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L^p-FOURIER ANALYSIS ASSOCIATED TO A FAMILY OF DIFFERENTIAL-REFLECTION OPERATORS

SALEM BEN SAID, ASMA BOUSSEN & MOHAMED SIFI

ABSTRACT. In [5] we introduced a family of differential-reflection operators $\Lambda_{A,\varepsilon}$ acting on smooth functions defined on \mathbb{R} , where the spectral problem for the operators $\Lambda_{A,\varepsilon}$ has been discussed. Here A is a Sturm-Liouville function with additional hypotheses and $\varepsilon \in \mathbb{R}$.

Via the eigenfunctions of $\Lambda_{A,\varepsilon}$, we introduce in this paper a generalized Fourier transform $\mathcal{F}_{A,\varepsilon}$. An L^p -harmonic analysis for $\mathcal{F}_{A,\varepsilon}$ is developed when $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ and $-1 \leq \varepsilon \leq 1$. In particular, an L^p -Schwartz space isomorphism theorem for $\mathcal{F}_{A,\varepsilon}$ is proved.

1. INTRODUCTION

In [5] we considered some aspects of harmonic analysis associated to the following family of one-dimensional (A, ε) -operators

$$\Lambda_{A,\varepsilon}f(x) = f'(x) + \frac{A'(x)}{A(x)} \left(\frac{f(x) - f(-x)}{2} \right) - \varepsilon \varrho f(-x), \quad (1.1)$$

where A is so-called a Chébli function on \mathbb{R} (i.e. A is a continuous \mathbb{R}^+ -valued function on \mathbb{R} satisfying certain regularity and convexity hypotheses), ϱ is the index of A , and $\varepsilon \in \mathbb{R}$. We note that $\varrho \geq 0$. The function A and the real number ε are the deformation parameters giving back three well known cases (as special examples) when:

- (1) $A(x) = A_\alpha(x) = |x|^{2\alpha+1}$ and ε arbitrary (Dunkl's operators [13]),
- (2) $A(x) = A_{\alpha,\beta}(x) = |\sinh x|^{2\alpha+1} (\cosh x)^{2\beta+1}$ and $\varepsilon = 0$ (Heckman's operators [18]),
- (3) $A(x) = A_{\alpha,\beta}(x) = |\sinh x|^{2\alpha+1} (\cosh x)^{2\beta+1}$ and $\varepsilon = 1$ (Cherednik's operators [11]).

We would like to mention that a differential-reflection operator built in terms of a Chébli function has been the first time done in [20]. This is one of the major themes of research at the "school of harmonic analysis" in Tunisia.

In [5] we proved that for $\lambda \in \mathbb{C}$, the equation

$$\Lambda_{A,\varepsilon}f(x) = i\lambda f(x), \quad (1.2)$$

where $f : \mathbb{R} \rightarrow \mathbb{C}$, admits a unique solution $\Psi_{A,\varepsilon}(\lambda, \cdot)$ satisfying $\Psi_{A,\varepsilon}(\lambda, 0) = 1$. Moreover, under the assumption $-1 \leq \varepsilon \leq 1$, we established in [5, Theorems 3.4 and 3.5] suitable estimates for the growth of the eigenfunction $\Psi_{A,\varepsilon}(\lambda, x)$ and of its partial derivatives.

In this paper we are concerned with a development of an L^p -harmonic analysis for a generalized Fourier transform $\mathcal{F}_{A,\varepsilon}$ when $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ and $-1 \leq \varepsilon \leq 1$. Here

$$\mathcal{F}_{A,\varepsilon}f(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{A,\varepsilon}(\lambda, -x)A(x)dx$$

for $f \in L^1(\mathbb{R}, A(x)dx)$.

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Using the estimates for the growth of $\Psi_{A,\varepsilon}(\lambda, x)$ we get holomorphic properties of $\mathcal{F}_{A,\varepsilon}$ on $L^p(\mathbb{R}, A(x)dx)$. A Riemann-Lebesgue lemma is also obtained for $1 \leq p < \frac{2}{1+\sqrt{1-\varepsilon^2}}$.

We then turn our attention to an L^p -Schwartz space isomorphism theorem for $\mathcal{F}_{A,\varepsilon}$. In [17] Harish-Chandra proved an L^2 -Schwartz space isomorphism for the spherical Fourier transform on non-compact Riemannian symmetric spaces. This result was extended to L^p -Schwartz spaces with $0 < p < 2$ by Trombli and Varadarajan [23] (see also [12, 15, 16]). In the early nighties, Anker gave a new and simple proof of their result, based on the Paley-Wiener theorem for the spherical Fourier transform on Riemannian symmetric spaces [1]. Recently, Anker's method was used in [21] to prove an L^p -Schwartz space isomorphism theorem for the Heckman-Opdam hypergeometric functions. Our approach is inspired by Anker's paper [loc. cit.]. More precisely, for $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$, put

$$\mathbb{C}_{p,\varepsilon} := \left\{ \lambda \in \mathbb{C} \mid |\operatorname{Im} \lambda| \leq \varrho \left((2/p) - 1 - \sqrt{1-\varepsilon^2} \right) \right\}. \quad (1.3)$$

Denote by $\mathcal{S}_p(\mathbb{R})$ the L^p -Schwartz space on \mathbb{R} , and by $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ the Schwartz space on the tube domain $\mathbb{C}_{p,\varepsilon}$. We prove that $\mathcal{F}_{A,\varepsilon}$ is a topological isomorphism between $\mathcal{S}_p(\mathbb{R})$ and $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ (see Theorem 3.12).

We close this paper by establishing a result in connection with pointwise multipliers of $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$. More precisely, for arbitrary $\alpha \geq 0$, a function ψ defined on the tube domain $\mathbb{C}_\alpha := \{ \lambda \in \mathbb{C} \mid |\operatorname{Im} \lambda| \leq \alpha \}$ is called a pointwise multiplier of $\mathcal{S}(\mathbb{C}_\alpha)$ if the mapping $\phi \mapsto \psi\phi$ is continuous from $\mathcal{S}(\mathbb{C}_\alpha)$ into itself. In [2] Betancor *et al.* characterized the set of pointwise multipliers of the Schwartz spaces $\mathcal{S}(\mathbb{C}_\alpha)$.

Under the assumptions $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ whenever $\varrho = 0$, and $\frac{2}{2+\sqrt{1-\varepsilon^2}} \leq p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ whenever $\varrho > 0$, we prove that if T is in the dual space $\mathcal{S}'_p(\mathbb{R})$ of $\mathcal{S}_p(\mathbb{R})$ such that $\psi := \mathcal{F}_{A,\varepsilon}(T)$ is a pointwise multiplier of $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$, then for any $s \in \mathbb{N}$ there exist $\ell \in \mathbb{N}$ and continuous functions f_m defined on \mathbb{R} , $m = 0, 1, \dots, \ell$, such that

$$T = \sum_{m=0}^{\ell} \Lambda_{A,\varepsilon}^m f_m$$

and, for every such m ,

$$\sup_{x \in \mathbb{R}} (|x| + 1)^s e^{\left(\frac{2}{p} - \sqrt{1-\varepsilon^2}\right)\varrho|x|} |f_m(x)| < \infty. \quad (1.4)$$

The organization of this paper is as follows: In Section 2 we recapitulate some definitions and basic notations, as well as some results from literature. Further, we recall from [5] some results on $\Psi_{A,\varepsilon}(\lambda, x)$. In Section 3 we develop an L^p -harmonic analysis for the Fourier transform $\mathcal{F}_{A,\varepsilon}$, where we mainly prove an L^p -Schwartz space isomorphism theorem for $\mathcal{F}_{A,\varepsilon}$. Finally, in Section 4 we characterize the distributions $T \in \mathcal{S}'_p(\mathbb{R})$ so that $\mathcal{F}_{A,\varepsilon}(T)$ is a pointwise multiplier of the Schwartz space $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$.

2. BACKGROUND

2.1. The Chébli transform. In this subsection we present results from [9, 10, 24, 25].

Throughout this paper we will denote by A a function on \mathbb{R} satisfying the following hypotheses:

- (H1) $A(x) = |x|^{2\alpha+1} B(x)$, where $\alpha > -\frac{1}{2}$ and B is any even, positive and smooth function on \mathbb{R} with $B(0) = 1$.
- (H2) A is increasing and unbounded on \mathbb{R}_+ .

(H3) A'/A is a decreasing and smooth function on \mathbb{R}_+^* , and hence the limit $2\varrho := \lim_{x \rightarrow +\infty} A'(x)/A(x) \geq 0$ exists.

Such a function A is called a Chébli function. From (H1) it follows that

$$\frac{A'(x)}{A(x)} = \frac{2\alpha + 1}{x} + C(x), \quad x \neq 0, \quad (2.1)$$

where $C := B'/B$ is an odd and smooth function on \mathbb{R} .

Let Δ_A , or simply Δ , be the following second order differential operator

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx}. \quad (2.2)$$

For $\mu \in \mathbb{C}$, we consider the Cauchy problem

$$\Delta f(x) = -(\mu^2 + \varrho^2)f(x) \quad \text{with} \quad f(0) = 1 \quad \text{and} \quad f'(0) = 0. \quad (2.3)$$

In [10] the author proved that the system (2.3) admits a unique solution φ_μ . For every $\mu \in \mathbb{C}$, the solution φ_μ is an even smooth function on \mathbb{R} and the map $\mu \mapsto \varphi_\mu(x)$ is analytic. The following Laplace type representation of φ_μ can be found in [10] (see also [24]).

Lemma 2.1. *For every $x \in \mathbb{R}^*$ there is a non-negative even continuous function $K(|x|, \cdot)$ supported in $[-|x|, |x|]$ such that for all $\mu \in \mathbb{C}$*

$$\varphi_\mu(x) = \int_0^{|x|} K(|x|, t) \cos(\mu t) dt. \quad (2.4)$$

The following estimates of the eigenfunctions φ_μ can be found in [9, 10, 25].

Lemma 2.2. *Let $\mu \in \mathbb{C}$ such that $|\operatorname{Im} \mu| \leq \varrho$. Then*

- 1) $\varphi_{\pm i\varrho}(x) = 1$, $\varphi_{-\mu}(x) = \varphi_\mu(x)$ and $|\varphi_\mu(x)| \leq 1$.
- 2) $e^{-\varrho|x|} \leq \varphi_0(x) \leq c(|x| + 1)e^{-\varrho|x|}$.
- 3) $|\varphi_\mu(x)| \leq \varphi_{i\operatorname{Im} \mu}(x) \leq e^{|\operatorname{Im} \mu||x|} \varphi_0(x)$.
- 4) $|\varphi'_\mu(x)| \leq c(\varrho^2 + |\mu|^2) e^{|\operatorname{Im} \mu||x|} \varphi_0(x)$.

The Chébli transform of $f \in L^1(\mathbb{R}_+, A(x)dx)$ is given by

$$\mathcal{F}_\Delta(f)(\mu) := \int_{\mathbb{R}_+} f(x) \varphi_\mu(x) A(x) dx. \quad (2.5)$$

The following Plancherel and inversion formulas for \mathcal{F}_Δ are proved in [10].

Theorem 2.3. *There exists a unique positive measure π with support \mathbb{R}_+ such that \mathcal{F}_Δ induces an isometric isomorphism from $L^2(\mathbb{R}_+, A(x)dx)$ onto $L^2(\mathbb{R}_+, \pi(d\mu))$, and for any $f \in L^1(\mathbb{R}_+, A(x)dx) \cap L^2(\mathbb{R}_+, A(x)dx)$ we have*

$$\int_{\mathbb{R}_+} |f(x)|^2 A(x) dx = \int_{\mathbb{R}_+} |\mathcal{F}_\Delta(f)(\mu)|^2 \pi(d\mu).$$

The inverse transform is given by

$$\mathcal{F}_\Delta^{-1} g(x) = \int_{\mathbb{R}_+} g(\mu) \varphi_\mu(x) \pi(d\mu). \quad (2.6)$$

To have a nice behavior for the Plancherel measure π we must add a further (growth) restriction on the function A . Following [24], we will assume that A'/A satisfies the following additional hypothesis:

(H4) There exists a constant $\delta > 0$ such that for all $x \in [x_0, \infty)$ (for some $x_0 > 0$),

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\varrho + e^{-\delta x}D(x) & \text{if } \varrho > 0, \\ \frac{2\alpha + 1}{x} + e^{-\delta x}D(x) & \text{if } \varrho = 0, \end{cases} \quad (2.7)$$

with D being a smooth function bounded together with its derivatives.

In these circumstances the Plancherel measure π is absolutely continuous with respect to the Lebesgue measure and has density $|c(\mu)|^{-2}$ where c is a continuous function on \mathbb{R}_+ and zero free on \mathbb{R}_+^* (see [3]). Moreover, by [25, Proposition 6.1.12 and Corollary 6.1.5] (see also [6]), for $\mu \in \mathbb{C}$ we have

- (i) If $\varrho \geq 0$ and $\alpha > -1/2$, then $|c(\mu)|^{-2} \sim |\mu|^{2\alpha+1}$ whenever $|\mu| \gg 1$.
- (ii) If $\varrho > 0$ and $\alpha > -1/2$, then $|c(\mu)|^{-2} \sim |\mu|^2$ whenever $|\mu| \ll 1$.
- (iii) If $\varrho = 0$ and $\alpha > 0$, then $|c(\mu)|^{-2} \sim |\mu|^{2\alpha+1}$ whenever $|\mu| \ll 1$.

In the literature, the function c is called Harish-Chandra's function of the operator Δ . We refer to [7] for more details on the c -function.

Henceforth we will assume that Chébli's function A satisfies the additional hypothesis (H4). It follows that for $|x|$ is large enough:

- (i) $A(x) = O(e^{2\varrho|x|})$ for $\varrho > 0$.
- (ii) $A(x) = O(|x|^{2\alpha+1})$ for $\varrho = 0$.

We close this section by giving some basic results of (the analogue of) the Abel transform associated to the second order differential operator Δ .

Denote by $\mathcal{D}_e(\mathbb{R})$ the space of even and compactly supported functions in $C^\infty(\mathbb{R})$. In [24] the author has proved that the Abel transform defined on $\mathcal{D}_e(\mathbb{R})$ by

$$\mathcal{A}f(y) = \frac{1}{2} \int_{|x|>|y|} K(|x|, y)f(x)A(x)dx \quad (2.8)$$

is an automorphism of $\mathcal{D}_e(\mathbb{R})$ satisfying

$$\mathcal{F}_\Delta = \mathcal{F}_{\text{euc}} \circ \mathcal{A}, \quad (2.9)$$

where \mathcal{F}_{euc} is the Euclidean Fourier transform.

2.2. A family of differential-reflection operators. Recall from (1.1) the differential-reflection operator $\Lambda_{A,\varepsilon}$. In view of (2.1) and the hypothesis (H4) on A'/A , the space $\mathcal{D}(\mathbb{R})$ (of smooth functions with compact support on \mathbb{R}) and the space $\mathcal{S}(\mathbb{R})$ (of Schwartz functions on \mathbb{R}) are invariant under the action of $\Lambda_{A,\varepsilon}$.

Let S denote the symmetry $(Sf)(x) := f(-x)$. For $f \in C^\infty(\mathbb{R})$ such that $\sup_{x \in \mathbb{R}} (1+|x|)^r e^{s|x|} |f^{(t)}(x)| < \infty$ for every $r, t \in \mathbb{N}$ and for some $2\varrho \leq s < \infty$, and for $g \in C^\infty(\mathbb{R})$ such that g and all its derivatives are at most of polynomial growth. Then

$$\int_{\mathbb{R}} \Lambda_{A,\varepsilon} f(x)g(x)A(x)dx = - \int_{\mathbb{R}} f(x)(\Lambda_{A,\varepsilon} + 2\varepsilon\varrho S)g(x)A(x)dx. \quad (2.10)$$

Let $\lambda \in \mathbb{C}$ and consider the initial data problem

$$\Lambda_{A,\varepsilon} f(x) = i\lambda f(x), \quad f(0) = 1, \quad (2.11)$$

where $f : \mathbb{R} \rightarrow \mathbb{C}$.

Theorem 2.4 (see [5, Theorem 3.2]). *Let $\lambda \in \mathbb{C}$. There exists a unique solution $\Psi_{A,\varepsilon}(\lambda, \cdot)$ to the problem (2.11). Further, for every $x \in \mathbb{R}$, the function $\lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x)$ is analytic on \mathbb{C} . More explicitly:*

(i) For $i\lambda \neq \varepsilon\rho$,

$$\Psi_{A,\varepsilon}(\lambda, x) = \varphi_{\mu_\varepsilon}(x) + \frac{1}{i\lambda - \varepsilon\rho} \varphi'_{\mu_\varepsilon}(x), \quad \text{where } \mu_\varepsilon^2 := \lambda^2 + (\varepsilon^2 - 1)\rho^2. \quad (2.12)$$

(ii) For $i\lambda = \varepsilon\rho$,

$$\Psi_{A,\varepsilon}(\lambda, x) = 1 + 2\varepsilon\rho \frac{\text{sg}(x)}{A(x)} \int_0^{|x|} A(t) dt. \quad (2.13)$$

The following theorem contains important estimates for the growth of the eigenfunction $\Psi_{A,\varepsilon}$.

Theorem 2.5 (see [5, Theorem 3.4]). *Suppose that $-1 \leq \varepsilon \leq 1$ and $x \in \mathbb{R}$. Then:*

- 1) For $\lambda \in \mathbb{R}$ we have $|\Psi_{A,\varepsilon}(\lambda, x)| \leq \sqrt{2}$.
- 2) For $\lambda = a + ib \in \mathbb{C}$ we have $|\Psi_{A,\varepsilon}(\lambda, x)| \leq \Psi_{A,\varepsilon}(0, x) e^{|b||x|}$.
- 3) For $\lambda = 0$ we distinguish the following two cases:
 - a) For $\varepsilon = 0$, we have $\Psi_{A,0}(0, x) = 1$.
 - b) For $\varepsilon \neq 0$, there is a constant $c_\varepsilon > 0$ such that $\Psi_{A,\varepsilon}(0, x) \leq c_\varepsilon(|x| + 1)e^{-\varrho(1 - \sqrt{1 - \varepsilon^2})|x|}$.

We remind the reader that $\varphi_{i\rho}(x) = 1$ and $\varphi_{i\sqrt{1 - \varepsilon^2}\rho}(x) \leq c(|x| + 1)e^{-\varrho(1 - \sqrt{1 - \varepsilon^2})|x|}$; see Lemma 2.2.

Theorem 2.6 (see [5, Theorem 3.5]). 1) *Assume that $\lambda \in \mathbb{C}$ and $|x| \geq x_0$ with $x_0 > 0$. Given $N \in \mathbb{N}$, there is a positive constant c such that*

$$\left| \frac{\partial^N}{\partial x^N} \Psi_{A,\varepsilon}(\lambda, x) \right| \leq c(|\lambda| + 1)^N e^{|\text{Im } \lambda| |x|} \varphi_{i\sqrt{1 - \varepsilon^2}\rho}(x). \quad (2.14)$$

2) *Assume that $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$. Given $M \in \mathbb{N}$, there is a positive constant c such that*

$$\left| \frac{\partial^M}{\partial \lambda^M} \Psi_{A,\varepsilon}(\lambda, x) \right| \leq c|x|^M e^{|\text{Im } \lambda| |x|} \varphi_{i\sqrt{1 - \varepsilon^2}\rho}(x). \quad (2.15)$$

2.3. Intertwining operators. First, let us recall the following Laplace type representation of $\Psi_{A,\varepsilon}(\lambda, \cdot)$ which can be found in [5, Corollary 4.4].

Theorem 2.7. *For every $x \in \mathbb{R}^*$, there is a non-negative continuous function $\mathbb{K}_\varepsilon(x, \cdot)$ supported in $[-|x|, |x|]$ such that for all $\lambda \in \mathbb{C}$,*

$$\Psi_{A,\varepsilon}(\lambda, x) = \int_{|y| < |x|} \mathbb{K}_\varepsilon(x, y) e^{i\lambda y} dy. \quad (2.16)$$

For $f \in C^\infty(\mathbb{R})$ we define $V_{A,\varepsilon}f$ by

$$V_{A,\varepsilon}f(x) = \int_{|y| < |x|} \mathbb{K}_\varepsilon(x, y) f(y) dy \quad \text{for } x \neq 0, \quad \text{and} \quad V_{A,\varepsilon}f(0) = f(0). \quad (2.17)$$

Observe that

$$\Psi_{A,\varepsilon}(\lambda, x) = V_{A,\varepsilon}(e^{i\lambda \cdot})(x). \quad (2.18)$$

Theorem 2.8 (see [5, Theorem 5.3]). *The operator $V_{A,\varepsilon}$ is the unique automorphism of $C^\infty(\mathbb{R})$ such that*

$$\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ \frac{d}{dx}, \quad (2.19)$$

where $\Lambda_{A,\varepsilon}$ is the differential-reflection operator (1.1).

Below we will deal with the dual operator ${}^tV_{A,\varepsilon}$ of $V_{A,\varepsilon}$ in the sense that

$$\int_{\mathbb{R}} V_{A,\varepsilon}f(x)g(x)A(x)dx = \int_{\mathbb{R}} f(y) {}^tV_{A,\varepsilon}g(y)dy. \quad (2.20)$$

This can be written

$${}^tV_{A,\varepsilon}g(y) = \int_{|x|>|y|} \mathbb{K}_{\varepsilon}(x,y)g(x)A(x)dx. \quad (2.21)$$

Denote by $C_e^{\infty}(\mathbb{R})$ the space of even functions in $C^{\infty}(\mathbb{R})$. For $f \in C_e^{\infty}(\mathbb{R})$ we set

$$\mathcal{E}_{\varepsilon}f(x) := f(x) - \frac{\varrho_{\varepsilon}|x|}{2} \int_{|y|<|x|} f(y) \frac{J_1(\varrho_{\varepsilon}\sqrt{x^2-y^2})}{\sqrt{x^2-y^2}} dy, \quad (2.22)$$

where J_1 is the Bessel function of the first kind, and $\varrho_{\varepsilon} := \sqrt{1-\varepsilon^2}\varrho$. If $\varepsilon = \pm 1$, then $\varrho_{\pm 1} = 0$, and therefore $\mathcal{E}_{\pm 1} = \text{id}$. By [22, Proposition 2.1] (see also [19, Theorem 5.1]) the integral transform $\mathcal{E}_{\varepsilon}$ is an automorphism of $C_e^{\infty}(\mathbb{R})$, and its inverse transform is given by

$$\mathcal{E}_{\varepsilon}^{-1}f(x) = f(x) + \frac{\varrho_{\varepsilon}|x|}{2} \int_{|y|<|x|} f(y) \frac{I_1(\varrho_{\varepsilon}\sqrt{x^2-y^2})}{\sqrt{x^2-y^2}} dy, \quad (2.23)$$

where I_1 is the modified Bessel function of the first kind.

Let $\mathcal{D}_e(\mathbb{R})$ be the space of even functions in $\mathcal{D}(\mathbb{R})$. For $g \in \mathcal{D}_e(\mathbb{R})$ put

$${}^t\mathcal{E}_{\varepsilon}g(y) := g(y) - \frac{\varrho_{\varepsilon}}{2} \int_{|x|>|y|} |x|g(x) \frac{J_1(\varrho_{\varepsilon}\sqrt{x^2-y^2})}{\sqrt{x^2-y^2}} dx. \quad (2.24)$$

Theorem 2.9 (see [5, Theorem 4.2]). *The transform integral ${}^t\mathcal{E}_{\varepsilon}$ is an automorphism of $\mathcal{D}_e(\mathbb{R})$, and its inverse transform is given by*

$${}^t\mathcal{E}_{\varepsilon}^{-1}g(y) = g(y) + \frac{\varrho_{\varepsilon}}{2} \int_{|x|>|y|} |x|g(x) \frac{I_1(\varrho_{\varepsilon}\sqrt{x^2-y^2})}{\sqrt{x^2-y^2}} dx. \quad (2.25)$$

Recall from (2.21) the dual operator ${}^tV_{A,\varepsilon}$ of $V_{A,\varepsilon}$. The operator ${}^tV_{A,\varepsilon}$ satisfies the following properties.

Theorem 2.10 (see [5, Lemma 5.4, Theorem 5.5]). 1) *The operator ${}^tV_{A,\varepsilon}$ can be expressed as*

$${}^tV_{A,\varepsilon}g(y) = {}^t\mathcal{E}_{\varepsilon}^{-1} \circ \mathcal{A}(g_e)(y) - \left(\varepsilon\varrho - \frac{d}{dy}\right) {}^t\mathcal{E}_{\varepsilon}^{-1} \circ \mathcal{A}(Jg_o)(y), \quad (2.26)$$

where $Jh(x) := \int_{-\infty}^x h(t)dt$ and \mathcal{A} is the Abel transform (2.8).

2) *The operator ${}^tV_{A,\varepsilon}$ is the unique automorphism of $\mathcal{D}(\mathbb{R})$ satisfying the intertwining property*

$$\frac{d}{dy} \circ {}^tV_{A,\varepsilon} = {}^tV_{A,\varepsilon} \circ (\Lambda_{A,\varepsilon} + 2\varepsilon\varrho S),$$

where S denotes the symmetry $(Sf)(x) := f(-x)$.

3. FOURIER TRANSFORM OF L^p -SCHWARTZ SPACES

Assume that $-1 \leq \varepsilon \leq 1$. For $f \in L^1(\mathbb{R}, A(x)dx)$ put

$$\mathcal{F}_{A,\varepsilon}f(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{A,\varepsilon}(\lambda, -x)A(x)dx. \quad (3.1)$$

To state its alleged inverse transform, let us introduce the following Plancherel measure

$$\pi_\varepsilon(d\lambda) = \frac{|\lambda|}{\sqrt{\lambda^2 - (1 - \varepsilon^2)\varrho^2} |c(\sqrt{\lambda^2 - (1 - \varepsilon^2)\varrho^2})|^2} \mathbf{1}_{\mathbb{R} \setminus]-\sqrt{1-\varepsilon^2}\varrho, \sqrt{1-\varepsilon^2}\varrho[}(\lambda)d\lambda, \quad (3.2)$$

where c is the Harish-Chandra's function associated to the second order differential operator Δ (see Section 2 for more details on the c -function). Below $\check{f}(x) := f(-x)$.

Theorem 3.1. *Let f be a smooth function with compact support on \mathbb{R} . Then*

1) (*Inversion formula*)

$$f(x) = \frac{1}{4} \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon}(f)(\lambda)\Psi_{A,\varepsilon}(\lambda, x)\left(1 - \frac{\varepsilon\varrho}{i\lambda}\right)\pi_\varepsilon(d\lambda). \quad (3.3)$$

2) (*Plancherel formula*)

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \frac{1}{4} \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon}(f)(\lambda)\overline{\mathcal{F}_{A,\varepsilon}(\check{f})(-\lambda)}\left(1 - \frac{\varepsilon\varrho}{i\lambda}\right)\pi_\varepsilon(d\lambda). \quad (3.4)$$

We may rewrite (3.4) for two smooth and compactly supported functions f and g as

$$\int_{\mathbb{R}} f(x)g(-x)A(x)dx = \frac{1}{4} \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon}(f)(\lambda)\mathcal{F}_{A,\varepsilon}(g)(\lambda)\left(1 - \frac{\varepsilon\varrho}{i\lambda}\right)\pi_\varepsilon(d\lambda). \quad (3.5)$$

Proof. For the sake of completeness, we provide a detailed proof.

1) Below $Jh(x) := \int_{-\infty}^x h(t)dt$. Using the superposition (2.12) of the eigenfunction $\Psi_\varepsilon(\lambda, x)$, we obtain

$$\mathcal{F}_{A,\varepsilon}f(\lambda) = 2\mathcal{F}_\Delta(f_e)(\mu_\varepsilon) + 2(i\lambda + \varepsilon\varrho)\mathcal{F}_\Delta(Jf_o)(\mu_\varepsilon),$$

where F_Δ is the Chébli transform (2.5). By the inversion formula (2.6) for \mathcal{F}_Δ we deduce that

$$f(x) = \int_{\mathbb{R}^+} \left\{ \mathcal{F}_\Delta(f_e)(\mu_\varepsilon)\varphi_{\mu_\varepsilon}(x) + \mathcal{F}_\Delta(Jf_o)(\mu_\varepsilon)\varphi'_{\mu_\varepsilon}(x) \right\} \pi(d\mu_\varepsilon). \quad (3.6)$$

Now, let us express $\varphi_{\mu_\varepsilon}$ and $\varphi'_{\mu_\varepsilon}$ in terms of $\Psi_{A,\varepsilon}$ as follows

$$\varphi_{\mu_\varepsilon}(x) = \frac{1}{2}(\Psi_{A,\varepsilon}(-\lambda, -x) + \Psi_{A,\varepsilon}(-\lambda, x)), \quad \varphi'_{\mu_\varepsilon}(x) = \frac{i\lambda + \varepsilon\varrho}{2}(\Psi_{A,\varepsilon}(-\lambda, -x) - \Psi_{A,\varepsilon}(-\lambda, x)).$$

Consequently, formula (3.6) becomes

$$\begin{aligned} f(x) &= \frac{1}{2} \int_{\mathbb{R}^+} \Psi_{A,\varepsilon}(-\lambda, -x) \left\{ \mathcal{F}_\Delta(f_e)(\mu_\varepsilon) + (i\lambda + \varepsilon\varrho)\mathcal{F}_\Delta(Jf_o)(\mu_\varepsilon) \right\} \pi(d\mu_\varepsilon) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^+} \Psi_{A,\varepsilon}(-\lambda, x) \left\{ \mathcal{F}_\Delta(f_e)(\mu_\varepsilon) - (i\lambda + \varepsilon\varrho)\mathcal{F}_\Delta(Jf_o)(\mu_\varepsilon) \right\} \pi(d\mu_\varepsilon) \\ &= \frac{1}{4} \int_{\mathbb{R}^+} \left\{ \Psi_{A,\varepsilon}(-\lambda, -x)\mathcal{F}_{A,\varepsilon}(f)(\lambda) + \Psi_{A,\varepsilon}(-\lambda, x)\mathcal{F}_{A,\varepsilon}(\check{f})(\lambda) \right\} \pi(d\mu_\varepsilon). \end{aligned} \quad (3.7)$$

Further, it is easy to check that

$$\Psi_{A,\varepsilon}(\lambda, x) = \left(1 + \frac{\varepsilon\varrho}{i\lambda}\right)\Psi_{A,\varepsilon}(-\lambda, -x) - \frac{\varepsilon\varrho}{i\lambda}\Psi_{A,\varepsilon}(\lambda, -x), \quad (3.8)$$

and therefore

$$\mathcal{F}_{A,\varepsilon}(\check{f})(\lambda) = \left(1 + \frac{\varepsilon Q}{i\lambda}\right) \mathcal{F}_{A,\varepsilon}(f)(-\lambda) - \frac{\varepsilon Q}{i\lambda} \mathcal{F}_{A,\varepsilon}(f)(\lambda). \quad (3.9)$$

In view of (3.8) and (3.9) we obtain

$$\begin{aligned} \int_{\mathbb{R}^+} \mathcal{F}_{A,\varepsilon}(f)(\lambda) \Psi_{A,\varepsilon}(-\lambda, -x) \pi(d\mu_\varepsilon) &= \int_{\mathbb{R}^+} \Psi_{A,\varepsilon}(\lambda, x) \mathcal{F}_{A,\varepsilon}(f)(\lambda) \left(1 - \frac{\varepsilon Q}{i\lambda}\right) \pi(d\mu_\varepsilon) \\ &\quad + \int_{\mathbb{R}^+} \Psi_{A,\varepsilon}(-\lambda, x) \mathcal{F}_{A,\varepsilon}(f)(\lambda) \left(\frac{\varepsilon Q}{i\lambda}\right) \pi(d\mu_\varepsilon), \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} \int_{\mathbb{R}^+} \mathcal{F}_{A,\varepsilon}(\check{f})(\lambda) \Psi_{A,\varepsilon}(-\lambda, x) \pi(d\mu_\varepsilon) &= \int_{\mathbb{R}^+} \Psi_{A,\varepsilon}(-\lambda, x) \mathcal{F}_{A,\varepsilon}(f)(-\lambda) \left(1 + \frac{\varepsilon Q}{i\lambda}\right) \pi(d\mu_\varepsilon) \\ &\quad + \int_{\mathbb{R}^+} \Psi_{A,\varepsilon}(-\lambda, x) \mathcal{F}_{A,\varepsilon}(f)(\lambda) \left(-\frac{\varepsilon Q}{i\lambda}\right) \pi(d\mu_\varepsilon). \end{aligned} \quad (3.11)$$

By substituting (3.10) and (3.11) into (3.7), we get the inversion formula (3.3).

2) On the one hand, using the fact that $\overline{\Psi_{A,\varepsilon}(\lambda, x)} = \Psi_{A,\varepsilon}(-\lambda, x)$ for $\lambda \in \mathbb{R}$, we have

$$\overline{\mathcal{F}_{A,\varepsilon}(\check{g})(\lambda)} = \int_{\mathbb{R}} \overline{g(x)} \Psi_\varepsilon(-\lambda, x) A(x) dx.$$

Applying the identity (3.7) for f implies

$$\int_{\mathbb{R}} f(x) \overline{g(x)} A(x) dx = \frac{1}{4} \int_{\mathbb{R}^+} \left\{ \mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(g)(\lambda)} + \mathcal{F}_{A,\varepsilon}(\check{f})(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(\check{g})(\lambda)} \right\} \pi_\varepsilon(d\lambda). \quad (3.12)$$

On the other hand, from (3.8) it follows that

$$\overline{\mathcal{F}_{A,\varepsilon}(\check{f})(-\lambda)} = \left(1 + \frac{\varepsilon Q}{i\lambda}\right) \overline{\mathcal{F}_{A,\varepsilon}(f)(\lambda)} - \frac{\varepsilon Q}{i\lambda} \overline{\mathcal{F}_{A,\varepsilon}(f)(-\lambda)}. \quad (3.13)$$

Therefore

$$\begin{aligned} \left(1 - \frac{\varepsilon Q}{i\lambda}\right) \mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(-\lambda)} \\ = \left(1 + \frac{\varepsilon^2 Q^2}{\lambda^2}\right) |\mathcal{F}_{A,\varepsilon}(f)(\lambda)|^2 - \frac{\varepsilon Q}{i\lambda} \left(1 - \frac{\varepsilon Q}{i\lambda}\right) \mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(f)(-\lambda)}. \end{aligned} \quad (3.14)$$

Now let us rewrite (3.13) as

$$\overline{\mathcal{F}_{A,\varepsilon}(f)(-\lambda)} = \frac{i\lambda}{i\lambda - \varepsilon Q} \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(\lambda)} + \frac{\varepsilon Q}{-i\lambda + \varepsilon Q} \overline{\mathcal{F}_{A,\varepsilon}(f)(\lambda)}.$$

Hence

$$\mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(f)(-\lambda)} = \frac{i\lambda}{i\lambda - \varepsilon Q} \mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(\lambda)} + \frac{\varepsilon Q}{-i\lambda + \varepsilon Q} |\mathcal{F}_{A,\varepsilon}(f)(\lambda)|^2,$$

which implies

$$\begin{aligned} \left(-\frac{\varepsilon Q}{i\lambda}\right) \left(\frac{i\lambda - \varepsilon Q}{i\lambda}\right) \mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(f)(-\lambda)} \\ = -\left(\frac{\varepsilon Q}{i\lambda}\right) \mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(\lambda)} - \left(\frac{\varepsilon^2 Q^2}{\lambda^2}\right) |\mathcal{F}_{A,\varepsilon}(f)(\lambda)|^2. \end{aligned} \quad (3.15)$$

Thus, (3.14) becomes

$$\left(1 - \frac{\varepsilon Q}{i\lambda}\right) \mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(-\lambda)} = |\mathcal{F}_{A,\varepsilon}(f)(\lambda)|^2 - \frac{\varepsilon Q}{i\lambda} \mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(\lambda)}. \quad (3.16)$$

This is the key identity towards the Plancherel formula (3.4). Moreover, from (3.16) we also have

$$\left(1 - \frac{\varepsilon Q}{i\lambda}\right) \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(-\lambda)} \mathcal{F}_{A,\varepsilon}(f)(\lambda) = |\mathcal{F}_{A,\varepsilon}(\check{f})(-\lambda)|^2 - \frac{\varepsilon Q}{i\lambda} \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(-\lambda)} \mathcal{F}_{A,\varepsilon}(f)(-\lambda). \quad (3.17)$$

Indeed, we obtain (3.17) in three steps:

1. replace f by \check{f} in (3.16).
2. substitute λ by $-\lambda$ in the resulting identity from step 1.
3. take the complex conjugates in the resulting identity from step 2.

By putting the pieces together we arrive at

$$\begin{aligned} & \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(-\lambda)} \left(1 - \frac{\varepsilon Q}{i\lambda}\right) \pi_{\varepsilon}(d\lambda) \\ &= \int_{\mathbb{R}^+} |\mathcal{F}_{A,\varepsilon}(f)(\lambda)|^2 \pi_{\varepsilon}(d\lambda) - \int_{\mathbb{R}^+} \mathcal{F}_{A,\varepsilon}(f)(\lambda) \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(\lambda)} \left(\frac{\varepsilon Q}{i\lambda}\right) \pi_{\varepsilon}(d\lambda) \\ & \quad + \int_{\mathbb{R}^-} |\mathcal{F}_{A,\varepsilon}(\check{f})(-\lambda)|^2 \pi_{\varepsilon}(d\lambda) - \int_{\mathbb{R}^-} \mathcal{F}_{A,\varepsilon}(f)(-\lambda) \overline{\mathcal{F}_{A,\varepsilon}(\check{f})(-\lambda)} \left(\frac{\varepsilon Q}{i\lambda}\right) \pi_{\varepsilon}(d\lambda) \\ &= \int_{\mathbb{R}^+} \left\{ |\mathcal{F}_{A,\varepsilon}(f)(\lambda)|^2 + |\mathcal{F}_{A,\varepsilon}(\check{f})(\lambda)|^2 \right\} \pi_{\varepsilon}(d\lambda), \end{aligned}$$

which compares very well with $4\|f\|_{L^2(\mathbb{R}, A(x)dx)}$ (see (3.12)). \square

Remarks 3.2. 1) For $\varepsilon = 1$, the Plancherel formula (3.4) corrects Theorem 5.13 in [8] (stated without a proof).

2) For $\varepsilon = 0$ we can prove the following stronger versions of the inversion and the Plancherel formulas:

(i) If $f \in L^1(\mathbb{R}, A(x)dx)$ and $\mathcal{F}_{A,0}(f) \in L^1(\mathbb{R}, \pi_0(d\lambda))$ then

$$f(x) = \frac{1}{4} \int_{\mathbb{R}} \mathcal{F}_{A,0}(f)(\lambda) \Psi_{A,0}(\lambda, x) \pi_0(d\lambda) \quad \text{almost everywhere.}$$

(ii) If $f \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$, then $\mathcal{F}_{A,0}f \in L^2(\mathbb{R}, \pi_0(d\lambda))$ and $\|\mathcal{F}_{A,0}f\|_{L^2_1} = 2\|f\|_{L^2_x}$.

(iii) There exists a unique isometry on $L^2(\mathbb{R}, A(x)dx)$ that coincides with $(1/2)\mathcal{F}_{A,0}$ on $L^1 \cap L^2(\mathbb{R}, A(x)dx)$.

The following lemma will be needed in the proof of a Paley-Wiener theorem for $\mathcal{F}_{A,\varepsilon}$.

Lemma 3.3. For $R > 0$, denote by $\mathcal{D}_R(\mathbb{R})$ the space of smooth functions with support inside $[-R, R]$. Then, $f \in \mathcal{D}_R(\mathbb{R})$ if and only if ${}^tV_{A,\varepsilon}f \in \mathcal{D}_R(\mathbb{R})$.

Proof. The direct statement follows from (2.21). The converse direction is more involved. On the one hand, one can prove that

$${}^tV_{A,\varepsilon}^{-1}g(y) = \mathcal{A}^{-1} \circ {}^t\mathcal{E}_{\varepsilon}(g_e)(y) + \left(\varepsilon Q + \frac{d}{dy}\right) \mathcal{A}^{-1} \circ {}^t\mathcal{E}_{\varepsilon}(Jg_o)(y), \quad (3.18)$$

where $Jh(x) := \int_{-\infty}^x h(t)dt$. On the other hand, from (2.24) and [4, Lemma 4.10] it follows that if $g_e \in \mathcal{D}_R(\mathbb{R})$ then $\mathcal{A}^{-1} \circ {}^t\mathcal{E}_{\varepsilon}(g_e) \in \mathcal{D}_R(\mathbb{R})$. Further, one may check that $g_o \in \mathcal{D}_R(\mathbb{R})$ if and only if $Jg_o \in \mathcal{D}_R(\mathbb{R})$. As Jg_o is an even function, it follows from above that $\mathcal{A}^{-1} \circ {}^t\mathcal{E}_{\varepsilon}(Jg_o) \in \mathcal{D}_R(\mathbb{R})$. \square

Let $PW_R(\mathbb{C})$ be the space of entire functions h on \mathbb{C} which are of exponential type and rapidly decreasing, i.e.

$$\exists R > 0, \forall t \in \mathbb{N}, \sup_{\lambda \in \mathbb{C}} (|\lambda| + 1)^t e^{-R|\operatorname{Im} \lambda|} |h(\lambda)| < \infty. \quad (3.19)$$

Theorem 3.4. Assume that $-1 \leq \varepsilon \leq 1$. The Fourier transform $\mathcal{F}_{A,\varepsilon}$ is a linear isomorphism between $\mathcal{D}_R(\mathbb{R})$ and the space of all entire functions h on \mathbb{C} satisfying (3.19).

Proof. The proof is standard. We shall only indicate how to proceed towards the statement. On the one hand, the Fourier transform $\mathcal{F}_{A,\varepsilon}$ can be written as $\mathcal{F}_{A,\varepsilon}(f) = \mathcal{F}_{\text{euc}} \circ {}^tV_{A,\varepsilon}(\check{f})$, where \mathcal{F}_{euc} is the Euclidean Fourier transform and ${}^tV_{A,\varepsilon}$ is the intertwining operator (2.21). This is a direct consequence of Theorem 2.7. Now, in view of Lemma 3.3, appealing to the Paley-Wiener theorem for the Euclidean Fourier transform \mathcal{F}_{euc} we get the desired statement. \square

For $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$, set $\vartheta_{p,\varepsilon} := \frac{2}{p} - 1 - \sqrt{1-\varepsilon^2}$. Observe that $1 \leq \frac{2}{1+\sqrt{1-\varepsilon^2}} \leq 2$. We introduce the tube domain

$$\mathbb{C}_{p,\varepsilon} := \{\lambda \in \mathbb{C} \mid |\operatorname{Im} \lambda| \leq \varrho \vartheta_{p,\varepsilon}\}.$$

For $\vartheta_{p,\varepsilon} = 0$ or $\varrho = 0$, the domain $\mathbb{C}_{p,\varepsilon}$ reduces to \mathbb{R} .

Proposition 3.5. For all $\lambda \in \mathbb{C}_{1,\varepsilon}$, the function $\lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x)$ is bounded for all $x \in \mathbb{R}$.

Proof. Let $R > 0$ be arbitrary but fixed and let $\mathbb{R}_{1,\varepsilon} := \{v \in \mathbb{R} \mid |v| \leq \varrho(1 - \sqrt{1-\varepsilon^2})\}$. Applying the maximum modulus principle together with the fact that $|\Psi_{A,\varepsilon}(\lambda, x)| \leq \Psi_{A,\varepsilon}(i \operatorname{Im} \lambda, x)$ in the domain $[-R, R] + i\mathbb{R}_{1,\varepsilon}$ implies that the maximum of $|\Psi_{A,\varepsilon}(\lambda, x)|$ is obtained when λ belongs to the boundary of $i\mathbb{R}_{1,\varepsilon}$, that is $\lambda = i\eta$ with $|\eta| = \varrho(1 - \sqrt{1-\varepsilon^2})$. Now, recall that $\Psi_{A,\varepsilon}(i\eta, x) + \Psi_{A,\varepsilon}(i\eta, -x) = 2\varphi_{\mu_\varepsilon}(x)$ when $\varepsilon \neq 0, \pm 1$, and $\Psi_{A,\varepsilon}(i\eta, x) + \Psi_{A,\varepsilon}(i\eta, -x) = 2$ when $\varepsilon = 0, \pm 1$. The parameter μ_ε satisfies $\mu_\varepsilon^2 = \lambda^2 - (1-\varepsilon^2)\varrho^2 = -\varrho^2(1-2\sqrt{1-\varepsilon^2}\{1-\sqrt{1-\varepsilon^2}\}) \leq 0$, and therefore $\mu_\varepsilon \in i\mathbb{R}$ with $|\mu_\varepsilon| \leq \varrho$. Using the fact that $\Psi_{A,\varepsilon}(i\eta, x) > 0$ for all $x \in \mathbb{R}$, together with the fact that $\varphi_{\mu_\varepsilon}(x) \leq 1$ for μ_ε as above (see Lemma 2.2), it follows that $\Psi_{A,\varepsilon}(i\eta, x) \leq 2$ for all $x \in \mathbb{R}$ and $-1 \leq \varepsilon \leq 1$. \square

Corollary 3.6. Let $f \in L^1(\mathbb{R}, A(x)dx)$. Then the following properties hold.

1) The Fourier transform $\mathcal{F}_{A,\varepsilon}(f)(\lambda)$ is well defined for all $\lambda \in \mathbb{C}_{1,\varepsilon}$. Moreover,

$$|\mathcal{F}_{A,\varepsilon}(f)(\lambda)| \leq 2\|f\|_1, \quad \lambda \in \mathbb{C}_{1,\varepsilon}.$$

2) The function $\mathcal{F}_{A,\varepsilon}(f)$ is holomorphic on $\mathring{\mathbb{C}}_{1,\varepsilon}$, the interior of $\mathbb{C}_{1,\varepsilon}$.

3) (Riemann-Lebesgue lemma)

$$\lim_{\lambda \in \mathbb{C}_{1,\varepsilon}, |\lambda| \rightarrow \infty} |\mathcal{F}_{A,\varepsilon}(f)(\lambda)| = 0. \quad (3.20)$$

Proof. The first two statements are direct consequences of Proposition 3.5, the fact that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is holomorphic in λ , and Morera's theorem. For the Riemann-Lebesgue lemma, a classical proof for the Euclidean Fourier transform carries over. More precisely, assume that $f \in \mathcal{D}(\mathbb{R})$ (the space of smooth functions with compact support on \mathbb{R}). Now, use the Paley-Wiener Theorem 3.4 to conclude that the limit (3.20) holds for test functions; the general case then follows from the fact that $\mathcal{D}(\mathbb{R})$ is dense in $L^1(\mathbb{R}, A(x)dx)$. \square

Next we discuss some properties of the Fourier transform $\mathcal{F}_{A,\varepsilon}$ on $L^p(\mathbb{R}, A(x)dx)$ with $p > 1$.

Lemma 3.7. Let $f \in L^p(\mathbb{R}, A(x)dx)$ with $1 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$. Then the following properties hold.

1) The Fourier transform $\mathcal{F}_{A,\varepsilon}(f)(\lambda)$ is well defined for all λ in $\mathring{\mathbb{C}}_{p,\varepsilon}$. Moreover,

$$|\mathcal{F}_{A,\varepsilon}(f)(\lambda)| \leq c\|f\|_p, \quad \lambda \in \mathring{\mathbb{C}}_{p,\varepsilon}.$$

2) The function $\mathcal{F}_{A,\varepsilon}(f)$ is holomorphic on $\mathring{\mathbb{C}}_{p,\varepsilon}$.

3) (Riemann-Lebesgue lemma)

$$\lim_{\lambda \in \mathbb{C}_{p,\varepsilon}, |\lambda| \rightarrow \infty} |\mathcal{F}_{A,\varepsilon}(f)(\lambda)| = 0. \quad (3.21)$$

Proof. The first two statements follow easily from the estimate

$$\Psi_{A,\varepsilon}(\lambda, x) \leq \begin{cases} c(|x| + 1)e^{|\operatorname{Im} \lambda||x|} e^{-\varrho|x|(1 - \sqrt{1 - \varepsilon^2})} & \text{for } \varrho > 0 \\ ce^{|\operatorname{Im} \lambda||x|}, & \text{for } \varrho = 0 \end{cases}$$

the fact that $A(x) \leq c|x|^\beta e^{2\varrho|x|}$ (a consequence of the hypothesis (H4) on Chébli's function A), the fact that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is holomorphic in λ , and Morera's theorem. The Riemann-Lebesgue lemma is established exactly as for (3.20) by approximating any function in $L^p(\mathbb{R}, A(x)dx)$ by compactly supported smooth functions for all $1 < p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}}$. \square

Theorem 3.8. *The Fourier transform $\mathcal{F}_{A,\varepsilon}$ is injective on $L^p(\mathbb{R}, A(x)dx)$ for $1 \leq p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}}$.*

Proof. Take q such that $p + q = pq$. For $f \in L^p(\mathbb{R}, A(x)dx)$ et $g \in \mathcal{D}(\mathbb{R})$ we have the inequalities

$$|(f, g)_A| := \left| \int_{\mathbb{R}} f(x)g(-x)A(x)dx \right| \leq \|f\|_{L^p_x} \|g\|_{L^q_x}$$

and

$$\begin{aligned} |(\mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(g))_{\pi_\varepsilon}| &:= \left| \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon}(f)(\lambda) \mathcal{F}_{A,\varepsilon}(g)(\lambda) \left(1 - \frac{\varepsilon \varrho}{i\lambda}\right) \pi_\varepsilon(d\lambda) \right| \\ &\leq \|\mathcal{F}_{A,\varepsilon}(f)\|_{L^\infty_\lambda} \|\mathcal{F}_{A,\varepsilon}(g)\|_{L^1_\lambda} \leq c \|f\|_{L^p_x} \|\mathcal{F}_{A,\varepsilon}(g)\|_{L^1_\lambda}. \end{aligned} \quad (3.22)$$

Above we have used Corollary 3.6 and Lemma 3.7 to get (3.22). Therefore the mapping $f \mapsto (f, g)_A$ and $f \mapsto (\mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(g))_{\pi_\varepsilon}$ are continuous functionals on $L^p(\mathbb{R}, A(x)dx)$. Now $(f, g)_A = (\mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(g))_{\pi_\varepsilon}$ for all $f \in \mathcal{D}(\mathbb{R})$ and by continuity for all $f \in L^p(\mathbb{R}, A(x)dx)$. Assume that $f \in L^p(\mathbb{R}, A(x)dx)$ and that $\mathcal{F}_{A,\varepsilon}(f) = 0$, then for all $g \in \mathcal{D}(\mathbb{R})$ we have $(f, g)_A = (\mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(g))_{\pi_\varepsilon} = 0$ and therefore $f = 0$. \square

For $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}}$, let $\mathcal{S}_p(\mathbb{R})$ be the space consisting of all functions $f \in C^\infty(\mathbb{R})$ such that

$$\sup_{x \in \mathbb{R}} (|x| + 1)^s \varphi_0(x)^{-2/p} |f^{(k)}(x)| < \infty, \quad (3.23)$$

for any $s \in \mathbb{N}$ and any $k \in \mathbb{N}$. The topology of $\mathcal{S}_p(\mathbb{R})$ is defined by the seminorms

$$\sigma_{s,k}^{(p)}(f) = \sup_{x \in \mathbb{R}} (|x| + 1)^s \varphi_0(x)^{-2/p} |f^{(k)}(x)|.$$

We pin down that $\mathcal{S}_p(\mathbb{R})$ is a dense subspace of $L^q(\mathbb{R}, A(x)dx)$ for $p \leq q < \infty$, while it is not contained in $L^q(\mathbb{R}, A(x)dx)$ for $0 < q < p$.

The following facts are standard; see for instance [14, Appendix A].

Lemma 3.9. 1) $\mathcal{S}_p(\mathbb{R})$ is a Fréchet space with respect to the seminorms $\sigma_{s,k}^{(p)}$.
2) $\mathcal{D}(\mathbb{R})$ is a dense subspace of $\mathcal{S}_p(\mathbb{R})$.

Recall from above the tube domain

$$\mathbb{C}_{p,\varepsilon} := \{\lambda \in \mathbb{C} \mid |\operatorname{Im} \lambda| \leq \varrho \vartheta_{p,\varepsilon}\},$$

where $\vartheta_{p,\varepsilon} = \frac{2}{p} - 1 - \sqrt{1 - \varepsilon^2}$.

The Schwartz space $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ consists of all complex valued functions h that are analytic in the interior of $\mathbb{C}_{p,\varepsilon}$, and such that h together with all its derivatives extend continuously to $\mathbb{C}_{p,\varepsilon}$ and satisfy

$$\sup_{\lambda \in \mathbb{C}_{p,\varepsilon}} (|\lambda| + 1)^t |h^{(\ell)}(\lambda)| < \infty, \quad (3.24)$$

for any $t \in \mathbb{N}$ and any $\ell \in \mathbb{N}$. The topology of $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ is defined by the seminorms

$$\tau_{t,\ell}^{(\vartheta_{p,\varepsilon})}(h) := \sup_{\lambda \in \mathbb{C}_{p,\varepsilon}} (|\lambda| + 1)^t |h^{(\ell)}(\lambda)|. \quad (3.25)$$

For $\vartheta_{p,\varepsilon} = 0$ or $\varrho = 0$, $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ is the classical Schwartz space on \mathbb{R} . By [4, Lemma 4.17] the Paley-Wiener space $PW(\mathbb{C})$ is dense in the Schwartz space $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$.

Lemma 3.10. *The Fourier transform $\mathcal{F}_{A,\varepsilon}$ maps $\mathcal{S}_p(\mathbb{R})$ continuously into $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ and is injective.*

Proof. Let $f \in \mathcal{S}_p(\mathbb{R})$. For $\lambda \in \mathbb{C}_{p,\varepsilon}$ we have

$$\begin{aligned} |\mathcal{F}_{A,\varepsilon}(f)(\lambda)| &\leq \int_{\mathbb{R}} |f(x)| |\Psi_{A,\varepsilon}(\lambda, -x)| A(x) dx \\ &\leq \int_{\mathbb{R}} |f(x)| \varphi_0(x)^{-2/p} \varphi_0(x)^{2/p} \Psi_{A,\varepsilon}(0, -x) e^{|\operatorname{Im} \lambda||x|} A(x) dx \\ &\leq c_1 \int_{\mathbb{R}} |f(x)| \varphi_0(x)^{-2/p} (|x| + 1)^{2/p+1} e^{-2\varrho|x|} A(x) dx. \end{aligned}$$

Under the hypothesis (H4) on Chébli's function A , there exists a $\beta > 0$ such that $A(x) \leq c|x|^\beta e^{2\varrho|x|}$. Hence,

$$|\mathcal{F}_{A,\varepsilon}(f)(\lambda)| \leq c_2 \int_{\mathbb{R}} |f(x)| \varphi_0(x)^{-2/p} (|x| + 1)^{2/p+1} |x|^\beta dx < \infty.$$

This proves that $\mathcal{F}_{A,\varepsilon}(f)$ is well defined for all $f \in \mathcal{S}_p(\mathbb{R})$ when $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}}$. Moreover, since the map $\lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x)$ is holomorphic on \mathbb{C} , it follows that for all $f \in \mathcal{S}_p(\mathbb{R})$, the function $\mathcal{F}_{A,\varepsilon}(f)$ is analytic in the interior of $\mathbb{C}_{p,\varepsilon}$, and continuous on $\mathbb{C}_{p,\varepsilon}$. Furthermore, by Theorem 2.6.2, we have

$$|\mathcal{F}_{A,\varepsilon}(f)(\lambda)^{(k)}| \leq c_3 \int_{\mathbb{R}} |f(x)| \varphi_0(x)^{-2/p} (|x| + 1)^{2/p+k+1} |x|^\beta dx < \infty.$$

Thus, all derivatives of $\mathcal{F}_{A,\varepsilon}(f)$ extend continuously to $\mathbb{C}_{p,\varepsilon}$. Next, we will prove that given a continuous seminorm τ on $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$, there exists a continuous seminorm σ on $\mathcal{S}_p(\mathbb{R})$ such that

$$\tau(\mathcal{F}_{A,\varepsilon}(f)) \leq c_4 \sigma(f), \quad \forall f \in \mathcal{S}_p(\mathbb{R}).$$

Note that the space $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ and its topology are also determined by the seminorms

$$h \mapsto \tilde{\tau}_{t,\ell}^{(\vartheta_{p,\varepsilon})}(h) := \sup_{\lambda \in \mathbb{C}_{p,\varepsilon}} \left| \left\{ (\lambda + 1)^t h(\lambda) \right\}^{(\ell)} \right|, \quad (3.26)$$

where t and ℓ are two arbitrary positive integers. By invoking the identity (2.10) we have for $r \in \mathbb{N}$,

$$\begin{aligned}
(i\lambda)^r \mathcal{F}_{A,\varepsilon}(f)(\lambda) &= (i\lambda)^r \int_{\mathbb{R}} \check{f}(x) \Psi_{A,\varepsilon}(\lambda, x) A(x) dx \\
&= \int_{\mathbb{R}} \check{f}(x) \Lambda_{A,\varepsilon}^r \Psi_{A,\varepsilon}(\lambda, x) A(x) dx \\
&= (-1)^r \int_{\mathbb{R}} (\Lambda_{A,\varepsilon} + 2\varepsilon \varrho S)^r \check{f}(x) \Psi_{A,\varepsilon}(\lambda, x) A(x) dx \\
&= \int_{\mathbb{R}} \Lambda_{A,\varepsilon}^r f(-x) \Psi_{A,\varepsilon}(\lambda, x) A(x) dx \\
&= \mathcal{F}_{A,\varepsilon}(\Lambda_{A,\varepsilon}^r f)(\lambda),
\end{aligned}$$

where S denotes the symmetry $Sf(x) = f(-x)$. Above we have used $(\Lambda_{A,\varepsilon} + 2\varepsilon \varrho S)^r \circ S = (-1)^r S \circ \Lambda_{A,\varepsilon}^r$. Thus

$$\left\{ (i\lambda)^r \mathcal{F}_{A,\varepsilon}(f)(\lambda) \right\}^{(\ell)} = \int_{\mathbb{R}} \Lambda_{A,\varepsilon}^r f(x) \partial_\lambda^\ell \Psi_{A,\varepsilon}(\lambda, -x) A(x) dx.$$

On the one hand, using Theorem 2.6.2 we obtain

$$\begin{aligned}
\left| \left\{ (i\lambda)^r \mathcal{F}_{A,\varepsilon}(f)(\lambda) \right\}^{(\ell)} \right| &\leq c_5 \int_{\mathbb{R}} |\Lambda_{A,\varepsilon}^r f(x)| (|x| + 1)^\ell \varphi_{i\sqrt{1-\varepsilon^2}\varrho}(x) e^{|\operatorname{Im} \lambda||x|} A(x) dx \\
&= c_5 \int_{|x| \leq a} |\Lambda_{A,\varepsilon}^r f(x)| (|x| + 1)^\ell \varphi_{i\sqrt{1-\varepsilon^2}\varrho}(x) e^{|\operatorname{Im} \lambda||x|} A(x) dx \\
&\quad + c_5 \int_{|x| > a} |\Lambda_{A,\varepsilon}^r f(x)| (|x| + 1)^\ell \varphi_{i\sqrt{1-\varepsilon^2}\varrho}(x) e^{|\operatorname{Im} \lambda||x|} A(x) dx \\
&\leq c_6 \int_{|x| \leq a} |\Lambda_{A,\varepsilon}^r f(x)| \varphi_0(x)^{-2/p} (|x| + 1)^{2/p+\ell+1} e^{-2\varrho|x|} A(x) dx \\
&\quad + c_6 \int_{|x| > a} |\Lambda_{A,\varepsilon}^r f(x)| \varphi_0(x)^{-2/p} (|x| + 1)^{2/p+\ell+1} e^{-2\varrho|x|} A(x) dx.
\end{aligned}$$

On the other hand, by mimicking the proof of [4, Lemma 4.18] we have:

(i) For $|x| \leq a$,

$$|\Lambda_{A,\varepsilon}^r f(x)| \leq c_7 \left(\sum_{i=0}^r |f^{(i)}(x)| + \sum_{i=0}^{r-1} |f^{(i)}(-x)| + \sum_{i=0}^r \sum_{m=0}^{n_r} |f^{(i)}(\xi_m)| \right),$$

where $\xi_m = \xi_m(x, r) \in]-|x|, |x|[$.

(ii) For $|x| > a$,

$$|\Lambda_{A,\varepsilon}^r f(x)| \leq c'_7 \left(\sum_{i=0}^r |f^{(i)}(x)| + \sum_{i=0}^{r-1} |f^{(i)}(-x)| \right).$$

The estimate

$$\tau(\mathcal{F}_{A,\varepsilon}(f)) \leq c_8 \sum_{\text{finite}} \sigma(f), \quad \forall f \in \mathcal{S}_p(\mathbb{R})$$

is now a matter of putting the pieces together.

The injectivity of the transform $\mathcal{F}_{A,\varepsilon}$ on $\mathcal{S}_p(\mathbb{R})$ is clear, by the fact that $\mathcal{F}_{A,\varepsilon}$ is injective on $L^q(\mathbb{R}, A(x)dx)$ for $1 \leq q \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ (see Theorem 3.8) and the fact that $\mathcal{S}_p(\mathbb{R})$ is a dense subspace of $L^q(\mathbb{R}, A(x)dx)$ for all $q < \infty$ so that $p \leq q$.

This concludes the proof of Lemma 3.10. \square

Lemma 3.11. *Let $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$. The inverse Fourier transform $\mathcal{F}_{A,\varepsilon}^{-1} : PW(\mathbb{C}) \rightarrow \mathcal{D}(\mathbb{R})$ given by*

$$\mathcal{F}_{A,\varepsilon}^{-1}h(x) = \frac{1}{4} \int_{\mathbb{R}} h(\lambda) \Psi_{A,\varepsilon}(\lambda, x) \left(1 - \frac{\varepsilon\varrho}{i\lambda}\right) \pi_\varepsilon(d\lambda)$$

is continuous for the topologies induced by $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ and $\mathcal{S}_p(\mathbb{R})$.

Proof. Let $f \in \mathcal{D}(\mathbb{R})$ and let $h \in PW(\mathbb{C})$ so that $f = \mathcal{F}_{A,\varepsilon}^{-1}(h)$. Given a seminorm σ on $\mathcal{S}_p(\mathbb{R})$ we should find a continuous seminorm τ on $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ such that $\sigma(f) \leq c \tau(h)$.

Denote by g the image of h by the inverse Euclidean Fourier transform $\mathcal{F}_{\text{euc}}^{-1}$. Making use of the Paley-Wiener Theorem 3.4 for $\mathcal{F}_{A,\varepsilon}$ and the classical Paley-Wiener theorem for \mathcal{F}_{euc} , we have the following support conservation property: $\text{supp}(f) \subset I_R := [-R, R] \Leftrightarrow \text{supp}(g) \subset I_R$.

For $j \in \mathbb{N}_{\geq 1}$, let $\omega_j \in C^\infty(\mathbb{R})$ with $\omega_j = 0$ on I_{j-1} and $\omega_j = 1$ outside of I_j . Assume that ω_j and all its derivatives are bounded, uniformly in j . We will write $g_j = \omega_j g$, and define $h_j := \mathcal{F}_{\text{euc}}(g_j)$ and $f_j := \mathcal{F}_{A,\varepsilon}^{-1}(h_j)$. Note that $g_j = g$ outside I_j . Hence, by the above support property, $f_j = f$ outside I_j . We shall estimate the function

$$x \mapsto (|x| + 1)^s \varphi_0(x)^{-2/p} |f_j^{(k)}(x)|$$

on $I_{j+1} \setminus I_j$ with $j \in \mathbb{N}_{\geq 1}$. Recall that $f_j = f$ on $I_{j+1} \setminus I_j$. In view of Theorem 2.6.1 we have

$$\begin{aligned} |f_j^{(k)}(x)| &\leq \int_{\mathbb{R}} |h_j(\lambda)| |\partial_x^k \Psi_{A,\varepsilon}(\lambda, x)| \left|1 - \frac{\varepsilon\varrho}{i\lambda}\right| \pi_\varepsilon(d\lambda) \\ &\leq \varphi_{i\sqrt{1-\varepsilon^2}\varrho}(x) \int_{\mathbb{R}} |h_j(\lambda)| (|\lambda| + 1)^k \left|1 - \frac{\varepsilon\varrho}{i\lambda}\right| \pi_\varepsilon(d\lambda), \end{aligned}$$

where

$$\left|1 - \frac{\varepsilon\varrho}{i\lambda}\right| \pi_\varepsilon(d\lambda) = \frac{\sqrt{\lambda^2 + \varepsilon^2\varrho^2}}{\sqrt{\lambda^2 - (1-\varepsilon^2)\varrho^2}} \frac{1}{|c(\sqrt{\lambda^2 - (1-\varepsilon^2)\varrho^2})|^2} \mathbf{1}_{\mathbb{R} \setminus]-\sqrt{1-\varepsilon^2}\varrho, \sqrt{1-\varepsilon^2}\varrho[}(\lambda) d\lambda.$$

By knowing about the asymptotic behavior of the c -function (see Section 2), one comes to

$$|f_j^{(k)}(x)| \leq c_1 \varphi_{i\sqrt{1-\varepsilon^2}\varrho}(x) \tau_{t_1,0}^{(0)}(h_j),$$

for some integer $t_1 > 0$. It follows that

$$\sup_{x \in I_{j+1} \setminus I_j} (|x| + 1)^s \varphi_0(x)^{-2/p} |f_j^{(k)}(x)| \leq c_2 j^{s+1} e^{\varrho j(\frac{2}{p}-1+\sqrt{1-\varepsilon^2})} \tau_{t_1,0}^{(0)}(h_j).$$

Recall that the two seminorms $\tau_{t,\ell}^{(\vartheta_{p,\varepsilon})}$ (see (3.25)) and $\tilde{\tau}_{t,\ell}^{(\vartheta_{p,\varepsilon})}$ (see (3.26)) are equivalent on $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$. Since $h_j = \mathcal{F}_{\text{euc}}(g_j)$, it follows that

$$(1 + \lambda)^{t_1} h_j(\lambda) = \sum_{\ell=0}^{t_1} \binom{t_1}{\ell} \lambda^\ell \mathcal{F}_{\text{euc}}(g_j)(\lambda).$$

Thus

$$\begin{aligned}
\tilde{\tau}_{t_1,0}^{(0)}(h_j) &\leq \sum_{\ell=0}^{t_1} \binom{t_1}{\ell} \int_{\mathbb{R}} |g_j^{(\ell)}(y)| dy \\
&\leq c_3 \sum_{\ell=0}^{t_1} \sup_{y \in \mathbb{R}} (|y| + 1)^2 |g_j^{(\ell)}(y)| \\
&= c_3 \sum_{\ell=0}^{t_1} \sup_{w \in \{\pm 1\}} \sup_{y \in \mathbb{R}_+} (y + 1)^2 |g_j^{(\ell)}(wy)|.
\end{aligned}$$

Now one uses the Leibniz rule to compute the derivatives of $g_j = \omega_j g$. Since $\omega_j = 0$ on I_{j-1} and is bounded, together with all its derivatives uniformly in j , then we have

$$\tilde{\tau}_{t_1,0}^{(0)}(h_j) \leq c_4 \sum_{\ell=0}^{t_1} \sup_{w \in \{\pm 1\}} \sup_{y \in \mathbb{R}_+ \setminus I_{j-1}} (y + 1)^2 |g^{(\ell)}(wy)|.$$

Hence

$$j^{s+1} e^{\varrho j(\frac{2}{p}-1+\sqrt{1-\varepsilon^2})} \tilde{\tau}_{t_1,0}^{(0)}(h_j) \leq c_5 \sum_{\ell=0}^{t_1} \sup_{w \in \{\pm 1\}} \sup_{y \in \mathbb{R}_+ \setminus I_{j-1}} (y + 1)^{s+3} e^{\varrho y(\frac{2}{p}-1+\sqrt{1-\varepsilon^2})} |g^{(\ell)}(wy)|.$$

Recall that $g(x) = \mathcal{F}_{\text{euc}}^{-1}(h)(x)$, where \mathcal{F}_{euc} is the Euclidean Fourier transform and $h \in PW(\mathbb{C})$. By Cauchy's integral theorem, it is known that

$$p(u) e^{au} g^{(\ell)}(u) = c \int_{\mathbb{R}} p(i\partial_\lambda) \{(i\lambda - \alpha)^\ell h(\lambda + i\alpha)\} e^{i\lambda u} d\lambda,$$

for any polynomial $p \in \mathbb{R}[u]$. Hence,

$$\begin{aligned}
\sum_{\ell=0}^{t_1} \sup_{w \in \{\pm 1\}} \sup_{y \in \mathbb{R}_+ \setminus I_{j-1}} (y + 1)^{s+3} e^{\varrho y(\frac{2}{p}-1+\sqrt{1-\varepsilon^2})} |g^{(\ell)}(wy)| &\leq c_6 \sum_{r=0}^{s+3} \sup_{|\text{Im } \lambda| \leq \varrho \vartheta_{p,\varepsilon}} (|\lambda| + 1)^2 |h^{(r)}(\lambda)| \\
&= c_6 \sum_{r=0}^{s+3} \tau_{t_2,r}^{(\vartheta_{p,\varepsilon})}(h),
\end{aligned}$$

for some integer $t_2 > 0$.

It remains for us to estimate the function

$$x \mapsto (|x| + 1)^s \varphi_0(x)^{-2/p} |f^{(k)}(x)|$$

on $I_1 = [-1, 1]$. First, it is not hard to prove that for $|x| \leq 1$, there is a positive constant c and an integer $m_k \geq 1$ such that

$$\left| \frac{\partial^k}{\partial x^k} \Psi_{A,\varepsilon}(\lambda, x) \right| \leq c \frac{(|\lambda| + 1)^{m_k}}{|\lambda - \varepsilon \varrho|} \varphi_0(x) \quad (3.27)$$

for $\lambda \in \mathbb{R}$ such that $|\lambda| \geq \sqrt{1 - \varepsilon^2} \varrho$. Now, arguing as above, we have

$$|f^{(k)}(x)| \leq c_1 \varphi_0(x) \int_{\mathbb{R}} |h(\lambda)| \frac{(|\lambda| + 1)^{m_k}}{|\lambda - \varepsilon \varrho|} \left| 1 - \frac{\varepsilon \varrho}{i\lambda} \right| \pi_\varepsilon(d\lambda).$$

Since I_1 is compact, it follows that

$$\sup_{x \in I_1} (|x| + 1)^s \varphi_0(x)^{-2/p} |f^{(k)}(x)| \leq c_2 \int_{\mathbb{R}} |h(\lambda)| \frac{(|\lambda| + 1)^{m_k}}{|\lambda - \varepsilon \varrho|} \left| 1 - \frac{\varepsilon \varrho}{i\lambda} \right| \pi_\varepsilon(d\lambda)$$

$$\leq c_3 \tau_{t,0}^{(0)}(h),$$

for some integer $t > 0$.

This finishes the proof of Lemma 3.11. \square

In summary, we have proved:

Theorem 3.12. *Let $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$. Then the Fourier transform $\mathcal{F}_{A,\varepsilon}$ is a topological isomorphism between $\mathcal{S}_p(\mathbb{R})$ and $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$.*

4. POINTWISE MULTIPLIERS

For $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$, denote by $\mathcal{S}'_p(\mathbb{R})$ and by $\mathcal{S}'(\mathbb{C}_{p,\varepsilon})$ the topological dual spaces of $\mathcal{S}_p(\mathbb{R})$ and $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$, respectively.

Let f be a Lebesgue measurable function on \mathbb{R} such that

$$\int_{\mathbb{R}} |f(x)|\varphi_0(x)^{2/p}(|x|+1)^{-\ell}A(x)dx < \infty$$

for some $\ell \in \mathbb{N}$. Then the functional T_f defined on $\mathcal{S}_p(\mathbb{R})$ by

$$\langle T_f, \phi \rangle = \int_{\mathbb{R}} f(x)\phi(-x)A(x)dx, \quad \phi \in \mathcal{S}_p(\mathbb{R})$$

is in $\mathcal{S}'_p(\mathbb{R})$. Indeed,

$$|\langle T_f, \phi \rangle| \leq \sigma_{\ell,0}^{(p)}(\phi) \int_{\mathbb{R}} |f(x)|\varphi_0(x)^{2/p}(|x|+1)^{-\ell}A(x)dx < \infty.$$

Further, since $p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}} \leq 2$, the Schwartz space $\mathcal{S}_p(\mathbb{R})$ can be seen as a subspace of $\mathcal{S}'_p(\mathbb{R})$ by identifying $f \in \mathcal{S}_p(\mathbb{R})$ with $T_f \in \mathcal{S}'_p(\mathbb{R})$.

Now let h be a measurable function on \mathbb{R} such that

$$\int_{\mathbb{R}} |h(\lambda)|(|\lambda|+1)^{-\ell} \left| 1 - \frac{\varepsilon \varrho}{i\lambda} \right| \pi_{\varepsilon}(d\lambda) < \infty$$

for some $\ell \in \mathbb{N}$. Here $\pi_{\varepsilon}(d\lambda)$ denotes the Plancherel measure (3.2),

$$\pi_{\varepsilon}(d\lambda) = \frac{|\lambda|}{\sqrt{\lambda^2 - (1-\varepsilon^2)\varrho^2} \left| c(\sqrt{\lambda^2 - (1-\varepsilon^2)\varrho^2}) \right|^2} \mathbf{1}_{\mathbb{R} \setminus]-\sqrt{1-\varepsilon^2}\varrho, \sqrt{1-\varepsilon^2}\varrho[}(\lambda)d\lambda,$$

where c is the Harish-Chandra's function associated to the operator Δ (see Section 2). Then the functional \mathcal{T}_h defined on $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ by

$$\langle \mathcal{T}_h, \psi \rangle = \int_{\mathbb{R}} h(\lambda)\psi(\lambda) \left(1 - \frac{\varepsilon \varrho}{i\lambda} \right) \pi_{\varepsilon}(d\lambda), \quad \psi \in \mathcal{S}(\mathbb{C}_{p,\varepsilon})$$

is in the dual space $\mathcal{S}'(\mathbb{C}_{p,\varepsilon})$. In fact,

$$|\langle \mathcal{T}_h, \psi \rangle| \leq c \tau_{0,\ell}^{(0)}(\psi) \int_{\mathbb{R}} |h(\lambda)|(|\lambda|+1)^{-\ell} \left| 1 - \frac{\varepsilon \varrho}{i\lambda} \right| \pi_{\varepsilon}(d\lambda) < \infty.$$

Moreover, since $|c(\mu)|^{-2} \sim |\mu|^{2\alpha+1}$ for $|\mu|$ large (with $\alpha > -1/2$) and

$$|c(\mu)|^{-2} \sim \begin{cases} |\mu|^2 & \text{for } |\mu| \ll 1 \text{ and } \varrho > 0, \\ |\mu|^{2\alpha+1} & \text{for } |\mu| \ll 1 \text{ and } \varrho = 0, \end{cases}$$

it follows that the Schwartz space $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ can be identified with a subspace of $\mathcal{S}'(\mathbb{C}_{p,\varepsilon})$.

For T in $\mathcal{S}'_p(\mathbb{R})$, we define the distributional Fourier transform $\mathcal{F}_{A,\varepsilon}(T)$ of T on $\mathcal{S}(\mathbb{C}_{p,\varepsilon}) = \mathcal{F}_{A,\varepsilon}(\mathcal{S}_p(\mathbb{R}))$ by

$$\langle \mathcal{F}_{A,\varepsilon}(T), \mathcal{F}_{A,\varepsilon}(\phi) \rangle = \langle T, \phi \rangle, \quad \phi \in \mathcal{S}_p(\mathbb{R}). \quad (4.1)$$

This can be written,

$$\langle \mathcal{F}_{A,\varepsilon}(T), \psi \rangle = \langle T, \mathcal{F}_{A,\varepsilon}^{-1}(\psi) \rangle, \quad \psi \in \mathcal{S}(\mathbb{C}_{p,\varepsilon}).$$

This definition is an extension of the Fourier transform on $\mathcal{S}_p(\mathbb{R})$. Indeed, let $f \in \mathcal{S}_p(\mathbb{R})$ with $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}} \leq 2$. Applying Fubini's theorem, then, for every $\phi \in \mathcal{S}_p(\mathbb{R})$, we have

$$\begin{aligned} \langle \mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(\phi) \rangle &= \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon}(f)(\lambda) \mathcal{F}_{A,\varepsilon}(\phi)(\lambda) \left(1 - \frac{\varepsilon \varrho}{i\lambda}\right) \pi_\varepsilon(d\lambda) \\ &= \int_{\mathbb{R}} f(x) \left\{ \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon}(\phi)(\lambda) \Psi_{A,\varepsilon}(\lambda, -x) \left(1 - \frac{\varepsilon \varrho}{i\lambda}\right) \pi_\varepsilon(d\lambda) \right\} A(x) dx \\ &= \int_{\mathbb{R}} f(x) \phi(-x) A(x) dx \\ &= \langle T_f, \phi \rangle. \end{aligned}$$

Hence $\mathcal{F}_{A,\varepsilon}(T_f) = \mathcal{F}_{A,\varepsilon}(T_f)$.

A function ψ defined on $\mathbb{C}_{p,\varepsilon}$ is called a pointwise multiplier of $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ if the mapping $\phi \mapsto \psi\phi$ is continuous from $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ into itself. The following statement comes from [2, Proposition 3.2], with changes appropriate to our setting.

Lemma 4.1. *Let ψ be a function defined on $\mathbb{C}_{p,\varepsilon}$. Then, ψ is a pointwise multiplier of $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ if and only if ψ satisfies the following three conditions:*

- (i) ψ is holomorphic in the interior of $\mathbb{C}_{p,\varepsilon}$.
- (ii) For every $t \in \mathbb{N}$, the derivatives $\psi^{(t)}$ extend continuously to $\mathbb{C}_{p,\varepsilon}$.
- (iii) For every $t \in \mathbb{N}$, there exists $n_t \in \mathbb{N}$, such that

$$\sup_{\lambda \in \mathbb{C}_{p,\varepsilon}} (|\lambda| + 1)^{-n_t} |\psi^{(t)}(\lambda)| < \infty. \quad (4.2)$$

Theorem 4.2. *Suppose that $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ whenever $\varrho = 0$, and $\frac{2}{2+\sqrt{1-\varepsilon^2}} \leq p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$ whenever $\varrho > 0$. If $T \in \mathcal{S}'_p(\mathbb{R})$ such that $\psi := \mathcal{F}_{A,\varepsilon}(T)$ is a pointwise multiplier of $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$, then for any $s \in \mathbb{N}$ there exist $\ell \in \mathbb{N}$ and continuous functions f_m defined on \mathbb{R} , $m = 0, 1, \dots, \ell$, such that*

$$T = \sum_{m=0}^{\ell} \Lambda_{A,\varepsilon}^m f_m$$

and, for every such m ,

$$\sup_{x \in \mathbb{R}} (|x| + 1)^s \varphi_0(x)^{-\frac{2}{p} + \sqrt{1-\varepsilon^2}} |f_m(x)| < \infty. \quad (4.3)$$

Here $\Lambda_{A,\varepsilon}$ is the differential-reflection operator (1.1).

Proof. It is assumed that $\psi = \mathcal{F}_{A,\varepsilon}(T)$ is a pointwise multiplier of $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$. Then by Lemma 4.1, for all $t \in \mathbb{N}$ there is an integer $n_t \in \mathbb{N}$ such that

$$\sup_{\lambda \in \mathbb{C}_{p,\varepsilon}} (|\lambda| + 1)^{-n_t} |\psi^{(t)}(\lambda)| < \infty. \quad (4.4)$$

Fix $s \in \mathbb{N}$ and consider an integer ℓ that will be later specified. Define the function κ on $\mathbb{C}_{p,\varepsilon}$ by

$$\kappa(\lambda) = (i\lambda + \varrho + 1)^{-\ell} \psi(\lambda).$$

In view of our assumption on p , the function κ satisfies the first and the second conditions in the definition of the space $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$. Further, since $|\Psi_{A,\varepsilon}(\lambda, x)| \leq \sqrt{2}$ for all $\lambda \in \mathbb{R}$, we have

$$|\mathcal{F}_{A,\varepsilon}^{-1}(\kappa)(x)| := \left| c \int_{\mathbb{R}} \kappa(\lambda) \Psi_{A,\varepsilon}(\lambda, x) \left(1 - \frac{\varepsilon \varrho}{i\lambda}\right) \pi_\varepsilon(d\lambda) \right| \leq c_1 \int_{\mathbb{R}} |\kappa(\lambda)| \left|1 - \frac{\varepsilon \varrho}{i\lambda}\right| \pi_\varepsilon(d\lambda),$$

where

$$\left|1 - \frac{\varepsilon \varrho}{i\lambda}\right| \pi_\varepsilon(d\lambda) = \frac{\sqrt{\lambda^2 + \varepsilon^2 \varrho^2}}{\sqrt{\lambda^2 - (1 - \varepsilon^2) \varrho^2}} \frac{1}{|c(\sqrt{\lambda^2 - (1 - \varepsilon^2) \varrho^2})|^2} \mathbf{1}_{\mathbb{R} \setminus]-\sqrt{1 - \varepsilon^2} \varrho, \sqrt{1 - \varepsilon^2} \varrho[}(\lambda) d\lambda.$$

Thus, in view of the estimate (4.4) and the behavior of $|c(\mu)|^{-2}$ for small and large $|\mu|$, it follows that $\mathcal{F}_{A,\varepsilon}^{-1}(\kappa)(x)$ exists for all $x \in \mathbb{R}$ provided that $\ell > n_0 + 2\alpha + 2$. Moreover, for all $\phi \in \mathcal{S}_p(\mathbb{R})$, Fubini's theorem leads to

$$\begin{aligned} \int_{\mathbb{R}} \phi(-x) \mathcal{F}_{A,\varepsilon}^{-1}(\kappa)(x) A(x) dx &= c_1 \int_{\mathbb{R}} \phi(-x) \left(\int_{\mathbb{R}} \kappa(\lambda) \Psi_{A,\varepsilon}(\lambda, x) \left(1 - \frac{\varepsilon \varrho}{i\lambda}\right) \pi_\varepsilon(d\lambda) \right) A(x) dx \\ &= c_1 \int_{\mathbb{R}} \kappa(\lambda) \left(\int_{\mathbb{R}} \phi(-x) \Psi_{A,\varepsilon}(\lambda, x) A(x) dx \right) \left(1 - \frac{\varepsilon \varrho}{i\lambda}\right) \pi_\varepsilon(d\lambda) \\ &= c_1 \int_{\mathbb{R}} \kappa(\lambda) \mathcal{F}_{A,\varepsilon}(\phi)(\lambda) \left(1 - \frac{\varepsilon \varrho}{i\lambda}\right) \pi_\varepsilon(d\lambda). \end{aligned}$$

It follows that the inverse Fourier transform $\mathcal{F}_{A,\varepsilon}^{-1}(\kappa)$ of κ as an element of $\mathcal{S}'(\mathbb{C}_{p,\varepsilon})$ concurs with the classical Fourier transform of κ . Further

$$T = \mathcal{F}_{A,\varepsilon}^{-1}((i\lambda + \varrho + 1)^\ell \kappa) = \sum_{m=0}^{\ell} \binom{\ell}{m} (\varrho + 1)^{\ell-m} \Lambda_{A,\varepsilon}^m \mathcal{F}_{A,\varepsilon}^{-1}(\kappa) := \sum_{m=0}^{\ell} \Lambda_{A,\varepsilon}^m f_m.$$

It remains for us to show that, given $s \in \mathbb{N}$, the functions f_m satisfy (4.3), provided that ℓ is large enough. To do so, we will use a similar approach to that in the proof of Lemma 3.11.

Denote by $\xi := \mathcal{F}_{A,\varepsilon}^{-1}(\kappa)$ and by $g := \mathcal{F}_{\text{euc}}^{-1}(\kappa)$, where \mathcal{F}_{euc} denotes the Euclidean Fourier transform. Observe that if ℓ is large enough, then g is well defined. For $j \in \mathbb{N}_{\geq 1}$, let $\omega_j \in C^\infty(\mathbb{R})$ such that $\omega_j = 0$ on $I_{j-1} := [-(j-1), j-1]$ and $\omega_j = 1$ outside of I_j . We shall assume that ω_j together with all its derivatives are bounded, uniformly in j .

We set $g_j := \omega_j g$, and define $\kappa_j := \mathcal{F}_{\text{euc}}(g_j)$ and $\xi_j = \mathcal{F}_{A,\varepsilon}^{-1}(\kappa_j)$. Since $\omega_j = 1$ outside of I_j , it follows that $g_j - g = 0$ outside of I_j , that is $\text{supp}(g_j - g) \subset I_j$. Using the support conservation property from the proof of Lemma 3.11, we deduce that ξ may differ from ξ_j only inside I_j . Now, we will estimate the function

$$x \mapsto (|x| + 1)^s \varphi_0(x)^{-\frac{2}{p} + \sqrt{1 - \varepsilon^2}} \xi(x), \quad (4.5)$$

first on I_1 and next on $I_{j+1} \setminus I_j$ for $j \in \mathbb{N}_{\geq 1}$.

We claim that $|\Psi_{A,\varepsilon}(\lambda, x)| \leq c_2(|\lambda| + 1)\varphi_0(x)$ for $\lambda \in \mathbb{R}$ such that $|\lambda| \geq \sqrt{1 - \varepsilon^2} \varrho$. Indeed, as $\lambda \in \mathbb{R}$ is such that $|\lambda| \geq \sqrt{1 - \varepsilon^2} \varrho$, it follows that $\mu_\varepsilon \in \mathbb{R}$. Thus, the claim follows from the superposition (2.12) of $\Psi_{A,\varepsilon}(\lambda, x)$ and the facts that $|\varphi_{\mu_\varepsilon}(x)| \leq \varphi_0(x)$ and $|\varphi'_{\mu_\varepsilon}(x)| \leq c(\mu_\varepsilon^2 + \varrho^2)\varphi_0(x)$ (see Lemma 2.2).

From the claim above we have

$$\begin{aligned} |\xi(x)| &\leq c_3 \int_{\mathbb{R}} |\kappa(\lambda)| |\Psi_{A,\varepsilon}(\lambda, x)| \left|1 - \frac{\varepsilon \varrho}{i\lambda}\right| \pi_\varepsilon(d\lambda) \\ &\leq c_4 \varphi_0(x) \int_{\mathbb{R}} |\kappa(\lambda)| (|\lambda| + 1) \left|1 - \frac{\varepsilon \varrho}{i\lambda}\right| \pi_\varepsilon(d\lambda). \end{aligned}$$

Since I_1 is compact, we deduce that for every $s \in \mathbb{N}$

$$\sup_{x \in I_1} (|x| + 1)^s \varphi_0(x)^{-\frac{2}{p} + \sqrt{1-\varepsilon^2}} |\xi(x)| < \infty$$

whenever $\ell > n_0 + 2\alpha + 3$. Here the parameter n_0 comes from (4.4). Now we consider the estimate of the function (4.5) on $I_{j+1} \setminus I_j$ for $j \in \mathbb{N}_{\geq 1}$. Recall that $\xi = \xi_j$ outside of I_j .

Arguing as above, we obtain

$$|\xi_j(x)| \leq c_5 \varphi_0(x) \sup_{\lambda \in \mathbb{R} \setminus]-\sqrt{1-\varepsilon^2}\varrho, \sqrt{1-\varepsilon^2}\varrho[} |(\lambda + 1)^{t_1} \kappa_j(\lambda)|$$

for some integer $t_1 > 2\alpha + 3$. It follows that

$$\sup_{x \in I_{j+1} \setminus I_j} (|x| + 1)^s \varphi_0(x)^{-\frac{2}{p} + \sqrt{1-\varepsilon^2}} |\xi_j(x)| \leq c_6 j^s e^{(\frac{2}{p}-1-\sqrt{1-\varepsilon^2})\varrho j} \sup_{\lambda \in \mathbb{R} \setminus]-\sqrt{1-\varepsilon^2}\varrho, \sqrt{1-\varepsilon^2}\varrho[} |(\lambda + 1)^{t_1} \kappa_j(\lambda)|.$$

Since $\kappa_j = \mathcal{F}_{\text{euc}}(g_j)$ with $g_j = \omega_j g$, we claim that

$$|(\lambda + 1)^{t_1} \kappa_j(\lambda)| \leq c_7 \sum_{q=0}^{t_1} \sup_{w \in \{\pm 1\}} \sup_{x \in \mathbb{R}^+ \setminus I_{j-1}} (x + 1)^2 |g^{(q)}(wx)|. \quad (4.6)$$

Indeed, on the one hand

$$\begin{aligned} (\lambda + 1)^{t_1} \kappa_j(\lambda) &= \sum_{r=0}^{t_1} c_r \int_{\mathbb{R}} g_j(x) \partial_x^r e^{i\lambda x} dx \\ &= \sum_{r=0}^{t_1} c_r \int_{\mathbb{R}} \omega_j(x) g(x) \partial_x^r e^{i\lambda x} dx. \end{aligned} \quad (4.7)$$

On the other hand, we have

$$(\omega_j g)^{(r)}(x) = \sum_{q=0}^r c_q g^{(q)}(x) \omega_j^{(r-q)}(x) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty. \quad (4.8)$$

In fact, starting from $g = \mathcal{F}_{\text{euc}}^{-1}(\kappa)$, we obtain

$$g^{(q)}(x) = c \int_{\mathbb{R}} \kappa(\lambda) (i\lambda)^q e^{i\lambda x} d\lambda. \quad (4.9)$$

Thus, if $\ell > n_0 + t_1 + 1$ then by Riemann-Lebesgue lemma for the Euclidean Fourier transform, $g^{(q)}(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Thus (4.8) holds true. Now, in view of (4.8) we may rewrite (4.7) as

$$(\lambda + 1)^{t_1} \kappa_j(\lambda) = \sum_{r=0}^{t_1} \sum_{q=0}^r c_{q,r} \int_{\mathbb{R}} g^{(q)}(x) \omega_j^{(r-q)}(x) e^{i\lambda x} dx.$$

Recall that the function ω_j vanishes on I_{j-1} and is bounded, together with all its derivatives, uniformly in j . Therefore,

$$\begin{aligned} |(\lambda + 1)^{t_1} \kappa_j(\lambda)| &\leq c \sum_{q=0}^{t_1} \int_{\mathbb{R} \setminus I_{j-1}} |g^{(q)}(x)| dx \\ &\leq c \sum_{q=0}^{t_1} \sup_{x \in \mathbb{R} \setminus I_{j-1}} (|x| + 1)^2 |g^{(q)}(x)|. \end{aligned}$$

This finishes the proof of our claim (4.6).

It follows that

$$\begin{aligned} & j^s e^{\left(\frac{2}{p}-1-\sqrt{1-\varepsilon^2}\right)\varrho j} \sup_{\lambda \in \mathbb{R} \setminus]-\sqrt{1-\varepsilon^2}\varrho, \sqrt{1-\varepsilon^2}\varrho[} |(\lambda+1)^{\ell_1} \kappa_j(\lambda)| \\ & \leq c_7 \sum_{q=0}^{\ell_1} \sup_{w \in \{\pm 1\}} \sup_{x \in \mathbb{R}^+ \setminus I_{j-1}} (x+1)^{s+2} e^{\left(\frac{2}{p}-1-\sqrt{1-\varepsilon^2}\right)\varrho x} |g^{(q)}(wx)|. \end{aligned}$$

Next we shall prove that the right hand is finite. Assume first that $\varrho = 0$. By (4.9) we have

$$(x+1)^{s+2} g^{(q)}(wx) = \sum_{r=0}^{s+2} c_{q,r} \int_{\mathbb{R}} \kappa(\lambda) \lambda^q \partial_{\lambda}^r e^{i\lambda wx} d\lambda. \quad (4.10)$$

We claim that

$$(\kappa(\lambda) \lambda^q)^{(r)} \rightarrow 0 \quad \text{as} \quad |\lambda| \rightarrow +\infty \quad (4.11)$$

provided that ℓ is large enough. Indeed, this claim follows immediately from the fact that

$$\begin{aligned} (\kappa(\lambda) \lambda^q)^{(r)} &= \sum_{a=0}^r c_a \lambda^{q-r+a} \kappa^{(a)}(\lambda) \quad (\text{with } r-a \leq q) \\ &= \sum_{a=0}^r \sum_{b=0}^a c_{a,b} \lambda^{q-r+a} (i\lambda+1)^{-\ell-a+b} \psi^{(b)}(\lambda), \end{aligned} \quad (4.12)$$

together with the fact that ψ satisfies (4.4). Thus, by (4.11) we may rewrite (4.10) as

$$(x+1)^{s+2} g^{(q)}(wx) = \sum_{r=0}^{s+2} c'_{q,r} \int_{\mathbb{R}} (\kappa(\lambda) \lambda^q)^{(r)} e^{i\lambda wx} d\lambda. \quad (4.13)$$

Using again the fact that ψ satisfies (4.4) together with the double sum (4.12), it follows from (4.13) that for $\varrho = 0$

$$\sup_{w \in \{\pm 1\}} \sup_{x \in \mathbb{R}^+ \setminus I_{j-1}} (x+1)^{s+2} |g^{(q)}(wx)| < \infty$$

provided that ℓ is large enough.

Now assume that $\varrho > 0$. Since $g = \mathcal{F}_{\text{euc}}^{-1}(\kappa)$ and κ is holomorphic in the interior of $\mathbb{C}_{p,\varepsilon}$, Cauchy's integral theorem gives

$$p(u) e^{\alpha u} g^{(q)}(u) = \text{cst} \int_{\mathbb{R}} p(i\partial_{\lambda}) \{(i\lambda - \alpha)^q \kappa(\lambda + i\alpha)\} e^{i\lambda u} d\lambda,$$

with $p(x) = (x+1)^{s+2}$ and $\alpha = \left(\frac{2}{p}-1-\sqrt{1-\varepsilon^2}\right)\varrho$. The same argument as above implies that

$$\sup_{w \in \{\pm 1\}} \sup_{x \in \mathbb{R}^+ \setminus I_{j-1}} (x+1)^{s+2} e^{\left(\frac{2}{p}-1-\sqrt{1-\varepsilon^2}\right)\varrho x} |g^{(q)}(wx)| < \infty$$

provided that ℓ is large enough.

Putting the pieces together we conclude that

$$\sup_{x \in I_{j+1} \setminus I_j} (|x|+1)^s \varphi_0(x)^{-\frac{2}{p} + \sqrt{1-\varepsilon^2}} |\xi_j(x)| < \infty$$

for ℓ large enough. □

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