $[1, n]$ . The cardinality of a finite set  $S$  is denoted by  $\#S$ . If  $u$  is a word, its letters are indexed from left to right from 1 to its length  $|u|$ . For any  $i \in [|u|]$ ,  $u_i$  is the letter of  $u$  at position  $i$ . If  $a$  is a letter and  $n$  is a nonnegative integer,  $a^n$  denotes the word consisting in  $n$  occurrences of  $a$ . Notice that  $a^0$  is the empty word  $\epsilon$ .

## 1. PRELIMINARIES: ALGEBRAIC STRUCTURES AND MAIN TOOLS

This preliminary section sets our conventions and notations about operads and algebras over an operad, and describes the main tools we will use. The definitions and some properties of the diassociative operad are also recalled. This section does not contains new results but it is a self-contained set of definitions about operads intended to readers familiar with algebra or combinatorics but not necessarily with operadic theory.

**1.1. Operads and algebras over an operad.** We list here several staple definitions about operads and algebras over an operad. We present also an important tool for this work: the construction  $\mathbb{T}$  producing operads from monoids.

1.1.1. *Operads.* A *nonsymmetric operad in the category of vector spaces*, or a *nonsymmetric operad* for short, is a graded vector space  $\mathcal{O} := \bigoplus_{n \geq 1} \mathcal{O}(n)$  together with linear maps

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \rightarrow \mathcal{O}(n + m - 1), \quad n, m \geq 1, i \in [n], \quad (1.1.1)$$

called *partial compositions*, and a distinguished element  $\mathbb{1} \in \mathcal{O}(1)$ , the *unit* of  $\mathcal{O}$ . This data has to satisfy the three relations

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z), \quad x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}(k), i \in [n], j \in [m], \quad (1.1.2a)$$

$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y, \quad x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}(k), i < j \in [n], \quad (1.1.2b)$$

$$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}, \quad x \in \mathcal{O}(n), i \in [n]. \quad (1.1.2c)$$

Since we shall consider in this paper mainly nonsymmetric operads, we shall call these simply *operads*. Moreover, all considered operads are such that  $\mathcal{O}(1)$  has dimension 1.

If  $x$  is an element of  $\mathcal{O}$  such that  $x \in \mathcal{O}(n)$  for a  $n \geq 1$ , we say that  $n$  is the *arity* of  $x$  and we denote it by  $|x|$ . An element  $x$  of  $\mathcal{O}$  of arity 2 is *associative* if  $x \circ_1 x = x \circ_2 x$ . If  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are operads, a linear map  $\phi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  is an *operad morphism* if it respects arities, sends the unit of  $\mathcal{O}_1$  to the unit of  $\mathcal{O}_2$ , and commutes with partial composition maps. We say that  $\mathcal{O}_2$  is a *suboperad* of  $\mathcal{O}_1$  if  $\mathcal{O}_2$  is a graded subspace of  $\mathcal{O}_1$ , and  $\mathcal{O}_1$  and  $\mathcal{O}_2$  have the same unit and the same partial compositions. For any set  $G \subseteq \mathcal{O}$ , the *operad generated by  $G$*  is the smallest suboperad of  $\mathcal{O}$  containing  $G$ . When the operad generated by  $G$  is  $\mathcal{O}$  itself and  $G$  is minimal with respect to inclusion among the subsets of  $\mathcal{O}$  satisfying this property,  $G$  is a *generating set* of  $\mathcal{O}$  and its elements are *generators* of  $\mathcal{O}$ . An *operad ideal* of  $\mathcal{O}$  is a graded subspace  $I$  of  $\mathcal{O}$  such that, for any  $x \in \mathcal{O}$  and  $y \in I$ ,  $x \circ_i y$  and  $y \circ_j x$  are in  $I$  for all valid integers  $i$  and  $j$ . Given an operad ideal  $I$  of  $\mathcal{O}$ , one can define the *quotient operad*  $\mathcal{O}/I$  of  $\mathcal{O}$  by  $I$  in the usual

way. When  $\mathcal{O}$  is such that all  $\mathcal{O}(n)$  are finite for all  $n \geq 1$ , the *Hilbert series* of  $\mathcal{O}$  is the series  $\mathcal{H}_{\mathcal{O}}(t)$  defined by

$$\mathcal{H}_{\mathcal{O}}(t) := \sum_{n \geq 1} \dim \mathcal{O}(n) t^n. \quad (1.1.3)$$

Instead of working with the partial composition maps of  $\mathcal{O}$ , it is something useful to work with the maps

$$\circ : \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \rightarrow \mathcal{O}(m_1 + \cdots + m_n), \quad n, m_1, \dots, m_n \geq 1, \quad (1.1.4)$$

linearly defined for any  $x \in \mathcal{O}$  of arity  $n$  and  $y_1, \dots, y_{n-1}, y_n \in \mathcal{O}$  by

$$x \circ (y_1, \dots, y_{n-1}, y_n) := (\dots ((x \circ_n y_n) \circ_{n-1} y_{n-1}) \dots) \circ_1 y_1. \quad (1.1.5)$$

These maps are called *composition maps* of  $\mathcal{O}$ .

**1.1.2. Set-operads.** Instead of being a direct sum of vector spaces  $\mathcal{O}(n)$ ,  $n \geq 1$ ,  $\mathcal{O}$  can be a graded disjoint union of sets. In this context,  $\mathcal{O}$  is a *set-operad*. All previous definitions remain valid by replacing direct sums  $\oplus$  by disjoint unions  $\sqcup$ , tensor products  $\otimes$  by Cartesian products  $\times$ , and vector space dimensions  $\dim$  by set cardinalities  $\#$ . Moreover, in the context of set-operads, we work with *operad congruences* instead of operad ideals. An operad congruence on a set-operad  $\mathcal{O}$  is an equivalence relation  $\equiv$  on  $\mathcal{O}$  such that all elements of a same  $\equiv$ -equivalence class have the same arity and for all elements  $x, x', y$ , and  $y'$  of  $\mathcal{O}$ ,  $x \equiv x'$  and  $y \equiv y'$  imply  $x \circ_i y \equiv x' \circ_i y'$  for all valid integers  $i$ . The *quotient operad*  $\mathcal{O}/\equiv$  of  $\mathcal{O}$  by  $\equiv$  is the set-operad defined in the usual way.

Any set-operad  $\mathcal{O}$  gives naturally rise to an operad on  $\text{Vect}(\mathcal{O})$  by extending the partial compositions of  $\mathcal{O}$  by linearity. Besides this, any equivalence relation  $\leftrightarrow$  of  $\mathcal{O}$  such that all elements of a same  $\leftrightarrow$ -equivalence class have the same arity induces a subspace of  $\text{Vect}(\mathcal{O})$  generated by all  $x - x'$  such that  $x \leftrightarrow x'$ , called *space induced* by  $\leftrightarrow$ . In particular, any operad congruence  $\equiv$  on  $\mathcal{O}$  induces an operad ideal of  $\text{Vect}(\mathcal{O})$ .

**1.1.3. From monoids to operads.** In a previous work [Gir12, Gir15], the author introduced a construction which, from any monoid, produces an operad. This construction is described as follows. Let  $\mathcal{M}$  be a monoid with an associative product  $\bullet$  admitting a unit  $1$ . We denote by  $\mathcal{T}\mathcal{M}$  the operad  $\mathcal{T}\mathcal{M} := \bigoplus_{n \geq 1} \mathcal{T}\mathcal{M}(n)$  where for all  $n \geq 1$ ,

$$\mathcal{T}\mathcal{M}(n) := \text{Vect}(\{u_1 \dots u_n : u_i \in \mathcal{M} \text{ for all } i \in [n]\}). \quad (1.1.6)$$

The partial composition of two words  $u \in \mathcal{T}\mathcal{M}(n)$  and  $v \in \mathcal{T}\mathcal{M}(m)$  is linearly defined by

$$u \circ_i v := u_1 \dots u_{i-1} (u_i \bullet v_1) \dots (u_i \bullet v_m) u_{i+1} \dots u_n, \quad i \in [n]. \quad (1.1.7)$$

The unit of  $\mathcal{T}\mathcal{M}$  is  $\mathbb{1} := 1$ . In other words,  $\mathcal{T}\mathcal{M}$  is the vector space of words on  $\mathcal{M}$  seen as an alphabet and the partial composition returns to insert a word  $v$  onto the  $i$ th letter  $u_i$  of a word  $u$  together with a left multiplication by  $u_i$ .

1.1.4. *Algebras over an operad.* Any operad  $\mathcal{O}$  encodes a category of algebras whose objects are called  $\mathcal{O}$ -algebras. An  $\mathcal{O}$ -algebra  $\mathcal{A}_{\mathcal{O}}$  is a vector space endowed with a right action

$$\cdot : \mathcal{A}_{\mathcal{O}}^{\otimes n} \otimes \mathcal{O}(n) \rightarrow \mathcal{A}_{\mathcal{O}}, \quad n \geq 1, \quad (1.1.8)$$

satisfying the relations imposed by the structure of  $\mathcal{O}$ , that are

$$(e_1 \otimes \cdots \otimes e_{n+m-1}) \cdot (x \circ_i y) = (e_1 \otimes \cdots \otimes e_{i-1} \otimes (e_i \otimes \cdots \otimes e_{i+m-1}) \cdot y \otimes e_{i+m} \otimes \cdots \otimes e_{n+m-1}) \cdot x, \quad (1.1.9)$$

for all  $e_1 \otimes \cdots \otimes e_{n+m-1} \in \mathcal{A}_{\mathcal{O}}^{\otimes n+m-1}$ ,  $x \in \mathcal{O}(n)$ ,  $y \in \mathcal{O}(m)$ , and  $i \in [n]$ . Notice that, by (1.1.9), if  $G$  is a generating set of  $\mathcal{O}$ , it is enough to define the action of each  $x \in G$  on  $\mathcal{A}_{\mathcal{O}}^{\otimes |x|}$  to wholly define  $\cdot$ .

In other words, any element  $x$  of  $\mathcal{O}$  of arity  $n$  plays the role of a linear operation

$$x : \mathcal{A}_{\mathcal{O}}^{\otimes n} \rightarrow \mathcal{A}_{\mathcal{O}}, \quad (1.1.10)$$

taking  $n$  elements of  $\mathcal{A}_{\mathcal{O}}$  as inputs and computing an element of  $\mathcal{A}_{\mathcal{O}}$ . By a slight but convenient abuse of notation, for any  $x \in \mathcal{O}(n)$ , we shall denote by  $x(e_1, \dots, e_n)$ , or by  $e_1 x e_2$  if  $x$  has arity 2, the element  $(e_1 \otimes \cdots \otimes e_n) \cdot x$  of  $\mathcal{A}_{\mathcal{O}}$ , for any  $e_1 \otimes \cdots \otimes e_n \in \mathcal{A}_{\mathcal{O}}^{\otimes n}$ . Observe that by (1.1.9), any associative element of  $\mathcal{O}$  gives rise to an associative operation on  $\mathcal{A}_{\mathcal{O}}$ .

Arrows in the category of  $\mathcal{O}$ -algebras are  $\mathcal{O}$ -algebra morphisms, that are linear maps  $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  between two  $\mathcal{O}$ -algebras  $\mathcal{A}_1$  and  $\mathcal{A}_2$  such that

$$\phi(x(e_1, \dots, e_n)) = x(\phi(e_1), \dots, \phi(e_n)), \quad (1.1.11)$$

for all  $e_1, \dots, e_n \in \mathcal{A}_1$  and  $x \in \mathcal{O}(n)$ . We say that  $\mathcal{A}_2$  is an  $\mathcal{O}$ -subalgebra of  $\mathcal{A}_1$  if  $\mathcal{A}_2$  is a subspace of  $\mathcal{A}_1$  and  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are endowed with the same right action of  $\mathcal{O}$ . If  $G$  is a set of elements of an  $\mathcal{O}$ -algebra  $\mathcal{A}$ , the  $\mathcal{O}$ -algebra generated by  $G$  is the smallest  $\mathcal{O}$ -subalgebra of  $\mathcal{A}$  containing  $G$ . When the  $\mathcal{O}$ -algebra generated by  $G$  is  $\mathcal{A}$  itself and  $G$  is minimal with respect to inclusion among the subsets of  $\mathcal{A}$  satisfying this property,  $G$  is a *generating set* of  $\mathcal{A}$  and its elements are *generators* of  $\mathcal{A}$ . An  $\mathcal{O}$ -algebra ideal of  $\mathcal{A}$  is a subspace  $I$  of  $\mathcal{A}$  such that for all operation  $x$  of  $\mathcal{O}$  of arity  $n$  and elements  $e_1, \dots, e_n$  of  $\mathcal{O}$ ,  $x(e_1, \dots, e_n)$  is in  $I$  whenever there is a  $i \in [n]$  such that  $e_i$  is in  $I$ .

The *free  $\mathcal{O}$ -algebra over one generator* is the  $\mathcal{O}$ -algebra  $\mathcal{F}_{\mathcal{O}}$  defined in the following way. We set  $\mathcal{F}_{\mathcal{O}} := \bigoplus_{n \geq 1} \mathcal{F}_{\mathcal{O}}(n) := \bigoplus_{n \geq 1} \mathcal{O}(n)$ , and for any  $e_1, \dots, e_n \in \mathcal{F}_{\mathcal{O}}$  and  $x \in \mathcal{O}(n)$ , the right action of  $x$  on  $e_1 \otimes \cdots \otimes e_n$  is defined by

$$x(e_1, \dots, e_n) := x \circ (e_1, \dots, e_n). \quad (1.1.12)$$

Then, any element  $x$  of  $\mathcal{O}(n)$  endows  $\mathcal{F}_{\mathcal{O}}$  with an operation

$$x : \mathcal{F}_{\mathcal{O}}(m_1) \otimes \cdots \otimes \mathcal{F}_{\mathcal{O}}(m_n) \rightarrow \mathcal{F}_{\mathcal{O}}(m_1 + \cdots + m_n) \quad (1.1.13)$$

respecting the graduation of  $\mathcal{F}_{\mathcal{O}}$ .



**1.2. Free operads, rewrite rules, and Koszulity.** We recall here a description of free operads through syntax trees and presentations of operads by generators and relations. The Koszul property for operads is a very important notion in this paper and its sequel [Gir16]. We recall it and describe an already known criterion to prove that a set-operad is Koszul by passing by rewrite rules on syntax trees.

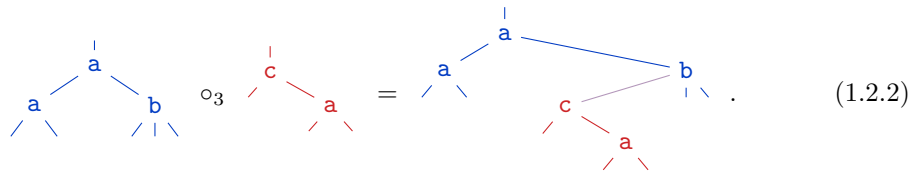
**1.2.1. Syntax trees.** Unless otherwise specified, we use in the sequel the standard terminology (*i.e.*, *node*, *edge*, *root*, *parent*, *child*, *path*, *ancestor*, *etc.*) about planar rooted trees [Knu97]. Let  $\mathfrak{t}$  be a planar rooted tree. The *arity* of a node of  $\mathfrak{t}$  is its number of children. An *internal node* (resp. a *leaf*) of  $\mathfrak{t}$  is a node with a nonzero (resp. null) arity. Given an internal node  $x$  of  $\mathfrak{t}$ , due to the planarity of  $\mathfrak{t}$ , the children of  $x$  are totally ordered from left to right and are thus indexed from 1 to the arity of  $x$ . If  $y$  is a child of  $x$ ,  $y$  defines a *subtree* of  $\mathfrak{t}$ , that is the planar rooted tree with root  $y$  and consisting in the nodes of  $\mathfrak{t}$  that have  $y$  as ancestor. We shall call  *$i$ th subtree* of  $x$  the subtree of  $\mathfrak{t}$  rooted at the  $i$ th child of  $x$ . A *partial subtree* of  $\mathfrak{t}$  is a subtree of  $\mathfrak{t}$  in which some internal nodes have been replaced by leaves and its descendants has been forgotten. Besides, due to the planarity of  $\mathfrak{t}$ , its leaves are totally ordered from left to right and thus are indexed from 1 to the arity of  $\mathfrak{t}$ . In our graphical representations, each tree is depicted so that its root is the uppermost node.

Let  $S := \sqcup_{n \geq 1} S(n)$  be a graded set. By extension, we say that the *arity* of an element  $x$  of  $S$  is  $n$  provided that  $x \in S(n)$ . A *syntax tree on  $S$*  is a planar rooted tree such that its internal nodes of arity  $n$  are labeled on elements of arity  $n$  of  $S$ . The *degree* (resp. *arity*) of a syntax tree is its number of internal nodes (resp. leaves). For instance, if  $S := S(2) \sqcup S(3)$  with  $S(2) := \{\mathbf{a}, \mathbf{c}\}$  and  $S(3) := \{\mathbf{b}\}$ ,



is a syntax tree on  $S$  of degree 5 and arity 8. Its root is labeled by  $\mathbf{b}$  and has arity 3.

**1.2.2. Free operads.** Let  $S$  be a graded set. The *free operad  $\mathbf{Free}(S)$  over  $S$*  is the operad wherein for any  $n \geq 1$ ,  $\mathbf{Free}(S)(n)$  is the vector space of syntax trees on  $S$  of arity  $n$ , the partial composition  $\mathfrak{s} \circ_i \mathfrak{t}$  of two syntax trees  $\mathfrak{s}$  and  $\mathfrak{t}$  on  $S$  consists in grafting the root of  $\mathfrak{t}$  on the  $i$ th leaf of  $\mathfrak{s}$ , and its unit is the tree consisting in one leaf. For instance, if  $S := S(2) \sqcup S(3)$  with  $S(2) := \{\mathbf{a}, \mathbf{c}\}$  and  $S(3) := \{\mathbf{b}\}$ , one has in  $\mathbf{Free}(S)$ ,



We denote by  $\text{cor} : S \rightarrow \mathbf{Free}(S)$  the inclusion map, sending any  $x$  of  $S$  to the *corolla* labeled by  $x$ , that is the syntax tree consisting in one internal node labeled by  $x$  attached to a required number of leaves. In the sequel, if required by the context, we shall implicitly see any element  $x$  of  $S$  as the corolla  $\text{cor}(x)$  of  $\mathbf{Free}(S)$ . For instance, when  $x$  and  $y$  are two elements of  $S$ , we shall simply denote by  $x \circ_i y$  the syntax tree  $\text{cor}(x) \circ_i \text{cor}(y)$  for all valid integers  $i$ .

For any operad  $\mathcal{O}$ , by seeing  $\mathcal{O}$  as a graded set,  $\mathbf{Free}(\mathcal{O})$  is the free operad of the syntax trees linearly labeled by elements of  $\mathcal{O}$ . The *evaluation map* of  $\mathcal{O}$  is the map

$$\text{eval}_{\mathcal{O}} : \mathbf{Free}(\mathcal{O}) \rightarrow \mathcal{O}, \quad (1.2.3)$$

recursively defined by

$$\text{eval}_{\mathcal{O}}(\mathfrak{t}) := \begin{cases} \mathbf{1} & \text{if } \mathfrak{t} \text{ is the leaf,} \\ x \circ (\text{eval}_{\mathcal{O}}(\mathfrak{s}_1), \dots, \text{eval}_{\mathcal{O}}(\mathfrak{s}_n)) & \text{otherwise,} \end{cases} \quad (1.2.4)$$

where  $\mathbf{1}$  is the unit of  $\mathcal{O}$ ,  $x$  is the label of the root of  $\mathfrak{t}$ , and  $\mathfrak{s}_1, \dots, \mathfrak{s}_n$  are, from left to right, the subtrees of the root of  $\mathfrak{t}$ . In other words, any tree  $\mathfrak{t}$  of  $\mathbf{Free}(\mathcal{O})$  can be seen as a tree-like expression for an element  $\text{eval}_{\mathcal{O}}(\mathfrak{t})$  of  $\mathcal{O}$ . Moreover, by induction on the degree of  $\mathfrak{t}$ , it appears that  $\text{eval}_{\mathcal{O}}$  is a well-defined surjective operad morphism.

1.2.3. *Presentations by generators and relations.* A *presentation* of an operad  $\mathcal{O}$  consists in a pair  $(\mathfrak{G}, \mathfrak{R})$  such that  $\mathfrak{G} := \sqcup_{n \geq 1} \mathfrak{G}(n)$  is a graded set,  $\mathfrak{R}$  is a subspace of  $\mathbf{Free}(\mathfrak{G})$ , and  $\mathcal{O}$  is isomorphic to  $\mathbf{Free}(\mathfrak{G})/\langle \mathfrak{R} \rangle$ , where  $\langle \mathfrak{R} \rangle$  is the operad ideal of  $\mathbf{Free}(\mathfrak{G})$  generated by  $\mathfrak{R}$ . We call  $\mathfrak{G}$  the *set of generators* and  $\mathfrak{R}$  the *space of relations* of  $\mathcal{O}$ . We say that  $\mathcal{O}$  is *quadratic* if one can exhibit a presentation  $(\mathfrak{G}, \mathfrak{R})$  of  $\mathcal{O}$  such that  $\mathfrak{R}$  is a homogeneous subspace of  $\mathbf{Free}(\mathfrak{G})$  consisting in syntax trees of degree 2. Besides, we say that  $\mathcal{O}$  is *binary* if one can exhibit a presentation  $(\mathfrak{G}, \mathfrak{R})$  of  $\mathcal{O}$  such that  $\mathfrak{G}$  is concentrated in arity 2.

With knowledge of a presentation  $(\mathfrak{G}, \mathfrak{R})$  of  $\mathcal{O}$ , it is easy to describe the category of the  $\mathcal{O}$ -algebras. Indeed, by denoting by  $\pi : \mathbf{Free}(\mathfrak{G}) \rightarrow \mathbf{Free}(\mathfrak{G})/\langle \mathfrak{R} \rangle$  the canonical surjection map, the category of  $\mathcal{O}$ -algebras is the category of vector spaces  $\mathcal{A}_{\mathcal{O}}$  endowed with maps  $\pi(g)$ ,  $g \in \mathfrak{G}$ , satisfying for all  $r \in \mathfrak{R}$  the relations

$$r(e_1, \dots, e_n) = 0, \quad (1.2.5)$$

for all  $e_1, \dots, e_n \in \mathcal{A}_{\mathcal{O}}$ , where  $n$  is the arity of  $r$ .

1.2.4. *Rewrite rules.* Let  $S$  be a graded set. A *rewrite rule* on syntax trees on  $S$  is a binary relation  $\rightarrow$  on  $\mathbf{Free}(S)$  whenever for all trees  $\mathfrak{s}$  and  $\mathfrak{t}$  of  $\mathbf{Free}(S)$ ,  $\mathfrak{s} \rightarrow \mathfrak{t}$  only if  $\mathfrak{s}$  and  $\mathfrak{t}$  have the same arity. When  $\rightarrow$  involves only syntax trees of degree two,  $\rightarrow$  is *quadratic*. We say that a syntax tree  $\mathfrak{s}'$  can be *rewritten* by  $\rightarrow$  into  $\mathfrak{t}'$  if there exist two syntax trees  $\mathfrak{s}$  and  $\mathfrak{t}$  satisfying  $\mathfrak{s} \rightarrow \mathfrak{t}$  and  $\mathfrak{s}'$  has a partial subtree equal to  $\mathfrak{s}$  such that, by replacing it by  $\mathfrak{t}$  in  $\mathfrak{s}'$ , we obtain  $\mathfrak{t}'$ . By a slight but convenient abuse of notation, we denote by  $\mathfrak{s}' \rightarrow \mathfrak{t}'$  this property. When a syntax tree  $\mathfrak{t}$  can be obtained by performing a sequence of  $\rightarrow$ -rewritings from a syntax tree  $\mathfrak{s}$ , we say that  $\mathfrak{s}$  is *rewritable* by  $\rightarrow$  into  $\mathfrak{t}$  and we denote this property by  $\mathfrak{s} \xrightarrow{*} \mathfrak{t}$ . For instance, for

$S := S(2) \sqcup S(3)$  with  $S(2) := \{a, c\}$  and  $S(3) := \{b\}$ , consider the rewrite rule  $\rightarrow$  on  $\mathbf{Free}(S)$  satisfying

$$\begin{array}{c} \dot{b} \\ \diagup \quad \diagdown \\ \phantom{a} \end{array} \rightarrow \begin{array}{c} \phantom{b} \\ \diagup \quad \diagdown \\ \dot{a} \end{array} \quad \text{and} \quad \begin{array}{c} \phantom{b} \\ \diagup \quad \diagdown \\ \dot{c} \end{array} \rightarrow \begin{array}{c} \dot{a} \\ \diagup \quad \diagdown \\ \phantom{b} \end{array} \quad \dot{c} \quad . \quad (1.2.6)$$

We then have the following sequence of rewritings

$$\begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{b} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{c} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{a} \end{array} \rightarrow \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{a} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{b} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{c} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{a} \end{array} \rightarrow \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{a} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{c} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{b} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{a} \end{array} \rightarrow \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{a} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{c} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{a} \end{array} \begin{array}{c} \phantom{a} \\ \diagup \quad \diagdown \\ \dot{a} \end{array} \quad . \quad (1.2.7)$$

We shall use the standard terminology (*confluent*, *terminating*, *convergent*, *normal form*, *critical pair*, *etc.*) about rewrite rules (see [BN98]).

Any rewrite rule  $\rightarrow$  on  $\mathbf{Free}(S)$  defines an operad congruence  $\equiv_{\rightarrow}$  on  $\mathbf{Free}(S)$  seen as a set-operad, the *operad congruence induced* by  $\rightarrow$ , as the finest operad congruence on  $\mathbf{Free}(S)$  containing the reflexive, symmetric, and transitive closure of  $\rightarrow$ .

**1.2.5. Koszulity.** A quadratic operad  $\mathcal{O}$  is *Koszul* if its Koszul complex is acyclic [GK94, LV12]. In this work, to prove the Koszulity of an operad  $\mathcal{O}$ , we shall make use of a combinatorial tool introduced by Hoffbeck [Hof10] (see also [LV12]) consisting in exhibiting a particular basis of  $\mathcal{O}$ , a so-called *Poincaré-Birkhoff-Witt basis*.

In this paper, we shall use this tool only in the context of set-operads, which reformulates, thanks to the work of Dotsenko and Khoroshkin [DK10], as follows. A set-operad  $\mathcal{O}$  is Koszul if there is a graded set  $S$  and a rewrite rule  $\rightarrow$  on  $\mathbf{Free}(S)$  such that  $\mathcal{O}$  is isomorphic to  $\mathbf{Free}(S)/\equiv_{\rightarrow}$  and  $\rightarrow$  is a convergent quadratic rewrite rule. Moreover, the set of normal forms of  $\rightarrow$  forms a Poincaré-Birkhoff-Witt basis of  $\mathcal{O}$ .

**1.3. Diassociative operad.** We recall here, by using the notions presented during the previous sections, the definition and some properties of the diassociative operad.

The *diassociative operad*  $\mathbf{Dias}$  was introduced by Loday [Lod01] as the operad admitting the presentation  $(\mathfrak{G}_{\mathbf{Dias}}, \mathfrak{R}_{\mathbf{Dias}})$  where  $\mathfrak{G}_{\mathbf{Dias}} := \mathfrak{G}_{\mathbf{Dias}}(2) := \{-, \vdash\}$  and  $\mathfrak{R}_{\mathbf{Dias}}$  is the space induced by the equivalence relation  $\equiv$  satisfying

$$\dashv \circ_1 \vdash \equiv \vdash \circ_2 \dashv, \quad (1.3.1a)$$

$$\dashv \circ_1 \dashv \equiv \dashv \circ_2 \dashv \equiv \dashv \circ_2 \vdash, \quad (1.3.1b)$$

$$\vdash \circ_1 \dashv \equiv \vdash \circ_1 \vdash \equiv \vdash \circ_2 \vdash. \quad (1.3.1c)$$

Note that  $\mathbf{Dias}$  is a binary and quadratic operad.

This operad admits the following realization [Cha05]. For any  $n \geq 1$ ,  $\text{Dias}(n)$  is the linear span of the  $\mathbf{e}_{n,k}$ ,  $k \in [n]$ , and the partial compositions linearly satisfy, for all  $n, m \geq 1$ ,  $k \in [n]$ ,  $\ell \in [m]$ , and  $i \in [n]$ ,

$$\mathbf{e}_{n,k} \circ_i \mathbf{e}_{m,\ell} = \begin{cases} \mathbf{e}_{n+m-1, k+m-1} & \text{if } i < k, \\ \mathbf{e}_{n+m-1, k+\ell-1} & \text{if } i = k, \\ \mathbf{e}_{n+m-1, k} & \text{otherwise } (i > k). \end{cases} \quad (1.3.2)$$

Since the partial composition of two basis elements of  $\text{Dias}$  produces exactly one basis element,  $\text{Dias}$  is well-defined as a set-operad. Moreover, this realization shows that  $\dim \text{Dias}(n) = n$  and hence, the Hilbert series of  $\text{Dias}$  satisfies

$$\mathcal{H}_{\text{Dias}}(t) = \frac{t}{(1-t)^2}. \quad (1.3.3)$$

From the presentation of  $\text{Dias}$ , we deduce that any  $\text{Dias}$ -algebra, also called *diassociative algebra*, is a vector space  $\mathcal{A}_{\text{Dias}}$  endowed with linear operations  $\dashv$  and  $\vdash$  satisfying the relations encoded by (1.3.1a)–(1.3.1c).

From the realization of  $\text{Dias}$ , we deduce that the free diassociative algebra  $\mathcal{F}_{\text{Dias}}$  over one generator is the vector space  $\text{Dias}$  endowed with the linear operations

$$\dashv, \vdash: \mathcal{F}_{\text{Dias}} \otimes \mathcal{F}_{\text{Dias}} \rightarrow \mathcal{F}_{\text{Dias}}, \quad (1.3.4)$$

satisfying, for all  $n, m \geq 1$ ,  $k \in [n]$ ,  $\ell \in [m]$ ,

$$\mathbf{e}_{n,k} \dashv \mathbf{e}_{m,\ell} = (\mathbf{e}_{n,k} \otimes \mathbf{e}_{m,\ell}) \cdot \mathbf{e}_{2,1} = (\mathbf{e}_{2,1} \circ_2 \mathbf{e}_{m,\ell}) \circ_1 \mathbf{e}_{n,k} = \mathbf{e}_{n+m,k}, \quad (1.3.5)$$

and

$$\mathbf{e}_{n,k} \vdash \mathbf{e}_{m,\ell} = (\mathbf{e}_{n,k} \otimes \mathbf{e}_{m,\ell}) \cdot \mathbf{e}_{2,2} = (\mathbf{e}_{2,2} \circ_2 \mathbf{e}_{m,\ell}) \circ_1 \mathbf{e}_{n,k} = \mathbf{e}_{n+m, n+\ell}. \quad (1.3.6)$$

As shown in [Gir12, Gir15], the diassociative operad is isomorphic to the suboperad of  $\mathcal{TM}$  generated by 01 and 10 where  $\mathcal{M}$  is the multiplicative monoid on  $\{0, 1\}$ . The concerned isomorphism sends any  $\mathbf{e}_{n,k}$  of  $\text{Dias}$  to the word  $0^{k-1} 1 0^{n-k}$  of  $\mathcal{TM}$ .

## 2. PLURIASOCIATIVE OPERADS

In this section, we define the main object of this work: a generalization on a nonnegative integer parameter  $\gamma$  of the diassociative operad. We provide a complete study of this new operad.

**2.1. Construction and first properties.** We define here our generalization of the diassociative operad using the functor  $\mathbb{T}$  (whose definition is recalled in Section 1.1.3). We then describe the elements and establish the Hilbert series of our generalization.

2.1.1. *Construction.* For any integer  $\gamma \geq 0$ , let  $\mathcal{M}_\gamma$  be the monoid  $\{0\} \cup [\gamma]$  with the binary operation  $\max$  as product, denoted by  $\uparrow$ . We define  $\text{Dias}_\gamma$  as the suboperad of  $\text{T}\mathcal{M}_\gamma$  generated by

$$\{0a, a0 : a \in [\gamma]\}. \quad (2.1.1)$$

By definition,  $\text{Dias}_\gamma$  is the vector space of words that can be obtained by partial compositions of words of (2.1.1). We have, for instance,

$$\text{Dias}_2(1) = \text{Vect}(\{0\}), \quad (2.1.2)$$

$$\text{Dias}_2(2) = \text{Vect}(\{01, 02, 10, 20\}), \quad (2.1.3)$$

$$\text{Dias}_2(3) = \text{Vect}(\{011, 012, 021, 022, 101, 102, 201, 202, 110, 120, 210, 220\}), \quad (2.1.4)$$

and

$$211201 \circ_4 31103 = 2113222301, \quad (2.1.5)$$

$$111101 \circ_3 20 = 1121101, \quad (2.1.6)$$

$$1013 \circ_2 210 = 121013. \quad (2.1.7)$$

It follows immediately from the definition of  $\text{Dias}_\gamma$  as a suboperad of  $\text{T}\mathcal{M}_\gamma$  that  $\text{Dias}_\gamma$  is a set-operad. Indeed, any partial composition of two basis elements of  $\text{Dias}_\gamma$  gives rise to exactly one basis element. We then shall see  $\text{Dias}_\gamma$  as a set-operad over all Section 2.

Notice that  $\text{Dias}_\gamma(2)$  is the set (2.1.1) of generators of  $\text{Dias}_\gamma$ . Besides, observe that  $\text{Dias}_0$  is the trivial operad and that  $\text{Dias}_\gamma$  is a suboperad of  $\text{Dias}_{\gamma+1}$ . We call  $\text{Dias}_\gamma$  the  $\gamma$ -*pluriassociative operad*.

### 2.1.2. Elements and dimensions.

**Proposition 2.1.1.** *For any integer  $\gamma \geq 0$ , as a set-operad, the underlying set of  $\text{Dias}_\gamma$  is the set of the words on the alphabet  $\{0\} \cup [\gamma]$  containing exactly one occurrence of 0.*

*Proof.* Let us show that any word  $x$  of  $\text{Dias}_\gamma$  satisfies the statement of the proposition by induction on the length  $n$  of  $x$ . This is true when  $n = 1$  because we necessarily have  $x = 0$ . Otherwise, when  $n \geq 2$ , there is a word  $y$  of  $\text{Dias}_\gamma$  of length  $n - 1$  and a generator  $g$  of  $\text{Dias}_\gamma$  such that  $x = y \circ_i g$  for a  $i \in [n - 1]$ . Then,  $x$  is obtained by replacing the  $i$ th letter  $a$  of  $y$  by the factor  $u := u_1 u_2$  where  $u_1 := a \uparrow g_1$  and  $u_2 := a \uparrow g_2$ . Since  $g$  contains exactly one 0, this operation consists in inserting a nonzero letter of  $[\gamma]$  into  $y$ . Since by induction hypothesis  $y$  contains exactly one 0, it follows that  $x$  satisfies the statement of the proposition.

Conversely, let us show that any word  $x$  satisfying the statement of the proposition belongs to  $\text{Dias}_\gamma$  by induction on the length  $n$  of  $x$ . This is true when  $n = 1$  because we necessarily have  $x = 0$  and 0 belongs to  $\text{Dias}_\gamma$  since it is its unit. Otherwise, when  $n \geq 2$ , there is an integer  $i \in [n - 1]$  such that  $x_i x_{i+1} \in \{0a, a0\}$  for an  $a \in [\gamma]$ . Let us suppose without loss of generality that  $x_i x_{i+1} = a0$ . By setting  $y$  as the word obtained by erasing the  $i$ th letter of  $x$ , we have  $x = y \circ_i a0$ . Thus, since by induction hypothesis  $y$  is an element of  $\text{Dias}_\gamma$ , it follows that  $x$  also is.  $\square$

We deduce from Proposition 2.1.1 that the Hilbert series of  $\text{Dias}_\gamma$  satisfies

$$\mathcal{H}_{\text{Dias}_\gamma}(t) = \frac{t}{(1 - \gamma t)^2} \quad (2.1.8)$$

and that for all  $n \geq 1$ ,  $\dim \text{Dias}_\gamma(n) = n\gamma^{n-1}$ . For instance, the first dimensions of  $\text{Dias}_1$ ,  $\text{Dias}_2$ ,  $\text{Dias}_3$ , and  $\text{Dias}_4$  are respectively

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \quad (2.1.9)$$

$$1, 4, 12, 32, 80, 192, 448, 1024, 2304, 5120, 11264, \quad (2.1.10)$$

$$1, 6, 27, 108, 405, 1458, 5103, 17496, 59049, 196830, 649539, \quad (2.1.11)$$

$$1, 8, 48, 256, 1280, 6144, 28672, 131072, 589824, 2621440, 11534336. \quad (2.1.12)$$

The second one is Sequence [A001787](#), the third one is Sequence [A027471](#), and the last one is Sequence [A002697](#) of [Slo].

**2.2. Presentation by generators and relations.** To establish a presentation of  $\text{Dias}_\gamma$ , we shall start by defining a morphism  $\text{word}_\gamma$  from a free operad to  $\text{Dias}_\gamma$ . Then, after showing that  $\text{word}_\gamma$  is a surjection, we will show that  $\text{word}_\gamma$  induces an operad isomorphism between a quotient of a free operad by a certain operad congruence  $\equiv_\gamma$  and  $\text{Dias}_\gamma$ . The space of relations of  $\text{Dias}_\gamma$  of its presentation will be induced by  $\equiv_\gamma$ .

2.2.1. *From syntax trees to words.* For any integer  $\gamma \geq 0$ , let  $\mathfrak{G}_{\text{Dias}_\gamma} := \mathfrak{G}_{\text{Dias}_\gamma}(2)$  be the graded set where

$$\mathfrak{G}_{\text{Dias}_\gamma}(2) := \{\dashv_a, \vdash_a : a \in [\gamma]\}. \quad (2.2.1)$$

Let  $\mathfrak{t}$  be a syntax tree of  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  and  $x$  be a leaf of  $\mathfrak{t}$ . We say that an integer  $a \in \{0\} \cup [\gamma]$  is *eligible* for  $x$  if  $a = 0$  or there is an ancestor  $y$  of  $x$  labeled by  $\dashv_a$  (resp.  $\vdash_a$ ) and  $x$  is in the right (resp. left) subtree of  $y$ . The *image* of  $x$  is its greatest eligible integer. Moreover, let

$$\text{word}_\gamma : \mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})(n) \rightarrow \text{Dias}_\gamma(n), \quad n \geq 1, \quad (2.2.2)$$

the map where  $\text{word}_\gamma(\mathfrak{t})$  is the word obtained by considering, from left to right, the images of the leaves of  $\mathfrak{t}$  (see Figure 1).

**Lemma 2.2.1.** *For any integer  $\gamma \geq 0$ , the map  $\text{word}_\gamma$  is an operad morphism from  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  to  $\text{Dias}_\gamma$ .*

*Proof.* Let us first show that  $\text{word}_\gamma$  is a well-defined map. Let  $\mathfrak{t}$  be a syntax tree of  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  of arity  $n$ . Observe that by starting from the root of  $\mathfrak{t}$ , there is a unique maximal path obtained by following the directions specified by its internal nodes (a  $\dashv_a$  means to go the left child while a  $\vdash_a$  means to go to the right child). Then, the leaf at the end of this path is the only leaf with 0 as image. Others  $n - 1$  leaves have integers of  $[\gamma]$  as images. By Proposition 2.1.1, this implies that  $\text{word}_\gamma(\mathfrak{t})$  is an element of  $\text{Dias}_\gamma(n)$ .

To prove that  $\text{word}_\gamma$  is an operad morphism, we consider its following alternative description. If  $\mathfrak{t}$  is a syntax tree of  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$ , we can consider the tree  $\mathfrak{t}'$  obtained by replacing in  $\mathfrak{t}$  each label  $\dashv_a$  (resp.  $\vdash_a$ ) by the word  $0a$  (resp.  $a0$ ), where  $a \in [\gamma]$ . Then, by a straightforward

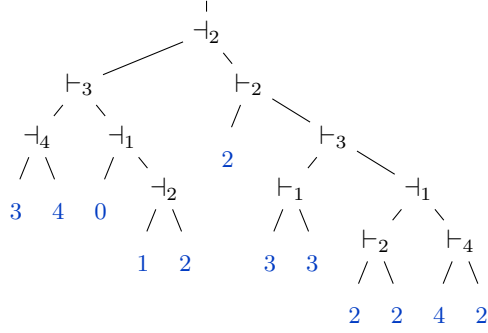


FIGURE 1. A syntax tree  $t$  of  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  where images of its leaves are shown. This tree satisfies  $\text{word}_\gamma(t) = 340122332242$ .

induction on the number of internal nodes of  $t$ , we obtain that  $\text{eval}_{\text{Dias}_\gamma}(t')$ , where  $t'$  is seen as a syntax tree of  $\mathbf{Free}(\text{Dias}_\gamma(2))$ , is  $\text{word}_\gamma(t)$ . It then follows that  $\text{word}_\gamma$  is an operad morphism.  $\square$

2.2.2. *Hook syntax trees.* Let us now consider the map

$$\text{hook}_\gamma : \text{Dias}_\gamma(n) \rightarrow \mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})(n), \quad n \geq 1, \quad (2.2.3)$$

defined for any word  $x$  of  $\text{Dias}_\gamma$  by

$$\text{hook}_\gamma(x) := \begin{array}{c} \vdash_{u_1} \\ / \quad \backslash \\ \vdash_{u_1} \quad \vdash_{u_{|u|}} \\ / \quad \backslash \end{array}, \quad (2.2.4)$$

where  $x$  decomposes, by Proposition 2.1.1, uniquely in  $x = u0v$  where  $u$  and  $v$  are words on the alphabet  $[\gamma]$ . The dashed edges denote, depending on their orientation, a right comb (wherein internal nodes are labeled, from top to bottom by  $\vdash_{u_1}, \dots, \vdash_{u_{|u|}}$ ) or a left comb (wherein internal nodes are labeled, from bottom to top, by  $\dashv_{v_1}, \dots, \dashv_{v_{|v|}}$ ). We shall call any syntax tree of the form (2.2.4) a *hook syntax tree*.

**Lemma 2.2.2.** *For any integer  $\gamma \geq 0$ , the map  $\text{word}_\gamma$  is a surjective operad morphism from  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  onto  $\text{Dias}_\gamma$ . Moreover, for any element  $x$  of  $\text{Dias}_\gamma$ ,  $\text{hook}_\gamma(x)$  belongs to the fiber of  $x$  under  $\text{word}_\gamma$ .*

*Proof.* The fact that  $x$  belongs to the fiber of  $x$  under  $\text{word}_\gamma$  is an immediate consequence of the definitions of  $\text{word}_\gamma$  and  $\text{hook}_\gamma$ , and the fact that by Proposition 2.1.1, any word  $x$  of  $\text{Dias}_\gamma$  decomposes uniquely in  $x = u0v$  where  $u$  and  $v$  are words on the alphabet  $[\gamma]$ . Then,  $\text{word}_\gamma$

is surjective as a map. Moreover, since by Lemma 2.2.1,  $\text{word}_\gamma$  is an operad morphism, it is a surjective operad morphism.  $\square$

2.2.3. *A rewrite rule on syntax trees.* Let  $\rightarrow_\gamma$  be the quadratic rewrite rule on  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  satisfying

$$\vdash_{a'} \circ_2 \dashv_a \rightarrow_\gamma \dashv_a \circ_1 \vdash_{a'}, \quad a, a' \in [\gamma], \quad (2.2.5a)$$

$$\dashv_a \circ_2 \vdash_b \rightarrow_\gamma \dashv_a \circ_1 \dashv_b, \quad a < b \in [\gamma], \quad (2.2.5b)$$

$$\vdash_a \circ_1 \dashv_b \rightarrow_\gamma \vdash_a \circ_2 \vdash_b, \quad a < b \in [\gamma], \quad (2.2.5c)$$

$$\dashv_a \circ_2 \dashv_b \rightarrow_\gamma \dashv_b \circ_1 \dashv_a, \quad a < b \in [\gamma], \quad (2.2.5d)$$

$$\vdash_a \circ_1 \vdash_b \rightarrow_\gamma \vdash_b \circ_2 \vdash_a, \quad a < b \in [\gamma], \quad (2.2.5e)$$

$$\dashv_d \circ_2 \dashv_c \rightarrow_\gamma \dashv_d \circ_1 \dashv_d, \quad c \leq d \in [\gamma], \quad (2.2.5f)$$

$$\dashv_d \circ_2 \vdash_c \rightarrow_\gamma \dashv_d \circ_1 \dashv_d, \quad c \leq d \in [\gamma], \quad (2.2.5g)$$

$$\vdash_d \circ_1 \dashv_c \rightarrow_\gamma \vdash_d \circ_2 \vdash_d, \quad c \leq d \in [\gamma], \quad (2.2.5h)$$

$$\vdash_d \circ_1 \vdash_c \rightarrow_\gamma \vdash_d \circ_2 \vdash_d, \quad c \leq d \in [\gamma], \quad (2.2.5i)$$

and denote by  $\equiv_\gamma$  the operadic congruence on  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  induced by  $\rightarrow_\gamma$ .

**Lemma 2.2.3.** *For any integer  $\gamma \geq 0$  and any syntax trees  $\mathfrak{t}_1$  and  $\mathfrak{t}_2$  of  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$ ,  $\mathfrak{t}_1 \equiv_\gamma \mathfrak{t}_2$  implies  $\text{word}_\gamma(\mathfrak{t}_1) = \text{word}_\gamma(\mathfrak{t}_2)$ .*

*Proof.* Let us denote by  $\leftrightarrow_\gamma$  the symmetric closure of  $\rightarrow_\gamma$ . In the first place, observe that for any relation  $\mathfrak{s}_1 \leftrightarrow_\gamma \mathfrak{s}_2$  where  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  are syntax trees of  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  (3), for any  $i \in [3]$ , the eligible integers for the  $i$ th leaves of  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  are the same. Besides, by definition of  $\equiv_\gamma$ , since  $\mathfrak{t}_1 \equiv_\gamma \mathfrak{t}_2$ , one can obtain  $\mathfrak{t}_2$  from  $\mathfrak{t}_1$  by performing a sequence of  $\leftrightarrow_\gamma$ -rewritings. According to the previous observation, a  $\leftrightarrow_\gamma$ -rewriting preserve the eligible integers of all leaves of the tree on which they are performed. Therefore, the images of the leaves of  $\mathfrak{t}_2$  are, from left to right, the same as the images of the leaves of  $\mathfrak{t}_1$  and hence,  $\text{word}_\gamma(\mathfrak{t}_1) = \text{word}_\gamma(\mathfrak{t}_2)$ .  $\square$

Lemma 2.2.3 implies that the map

$$\bar{\text{word}}_\gamma : \mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})(n) / \equiv_\gamma \rightarrow \text{Dias}_\gamma(n), \quad n \geq 1, \quad (2.2.6)$$

satisfying, for any  $\equiv_\gamma$ -equivalence class  $[\mathfrak{t}]_{\equiv_\gamma}$ ,

$$\bar{\text{word}}_\gamma([\mathfrak{t}]_\gamma) = \text{word}_\gamma(\mathfrak{t}), \quad (2.2.7)$$

where  $\mathfrak{t}$  is any tree of  $[\mathfrak{t}]_{\equiv_\gamma}$  is well-defined.

**Lemma 2.2.4.** *For any integer  $\gamma \geq 0$ , any syntax tree  $\mathfrak{t}$  of  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  can be rewritten, by a sequence of  $\rightarrow_\gamma$ -rewritings, into a hook syntax tree. Moreover, this hook syntax tree is  $\text{hook}_\gamma(\text{word}_\gamma(\mathfrak{t}))$ .*



*Proof.* In the following, to gain readability, we shall denote by  $\dashv_*$  (resp.  $\vdash_*$ ) any element  $\dashv_a$  (resp.  $\vdash_a$ ) of  $\mathfrak{G}_{\text{Dias}_\gamma}$  when taking into account the value of  $a \in [\gamma]$  is not necessary. Using this notation, from (2.2.5a)—(2.2.5i), we observe that  $\rightarrow_\gamma$  expresses as

$$\vdash_* \circ_2 \dashv_* \rightarrow_\gamma \dashv_* \circ_1 \vdash_*, \quad (2.2.8a)$$

$$\dashv_* \circ_2 \vdash_* \rightarrow_\gamma \dashv_* \circ_1 \dashv_*, \quad (2.2.8b)$$

$$\vdash_* \circ_1 \dashv_* \rightarrow_\gamma \vdash_* \circ_2 \vdash_*, \quad (2.2.8c)$$

$$\dashv_* \circ_2 \dashv_* \rightarrow_\gamma \dashv_* \circ_1 \dashv_*, \quad (2.2.8d)$$

$$\vdash_* \circ_1 \vdash_* \rightarrow_\gamma \vdash_* \circ_2 \vdash_* . \quad (2.2.8e)$$

Let us first focus on the first part of the statement of the lemma to show that  $\mathfrak{t}$  is rewritable by  $\rightarrow_\gamma$  into a hook syntax tree. We reason by induction on the arity  $n$  of  $\mathfrak{t}$ . When  $n \leq 2$ ,  $\mathfrak{t}$  is immediately a hook syntax tree. Otherwise,  $\mathfrak{t}$  has at least two internal nodes. Then,  $\mathfrak{t}$  is made of a root connected to a first subtree  $\mathfrak{t}_1$  and a second subtree  $\mathfrak{t}_2$ . By induction hypothesis,  $\mathfrak{t}$  is rewritable by  $\rightarrow_\gamma$  into a tree made of a root  $r$  of the same label as the one of the root of  $\mathfrak{t}$ , connected to a first subtree  $\mathfrak{s}_1$  such that  $\mathfrak{t}_1 \xrightarrow{*}_\gamma \mathfrak{s}_1$  and a second subtree  $\mathfrak{s}_2$  such that  $\mathfrak{t}_2 \xrightarrow{*}_\gamma \mathfrak{s}_2$ , both being hook syntax trees. We have to deal two cases following the number of internal nodes of  $\mathfrak{t}_1$ .

*Case 1.* If  $\mathfrak{t}_1$  has at least one internal node, we have the two  $\xrightarrow{*}_\gamma$ -relations

$$(2.2.9)$$

The first  $\xrightarrow{*}_\gamma$ -relation of (2.2.9) has just been explained. The second one comes from the application of the induction hypothesis on the upper part of the tree of the middle of (2.2.9) obtained by cutting the edge connecting the node  $x$  to its father. When the rightmost tree of (2.2.9) is not already a hook syntax tree, one has two cases following the label of  $x$ .

*Case 1.1.* If  $x$  is labeled by  $\vdash_*$ , by (2.2.8e), the bottom part of the rightmost tree of (2.2.9) consisting in internal nodes labeled by  $\vdash_*$  is rewritable by  $\rightarrow_\gamma$  into a right comb tree wherein internal nodes are labeled by  $\vdash_*$ . Then, the rightmost tree of (2.2.9) is rewritable by  $\rightarrow_\gamma$  into a hook syntax tree, and then  $\mathfrak{t}$  also is.

*Case 1.2.* Otherwise,  $x$  is labeled by  $\dashv_*$ . By definition of  $\text{hook}_\gamma$ , the second subtree of  $x$  is a leaf. By (2.2.8c), the bottom part of the rightmost tree of (2.2.9) consisting in  $x$  and internal nodes labeled by  $\vdash_*$  can be rewritten by  $\rightarrow_\gamma$  into a right comb tree wherein internal nodes are labeled by  $\vdash_*$ . Then, the rightmost tree of (2.2.9) is rewritable by  $\rightarrow_\gamma$  into a hook syntax tree, and then  $\mathfrak{t}$  also is.

*Case 2.* Otherwise,  $\mathfrak{t}_1$  is the leaf. We then have the  $\overset{*}{\rightarrow}_\gamma$ -relation

$$\mathfrak{t} \xrightarrow{\overset{*}{\rightarrow}_\gamma} \begin{array}{c} | \\ r \\ / \quad \backslash \\ \mathfrak{s}_{21} \quad r' \\ / \quad \backslash \\ \mathfrak{s}_{21} \quad \mathfrak{s}_{22} \end{array}, \quad (2.2.10)$$

where  $\mathfrak{s}_{21}$  is the first subtree of the root of  $\mathfrak{s}_2$ ,  $\mathfrak{s}_{22}$  is the second subtree of the root of  $\mathfrak{s}_2$ , and  $r'$  is a node with the same label as the root of  $\mathfrak{s}_2$ .

*Case 2.1.* If  $r \circ_2 r'$  is equal to  $\vdash_* \circ_2 \dashv_*$ ,  $\dashv_* \circ_2 \vdash_*$ , or  $\dashv_* \circ_2 \dashv_*$ , respectively by (2.2.8a), (2.2.8b), and (2.2.8d), the rightmost tree of (2.2.10) can be rewritten by  $\rightarrow_\gamma$  into a tree  $\mathfrak{r}$  having a first subtree with at least one internal node. Hence,  $\mathfrak{r}$  is of the form required to be treated by *Case 1.*, implying that  $\mathfrak{t}$  is rewritable by  $\rightarrow_\gamma$  into a hook syntax tree.

*Case 2.2.* Otherwise,  $r \circ_2 r'$  is equal to  $\vdash_* \circ_2 \vdash_*$ . Since  $\mathfrak{s}_2$  is by hypothesis a hook syntax tree, it is necessarily a right comb tree whose internal nodes are labeled by  $\vdash_*$ . Hence, the rightmost tree of (2.2.10) is already a hook syntax tree, showing that  $\mathfrak{t}$  is rewritable by  $\rightarrow_\gamma$  into a hook syntax tree.

Let us finally show the last part of the statement of the lemma. Observe that, by definition of  $\text{hook}_\gamma$  and  $\text{word}_\gamma$ , if  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  are two different hook syntax trees,  $\text{word}_\gamma(\mathfrak{s}_1) \neq \text{word}_\gamma(\mathfrak{s}_2)$ . We have just shown that  $\mathfrak{t}$  is rewritable by  $\rightarrow_\gamma$  into a hook syntax tree  $\mathfrak{s}$ . Besides, by Lemma 2.2.3, one has  $\text{word}_\gamma(\mathfrak{t}) = \text{word}_\gamma(\mathfrak{s})$ . Then,  $\mathfrak{s}$  is necessarily the hook syntax tree  $\text{hook}_\gamma(\text{word}_\gamma(\mathfrak{t}))$ .  $\square$

#### 2.2.4. Presentation by generators and relations.

**Lemma 2.2.5.** *For any integers  $\gamma \geq 0$  and  $n \geq 1$ , the map  $\overline{\text{word}}_\gamma$  defines a bijection between  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})(n)/\equiv_\gamma$  and  $\text{Dias}_\gamma(n)$ .*

*Proof.* Let us show that  $\overline{\text{word}}_\gamma$  is injective. Let  $\mathfrak{t}_1$  and  $\mathfrak{t}_2$  be two syntax trees of  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  such that  $\text{word}_\gamma(\mathfrak{t}_1) = \text{word}_\gamma(\mathfrak{t}_2)$  and let  $\mathfrak{s} := \text{hook}_\gamma(\text{word}_\gamma(\mathfrak{t}_1)) = \text{hook}_\gamma(\text{word}_\gamma(\mathfrak{t}_2))$ . By Lemma 2.2.4, one has  $\mathfrak{t}_1 \xrightarrow{\overset{*}{\rightarrow}_\gamma} \mathfrak{s}$  and  $\mathfrak{t}_2 \xrightarrow{\overset{*}{\rightarrow}_\gamma} \mathfrak{s}$ , and hence,  $\mathfrak{t}_1 \equiv_\gamma \mathfrak{t}_2$ . By the definition of the map  $\overline{\text{word}}_\gamma$  from the map  $\text{word}_\gamma$ , this show that  $\overline{\text{word}}_\gamma$  is injective. Besides, by Lemma 2.2.2,  $\overline{\text{word}}_\gamma$  is surjective, whence the statement of the lemma.  $\square$

**Theorem 2.2.6.** *For any integer  $\gamma \geq 0$ , the operad  $\text{Dias}_\gamma$  admits the following presentation. It is generated by  $\mathfrak{G}_{\text{Dias}_\gamma}$  and its space of relations  $\mathfrak{R}_{\text{Dias}_\gamma}$  is the space induced by the equivalence relation  $\leftrightarrow_\gamma$  satisfying*

$$\dashv_a \circ_1 \vdash_{a'} \leftrightarrow_\gamma \vdash_{a'} \circ_2 \dashv_a, \quad a, a' \in [\gamma], \quad (2.2.11a)$$

$$\dashv_a \circ_1 \dashv_b \leftrightarrow_\gamma \dashv_a \circ_2 \vdash_b, \quad a < b \in [\gamma], \quad (2.2.11b)$$

$$\vdash_a \circ_1 \dashv_b \leftrightarrow_\gamma \vdash_a \circ_2 \vdash_b, \quad a < b \in [\gamma], \quad (2.2.11c)$$

$$\dashv_b \circ_1 \dashv_a \leftrightarrow_\gamma \dashv_a \circ_2 \dashv_b, \quad a < b \in [\gamma], \quad (2.2.11d)$$

$$\vdash_a \circ_1 \vdash_b \leftrightarrow_\gamma \vdash_b \circ_2 \vdash_a, \quad a < b \in [\gamma], \quad (2.2.11e)$$

$$\dashv_d \circ_1 \dashv_d \leftrightarrow_\gamma \dashv_d \circ_2 \dashv_c \leftrightarrow_\gamma \dashv_d \circ_2 \vdash_c, \quad c \leq d \in [\gamma], \quad (2.2.11f)$$

$$\vdash_d \circ_1 \dashv_c \leftrightarrow_\gamma \vdash_d \circ_1 \vdash_c \leftrightarrow_\gamma \vdash_d \circ_2 \vdash_d, \quad c \leq d \in [\gamma]. \quad (2.2.11g)$$

*Proof.* By Lemma 2.2.5, the map  $\overline{\text{word}}_\gamma$  is, for any  $n \geq 1$ , a bijection between the sets  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})(n)/\equiv_\gamma$  and  $\text{Dias}_\gamma(n)$ . Moreover, by Lemma 2.2.1,  $\text{word}_\gamma$  is an operad morphism, and then  $\overline{\text{word}}_\gamma$  also is. Hence,  $\overline{\text{word}}_\gamma$  is an operad isomorphism between  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})/\equiv_\gamma$  and  $\text{Dias}_\gamma$ . Therefore, since  $\mathfrak{A}_{\text{Dias}_\gamma}$  is the space induced by  $\equiv_\gamma$ ,  $\text{Dias}_\gamma$  admits the stated presentation.  $\square$

The space of relations  $\mathfrak{A}_{\text{Dias}_\gamma}$  of  $\text{Dias}_\gamma$  exhibited by Theorem 2.2.6 can be rephrased in a more compact way as the space generated by

$$\dashv_a \circ_1 \vdash_{a'} - \vdash_{a'} \circ_2 \dashv_a, \quad a, a' \in [\gamma], \quad (2.2.12a)$$

$$\dashv_a \circ_1 \dashv_{a\uparrow a'} - \dashv_a \circ_2 \vdash_{a'}, \quad a, a' \in [\gamma], \quad (2.2.12b)$$

$$\vdash_a \circ_1 \dashv_{a'} - \vdash_a \circ_2 \vdash_{a\uparrow a'}, \quad a, a' \in [\gamma], \quad (2.2.12c)$$

$$\dashv_{a\uparrow a'} \circ_1 \dashv_a - \dashv_a \circ_2 \dashv_{a'}, \quad a, a' \in [\gamma], \quad (2.2.12d)$$

$$\vdash_a \circ_1 \vdash_{a'} - \vdash_{a\uparrow a'} \circ_2 \vdash_a, \quad a, a' \in [\gamma]. \quad (2.2.12e)$$

Observe that, by Theorem 2.2.6,  $\text{Dias}_1$  and the diassociative operad (see [Lod01] or Section 1.3) admit the same presentation. Then, for all integers  $\gamma \geq 0$ , the operads  $\text{Dias}_\gamma$  are generalizations of the diassociative operad.

**2.3. Miscellaneous properties.** From the description of the elements of  $\text{Dias}_\gamma$  and its structure revealed by its presentation, we develop here some of its properties. Unless otherwise specified,  $\text{Dias}_\gamma$  is still considered in this section as a set-operad.

### 2.3.1. Koszulity.

**Theorem 2.3.1.** *For any integer  $\gamma \geq 0$ ,  $\text{Dias}_\gamma$  is a Koszul operad. Moreover, the set of hook syntax trees of  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  forms a Poincaré-Birkhoff-Witt basis of  $\text{Dias}_\gamma$ .*

*Proof.* From the definition of hook syntax trees, it appears that no hook syntax tree can be rewritten by  $\rightarrow_\gamma$  into another syntax tree. Hence, and by Lemma 2.2.4,  $\rightarrow_\gamma$  is a terminating rewrite rule and its normal forms are hook syntax trees. Moreover, again by Lemma 2.2.4, since any syntax tree is rewritable by  $\rightarrow_\gamma$  into a unique hook syntax tree,  $\rightarrow_\gamma$  is a confluent rewrite rule, and hence,  $\rightarrow_\gamma$  is convergent. Now, since by Theorem 2.2.6, the space of relations of  $\text{Dias}_\gamma$  is the space induced by the operad congruence induced by  $\rightarrow_\gamma$ , by the Koszulity criterion [Hof10, DK10, LV12] we have reformulated in Section 1.2.5,  $\text{Dias}_\gamma$  is a Koszul operad and the set of hook syntax trees of  $\mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})$  forms a Poincaré-Birkhoff-Witt basis of  $\text{Dias}_\gamma$ .  $\square$

2.3.2. *Symmetries.* If  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are two operads, a linear map  $\phi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  is an *operad antimorphism* if it respects arities and anticommutes with partial composition maps, that is,

$$\phi(x \circ_i y) = \phi(x) \circ_{n-i+1} \phi(y), \quad x \in \mathcal{O}(n), y \in \mathcal{O}, i \in [n]. \quad (2.3.1)$$

A *symmetry* of an operad  $\mathcal{O}$  is either an automorphism or an antiautomorphism. The set of all symmetries of  $\mathcal{O}$  form a group for the composition, called the *group of symmetries* of  $\mathcal{O}$ .

**Proposition 2.3.2.** *For any integer  $\gamma \geq 0$ , the group of symmetries of  $\text{Dias}_\gamma$  as a set-operad contains two elements: the identity map and the linear map sending any word of  $\text{Dias}_\gamma$  to its mirror image.*

*Proof.* Let us denote by  $\mathbb{G}_\gamma$  the set  $\{0a, a0 : a \in [\gamma]\}$ . Since  $\text{Dias}_\gamma$  is generated by  $\mathbb{G}_\gamma$ , any automorphism or antiautomorphism  $\phi$  of  $\text{Dias}_\gamma$  is wholly determined by the images of the elements of  $\mathbb{G}_\gamma$ . Besides let us observe that  $\phi$  is in particular a permutation of  $\mathbb{G}_\gamma$ .

By contradiction, assume that  $\phi$  is an automorphism of  $\text{Dias}_\gamma$  different from the identity map. We have two cases to explore.

*Case 1.* If there are  $a, a' \in [\gamma]$  satisfying  $\phi(0a) = a'0$ , since  $\phi$  is a permutation of  $\mathbb{G}_\gamma$ , there are  $b, b' \in [\gamma]$  satisfying  $\phi(b0) = 0b'$ . Then, we have at the same time  $b0 \circ_2 0a = b0a = 0a \circ_1 b0$ ,

$$\phi(b0 \circ_2 0a) = \phi(b0) \circ_2 \phi(0a) = 0b' \circ_2 a'0 = 0(b' \uparrow a')b', \quad (2.3.2)$$

and

$$\phi(0a \circ_1 b0) = \phi(0a) \circ_1 \phi(b0) = a'0 \circ_1 0b' = a'(a' \uparrow b')0. \quad (2.3.3)$$

This shows that  $\phi(b0 \circ_2 0a) \neq \phi(0a \circ_1 b0)$  and hence,  $\phi$  is not an operad morphism. By a similar argument, one can show that there are no  $a, a' \in [\gamma]$  such that  $\phi(a0) = 0a'$ .

*Case 2.* Otherwise, for all  $a \in [\gamma]$ , we have  $\phi(0a) = 0a'$  and  $\phi(a0) = a''0$  for some  $a', a'' \in [\gamma]$ . Since, by hypothesis,  $\phi$  is not the identity map, there exist  $a \neq a' \in [\gamma]$  such that  $\phi(0a) = 0a'$  or  $\phi(a0) = a'0$ . Let us assume, without loss of generality, that  $\phi(0a) = 0a'$ . Since  $\phi$  is a permutation of  $\mathbb{G}_\gamma$ , there exist  $b \neq b' \in [\gamma]$  such that  $\phi(0b) = 0b'$ . One can assume, without loss of generality, that  $a < b$  and  $b' < a'$ . Then, we have at the same time  $0a \circ_2 0b = 0ab = 0b \circ_1 0a$ ,

$$\phi(0a \circ_2 0b) = \phi(0a) \circ_2 \phi(0b) = 0a' \circ_2 0b' = 0a'a', \quad (2.3.4)$$

and

$$\phi(0b \circ_1 0a) = \phi(0b) \circ_1 \phi(0a) = 0b' \circ_1 0a' = 0a'b'. \quad (2.3.5)$$

This shows that  $\phi(0a \circ_2 0b) \neq \phi(0b \circ_1 0a)$  and hence, that  $\phi$  is not an operad morphism. By a similar argument, one can show that there are no  $a \neq a' \in [\gamma]$  such that  $\phi(a0) = \phi(a'0)$ .

We then have shown that if  $\phi$  is an automorphism of  $\text{Dias}_\gamma$ , it is necessarily the identity map.

Finally, by Proposition 2.1.1, if  $x$  is an element of  $\text{Dias}_\gamma$ , its mirror image also is in  $\text{Dias}_\gamma$ . Moreover, it is immediate to see that the map sending a word to its mirror image is an antiautomorphism of  $\text{Dias}_\gamma$ . Similar arguments as the ones developed previously show that it is the only.  $\square$

2.3.3. *Basic operad.* A set-operad  $\mathcal{O}$  is *basic* if for all  $y_1, \dots, y_n \in \mathcal{O}$ , all the maps

$$\circ^{y_1, \dots, y_n} : \mathcal{O}(n) \rightarrow \mathcal{O}(|y_1| + \dots + |y_n|) \quad (2.3.6)$$

defined by

$$\circ^{y_1, \dots, y_n}(x) := x \circ (y_1, \dots, y_n), \quad x \in \mathcal{O}(n), \quad (2.3.7)$$

are injective. This property for set-operads introduced by Vallette [Val07] is a very relevant one since there is a general construction producing a family of posets (see [MY91] and [CL07]) from a basic set-operad. This family of posets leads to the definition of an incidence Hopf algebra by a construction of Schmitt [Sch94].

**Proposition 2.3.3.** *For any integer  $\gamma \geq 0$ ,  $\text{Dias}_\gamma$  is a basic operad.*

*Proof.* Let  $n \geq 1$ ,  $y_1, \dots, y_n$  be words of  $\text{Dias}_\gamma$ , and  $x$  and  $x'$  be two words of  $\text{Dias}_\gamma(n)$  such that  $\circ^{y_1, \dots, y_n}(x) = \circ^{y_1, \dots, y_n}(x')$ . Then, for all  $i \in [n]$  and  $j \in [|y_i|]$ , we have  $x_i \uparrow y_{i,j} = x'_i \uparrow y_{i,j}$  where  $y_{i,j}$  is the  $j$ th letter of  $y_i$ . Since by Proposition 2.1.1, any word  $y_i$  contains a 0, we have in particular  $x_i \uparrow 0 = x'_i \uparrow 0$  for all  $i \in [n]$ . This implies  $x = x'$  and thus, that  $\circ^{y_1, \dots, y_n}$  is injective.  $\square$

2.3.4. *Rooted operad.* We restate here a property on operads introduced by Chapoton [Cha14]. An operad  $\mathcal{O}$  is *rooted* if there is a map

$$\text{root} : \mathcal{O}(n) \rightarrow [n], \quad n \geq 1, \quad (2.3.8)$$

satisfying, for all  $x \in \mathcal{O}(n)$ ,  $y \in \mathcal{O}(m)$ , and  $i \in [n]$ ,

$$\text{root}(x \circ_i y) = \begin{cases} \text{root}(x) + m - 1 & \text{if } i \leq \text{root}(x) - 1, \\ \text{root}(x) + \text{root}(y) - 1 & \text{if } i = \text{root}(x), \\ \text{root}(x) & \text{otherwise } (i \geq \text{root}(x) + 1). \end{cases} \quad (2.3.9)$$

We call such a map a *root map*. More intuitively, the root map of a rooted operad associates a particular input with any of its elements and this input is preserved by partial compositions.

It is immediate that any operad  $\mathcal{O}$  is a rooted operad for the root maps  $\text{root}_L$  and  $\text{root}_R$ , which send respectively all elements  $x$  of arity  $n$  to 1 or to  $n$ . For this reason, we say that an operad  $\mathcal{O}$  is *nontrivially rooted* if it can be endowed with a root map different from  $\text{root}_L$  and  $\text{root}_R$ .

**Proposition 2.3.4.** *For any integer  $\gamma \geq 0$ ,  $\text{Dias}_\gamma$  is a nontrivially rooted operad for the root map sending any word of  $\text{Dias}_\gamma$  to the position of its 0.*

*Proof.* Thanks to Proposition 2.1.1, the map of the statement of the proposition is well-defined. The fact that 0 is the neutral element for the  $\uparrow$  operation and the fact that any word of  $\text{Dias}_\gamma$  contains exactly one 0 imply that this map satisfies (2.3.9). Finally, this map is obviously different from  $\text{root}_L$  and  $\text{root}_R$ , whence the statement of the proposition.  $\square$

2.3.5. *Alternative basis.* In this section,  $\text{Dias}_\gamma$  is considered as an operad in the category of vector spaces.

Let  $\preceq_\gamma$  be the order relation on the underlying set of  $\text{Dias}_\gamma(n)$ ,  $n \geq 1$ , where for all words  $x$  and  $y$  of  $\text{Dias}_\gamma$  of a same arity  $n$ , we have

$$x \preceq_\gamma y \quad \text{if } x_i \leq y_i \text{ for all } i \in [n]. \quad (2.3.10)$$

This order relation allows to define for all word  $x$  of  $\text{Dias}_\gamma$  the elements

$$\mathsf{K}_x^{(\gamma)} := \sum_{x \preceq_\gamma x'} \mu_\gamma(x, x') x', \quad (2.3.11)$$

where  $\mu_\gamma$  is the Möbius function of the poset defined by  $\preceq_\gamma$ . For instance,

$$\mathsf{K}_{102}^{(2)} = 102 - 202, \quad (2.3.12)$$

$$\mathsf{K}_{102}^{(3)} = \mathsf{K}_{102}^{(4)} = 102 - 103 - 202 + 203, \quad (2.3.13)$$

$$\mathsf{K}_{23102}^{(3)} = 23102 - 23103 - 23202 + 23203 - 33102 + 33103 + 33202 - 33203. \quad (2.3.14)$$

Since, by Möbius inversion, for any word  $x$  of  $\text{Dias}_\gamma$  one has

$$x = \sum_{x \preceq_\gamma x'} \mathsf{K}_{x'}^{(\gamma)}, \quad (2.3.15)$$

the family of all  $\mathsf{K}_x^{(\gamma)}$ , where the  $x$  are words of  $\text{Dias}_\gamma$ , forms by triangularity a basis of  $\text{Dias}_\gamma$ , called the *K-basis*.

If  $u$  and  $v$  are two words of a same length  $n$ , we denote by  $\text{ham}(u, v)$  the *Hamming distance* between  $u$  and  $v$  that is the number of positions  $i \in [n]$  such that  $u_i \neq v_i$ . Moreover, for any word  $x$  of  $\text{Dias}_\gamma$  of length  $n$  and any subset  $J$  of  $[n]$ , we denote by  $\text{Incr}_\gamma(x, J)$  the set of words obtained by incrementing by one some letters of  $x$  smaller than  $\gamma$  and greater than 0 whose positions are in  $J$ . We shall simply denote by  $\text{Incr}_\gamma(x)$  the set  $\text{Incr}_\gamma(x, [n])$ . Proposition 2.1.1 ensures that all  $\text{Incr}_\gamma(x, J)$  are sets of words of  $\text{Dias}_\gamma$ .

**Lemma 2.3.5.** *For any integer  $\gamma \geq 0$  and any word  $x$  of  $\text{Dias}_\gamma$ ,*

$$\mathsf{K}_x^{(\gamma)} = \sum_{x' \in \text{Incr}_\gamma(x)} (-1)^{\text{ham}(x, x')} x'. \quad (2.3.16)$$

*Proof.* Let  $n$  be the arity of  $x$ . To compute  $\mathsf{K}_x^{(\gamma)}$  from its definition (2.3.11), it is enough to know the Möbius function  $\mu_\gamma$  of the poset  $\mathbb{P}_x^{(\gamma)}$  consisting in the words  $x'$  of  $\text{Dias}_\gamma$  satisfying  $x \preceq_\gamma x'$ . Immediately from the definition of  $\preceq_\gamma$ , it appears that  $\mathbb{P}_x^{(\gamma)}$  is isomorphic to the Cartesian product poset

$$\mathbb{T}_x^{(\gamma)} := \mathbb{T}(\gamma - x_1) \times \cdots \times \mathbb{T}(\gamma - x_{r-1}) \times \mathbb{T}(0) \times \mathbb{T}(\gamma - x_{r+1}) \times \cdots \times \mathbb{T}(\gamma - x_n), \quad (2.3.17)$$

where for any nonnegative integer  $k$ ,  $\mathbb{T}(k)$  denotes the poset over  $\{0\} \cup [k]$  with the natural total order relation, and  $r$  is the position of, by Proposition 2.1.1, the only 0 of  $x$ . The map  $\phi_x^{(\gamma)} : \mathbb{P}_x^{(\gamma)} \rightarrow \mathbb{T}_x^{(\gamma)}$  defined for all words  $x'$  of  $\mathbb{P}_x^{(\gamma)}$  by

$$\phi_x^{(\gamma)}(x') := (x'_1 - x_1, \dots, x'_{r-1} - x_{r-1}, 0, x'_{r+1} - x_{r+1}, \dots, x'_n - x_n) \quad (2.3.18)$$

is an isomorphism of posets.

Recall that the Möbius function  $\mu$  of  $\mathbb{T}(k)$  satisfies, for all  $a, a' \in \mathbb{T}(k)$ ,

$$\mu(a, a') = \begin{cases} 1 & \text{if } a' = a, \\ -1 & \text{if } a' = a + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.3.19)$$

Moreover, since by [Sta11], the Möbius function of a Cartesian product poset is the product of the Möbius functions of the posets involved in the product, through the isomorphism  $\phi_x^{(\gamma)}$ , we obtain that when  $x'$  is in  $\text{Incr}_\gamma(x)$ ,  $\mu_\gamma(x, x') = (-1)^{\text{ham}(x, x')}$  and that when  $x'$  is not in  $\text{Incr}_\gamma$ ,  $\mu_\gamma(x, x') = 0$ . Therefore, (2.3.16) is established.  $\square$

**Lemma 2.3.6.** *For any integer  $\gamma \geq 0$ , any word  $x$  of  $\text{Dias}_\gamma$ , and any nonempty set  $J$  of positions of letters of  $x$  that are greater than 0 and smaller than  $\gamma$ ,*

$$\sum_{x' \in \text{Incr}_\gamma(x, J)} (-1)^{\text{ham}(x, x')} = 0. \quad (2.3.20)$$

*Proof.* The statement of the lemma follows by induction on the nonzero cardinality of  $J$ .  $\square$

To compute a direct expression for the partial composition of  $\text{Dias}_\gamma$  over the  $\mathbb{K}$ -basis, we have to introduce two notations. If  $x$  is a word of  $\text{Dias}_\gamma$  of length nonsmaller than 2, we denote by  $\min(x)$  the smallest letter of  $x$  among its letters different from 0. Proposition 2.1.1 ensures that  $\min(x)$  is well-defined. Moreover, for all words  $x$  and  $y$  of  $\text{Dias}_\gamma$ , a position  $i$  such that  $x_i \neq 0$ , and  $a \in [\gamma]$ , we denote by  $x \circ_{a, i} y$  the word  $x \circ_i y$  in which the 0 coming from  $y$  is replaced by  $a$  instead of  $x_i$ .

**Theorem 2.3.7.** *For any integer  $\gamma \geq 0$ , the partial composition of  $\text{Dias}_\gamma$  over the  $\mathbb{K}$ -basis satisfies, for all words  $x$  and  $y$  of  $\text{Dias}_\gamma$  of arities nonsmaller than 2,*

$$\mathbb{K}_x^{(\gamma)} \circ_i \mathbb{K}_y^{(\gamma)} = \begin{cases} \mathbb{K}_{x \circ_i y}^{(\gamma)} & \text{if } \min(y) > x_i, \\ \sum_{a \in [x_i, \gamma]} \mathbb{K}_{x \circ_{a, i} y}^{(\gamma)} & \text{if } \min(y) = x_i, \\ 0 & \text{otherwise } (\min(y) < x_i). \end{cases} \quad (2.3.21)$$

*Proof.* First of all, by Lemma 2.3.5 together with (2.3.15), we obtain

$$\begin{aligned} \mathbb{K}_x^{(\gamma)} \circ_i \mathbb{K}_y^{(\gamma)} &= \sum_{\substack{x' \in \text{Incr}_\gamma(x) \\ y' \in \text{Incr}_\gamma(y)}} (-1)^{\text{ham}(x, x') + \text{ham}(y, y')} \left( \sum_{x' \circ_i y' \preccurlyeq_\gamma z} \mathbb{K}_z^{(\gamma)} \right) \\ &= \sum_{x \circ_i y \preccurlyeq_\gamma z} \sum_{\substack{x' \in \text{Incr}_\gamma(x) \\ y' \in \text{Incr}_\gamma(y) \\ x' \circ_i y' \preccurlyeq_\gamma z}} (-1)^{\text{ham}(x, x') + \text{ham}(y, y')} \mathbb{K}_z^{(\gamma)}. \end{aligned} \quad (2.3.22)$$

Let us denote by  $n$  (resp.  $m$ ) the arity of  $x$  (resp.  $y$ ) and let  $z$  be a word of  $\text{Dias}_\gamma$  such that  $x \circ_i y \preceq_\gamma z$ . Let  $x' \in \text{Incr}_\gamma(x)$  and  $y' \in \text{Incr}_\gamma(y)$ . We have, by definition of the partial composition of  $\text{Dias}_\gamma$ ,

$$x \circ_i y = x_1 \dots x_{i-1} t_1 \dots t_{r-1} x_i t_{r+1} \dots t_m x_{i+1} \dots x_n, \quad (2.3.23)$$

and

$$x' \circ_i y' = x'_1 \dots x'_{i-1} t'_1 \dots t'_{r-1} x'_i t'_{r+1} \dots t'_m x'_{i+1} \dots x'_n, \quad (2.3.24)$$

where  $r$  denotes the position of the only, by Proposition 2.1.1, 0 of  $y$  and for all  $j \in [m] \setminus \{r\}$ ,  $t_j := x_i \uparrow y_j$  and  $t'_j := x'_i \uparrow y'_j$ . By (2.3.22), the pair  $(x', y')$  contributes to the coefficient of  $\mathbf{K}_z^{(\gamma)}$  in (2.3.22) if and only if  $x \circ_i y \preceq_\gamma x' \circ_i y' \preceq z$ . To compute this coefficient, we have three cases to consider following the value of  $\min(y)$  compared to the value of  $x_i$ .

*Case 1.* Assume first that  $\min(y) < x_i$ . Then, there is at least a  $s \in [m] \setminus \{r\}$  such that  $y_s < x_i$ . This implies that  $t_s = x_i$  and that  $y'_s$  has no influence on  $t'_s$  and then, on  $x' \circ_i y'$ . Thus, the word  $y'' := y'_1 \dots y'_{s-1} a y'_{s+1} \dots y'_m$  where  $a$  is the only possible letter such that  $y'' \in \text{Incr}_\gamma(y)$  and  $a \neq y'_s$  satisfies  $x' \circ_i y'' = x' \circ_i y'$ . Therefore, since  $\text{ham}(y', y'') = 1$ , the contribution of the pair  $(x', y')$  for the coefficient of  $\mathbf{K}_z^{(\gamma)}$  in (2.3.22) is compensated by the contribution of the pair  $(x', y'')$ . This shows that this coefficient is 0 and hence,  $\mathbf{K}_x^{(\gamma)} \circ_i \mathbf{K}_y^{(\gamma)} = 0$ .

*Case 2.* Assume now that  $\min(y) > x_i$ . Then, for all  $j \in [m] \setminus \{r\}$ , we have  $y_j > x_i$  and thus,  $t_j = y_j$ . When  $z = x \circ_i y$ , we necessarily have  $x' = x$  and  $y' = y$ . Hence, the coefficient of  $\mathbf{K}_{x \circ_i y}^{(\gamma)}$  in (2.3.22) is 1. Else, when  $z \neq x \circ_i y$ , we have  $x' \circ_i y' \in \text{Incr}_\gamma(x \circ_i y, J)$ , where  $J$  is the nonempty set of the positions of letters of  $z$  different from letters of  $x \circ_i y$ . Now, from (2.3.22), the coefficient of  $\mathbf{K}_z^{(\gamma)}$  in (2.3.22) is

$$\sum_{x' \circ_i y' \in \text{Incr}_\gamma(x \circ_i y, J)} (-1)^{\text{ham}(x, x') + \text{ham}(y, y')}. \quad (2.3.25)$$

Lemma 2.3.6 implies that this coefficient is 0. This shows that  $\mathbf{K}_x^{(\gamma)} \circ_i \mathbf{K}_y^{(\gamma)} = \mathbf{K}_{x \circ_i y}^{(\gamma)}$ .

*Case 3.* The last case occurs when  $\min(y) = x_i$ . Then, for all  $j \in [m] \setminus \{r\}$ , we have  $y_j \geq x_i$  and thus,  $t_j = y_j$ . Moreover, there is at least a  $s \in [m] \setminus \{r\}$  such that  $y_s = x_i$ . When  $z = x \circ_{a,i} y$  with  $a \in [x_i, \gamma]$ , we necessarily have  $x' = x$  and  $y' = y$ . Therefore, for all  $a \in [x_i, \gamma]$ , the  $\mathbf{K}_{x \circ_{a,i} y}^{(\gamma)}$  have coefficient 1 in (2.3.22). The same argument as the one exposed for *Case 2.* shows that when  $z \neq x \circ_{a,i} y$  for all  $a \in [x_i, \gamma]$ , the coefficient of  $\mathbf{K}_z^{(\gamma)}$  is zero. Hence,  $\mathbf{K}_x^{(\gamma)} \circ_i \mathbf{K}_y^{(\gamma)} = \sum_{a \in [x_i, \gamma]} \mathbf{K}_{x \circ_{a,i} y}^{(\gamma)}$ .

□

We have for instance

$$\mathbf{K}_{20413}^{(5)} \circ_1 \mathbf{K}_{304}^{(5)} = \mathbf{K}_{3240413}^{(5)}, \quad (2.3.26)$$

$$\mathbf{K}_{20413}^{(5)} \circ_2 \mathbf{K}_{304}^{(5)} = \mathbf{K}_{2304413}^{(5)}, \quad (2.3.27)$$

$$\mathbf{K}_{20413}^{(5)} \circ_3 \mathbf{K}_{304}^{(5)} = 0, \quad (2.3.28)$$

$$\mathbf{K}_{20413}^{(5)} \circ_4 \mathbf{K}_{304}^{(5)} = \mathbf{K}_{2043143}^{(5)}, \quad (2.3.29)$$

$$\mathbf{K}_{20413}^{(5)} \circ_5 \mathbf{K}_{304}^{(5)} = \mathbf{K}_{2041334}^{(5)} + \mathbf{K}_{2041344}^{(5)} + \mathbf{K}_{2041354}^{(5)}. \quad (2.3.30)$$



Theorem 2.3.7 implies in particular that the structure coefficients of the partial composition of  $\mathbf{Dias}_\gamma$  over the  $\mathbf{K}$ -basis are 0 or 1. It is possible to define another bases of  $\mathbf{Dias}_\gamma$  by reversing in (2.3.11) the relation  $\preceq_\gamma$  and by suppressing or keeping the Möbius function  $\mu_\gamma$ . This gives obviously rise to three other bases. It worth to note that, as small computations reveal, over all these additional bases, the structure coefficients of the partial composition of  $\mathbf{Dias}_\gamma$  can be negative or different from 1. This observation makes the  $\mathbf{K}$ -basis even more particular and interesting. It has some other properties, as next section will show.

2.3.6. *Alternative presentation.* The  $\mathbf{K}$ -basis introduced in the previous section leads to state a new presentation for  $\mathbf{Dias}_\gamma$  in the following way.

For any integer  $\gamma \geq 0$ , let  $\dashv_a$  and  $\vdash_a$ ,  $a \in [\gamma]$ , be the elements of  $\mathbf{Free}(\mathfrak{G}_{\mathbf{Dias}_\gamma})$  (2) defined by

$$\dashv_a := \begin{cases} \dashv_\gamma & \text{if } a = \gamma, \\ \dashv_a - \dashv_{a+1} & \text{otherwise,} \end{cases} \quad (2.3.31a)$$

and

$$\vdash_a := \begin{cases} \vdash_\gamma & \text{if } a = \gamma, \\ \vdash_a - \vdash_{a+1} & \text{otherwise.} \end{cases} \quad (2.3.31b)$$

Then, since for all  $a \in [\gamma]$  we have

$$\dashv_a = \sum_{a \leq b \in [\gamma]} \dashv_b \quad (2.3.32a)$$

and

$$\vdash_a = \sum_{a \leq b \in [\gamma]} \vdash_b, \quad (2.3.32b)$$

by triangularity, the family  $\mathfrak{G}'_{\mathbf{Dias}_\gamma} := \{\dashv_a, \vdash_a : a \in [\gamma]\}$  forms a basis of  $\mathbf{Free}(\mathfrak{G}_{\mathbf{Dias}_\gamma})$  (2) and then, generates  $\mathbf{Free}(\mathfrak{G}_{\mathbf{Dias}_\gamma})$  as an operad. This change of basis from  $\mathbf{Free}(\mathfrak{G}_{\mathbf{Dias}_\gamma})$  to  $\mathbf{Free}(\mathfrak{G}'_{\mathbf{Dias}_\gamma})$  comes from the change of basis from the usual basis of  $\mathbf{Dias}_\gamma$  to the  $\mathbf{K}$ -basis. Let us now express a presentation of  $\mathbf{Dias}_\gamma$  through the family  $\mathfrak{G}'_{\mathbf{Dias}_\gamma}$ .

**Proposition 2.3.8.** *For any integer  $\gamma \geq 0$ , the operad  $\mathbf{Dias}_\gamma$  admits the following presentation. It is generated by  $\mathfrak{G}'_{\mathbf{Dias}_\gamma}$  and its space of relations is  $\mathfrak{R}'_{\mathbf{Dias}_\gamma}$  is generated by*

$$\dashv_a \circ_1 \vdash_{a'} - \vdash_{a'} \circ_2 \dashv_a, \quad a, a' \in [\gamma], \quad (2.3.33a)$$

$$\vdash_b \circ_1 \vdash_a, \quad a < b \in [\gamma], \quad (2.3.33b)$$

$$\dashv_b \circ_2 \dashv_a, \quad a < b \in [\gamma], \quad (2.3.33c)$$

$$\vdash_b \circ_1 \dashv_a, \quad a < b \in [\gamma], \quad (2.3.33d)$$

$$\dashv_b \circ_2 \vdash_a, \quad a < b \in [\gamma], \quad (2.3.33e)$$

$$\vdash_a \circ_1 \vdash_b - \vdash_b \circ_2 \vdash_a, \quad a < b \in [\gamma], \quad (2.3.33f)$$

$$\dashv_b \circ_1 \dashv_a - \dashv_a \circ_2 \dashv_b, \quad a < b \in [\gamma], \quad (2.3.33g)$$

$$\vdash_a \circ_1 \dashv_b - \vdash_a \circ_2 \dashv_b, \quad a < b \in [\gamma], \quad (2.3.33h)$$

$$\dashv_a \circ_1 \dashv_b - \dashv_a \circ_2 \dashv_b, \quad a < b \in [\gamma], \quad (2.3.33i)$$

$$\vdash_a \circ_1 \vdash_a - \left( \sum_{a \leq b \in [\gamma]} \vdash_a \circ_2 \vdash_b \right), \quad a \in [\gamma], \quad (2.3.33j)$$

$$\left( \sum_{a \leq b \in [\gamma]} \dashv_a \circ_1 \dashv_b \right) - \dashv_a \circ_2 \dashv_a, \quad a \in [\gamma], \quad (2.3.33k)$$

$$\vdash_a \circ_1 \dashv_a - \left( \sum_{a \leq b \in [\gamma]} \vdash_b \circ_2 \vdash_a \right), \quad a \in [\gamma], \quad (2.3.33l)$$

$$\left( \sum_{a \leq b \in [\gamma]} \dashv_b \circ_1 \dashv_a \right) - \dashv_a \circ_2 \vdash_a, \quad a \in [\gamma]. \quad (2.3.33m)$$

*Proof.* Let us show that  $\mathfrak{R}'_{\text{Dias}_\gamma}$  is equal to the space of relations  $\mathfrak{R}_{\text{Dias}_\gamma}$  of  $\text{Dias}_\gamma$  defined in the statement of Theorem 2.2.6. First of all, recall that the map  $\text{word}_\gamma : \mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma}) \rightarrow \text{Dias}_\gamma$  defined in Section 2.2.1 satisfies  $\text{word}_\gamma(\dashv_a) = 0a$  and  $\text{word}_\gamma(\vdash_a) = a0$  for all  $a \in [\gamma]$ . By Theorem 2.2.6, for any  $x \in \mathbf{Free}(\mathfrak{G}_{\text{Dias}_\gamma})(3)$ ,  $x$  is in  $\mathfrak{R}_{\text{Dias}_\gamma}$  if and only if  $\text{word}_\gamma(x) = 0$ .

Besides, by definition of  $\dashv_a, \vdash_a, a \in [\gamma]$ , and by making use of the  $\mathbb{K}$ -basis of  $\text{Dias}_\gamma$ , we have  $\text{word}_\gamma(\dashv_a) = \mathbb{K}_{0a}^{(\gamma)}$  and  $\text{word}_\gamma(\vdash_a) = \mathbb{K}_{a0}^{(\gamma)}$ . By using the partial composition rules for  $\text{Dias}_\gamma$  over the  $\mathbb{K}$ -basis of Theorem 2.3.7, straightforward computations show that  $\text{word}_\gamma(x) = 0$  for all elements  $x$  among (2.3.33a)—(2.3.33m). This implies that  $\mathfrak{R}'_{\text{Dias}_\gamma}$  is a subspace of  $\mathfrak{R}_{\text{Dias}_\gamma}$ .

Now, one can observe that elements (2.3.33a)—(2.3.33m) are linearly independent. Then,  $\mathfrak{R}'_{\text{Dias}_\gamma}$  has dimension  $5\gamma^2$  which is also, by Theorem 2.2.6, the dimension of  $\mathfrak{R}_{\text{Dias}_\gamma}$ . Hence,  $\mathfrak{R}'_{\text{Dias}_\gamma}$  and  $\mathfrak{R}_{\text{Dias}_\gamma}$  are equal. The statement of the proposition follows.  $\square$

Despite the apparent complexity of the presentation of  $\text{Dias}_\gamma$  exhibited by Proposition 2.3.8, as we will see in Section 2 of [Gir16], the Koszul dual of  $\text{Dias}_\gamma$  computed from this presentation has a very simple and manageable expression.

### 3. PLURIASOCIATIVE ALGEBRAS

We now focus on algebras over  $\gamma$ -pluriassociative operads. For this purpose, we construct free  $\text{Dias}_\gamma$ -algebras over one generator, and define and study two notions of units for  $\text{Dias}_\gamma$ -algebras. We end this section by introducing a convenient way to define  $\text{Dias}_\gamma$ -algebras and give several examples of such algebras.

**3.1. Category of pluriassociative algebras and free objects.** Let us study the category of  $\text{Dias}_\gamma$ -algebras and the units for algebras in this category.

**3.1.1. Pluriassociative algebras.** We call  $\gamma$ -pluriassociative algebra any  $\text{Dias}_\gamma$ -algebra. From the presentation of  $\text{Dias}_\gamma$  provided by Theorem 2.2.6, any  $\gamma$ -pluriassociative algebra is a vector space endowed with linear operations  $\dashv_a, \vdash_a, a \in [\gamma]$ , satisfying the relations encoded by (2.2.12a)—(2.2.12e).

3.1.2. *General definitions.* Let  $\mathcal{P}$  be a  $\gamma$ -pluriassociative algebra. We say that  $\mathcal{P}$  is *commutative* if for all  $x, y \in \mathcal{P}$  and  $a \in [\gamma]$ ,  $x \dashv_a y = y \vdash_a x$ . Besides,  $\mathcal{P}$  is *pure* for all  $a, a' \in [\gamma]$ ,  $a \neq a'$  implies  $\dashv_a \neq \dashv_{a'}$  and  $\vdash_a \neq \vdash_{a'}$ .

Given a subset  $C$  of  $[\gamma]$ , one can keep on the vector space  $\mathcal{P}$  only the operations  $\dashv_a$  and  $\vdash_a$  such that  $a \in C$ . By renumbering the indexes of these operations from 1 to  $\#C$  by respecting their former relative numbering, we obtain a  $\#C$ -pluriassociative algebra. We call it the  $\#C$ -*pluriassociative subalgebra induced by  $C$*  of  $\mathcal{P}$ .

3.1.3. *Free pluriassociative algebras.* Recall that  $\mathcal{F}_{\text{Dias}_\gamma}$  denotes the free  $\text{Dias}_\gamma$ -algebra over one generator. By definition,  $\mathcal{F}_{\text{Dias}_\gamma}$  is the linear span of the set of the words on  $\{0\} \cup [\gamma]$  with exactly one occurrence of 0. Let us endow this space with the linear operations

$$\dashv_a, \vdash_a : \mathcal{F}_{\text{Dias}_\gamma} \otimes \mathcal{F}_{\text{Dias}_\gamma} \rightarrow \mathcal{F}_{\text{Dias}_\gamma}, \quad a \in [\gamma], \quad (3.1.1)$$

satisfying, for any such words  $u$  and  $v$ ,

$$u \dashv_a v := u h_a(v) \quad (3.1.2a)$$

and

$$u \vdash_a v := h_a(u) v, \quad (3.1.2b)$$

where  $h_a(u)$  (resp.  $h_a(v)$ ) is the word obtained by replacing in  $u$  (resp.  $v$ ) any occurrence of a letter smaller than  $a$  by  $a$ .

**Proposition 3.1.1.** *For any integer  $\gamma \geq 0$ , the vector space  $\mathcal{F}_{\text{Dias}_\gamma}$  of nonempty words on  $\{0\} \cup [\gamma]$  containing exactly one occurrence of 0 endowed with the operations  $\dashv_a, \vdash_a, a \in [\gamma]$ , is the free  $\gamma$ -pluriassociative algebra over one generator.*

*Proof.* The fact that  $\mathcal{F}_{\text{Dias}_\gamma}$  is the stated vector space is a consequence of the description of the elements of  $\text{Dias}_\gamma$  provided by Proposition 2.1.1. Since  $\text{Dias}_\gamma$  is by definition the suboperad of  $\text{TM}_\gamma$  generated by  $\{0a, a0 : a \in [\gamma]\}$ ,  $\mathcal{F}_{\text{Dias}_\gamma}$  is endowed with  $2\gamma$  binary operations where any generator  $0a$  (resp.  $a0$ ) gives rise to the operation  $\dashv_a$  (resp.  $\vdash_a$ ) of  $\mathcal{F}_{\text{Dias}_\gamma}$ . Moreover, by making use of the realization of  $\text{Dias}_\gamma$ , we have for all  $u, v \in \mathcal{F}_{\text{Dias}_\gamma}$  and  $a \in [\gamma]$ ,

$$u \dashv_a v = (u \otimes v) \cdot 0a = (0a \circ_2 v) \circ_1 u = u h_a(v) \quad (3.1.3a)$$

and

$$u \vdash_a v = (u \otimes v) \cdot a0 = (a0 \circ_2 v) \circ_1 u = h_a(u) v. \quad (3.1.3b)$$

□

One has for instance in  $\mathcal{F}_{\text{Dias}_4}$ ,

$$101241 \dashv_2 203 = 101241\mathbf{223} \quad (3.1.4)$$

and

$$101241 \vdash_3 203 = \mathbf{333343}203. \quad (3.1.5)$$

**3.2. Bar and wire-units.** Loday has defined in [Lod01] some notions of units in diassociative algebras. We generalize here these definitions to the context of  $\gamma$ -pluriassociative algebras.

**3.2.1. Bar-units.** Let  $\mathcal{P}$  be a  $\gamma$ -pluriassociative algebra and  $a \in [\gamma]$ . We say that an element  $e$  of  $\mathcal{P}$  is an  $a$ -bar-unit, or simply a bar-unit when taking into account the value of  $a$  is not necessary, of  $\mathcal{P}$  if for all  $x \in \mathcal{P}$ ,

$$x \dashv_a e = x = e \vdash_a x. \quad (3.2.1)$$

As we shall see below, a  $\gamma$ -pluriassociative algebra can have, for a given  $a \in [\gamma]$ , several  $a$ -bar-units. The  $a$ -halo of  $\mathcal{P}$ , denoted by  $\text{Halo}_a(\mathcal{P})$ , is the set of the  $a$ -bar-units of  $\mathcal{P}$ .

**3.2.2. Wire-units.** Let  $\mathcal{P}$  be a  $\gamma$ -pluriassociative algebra and  $a \in [\gamma]$ . We say that an element  $e$  of  $\mathcal{P}$  is an  $a$ -wire-unit, or simply a wire-unit when taking into account the value of  $a$  is not necessary, of  $\mathcal{P}$  if for all  $x \in \mathcal{P}$ ,

$$e \dashv_a x = x = x \vdash_a e. \quad (3.2.2)$$

As shows the following proposition, the presence of a wire-unit in  $\mathcal{P}$  has some implications.

**Proposition 3.2.1.** *Let  $\gamma \geq 0$  be an integer and  $\mathcal{P}$  be a  $\gamma$ -pluriassociative algebra admitting a  $b$ -wire-unit  $e$  for a  $b \in [\gamma]$ . Then*

- (i) for all  $a \in [b]$ , the operations  $\dashv_a$ ,  $\dashv_b$ ,  $\vdash_a$ , and  $\vdash_b$  of  $\mathcal{P}$  are equal;
- (ii)  $e$  is also an  $a$ -wire-unit for all  $a \in [b]$ ;
- (iii)  $e$  is the only wire-unit of  $\mathcal{P}$ ;
- (iv) if  $e'$  is an  $a$ -bar unit for a  $a \in [b]$ , then  $e' = e$ .

*Proof.* Let us show part (i). By Relation (2.2.12d) of  $\gamma$ -pluriassociative algebras and by the fact that  $e$  is a  $b$ -wire-unit of  $\mathcal{P}$ , we have for all elements  $y$  and  $z$  of  $\mathcal{P}$  and all  $a \in [b]$ ,

$$y \dashv_a z = e \dashv_b (y \dashv_a z) = e \dashv_b (y \vdash_a z) = y \vdash_a z. \quad (3.2.3)$$

Thus, the operations  $\dashv_a$  and  $\vdash_a$  of  $\mathcal{P}$  are equal. Moreover, for the same reasons, we have

$$y \dashv_a z = e \dashv_b (y \dashv_a z) = (e \dashv_b y) \dashv_b z = y \dashv_b z. \quad (3.2.4)$$

Then, the operations  $\dashv_a$  and  $\dashv_b$  of  $\mathcal{P}$  are equal, whence (i).

Now, by (i) and by the fact that  $e$  is a  $b$ -wire-unit, we have for all elements  $x$  of  $\mathcal{P}$  and all  $a \in [b]$ ,

$$e \dashv_a x = e \dashv_b x = x = x \vdash_b e = x \vdash_a e, \quad (3.2.5)$$

showing (ii).

To prove (iii), assume that  $e'$  is a  $b'$ -wire-unit of  $\mathcal{P}$  for a  $b' \in [\gamma]$ . By (i) and by the fact that  $e$  is a  $b$ -wire-unit, one has

$$e = e \vdash_{b'} e' = e \dashv_b e' = e', \quad (3.2.6)$$

showing (iii).

To establish (iv), let us first prove that  $e$  is a  $b$ -bar-unit. By (i) and by the fact that  $e$  is a  $b$ -wire-unit, we have for all elements  $x$  of  $\mathcal{P}$ ,

$$e \vdash_b x = e \dashv_b x = x = x \vdash_b e = x \dashv_b e. \quad (3.2.7)$$

Now, since  $e'$  is an  $a$ -bar-unit for an  $a \in [b]$ , by (i) and by the fact that  $e$  is a  $b$ -wire-unit,

$$e = e' \vdash_a e = e' \vdash_b e = e'. \quad (3.2.8)$$

This shows (iv).  $\square$

Relying on Proposition 3.2.1, we define the *height* of a  $\gamma$ -pluriassociative algebra  $\mathcal{P}$  as zero if  $\mathcal{P}$  has no wire-unit, otherwise as the greatest integer  $h \in [\gamma]$  such that the unique wire-unit  $e$  of  $\mathcal{P}$  is a  $h$ -wire-unit. Observe that any pure  $\gamma$ -pluriassociative algebra has height 0 or 1.

**3.3. Construction of pluriassociative algebras.** We now present a general way to construct  $\gamma$ -pluriassociative algebras. Our construction is a natural generalization of some constructions introduced by Loday [Lod01] in the context of diassociative algebras. We introduce in this section new algebraic structures, the so-called  $\gamma$ -multiprojection algebras, which are inputs of our construction.

3.3.1. *Multiassociative algebras.* For any integer  $\gamma \geq 0$ , a  $\gamma$ -multiassociative algebra is a vector space  $\mathcal{M}$  endowed with linear operations

$$\star_a : \mathcal{M} \otimes \mathcal{M} \rightarrow \mathcal{M}, \quad a \in [\gamma], \quad (3.3.1)$$

satisfying, for all  $x, y, z \in \mathcal{M}$ , the relations

$$(x \star_a y) \star_b z = (x \star_b y) \star_{a'} z = x \star_{a''} (y \star_b z) = x \star_b (y \star_{a'''} z), \quad a, a', a'', a''' \leq b \in [\gamma]. \quad (3.3.2)$$

These algebras are obvious generalizations of associative algebras since all of its operations are associative. Observe that by (3.3.2), all bracketings of an expression involving elements of a  $\gamma$ -multiassociative algebra and some of its operations are equal. Then, since the bracketings of such expressions are not significant, we shall denote these without parenthesis. In Section 3 of [Gir16], we will study the underlying operads of the category of  $\gamma$ -multiassociative algebras, called  $\text{As}_\gamma$ , for a very specific purpose.

If  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are two  $\gamma$ -multiassociative algebras, a linear map  $\phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$  is a  $\gamma$ -multiassociative algebra morphism if it commutes with the operations of  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . We say that  $\mathcal{M}$  is *commutative* when all operations of  $\mathcal{M}$  are commutative. Besides, for an  $a \in [\gamma]$ , an element  $\mathbb{1}$  of  $\mathcal{M}$  is an  $a$ -unit, or simply a *unit* when taking into account the value of  $a$  is not necessary, of  $\mathcal{M}$  if for all  $x \in \mathcal{M}$ ,  $\mathbb{1} \star_a x = x = x \star_a \mathbb{1}$ . When  $\mathcal{M}$  admits a unit, we say that  $\mathcal{M}$  is *unital*. As shows the following proposition, the presence of a unit in  $\mathcal{M}$  has some implications.

**Proposition 3.3.1.** *Let  $\gamma \geq 0$  be an integer and  $\mathcal{M}$  be a  $\gamma$ -multiassociative algebra admitting a  $b$ -unit  $\mathbb{1}$  for a  $b \in [\gamma]$ . Then*

- (i) *for all  $a \in [b]$ , the operations  $\star_a$  and  $\star_b$  of  $\mathcal{M}$  are equal;*
- (ii)  *$\mathbb{1}$  is also an  $a$ -unit for all  $a \in [b]$ ;*
- (iii)  *$\mathbb{1}$  is the only unit of  $\mathcal{M}$ .*

*Proof.* By Relation (3.3.2) of  $\gamma$ -multiassociative algebras and by the fact that  $\mathbf{1}$  is a  $b$ -unit of  $\mathcal{M}$ , we have for all elements  $y$  and  $z$  of  $\mathcal{M}$  and all  $a \in [b]$ ,

$$y \star_a z = y \star_a z \star_b \mathbf{1} = y \star_b z \star_b \mathbf{1} = y \star_b z. \quad (3.3.3)$$

Therefore,  $\star_a = \star_b$ , showing (i).

Now, by (i) and by the fact that  $\mathbf{1}$  is a  $b$ -unit, we have for all elements  $x$  of  $\mathcal{M}$  and all  $a \in [b]$ ,

$$\mathbf{1} \star_a x = \mathbf{1} \star_b x = x = x \star_b \mathbf{1} = x \star_a \mathbf{1}, \quad (3.3.4)$$

showing (ii).

To prove (iii), assume that  $\mathbf{1}'$  is a  $b'$ -unit of  $\mathcal{M}$  for a  $b' \in [\gamma]$ . By (i) and by the fact that  $\mathbf{1}$  is a  $b$ -unit, one has

$$\mathbf{1} = \mathbf{1} \star_{b'} \mathbf{1}' = \mathbf{1} \star_b \mathbf{1}' = \mathbf{1}', \quad (3.3.5)$$

establishing (iii).  $\square$

Relying on Proposition 3.3.1, similarly to the case of  $\gamma$ -pluriassociative algebras, we define the *height* of a  $\gamma$ -multiassociative algebra  $\mathcal{M}$  as zero if  $\mathcal{M}$  has no unit, otherwise as the greatest integer  $h \in [\gamma]$  such that the unit  $\mathbf{1}$  of  $\mathcal{M}$  is an  $h$ -unit.

**3.3.2. Multiprojection algebras.** We call  $\gamma$ -multiprojection algebra any  $\gamma$ -multiassociative algebra  $\mathcal{M}$  endowed with endomorphisms

$$\pi_a : \mathcal{M} \rightarrow \mathcal{M}, \quad a \in [\gamma], \quad (3.3.6)$$

satisfying

$$\pi_a \circ \pi_{a'} = \pi_{a \uparrow a'}, \quad a, a' \in [\gamma]. \quad (3.3.7)$$

By extension, the *height* of  $\mathcal{M}$  is its height as a  $\gamma$ -multiassociative algebra. We say that  $\mathcal{M}$  is *unital* as a  $\gamma$ -multiprojection algebra if  $\mathcal{M}$  is unital as a  $\gamma$ -multiassociative algebra and its unit, by Proposition 3.3.1, satisfies  $\pi_a(\mathbf{1}) = \mathbf{1}$  for all  $a \in [h]$  where  $h$  is the height of  $\mathcal{M}$ .

**3.3.3. From multiprojection algebras to pluriassociative algebras.** Next result describes how to construct  $\gamma$ -pluriassociative algebras from  $\gamma$ -multiprojection algebras.

**Theorem 3.3.2.** *For any integer  $\gamma \geq 0$  and any  $\gamma$ -multiprojection algebra  $\mathcal{M}$ , the vector space  $\mathcal{M}$  endowed with binary linear operations  $\dashv_a, \vdash_a$ ,  $a \in [\gamma]$ , defined for all  $x, y \in \mathcal{M}$  by*

$$x \dashv_a y := x \star_a \pi_a(y) \quad (3.3.8a)$$

and

$$x \vdash_a y := \pi_a(x) \star_a y, \quad (3.3.8b)$$

where the  $\star_a$ ,  $a \in [\gamma]$ , are the operations of  $\mathcal{M}$  and the  $\pi_a$ ,  $a \in [\gamma]$ , are its endomorphisms, is a  $\gamma$ -pluriassociative algebra, denoted by  $M(\mathcal{M})$ .

*Proof.* This is a verification of the relations of  $\gamma$ -pluriassociative algebras in  $M(\mathcal{M})$ . Let  $x, y$ , and  $z$  be three elements of  $M(\mathcal{M})$  and  $a, a' \in [\gamma]$ .

By (3.3.2), we have

$$(x \vdash_{a'} y) \dashv_a z = \pi_{a'}(x) \star_{a'} y \star_a \pi_a(z) = x \vdash_{a'} (y \dashv_a z), \quad (3.3.9)$$

showing that (2.2.12a) is satisfied in  $M(\mathcal{M})$ .

Moreover, by (3.3.2) and (3.3.7), we have

$$\begin{aligned} x \dashv_a (y \vdash_{a'} z) &= x \star_a \pi_a(\pi_{a'}(y) \star_{a'} z) \\ &= x \star_a \pi_{a \uparrow a'}(y) \star_{a'} \pi_a(z) \\ &= x \star_{a \uparrow a'} \pi_{a \uparrow a'}(y) \star_a \pi_a(z) \\ &= (x \dashv_{a \uparrow a'} y) \dashv_a z, \end{aligned} \quad (3.3.10)$$

so that (2.2.12b), and for the same reasons (2.2.12c), check out in  $M(\mathcal{M})$ .

Finally, again by (3.3.2) and (3.3.7), we have

$$\begin{aligned} x \dashv_a (y \dashv_{a'} z) &= x \star_a \pi_a(y \star_{a'} \pi_{a'}(z)) \\ &= x \star_a \pi_a(y) \star_{a'} \pi_{a \uparrow a'}(z) \\ &= x \star_a \pi_a(y) \star_{a \uparrow a'} \pi_{a \uparrow a'}(z) \\ &= (x \dashv_a y) \dashv_{a \uparrow a'} z, \end{aligned} \quad (3.3.11)$$

showing that (2.2.12d), and for the same reasons (2.2.12e), are satisfied in  $M(\mathcal{M})$ .  $\square$

When  $\mathcal{M}$  is commutative, since for all  $x, y \in M(\mathcal{M})$  and  $a \in [\gamma]$ ,

$$x \dashv_a y = x \star_a \pi_a(y) = \pi_a(y) \star_a x = y \vdash_a x, \quad (3.3.12)$$

it appears that  $M(\mathcal{M})$  is a commutative  $\gamma$ -pluriassociative algebra.

When  $\mathcal{M}$  is unital,  $M(\mathcal{M})$  has several properties, summarized in the next proposition.

**Proposition 3.3.3.** *Let  $\gamma \geq 0$  be an integer,  $\mathcal{M}$  be a unital  $\gamma$ -multiprojection algebra of height  $h$ . Then, by denoting by  $\mathbb{1}$  the unit of  $\mathcal{M}$  and by  $\pi_a, a \in [\gamma]$ , its endomorphisms,*

- (i) *for any  $a \in [h]$ ,  $\mathbb{1}$  is an  $a$ -bar-unit of  $M(\mathcal{M})$ ;*
- (ii) *for any  $a \leq b \in [h]$ ,  $\text{Halo}_a(M(\mathcal{M}))$  is a subset of  $\text{Halo}_b(M(\mathcal{M}))$ ;*
- (iii) *for any  $a \in [h]$ , the linear span of  $\text{Halo}_a(M(\mathcal{M}))$  forms an  $h - a + 1$ -pluriassociative subalgebra of the  $h - a + 1$ -pluriassociative subalgebra of  $M(\mathcal{M})$  induced by  $[a, h]$ ;*
- (iv) *for any  $a \in [h]$ ,  $\pi_a$  is the identity map if and only if  $\mathbb{1}$  is an  $a$ -wire-unit of  $M(\mathcal{M})$ .*

*Proof.* Let us denote by  $\star_a, a \in [\gamma]$ , the operations of  $\mathcal{M}$ .

Since  $\mathbb{1}$  is an  $h$ -unit of  $\mathcal{M}$ , for all elements  $x$  of  $M(\mathcal{M})$  and all  $a \in [h]$ ,

$$x \dashv_a \mathbb{1} = x \star_a \pi_a(\mathbb{1}) = x \star_a \mathbb{1} = x = \mathbb{1} \star_a x = \pi_a(\mathbb{1}) \star_a x = \mathbb{1} \vdash_a x, \quad (3.3.13)$$

showing (i).

Assume that  $e$  is an element of  $\text{Halo}_a(M(\mathcal{M}))$  for an  $a \in [h]$ , that is,  $e$  is an  $a$ -bar-unit of  $M(\mathcal{M})$ . Then, for all elements  $x$  of  $M(\mathcal{M})$ ,

$$x \dashv_a e = x \star_a \pi_a(e) = x = \pi_a(e) \star_a x = e \vdash_a x, \quad (3.3.14)$$

showing that  $\pi_a(e)$  is the unit for the operation  $\star_a$  on  $M(\mathcal{M})$  and therefore,  $\pi_a(e) = \mathbb{1}$ . Since  $\mathcal{M}$  is unital, we have  $\pi_b(\mathbb{1}) = \mathbb{1}$  for all  $b \in [h]$ . Hence, and by (3.3.7), for all  $a \leq b \in [h]$ ,

$$\pi_b(e) = \pi_b(\pi_a(e)) = \pi_b(\mathbb{1}) = \mathbb{1}. \quad (3.3.15)$$

Then, for all elements  $x$  of  $M(\mathcal{M})$  and all  $a \leq b \in [h]$ ,

$$x \dashv_b e = x \star_b \pi_b(e) = x \star_b \mathbb{1} = x = \mathbb{1} \star_b x = \pi_b(e) \star_b x = e \vdash_b x, \quad (3.3.16)$$

showing that  $e$  is also a  $b$ -bar-unit of  $M(\mathcal{M})$ , whence (ii).

Let  $a \in [\gamma]$  and  $e$  and  $e'$  be elements of  $\text{Halo}_a(M(\mathcal{M}))$ . By (ii),  $e$  and  $e'$  are  $b$ -bar-units of  $M(\mathcal{M})$  for all  $a \leq b \in [h]$  and hence,

$$e \dashv_b e' = e = e' \vdash_b e. \quad (3.3.17)$$

Therefore, the linear span of  $\text{Halo}_a(M(\mathcal{M}))$  is stable for the operations  $\dashv_b$  and  $\vdash_b$ . This implies (iii).

Finally, assume that  $\pi_a$  is the identity map for an  $a \in [h]$ . Then, for all elements  $x$  of  $M(\mathcal{M})$ ,

$$\mathbb{1} \dashv_a x = \mathbb{1} \star_a \pi_a(x) = \mathbb{1} \star_a x = x = x \star_a \mathbb{1} = \pi_a(x) \star_a \mathbb{1} = x \vdash_a \mathbb{1}, \quad (3.3.18)$$

showing that  $\mathbb{1}$  is an  $a$ -wire unit of  $M(\mathcal{M})$ . Conversely, if  $\mathbb{1}$  is an  $a$ -wire unit of  $M(\mathcal{M})$ , for all elements  $x$  of  $M(\mathcal{M})$ , the relations  $\mathbb{1} \dashv_a x = x = x \vdash_a \mathbb{1}$  imply  $\mathbb{1} \star_a \pi_a(x) = x = \pi_a(x) \star_a \mathbb{1}$  and hence,  $\pi_a(x) = x$ . This shows (iv).  $\square$

**3.3.4. Examples of constructions of pluriassociative algebras.** The construction  $M$  of Theorem 3.3.2 allows to build several  $\gamma$ -pluriassociative algebras. Here follows few examples.

**The  $\gamma$ -pluriassociative algebra of positive integers.** Let  $\gamma \geq 1$  be an integer and consider the vector space  $\text{Pos}$  of positive integers, endowed with the operations  $\star_a$ ,  $a \in [\gamma]$ , all equal to the operation  $\uparrow$  extended by linearity and with the endomorphisms  $\pi_a$ ,  $a \in [\gamma]$ , linearly defined for any positive integer  $x$  by  $\pi_a(x) := a \uparrow x$ . Then,  $\text{Pos}$  is a non-unital  $\gamma$ -multiprojection algebra. By Theorem 3.3.2,  $M(\text{Pos})$  is a  $\gamma$ -pluriassociative algebra. We have for instance

$$2 \dashv_3 5 = 5, \quad (3.3.19)$$

and

$$1 \vdash_3 2 = 3. \quad (3.3.20)$$

We can observe that  $M(\text{Pos})$  is commutative, pure, and its 1-halo is  $\{1\}$ . Moreover, when  $\gamma \geq 2$ ,  $M(\text{Pos})$  has no wire-unit and no  $a$ -bar-unit for  $a \geq 2 \in [\gamma]$ . This example is important because it provides a counterexample for (ii) of Proposition 3.3.3 in the case when the construction  $M$  is applied to a non-unital  $\gamma$ -multiprojection algebra.



**The  $\gamma$ -pluriassociative algebra of finite sets.** Let  $\gamma \geq 1$  be an integer and consider the vector space **Sets** of finite sets of positive integers, endowed with the operations  $\star_a$ ,  $a \in [\gamma]$ , all equal to the union operation  $\cup$  extended by linearity and with the endomorphisms  $\pi_a$ ,  $a \in [\gamma]$ , linearly defined for any finite set of positive integers  $x$  by  $\pi_a(x) := x \cap [a, \gamma]$ . Then, **Sets** is a  $\gamma$ -multiprojection algebra. By Theorem 3.3.2,  $M(\mathbf{Sets})$  is a  $\gamma$ -pluriassociative algebra. We have for instance

$$\{2, 4\} \dashv_3 \{1, 3, 5\} = \{2, 3, 4, 5\}, \quad (3.3.21)$$

and

$$\{1, 2, 4\} \vdash_3 \{1, 3, 5\} = \{1, 3, 4, 5\}. \quad (3.3.22)$$

We can observe that  $M(\mathbf{Sets})$  is commutative and pure. Moreover,  $\emptyset$  is a 1-wire-unit of  $M(\mathbf{Sets})$  and, by Proposition 3.2.1, it is its only wire-unit. Therefore,  $M(\mathbf{Sets})$  has height 1. Observe that for any  $a \in [\gamma]$ , the  $a$ -halo of  $M(\mathbf{Sets})$  consists in the subsets of  $[a - 1]$ . Besides, since **Sets** is a unital  $\gamma$ -multiprojection algebra,  $M(\mathbf{Sets})$  satisfies all properties exhibited by Proposition 3.3.3.

**The  $\gamma$ -pluriassociative algebra of words.** Let  $\gamma \geq 1$  be an integer and consider the vector space **Words** of the words of positive integers. Let us endow **Words** with the operations  $\star_a$ ,  $a \in [\gamma]$ , all equal to the concatenation operation extended by linearity and with the endomorphisms  $\pi_a$ ,  $a \in [\gamma]$ , where for any word  $x$  of positive integers,  $\pi_a(x)$  is the longest subword of  $x$  consisting in letters greater than or equal to  $a$ . Then, **Words** is a  $\gamma$ -multiprojection algebra. By Theorem 3.3.2,  $M(\mathbf{Words})$  is a  $\gamma$ -pluriassociative algebra. We have for instance

$$412 \dashv_3 14231 = 41243, \quad (3.3.23)$$

and

$$11 \vdash_2 323 = 323. \quad (3.3.24)$$

We can observe that  $M(\mathbf{Words})$  is not commutative and is pure. Moreover,  $\epsilon$  is a 1-wire-unit of  $M(\mathbf{Words})$  and by Proposition 3.2.1, it is its only wire-unit. Therefore,  $M(\mathbf{Words})$  has height 1. Observe that for any  $a \in [\gamma]$ , the  $a$ -halo of  $M(\mathbf{Words})$  consists in the words on the alphabet  $[a - 1]$ . Besides, since **Words** is a unital  $\gamma$ -multiprojection algebra,  $M(\mathbf{Words})$  satisfies all properties exhibited by Proposition 3.3.3.

The  $\gamma$ -pluriassociative algebras  $M(\mathbf{Sets})$  and  $M(\mathbf{Words})$  are related in the following way. Let  $I_{\text{com}}$  be the subspace of  $M(\mathbf{Words})$  generated by the  $x - x'$  where  $x$  and  $x'$  are words of positive integers and have the same commutative image. Since  $I_{\text{com}}$  is a  $\gamma$ -pluriassociative algebra ideal of  $M(\mathbf{Words})$ , one can consider the quotient  $\gamma$ -pluriassociative algebra  $\mathbf{CWords} := M(\mathbf{Words})/I_{\text{com}}$ . Its elements can be seen as commutative words of positive integers.

Moreover, let  $I_{\text{occ}}$  be the subspace of  $M(\mathbf{CWords})$  generated by the  $x - x'$  where  $x$  and  $x'$  are commutative words of positive integers and for any letter  $a \in [\gamma]$ ,  $a$  appears in  $x$  if and only if  $a$  appears in  $x'$ . Since  $I_{\text{occ}}$  is a  $\gamma$ -pluriassociative algebra ideal of  $M(\mathbf{CWords})$ , one can consider the quotient  $\gamma$ -pluriassociative algebra  $M(\mathbf{CWords})/I_{\text{occ}}$ . Its elements can be seen as finite subsets of positive integers and we observe that  $M(\mathbf{CWords})/I_{\text{occ}} = M(\mathbf{Sets})$ .

**The  $\gamma$ -pluriassociative algebra of marked words.** Let  $\gamma \geq 1$  be an integer and consider the vector space  $\text{MWords}$  of the words of positive integers where letters can be marked or not, with at least one occurrence of a marked letter. We denote by  $\bar{a}$  any *marked letter*  $a$  and we say that the *value* of  $\bar{a}$  is  $a$ . Let us endow  $\text{MWords}$  with the linear operations  $\star_a$ ,  $a \in [\gamma]$ , where for all words  $u$  and  $v$  of  $\text{MWords}$ ,  $u \star_a v$  is obtained by concatenating  $u$  and  $v$ , and by replacing therein all marked letters by  $\bar{c}$  where  $c := \max(u) \uparrow a \uparrow \max(v)$  where  $\max(u)$  (resp.  $\max(v)$ ) denotes the greatest value among the marked letters of  $u$  (resp.  $v$ ). For instance,

$$2\bar{1}31\bar{3} \star_2 3\bar{4}\bar{1}2\bar{1} = 2\bar{4}31\bar{4}3\bar{4}\bar{4}2\bar{1}, \quad (3.3.25)$$

and

$$\bar{2}11\bar{1} \star_3 3\bar{4}\bar{2} = \bar{3}11\bar{3}3\bar{4}\bar{3}. \quad (3.3.26)$$

We also endow  $\text{MWords}$  with the endomorphisms  $\pi_a$ ,  $a \in [\gamma]$ , where for any word  $u$  of  $\text{MWords}$ ,  $\pi_a(u)$  is obtained by replacing in  $u$  any occurrence of a nonmarked letter smaller than  $a$  by  $a$ . For instance,

$$\pi_3(2\bar{2}1\bar{4}\bar{3}\bar{5}) = 3\bar{2}3\bar{4}\bar{3}\bar{5}. \quad (3.3.27)$$

One can show without difficulty that  $\text{MWords}$  is a  $\gamma$ -multiprojection algebra. By Theorem 3.3.2,  $\text{M}(\text{MWords})$  is a  $\gamma$ -pluriassociative algebra. We have for instance

$$3\bar{2}\bar{5} \vdash_3 4\bar{4}\bar{1} = 3\bar{4}\bar{5}4\bar{4}\bar{3}, \quad (3.3.28)$$

and

$$1\bar{3}4\bar{1}\bar{3} \vdash_2 31\bar{2}3\bar{1}\bar{1} = 2\bar{3}4\bar{3}31\bar{3}3\bar{3}\bar{1}. \quad (3.3.29)$$

We can observe that  $\text{M}(\text{MWords})$  is not commutative, pure, and has no wire-units neither bar-units.

**The free  $\gamma$ -pluriassociative algebra over one generator.** Let  $\gamma \geq 0$  be an integer. We give here a construction of the free  $\gamma$ -pluriassociative algebra  $\mathcal{F}_{\text{Dias}_\gamma}$  over one generator described in Section 3.1.3 passing through the following  $\gamma$ -multiprojection algebra and the construction  $\text{M}$ . Consider the vector space of nonempty words on the alphabet  $\{0\} \cup [\gamma]$  with exactly one occurrence of 0, endowed with the operations  $\star_a$ ,  $a \in [\gamma]$ , all equal to the concatenation operation extended by linearity and with the endomorphisms  $h_a$ ,  $a \in [\gamma]$ , defined in Section 3.1.3. This vector space is a  $\gamma$ -multiprojection algebra. Therefore, by Theorem 3.3.2, it gives rise by the construction  $\text{M}$  to a  $\gamma$ -pluriassociative algebra and it appears that it is  $\mathcal{F}_{\text{Dias}_\gamma}$ . Besides, we can now observe that  $\mathcal{F}_{\text{Dias}_\gamma}$  is not commutative, pure, and has no wire-units neither bar-units.

#### 4. PLURITRIASSOCIATIVE OPERADS

Our original idea of using the  $\text{T}$  construction (see Sections 1.1.3 and 2.1.1) to obtain a generalization of the diassociative operad admits an analogue in the context of the triassociative operad [LR04]. We describe in this section a generalisation on a nonnegative integer parameter  $\gamma$  of the triassociative operad.

Since the proofs of the results contained in this section are very similar to the ones of Section 2, we omit proofs here.

**4.1. Construction and first properties.** For any integer  $\gamma \geq 0$ , we define  $\text{Trias}_\gamma$  as the suboperad of  $\mathcal{M}_\gamma$  generated by

$$\{0a, 00, a0 : a \in [\gamma]\}. \quad (4.1.1)$$

By definition,  $\text{Trias}_\gamma$  is the vector space of words that can be obtained by partial compositions of words of (4.1.1). We have, for instance,

$$\text{Trias}_2(1) = \text{Vect}(\{0\}), \quad (4.1.2)$$

$$\text{Trias}_2(2) = \text{Vect}(\{00, 01, 02, 10, 20\}), \quad (4.1.3)$$

$$\begin{aligned} \text{Trias}_2(3) = \text{Vect}(\{000, 001, 002, 010, 011, 012, 020, 021, \\ 022, 100, 101, 102, 110, 120, 200, 201, 202, 210, 220\}), \end{aligned} \quad (4.1.4)$$

It follows immediately from the definition of  $\text{Trias}_\gamma$  as a suboperad of  $\text{T}\mathcal{M}_\gamma$  that  $\text{Trias}_\gamma$  is a set-operad. Moreover, one can observe that  $\text{Trias}_\gamma$  is generated by the same generators as the ones of  $\text{Dias}_\gamma$  (see (2.1.1)), plus the word 00. Therefore,  $\text{Dias}_\gamma$  is a suboperad of  $\text{Trias}_\gamma$ . Besides, note that  $\text{Trias}_0$  is the associative operad and that  $\text{Trias}_\gamma$  is a suboperad of  $\text{Trias}_{\gamma+1}$ . We call  $\text{Trias}_\gamma$  the  $\gamma$ -*pluritriassociative operad*.

**Proposition 4.1.1.** *For any integer  $\gamma \geq 0$ , as a set-operad, the underlying set of  $\text{Trias}_\gamma$  is the set of the words on the alphabet  $\{0\} \cup [\gamma]$  containing at least one occurrence of 0.*

We deduce from Proposition 4.1.1 that the Hilbert series of  $\text{Trias}_\gamma$  satisfies

$$\mathcal{H}_{\text{Trias}_\gamma}(t) = \frac{t}{(1 - \gamma t)(1 - \gamma t - t)} \quad (4.1.5)$$

and that for all  $n \geq 1$ ,  $\dim \text{Trias}_\gamma(n) = (\gamma + 1)^n - \gamma^n$ . For instance, the first dimensions of  $\text{Trias}_1$ ,  $\text{Trias}_2$ ,  $\text{Trias}_3$ , and  $\text{Trias}_4$  are respectively

$$1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \quad (4.1.6)$$

$$1, 5, 19, 65, 211, 665, 2059, 6305, 19171, 58025, 175099, \quad (4.1.7)$$

$$1, 7, 37, 175, 781, 3367, 14197, 58975, 242461, 989527, 4017157, \quad (4.1.8)$$

$$1, 9, 61, 369, 2101, 11529, 61741, 325089, 1690981, 8717049, 44633821. \quad (4.1.9)$$

The first one is Sequence [A000225](#), the second one is Sequence [A001047](#), the third one is Sequence [A005061](#), and the last one is Sequence [A005060](#) of [Slo].

**4.2. Presentation by generators and relations.** We follow the same strategy as the one used in Section 2.2 to establish a presentation by generators and relations of  $\mathbf{Trias}_\gamma$  and prove that it is a Koszul operad. As announced above, we omit complete proofs here but we describe the analogue for  $\mathbf{Trias}_\gamma$  of the maps  $\text{word}_\gamma$  and  $\text{hook}_\gamma$  defined in Section 2.2 for the operad  $\mathbf{Dias}_\gamma$ .

For any integer  $\gamma \geq 0$ , let  $\mathfrak{G}_{\mathbf{Trias}_\gamma} := \mathfrak{G}_{\mathbf{Trias}_\gamma}(2)$  be the graded set where

$$\mathfrak{G}_{\mathbf{Trias}_\gamma}(2) := \{\neg_a, \perp, \vdash_a : a \in [\gamma]\}. \tag{4.2.1}$$

Let  $\mathfrak{t}$  be a syntax tree of  $\mathbf{Free}(\mathfrak{G}_{\mathbf{Trias}_\gamma})$  and  $x$  be a leaf of  $\mathfrak{t}$ . We say that an integer  $a \in \{0\} \cup [\gamma]$  is *eligible* for  $x$  if  $a = 0$  or there is an ancestor  $y$  of  $x$  labeled by  $\neg_a$  (resp.  $\vdash_a$ ) and  $x$  is in the right (resp. left) subtree of  $y$ . The *image* of  $x$  is its greatest eligible integer. Moreover, let

$$\text{wordt}_\gamma : \mathbf{Free}(\mathfrak{G}_{\mathbf{Trias}_\gamma})(n) \rightarrow \mathbf{Trias}_\gamma(n), \quad n \geq 1, \tag{4.2.2}$$

the map where  $\text{wordt}_\gamma(\mathfrak{t})$  is the word obtained by considering, from left to right, the images of the leaves of  $\mathfrak{t}$  (see Figure 2). Observe that  $\text{wordt}_\gamma$  is an extension of  $\text{word}_\gamma$  (see (2.2.2)).

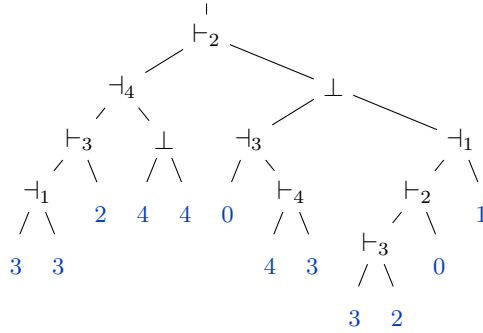


FIGURE 2. A syntax tree  $\mathfrak{t}$  of  $\mathbf{Free}(\mathfrak{G}_{\mathbf{Trias}_\gamma})$  where images of its leaves are shown. This tree satisfies  $\text{wordt}_\gamma(\mathfrak{t}) = 332440433201$ .

Consider now the map

$$\text{hookt}_\gamma : \mathbf{Trias}_\gamma(n) \rightarrow \mathbf{Free}(\mathfrak{G}_{\mathbf{Trias}_\gamma})(n), \quad n \geq 1, \tag{4.2.3}$$

defined for any word  $x$  of  $\text{Trias}_\gamma$  by

$$\text{hookt}_\gamma(x) := \begin{array}{c} \perp \\ \swarrow \quad \searrow \\ \text{hook}_\gamma(u) \quad \perp_{v_k^{(\ell)}} \\ \swarrow \quad \searrow \\ \perp \quad \perp_{v_k^{(1)}} \\ \swarrow \quad \searrow \\ \perp_{v_1^{(1)}} \quad \perp_{v_1^{(1)}} \end{array}, \quad (4.2.4)$$

where  $x$  decomposes, by Proposition 4.1.1, uniquely in  $x = u0v^{(1)} \dots 0v^{(\ell)}$  where  $u$  is a word of  $\text{Dias}_\gamma$  and for all  $i \in [\ell]$ , the  $v^{(i)}$  are words on the alphabet  $[\gamma]$ . The length  $|v^{(i)}|$  of any  $v_i$  is denoted by  $k^{(i)}$ . The dashed edges denote left comb trees wherein internal nodes are labeled as specified. Observe that  $\text{hookt}_\gamma$  is an extension of  $\text{hook}_\gamma$  (see (2.2.3)). We shall call any syntax tree of the form (4.2.4) an *extended hook syntax tree*.

**Theorem 4.2.1.** *For any integer  $\gamma \geq 0$ , the operad  $\text{Trias}_\gamma$  admits the following presentation. It is generated by  $\mathfrak{G}_{\text{Trias}_\gamma}$  and its space of relations  $\mathfrak{R}_{\text{Trias}_\gamma}$  is the space induced by the equivalence relation  $\leftrightarrow_\gamma$  satisfying*

$$\perp \circ_1 \perp \leftrightarrow_\gamma \perp \circ_2 \perp, \quad (4.2.5a)$$

$$\neg_a \circ_1 \perp \leftrightarrow_\gamma \perp \circ_2 \neg_a, \quad a \in [\gamma], \quad (4.2.5b)$$

$$\perp \circ_1 \vdash_a \leftrightarrow_\gamma \vdash_a \circ_2 \perp, \quad a \in [\gamma], \quad (4.2.5c)$$

$$\perp \circ_1 \neg_a \leftrightarrow_\gamma \perp \circ_2 \neg_a, \quad a \in [\gamma], \quad (4.2.5d)$$

$$\neg_a \circ_1 \vdash_{a'} \leftrightarrow_\gamma \vdash_{a'} \circ_2 \neg_a, \quad a, a' \in [\gamma], \quad (4.2.5e)$$

$$\neg_a \circ_1 \neg_b \leftrightarrow_\gamma \neg_a \circ_2 \neg_b, \quad a < b \in [\gamma], \quad (4.2.5f)$$

$$\vdash_a \circ_1 \neg_b \leftrightarrow_\gamma \vdash_a \circ_2 \neg_b, \quad a < b \in [\gamma], \quad (4.2.5g)$$

$$\neg_b \circ_1 \neg_a \leftrightarrow_\gamma \neg_b \circ_2 \neg_a, \quad a < b \in [\gamma], \quad (4.2.5h)$$

$$\vdash_a \circ_1 \vdash_b \leftrightarrow_\gamma \vdash_b \circ_2 \vdash_a, \quad a < b \in [\gamma], \quad (4.2.5i)$$

$$\neg_d \circ_1 \neg_d \leftrightarrow_\gamma \neg_d \circ_2 \perp \leftrightarrow_\gamma \neg_d \circ_2 \neg_c \leftrightarrow_\gamma \neg_d \circ_2 \vdash_c, \quad c \leq d \in [\gamma], \quad (4.2.5j)$$

$$\vdash_d \circ_1 \neg_c \leftrightarrow_\gamma \vdash_d \circ_1 \vdash_c \leftrightarrow_\gamma \vdash_d \circ_1 \perp \leftrightarrow_\gamma \vdash_d \circ_2 \vdash_d, \quad c \leq d \in [\gamma]. \quad (4.2.5k)$$

Observe that, by Theorem 4.2.1,  $\text{Trias}_1$  and the triassociative operad [LR04] admit the same presentation. Then, for all integers  $\gamma \geq 0$ , the operads  $\text{Trias}_\gamma$  are generalizations of the triassociative operad.

**Theorem 4.2.2.** *For any integer  $\gamma \geq 0$ ,  $\text{Trias}_\gamma$  is a Koszul operad. Moreover, the set of extended hook syntax trees of  $\mathbf{Free}(\mathfrak{G}_{\text{Trias}_\gamma})$  forms a Poincaré-Birkhoff-Witt basis of  $\text{Trias}_\gamma$ .*

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LABORATOIRE D'INFORMATIQUE GASPARD-MONGE, UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE, 5 BOULEVARD  
DESCARTES, CHAMPS-SUR-MARNE, 77454 MARNE-LA-VALLÉE CEDEX 2, FRANCE  
*E-mail address:* `samuele.giraudou-pem.fr`