SOURCE FIELDS RECONSTRUCTION ON A 3D STRUCTURE IN NOISY ENVIRONMENT

S Forget, N Totaro, J-L Guyader, M Schaeffer

NOVEM 2015
NOISE AND VIBRATION – EMERGING TECHNOLOGIES
Dubrovnik – Croatia
April 13-15 2015

SOURCE FIELDS RECONSTRUCTION ON A 3D STRUCTURE IN NOISY ENVIRONMENT

S. Forget[1*] , N. Totaro[2] , J-L. Guyader[2] and M. Schaeffer[1]

[1] Centre Technique Renault de Lardy,
1 Allée Cornuel, F-91510 Lardy, France
Email: sandra.forget@renault.com

[2] Laboratoire Vibrations Acoustique, INSA-Lyon,
25 bis Avenue Jean Capelle, F-69621 Villeurbanne Cedex, France
Email: nicolas.totaro@insa-lyon.fr

ABSTRACT

Nowadays, noise source identification remains a current challenge in industry to be able to act at the origin of noise. Since decades, many identification methods have been developed but most of them are not suitable for industrial applications. In this context, the presented method named Mixed - inverse Patch Transfer Functions (M-iPTF), could be promising as it is able to reconstruct acoustic fields on irregularly shaped sources, only needs pressure measurements and can be performed in a noisy environment. The present paper describes the theoretical background of the M-iPTF method, based on the application of the Green's identity on a virtual cavity surrounding the source, and its application results on a baffled oil pan excited with an electro-dynamic shaker, with and without the presence of a secondary correlated source.

1 INTRODUCTION

Since early 1990s, many noise source identification methods have been developed and some have matured and have been used throughout many industries [1]. Even though all these techniques have the same goal, that is to say to know from where the sound originates, it does not exist one method that prevails on other. A choice has to be made depending on the source object, the operating conditions and the available sensors and facilities. Usually, these methods are classified into three categories: Near-field Acoustic Holography (NAH) approaches, Beamforming techniques, and inverse methods.

The NAH, firstly introduced by Maynard *et al* [2], uses the 2D Fourier transform of the Helmholtz equation and pressure measurements from a microphones array to reconstruct the source fields. The main drawbacks are : (i) the back propagation can only be applied to simple geometries (plane, cylindrical or spherical [3]), (ii) measurements have to be done in the near field to prevent boundary effects and (iii) the source has to be placed in free field conditions.

The beamforming technique [4] uses a microphones array placed in the far field and process suitable time delays on signals depending on the microphones position, to determine by summation the source location. However, it is only a qualitative approach and the method presents some limitations at low frequencies.

In the third category, a promising approach, the inverse Patch Transfer Functions (iPTF) developed in [5, 6], could address the problems mentioned above. Indeed by defining a closed acoustic volume surrounding the source and by using the Green's identity, the iPTF method can separate the contribution of the source from what coming from its environment. In this sense, it is a separation method which allows to be applied in a non anechoic environement. Additionally, it uses the computation of eigen modes of this cavity by FEM to handle cavities and sources with complex shapes [7]. The back propagation can thus be made directly on the source surface. The challenge is the experimental process. It requires to measure particle velocities and pressures on the virtual surface surrounding the cavity. These can be addressed by using a p-U probe but its quite expensive cost could be a barrier to an industrial development.

The proposed method relies on the same concept than the iPTF method, i.e the definition of a virtual cavity surrounding the source and the application of the Green's identity. The difference comes from the boundary conditions taken for this cavity. Indeed, as they are independent from the real boundary conditions of the problem, they can be chosen arbitrarily. For the classical iPTF method, named Homogeneous-iPTF (H-iPTF), all surfaces of the cavity satisfy the homogenous Neumann condition. The developed method, named Mixed-iPTF (M-iPTF), uses mixed boundary conditions for the virtual cavity so that the measured data change. Here, instead of measuring pressures and particle velocities, the M-iPTF method only requires pressure measurements. This enables to consider a possible industrial measurement process by using a simple microphones array. Obviously, the M-iPTF method keeps the source separation property, which does not require free field conditions and the back propagation can be made directly on a complex 3D source.

2 MIXED - INVERSE PATCH TRANSFER FUNCTIONS

2.1 Theoretical background

Let's consider the vibro-acoustic problem presented in Figure 1. It consists in a vibrating surface Σ (source denoted S) radiating in any acoustic environment. In the present example, the source S is located near a rigid wall Σ''. To insulate the source S from its acoustic environment, which may contain secondary stationnary sources (denoted S' in Figure 1) or objects, ones defines a virtual surface Σ', arbitrarily chosen, surrounding the source to identify. It is important to notice here that this virtual surface has no physical meaning.

A "virtual" closed acoustic volume (denoted Ω in Figure 1) is then delimited by surfaces Σ, Σ' and Σ''. The Green's identity can be used to compute the pressure at any point in the volume knowing quantities (pressure and/or particle velocity) on boundaries. In its general form, the Green's identity can be written as :

$$\iiint_{\Omega} [\Phi\Delta\Psi - \Psi\Delta\Phi] \, d\Omega - \iint_{\Gamma} [\Phi\frac{\partial\Psi}{\partial n} - \Psi\frac{\partial\Phi}{\partial n}] \, d\Gamma = 0 \tag{1}$$

The functions Φ and Ψ of Equation (1) are replaced by $p(M)$ and $\phi_n(M)$ in Equation (2), which corresponds respectively to the pressure at any point M in the cavity and the mode shapes of this cavity expressed in pressure. This choice is arbitrary provided that both functions are twice differentiable. In that case the pressure and pressure eigenmodes of the cavity allow to simplify the

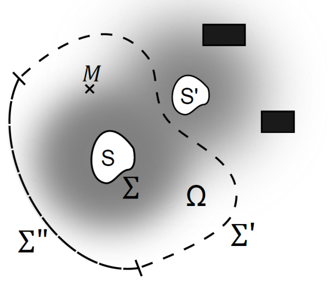


Figure 1: Definition of the virtual closed volume Ω and the boundary surfaces Σ (surface of the vibrating source), Σ' (virtual surface surrounding the source) and Σ'' (physically rigid wall).

left-hand side of the Green's identity in Equation (2), thanks to the use of the Helmholtz-Kirchhoff equations (3) and (4).

$$\begin{aligned}\iiint_{\Omega} [\phi_n(M)\Delta p(M) - p(M)\Delta\phi_n(M)]\, d\Omega = &\iint_{\Sigma} [\phi_n(Q)\frac{\partial p(Q)}{\partial n} - p(Q)\frac{\partial \phi_n(Q)}{\partial n}]\, d\Sigma \\ &+ \iint_{\Sigma'} [\phi_n(Q')\frac{\partial p(Q')}{\partial n} - p(Q')\frac{\partial \phi_n(Q')}{\partial n}]\, d\Sigma' \\ &+ \iint_{\Sigma''} [\phi_n(Q'')\frac{\partial p(Q'')}{\partial n} - p(Q'')\frac{\partial \phi_n(Q'')}{\partial n}]\, d\Sigma''\end{aligned} \tag{2}$$

$$\Delta p(M) + k^{*2} p(M) = 0 \qquad \forall M \in \Omega \tag{3}$$

where, $k^* = \omega/c^* = k(1 + \mathrm{i}\eta)^{-1}$ is the complex wave number, η the damping loss factor, ω the angular frequency and c the speed of sound in acoustic medium;

and

$$\Delta\phi_n(M) + k_n^2\, \phi_n(M) = 0 \qquad \forall M \in \Omega \tag{4}$$

where, k_n is the wave number at the eigen frequency ω_n.

Then, to simplify the right-hand side of the Green's identity (2), the Euler's condition is applied :

$$\frac{\partial p(Q)}{\partial n} = -\mathrm{i}\omega\rho_0\, V_N(Q) \qquad \forall Q \in \Sigma \tag{5}$$

$$\frac{\partial p(Q')}{\partial n} = -\mathrm{i}\omega\rho_0\, V_N(Q') \qquad \forall Q' \in \Sigma' \tag{6}$$

$$\frac{\partial p(Q'')}{\partial n} = 0 \qquad \forall Q'' \in \Sigma'' \tag{7}$$

where, V_N is the outgoing normal velocity of the concerned surface.

Finally, boundary conditions of the virtual cavity have to be defined. Again, this choice has no physical meaning, the boundary conditions are defined by convenience as the Green's identity allows it. The classical H-iPTF method relies on a rigid wall cavity satisfying the homogeneous Neumann's condition. For the presented method, the source surface Σ and rigid surface Σ'' fulfill the homogeneous Neumann's condition (8), while the virtual surface Σ' satisfies the Dirichlet's condition (9).

$$\frac{\partial \phi_n(Q)}{\partial n} = 0 \qquad \forall Q \in \Sigma, \Sigma'' \tag{8}$$

$$\phi_n(Q') = 0 \qquad \forall Q' \in \Sigma' \tag{9}$$

Then, by decomposing the pressure $p(M)$ on a basis composed of the cavity pressure eigenmodes $\phi_n(M)$ and by using their orthogonality property, the pressure at any point M in the cavity can be expressed by :

$$p(M) = \sum_{n=1}^{\infty} \mathrm{i}\rho_0 \frac{\phi_n(M)}{[k^{*2} - k_n^2]\,\Lambda_n} \left[\iint_{\Sigma} \omega\ \phi_n(Q)\ V_N(Q)\ d\Sigma - \iint_{\Sigma'} \omega_n\ \chi_n(Q')\ p(Q')\ d\Sigma' \right] \tag{10}$$

where, Λ_n represents the norm of the n-th mode, χ_n are the mode shapes of the cavity expressed in velocity given by the Euler's equation (11)

$$\frac{\partial \phi_n(Q')}{\partial n} = -\mathrm{i}\omega_n \rho_0\ \chi_n(Q') \tag{11}$$

Finally, to be computed, Equation (10) is written in a discrete form. All surfaces are divided into elementary surfaces named *patches* and the modal basis of the cavity is truncated to a mode N for which the convergence have to be ensured

$$p(M) = \sum_{n=1}^{N} \mathrm{i}\rho_0 \frac{\phi_n(M)}{[k^{*2} - k_n^2]\,\Lambda_n} \left[\sum_{j=1}^{N_j} \omega\ A_j\ \langle \phi_n \rangle_j\ \langle V_N \rangle_j - \sum_{k=1}^{N_k} \omega_n\ A_k\ \langle \chi_n \rangle_k\ \langle p \rangle_k \right] \tag{12}$$

where, $< X >_S$ represents the space average of the variable X on surface S, and A_S is the area of the surface S. Subscripts j and k denote respectively a patch on the source surface Σ and a patch on the virtual surface Σ'.

In Equation (12), the physically rigid surface no longer appears in the calculation. It has been already taken into account through the rigid boundary condition of the virtual cavity.

Considering the pressure computation of several points M in the cavity, Equation (12) can be written in a matrix form :

$$\mathbf{p}_i = \mathbf{Z}_{ij}\ \mathbf{V}_j - \mathbf{Y}_{ik}\ \mathbf{p}_k \tag{13}$$

where

$$\mathbf{Z}_{ij} = \sum_{n=1}^{N} \mathrm{i}\omega\rho_0 \frac{(\phi_n)_i \, \langle\phi_n\rangle_j \, A_j}{[k^{*2} - k_n^2] \, \Lambda_n} \qquad \text{and} \qquad \mathbf{Y}_{ik} = \sum_{n=1}^{N} \mathrm{i}\omega_n\rho_0 \frac{(\phi_n)_i \, \langle\chi_n\rangle_k \, A_k}{[k^{*2} - k_n^2] \, \Lambda_n} \tag{14}$$

subscript i denotes a measured point inside the cavity.

Equation (13) expresses the direct problem formulation with the computation of the radiation of the source S and its environment. Conversely, to retrieve the field coming from the source, the problem requires to be inverted :

$$\mathbf{V}_j = \mathbf{Z}_{ij}^{-1}(\mathbf{p}_i + \mathbf{Y}_{ik} \, \mathbf{p}_k) \tag{15}$$

Thus finally, by using Equation (15), it is possible to reconstruct the source velocity field from only pressure measurements performed on the virtual surface and in the cavity. These measurements can be done using a microphones array, allowing a cost-effective industrial measurement process. The use of the Green's identity has allowed to separate the contribution of the source from what coming from its environment, making the M-iPTF method usable in a non-anechoic environment. At last, the eigen-modes of the cavity expressed in pressure and in velocity can be computed using a finite elements solver, enabling the cavity and thus also the source to be of complex shapes .

Once the velocity field is known, the source is fully determined because all others acoustic quantities can be retrieved: the boundary pressure field can be computed using the direct problem formulation (16) and the normal intensity field can be obtained by multiplying the pressure and velocity fields (17). In parallel, other indicators like the radiated acoustic power or the radiation efficiency can be deduced. The source is then fully described by the proposed inverse method.

$$\mathbf{p}_j = \mathbf{Z}_{jj} \, \mathbf{V}_j - \mathbf{Y}_{jk} \, \mathbf{p}_k \tag{16}$$

$$\mathbf{I}_j = \frac{1}{2} \, \Re(\mathbf{p}_j \circ \mathbf{V}_j^*) \tag{17}$$

where, $\Re(\bullet)$ and $\bullet^*$ represent the real part and the complex conjugate of a complex number. $\circ$ is the Hadamard's product.

2.2 Tikhonov's regularization

The direct problem formulation of the M-iPTF method can be written on the general form of a linear system $\mathbf{Ax} = \mathbf{b}$:

$$\mathbf{Z}_{ij} \, \mathbf{V}_j = \tilde{\mathbf{p}}_i \quad \text{with} \quad \tilde{\mathbf{p}}_i = \mathbf{p}_i + \mathbf{Y}_{ik} \, \mathbf{p}_k \tag{18}$$

In acoustics and more generally in all domains of physics, inverse problems are often ill-posed problems in the Hadamard's sens [8, 9]. According to him, an inverse problem is well-posed when these three conditions are satisfied together : the solution exists, is unique and stable.

In general, the first condition will be satisfied, the M-iPTF method tries to identify an input from an observed output.

The second condition is relating to the problem determination, i.e the ratio between the number of measured points in the cavity and the number of source patches, and the rank of the system governed by the number of modes taken into account. In general, in order to limit the number of measurements, the system is under-determined. Thus, under these conditions, the solution is not unique and the $\mathbf{Z}_{ij}$ matrix is not square [10].

Finally, the solution have to be stable. Many sources of instabilities exist, in particular coming from the measurement process, which can disturb the identification. Indeed the $\tilde{\mathbf{p}_i}$ vector is obtained by measurements and thus can contain errors and measurement noise. Also, the size of patches can limit the maximal frequency of identification and the modal truncation can affect the convergence of the solution.

Thus, due to the discretization and the measurement process, the inverse problem of the M-iPTF method is ill-posed and requires a reguralization. Many regularization methods exist in the literature [11, 12], but only the so-called Tikhonov regularization has been used in this paper. The Tikhonov-regularized solution of Equation (18) is given by [7] :

$$\mathbf{V}_j^{\beta} = \sum_{m=1}^{rank(\mathbf{Z}_{ij})} \frac{\mathbf{v}_m \, \mathbf{u}_m^H}{\sigma_m} \left(\frac{\sigma_m^2}{\sigma_m^2 + \beta^2} \right) \tilde{\mathbf{p}_i} \tag{19}$$

where σ_m, $\mathbf{u}_m$ and $\mathbf{v}_m$ are respectively the singular values in the decreasing order, the left and right singular vector of the $\mathbf{Z}_{ij}$ matrix coming from its Singular Values Decomposition. $\bullet^H$ denotes the Hermitian transpose.

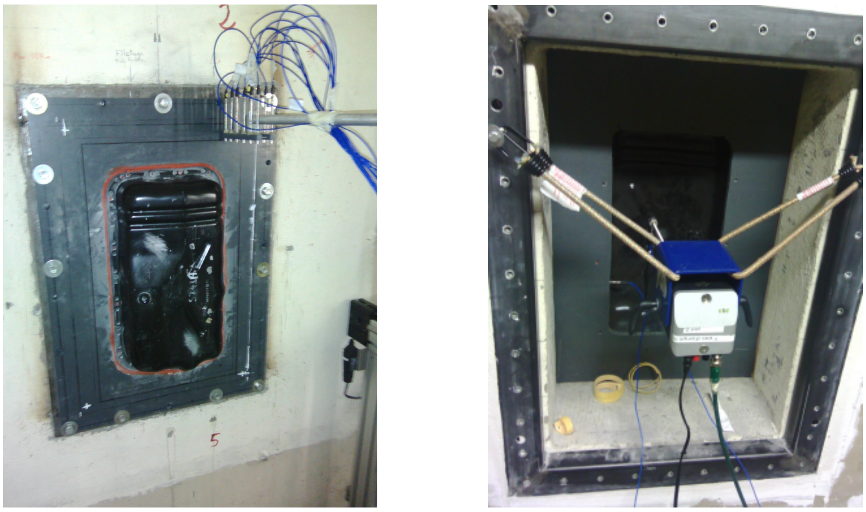
β is a real positive parameter that prevents small singular values, associated with small data values and thus dominated by noise, to amplify the norm of the regularised solution during the inversion process. It is a tradeoff between measurement error and error of regularization. The optimal β parameter can be determined by the two well-known methods, the so-called L-Curv (LCV) and Generalized Cross Validation (GCV) methods [11, 12]. These regularization methods were both applied to the M-iPTF method but only the LCV regularisation is presented thereafter due to similar results in between.

3 EXPERIMENTAL VALIDATION

3.1 System under study

A validation of the M-iPTF method was carried out on a real baffled oil pan (Figure 2(a)) excited by an electro-dynamic shaker (Figure 2(b)).

The measurement process was performed using a three axis robot and a 6x2 regular microphones array. The virtual surface was taken rectangular to simplify measurements (Figure 3(b)) and the FE modelisation of the cavity for its modal extraction (Figure 3(c)). As it can be seen in Equation (18), one needs the measurement of space averaged pressure $\mathbf{p}_k$ on patches of the virtual surface and pressure $\mathbf{p}_i$ at points inside the volume. In experiments, the space averaged pressure is approximated by the value at centers of patches. Also, points in the cavity were taken randomly in the cavity. Thus, combining pressure measurements and a numerical model, Equation (19) permits to reconstruct the velocity, pressure and intensity fields on the surface of the structure (Figure 3(a)). In parallel, to validate the present approach, some reference measurements have been done, by using a p-U probe, on the nearest rectangular box surrounding the oil pan (the measurement of pressure and particle velocity on the real surface of the oil pan was not possible).

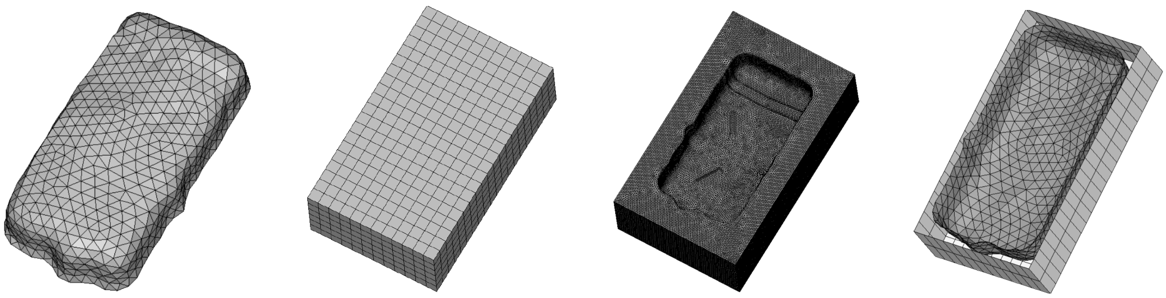


(a) Source : baffled oil pan (b) Position of the shaker

Figure 2. System under study : (a) a baffled oil pan excited by (b) an electro-dynamic shaker.

As it was mentioned in section 2.2, several factors can influence the identification results. A parametric study was carried out to assess their influence but it will be presented in further works. For the present paper, these parameters (Table 1) are taken optimal to ensure a proper identification and to illustrate the proposed method.

The ability of the M-iPTF method to correctly reconstruct the oil pan fields was tested with and without the presence of a secondary correlated source. Results are presented in sections 3.2 and 3.3.



(a) Oil pan patches model (b) Virtual surface patches model (c) Virtual cavity FE model (d) Surface for P-U measurements

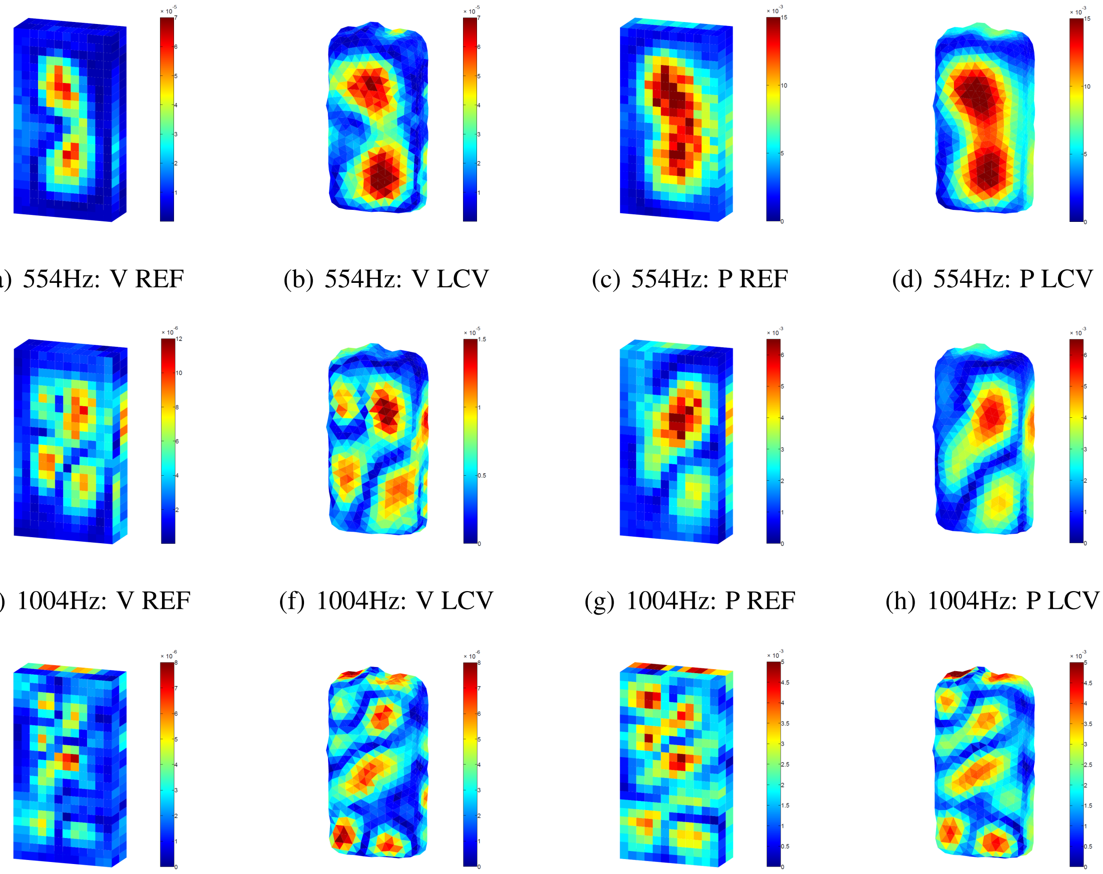
Figure 3: (a) Patches model for the source, (b) for the virtual surface, (c) FE model of the cavity and (d) surface of the reference measurements.

Virtual surface dimensions (m)	0.540 x 0.324 x 0.162
Frequency range of identification (Hz)	[0-3000]
Number of identification patches	1047
Number of virtual surface patches	624
Number of points taken in the cavity	700
Frequency range of modal extraction (Hz)	[0-8500]
Number of modes	1283

Table 1. Information concerning the identification process.

3.2 Identification results without a secondary correlated source

Figure 4 shows the comparison between the maps of the reference velocity and pressure fields measured with a p-U probe and those obtained from the computation of the M-iPTF method with the LCV - Tikhonov regularization at the frequencies of 554, 1004 and 1951 Hz. One has to remind that the "reference" measurements have been done on a rectangular box close to the surface of the oil pan. Considering this remark, Figure 4 demonstrates a very accurate reconstruction of the velocity field as well as the pressure field. The slight shift comes from the fact that the reference surface is slightly larger than the oil pan surface on which the identification is done.



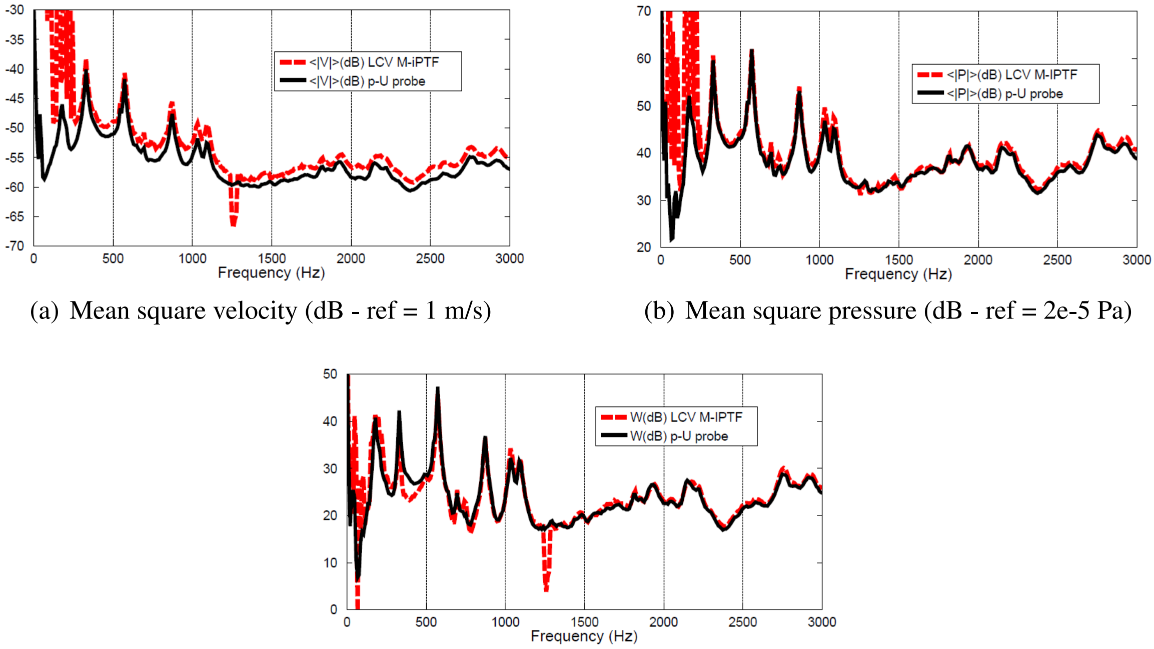
(a) 554Hz: V REF (b) 554Hz: V LCV (c) 554Hz: P REF (d) 554Hz: P LCV

(e) 1004Hz: V REF (f) 1004Hz: V LCV (g) 1004Hz: P REF (h) 1004Hz: P LCV

(i) 1951Hz: V REF (j) 1951Hz: V LCV (k) 1951Hz: P REF (l) 1951Hz: P LCV

Figure 4: Comparison between p-U probe measurements (velocities ((a)-(e)-(i)) and pressures ((c)-(g)-(k)) and identification results with LCV regularization (velocities ((b)-(f)-(j)) and pressures ((d)-(h)-(l))) respectively at 554 Hz, 1004 Hz and 1951 Hz.

In Figure 5, the mean square velocity, pressure and the radiated acoustic power are plotted. p-U probe measurements are compared to the fields identified using the LCV regularization. Below 300 Hz, due to a low signal-to-noise ratio, the LCV strategy of regularization seems to be inappropriate in that case. Above 300 Hz, results are very good even if a slight shift for the amplitude of the velocity can be observed. This quantity is highly sensitive to the difference between the "reference" surface and the real surface of the oil pan used for identification. On the contrary, as expected, the acoustic radiated power is conservative and not affected by this difference between surfaces as it can be seen in Figure 5(c).



(a) Mean square velocity (dB - ref = 1 m/s)

(b) Mean square pressure (dB - ref = 2e-5 Pa)

(c) Acoustic radiated power (dB - ref = 10e-12 W)

Figure 5: (a) Mean square velocity, (b) mean square pressure and (c) radiated acoustic power (black solid line : p-U measurements, red dotted line : identification with LCV regularization)

3.3 Identification results in a noisy environment

The oil pan environment is now considered as noisy by the presence of a secondary correlated source placed outside the cavity domain (Figure 6).

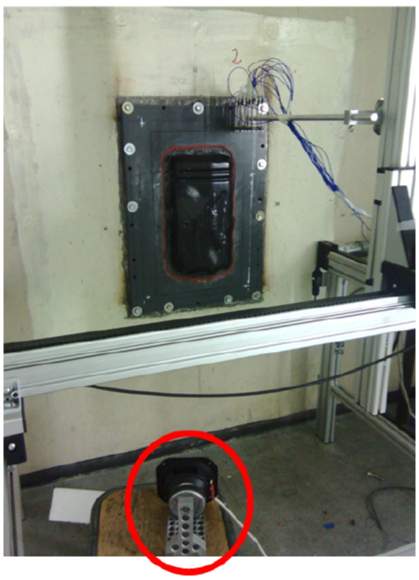
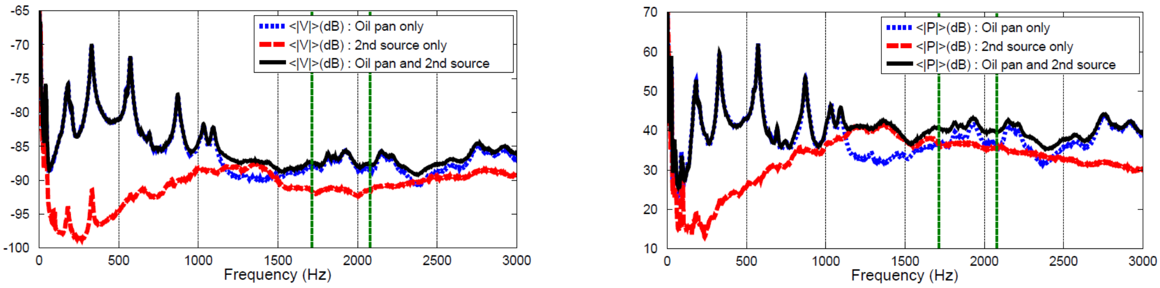


Figure 6. Position of the secondary correlated source.

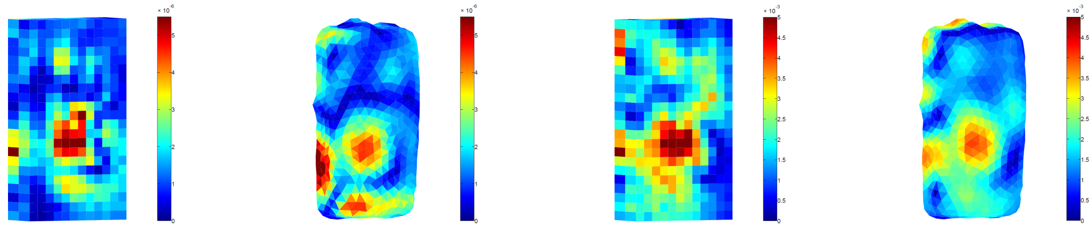
In Figure 7, the mean square velocity and pressure measured using the p-U probe are plotted in three different configurations : the oil pan radiates only, the secondary source is alone and both are activated. Here, depending on the frequency range, one source can prevail on the second. Thus, to validate the method in a noisy environment, only the configuration for which both sources generate an equivalent sound pressure level, i.e at the frequencies of 1713 Hz and 2080 Hz in Figure 7(b), is studied.



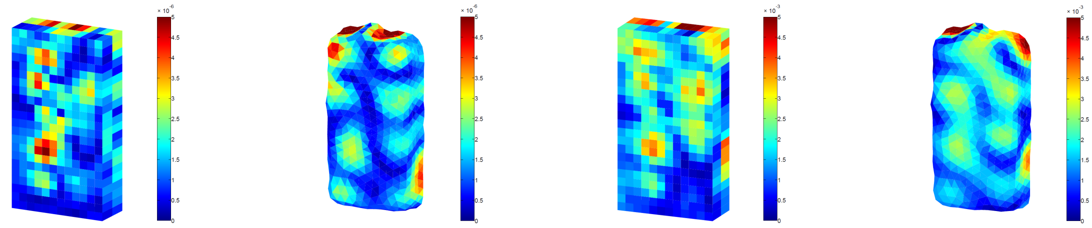
(a) Mean square velocity (dB - ref = 1 m/s) (b) Mean square pressure (dB - ref = 2e-5 Pa)

Figure 7: (a) Mean square velocity and (b) mean square pressure computed with the p-U probe measurements when : the oil pan radiates only (blue dotted line), the secondary source is alone (red dashed line) and both are activated (black solid line). Vertical green lines correspond to frequencies at 1713 Hz and 2080 Hz.

In Figure 8, the maps of the velocity and pressure fields measured by using the p-U probe are compared to those obtained from the computation of the M-iPTF method with the LCV - Tikhonov regularization at the frequencies of 1713 Hz and 2080 Hz. At these frequencies, for which the sound generated by the oil pan and the secondary source is equivalent, the reconstruction of the velocity and pressure fields is quite satisfactory. The slight difference of amplitude, lower for the identified fields, might come from an over regularization due to the important noise coming from the correlated source.



(a) 1713Hz: V REF (b) 1713Hz:V LCV (c) 1713Hz: P REF (d) 1713Hz: P LCV



(e) 2080Hz: V REF (f) 2080Hz: V LCV (g) 2080Hz: P REF (h) 2080Hz: P LCV

Figure 8: Comparison between p-U probe measurements (velocities ((a)-(e)) and pressures ((c)-(g))) and identification results (velocities ((b)-(f)) and pressures((d)-(h))) with the LCV strategy, respectively at 1713 Hz and 2080 Hz.

4 CONCLUDING REMARKS

In acoustics, many identification methods exist, due to the variety of applications. The proposed method in this paper belongs to the inverse methods domain and has been developed in particular for industrial applications. The method relies on the application of the Green's identity on a virtual cavity defined around the source and the computation by FEM of the eigen-modes of this cavity. That permits to the M-iPTF method to be a separation method, i.e to be performed in a non anechoic environment, and the source to be of complex shape. Likewise, the mixed chosen boundary conditions for the cavity allow to use only pressure measurements. However, as inverse problems, the method suffers from ill-conditioning and have to be regularized. The M-iPTF method has shown its ability to correctly reconstruct all acoustic source fields in a large frequency range in an experimental application with a complex 3D source and a noisy environment.

ACKNOWLEDGMENTS

This work was performed within the framework of Labex CeLyA of Université de Lyon, operated by the French National Research Agency (ANR-10-LABX-0060/ ANR-11-IDEX-0007).

REFERENCES

[1] Jeroen Lanslots, Filip Deblauwe, and Karl Janssens. Selecting sound source localization techniques for industrial applications. *Sound and Vibration*, 44(6):6, 2010.

[2] Julian D Maynard, Earl G Williams, and Y Lee. Nearfield acoustic holography: I. theory of generalized holography and the development of nah. *The Journal of the Acoustical Society of America*, 78(4):1395–1413, 1985.

[3] Earl G Williams. *Fourier acoustics: sound radiation and nearfield acoustical holography*. academic press, 1999.

[4] J Billingsley and R Kinns. The acoustic telescope. *Journal of Sound and Vibration*, 48(4):485–510, 1976.

[5] Mathieu Aucejo, Nicolas Totaro, and J-L Guyader. Identification of source velocities on 3d structures in non-anechoic environments: Theoretical background and experimental validation of the inverse patch transfer functions method. *Journal of Sound and Vibration*, 329(18):3691–3708, 2010.

[6] D. Vigoureux, N.Totaro, J. Lagneaux, and J.L. Guyader. Inverse patch transfer functions method as a tool for source field identification. *Journal of Vibration and Acoustics*, 137(2):12, 2015.

[7] N. Totaro, D. Vigoureux, Q. Leclère, J. Lagneaux, and J.L. Guyader. Sound fields separation and reconstruction of irregularly shaped sources. *Journal of Sound and Vibration*, 336(0):62 – 81, 2015.

[8] Jacques Hadamard. Sur les problèmes aux dérivées partielles et leur signification physique. *Princeton university bulletin*, 13(49-52):28, 1902.

[9] Andrej Nikolaevich Tikhonov and Vasiliy Yakovlevich Arsenin. Solutions of ill-posed problems. 1977.

[10] Per Christian Hansen. *Rank-deficient and discrete ill-posed problems: numerical aspects of linear inversion*, volume 4. Siam, 1998.

[11] Y. Kim and P.A. Nelson. Optimal regularisation for acoustic source reconstruction by inverse methods. *Journal of Sound and Vibration*, 275(35):463 – 487, 2004.

[12] Q. Leclère. Acoustic imaging using under-determined inverse approaches: Frequency limitations and optimal regularization. *Journal of Sound and Vibration*, 321(35):605 – 619, 2009.