

Convex Super-Resolution Detection of Lines in Images

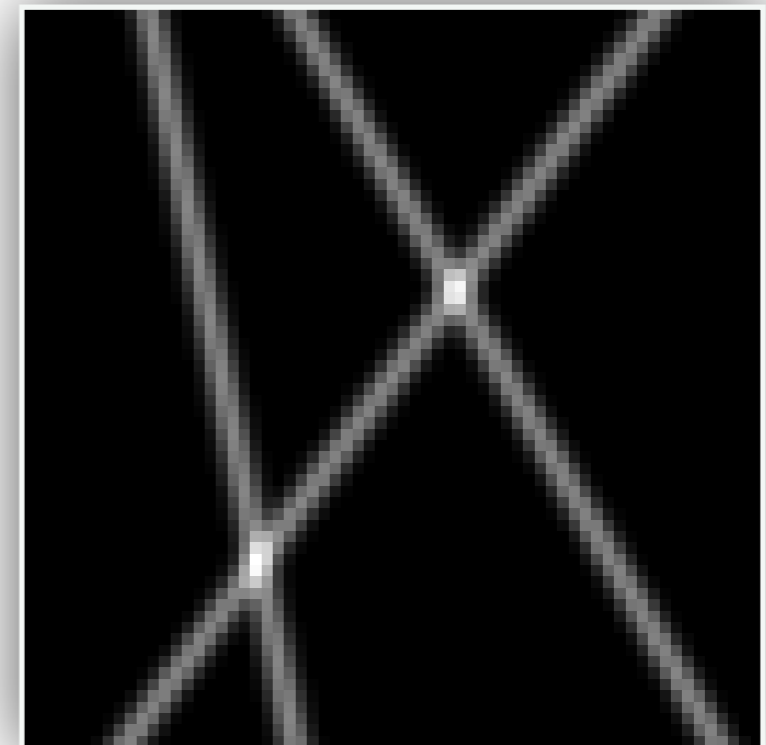
Kévin Polisano / kevin.polisano@imag.fr

joint work with

Laurent Condat

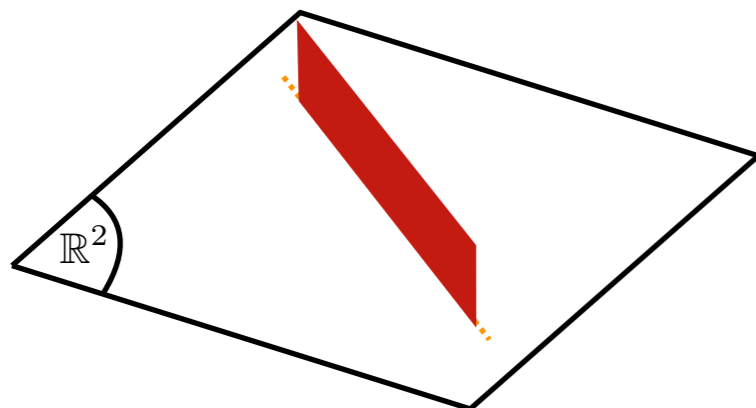
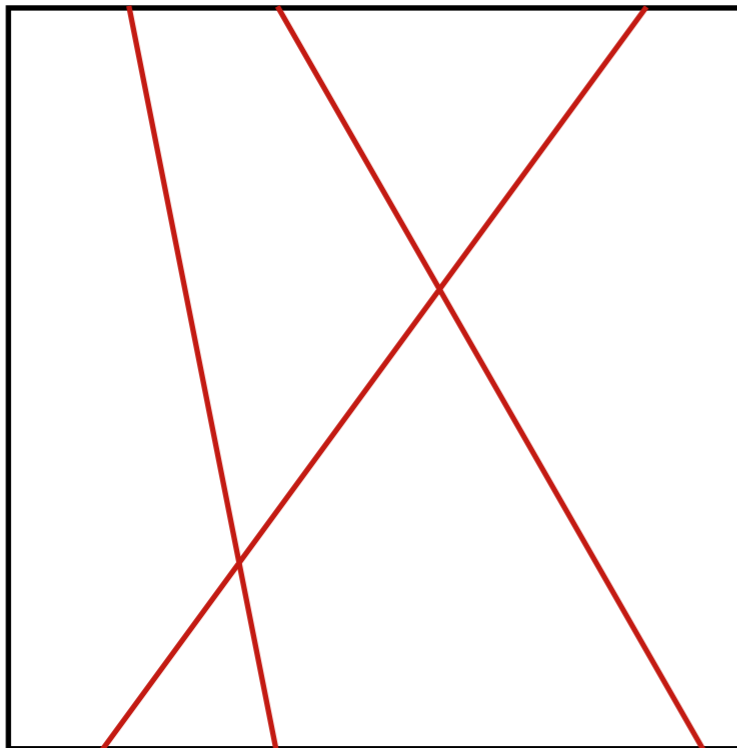
Valérie Perrier

Marianne Clausel

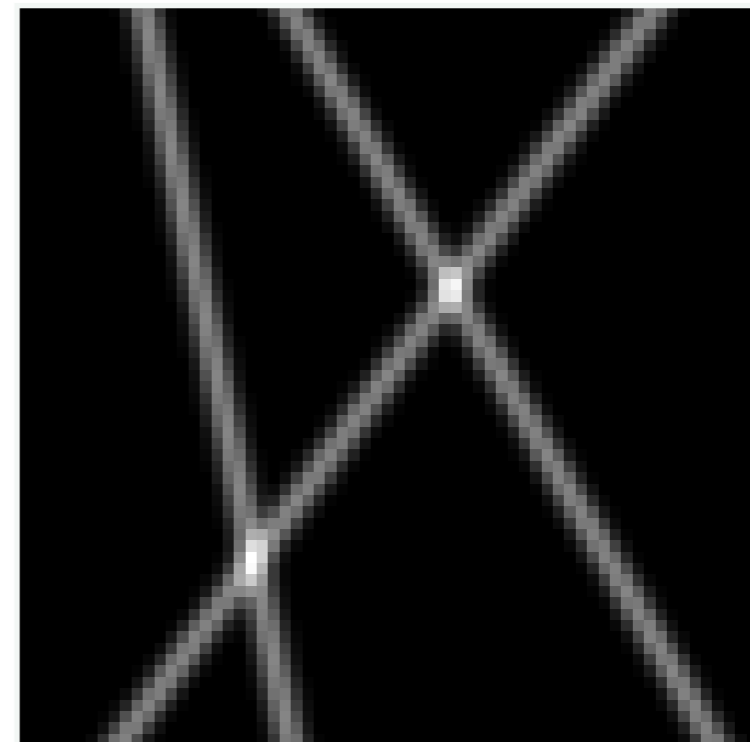


Super-Resolution Detection of Lines

Objective f



Data

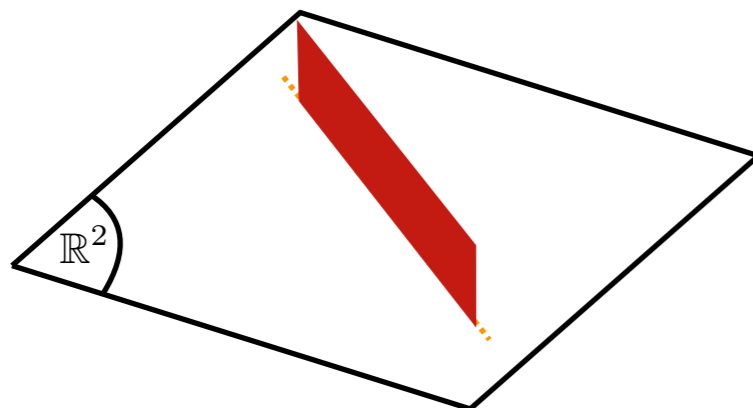
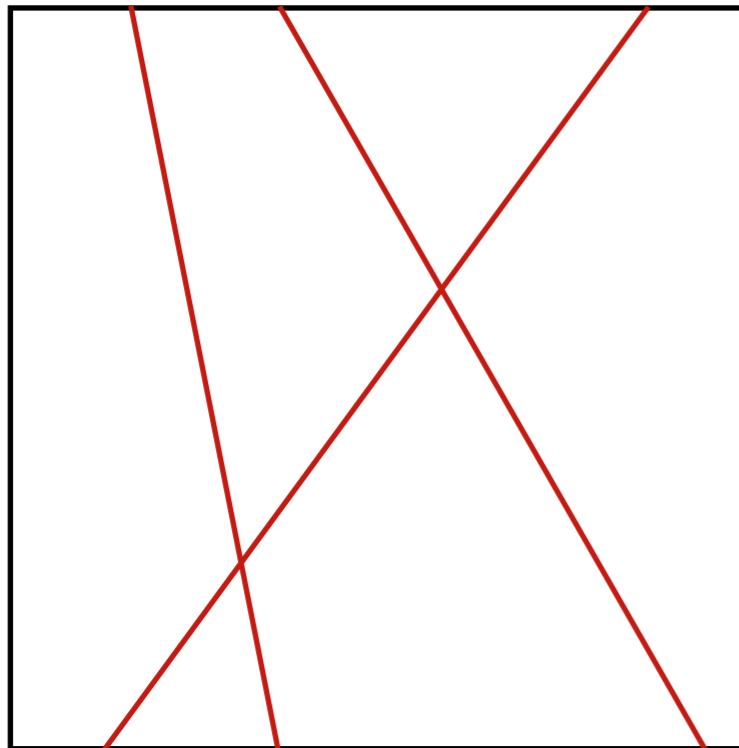


$$f_{\text{obs}} = f * h$$

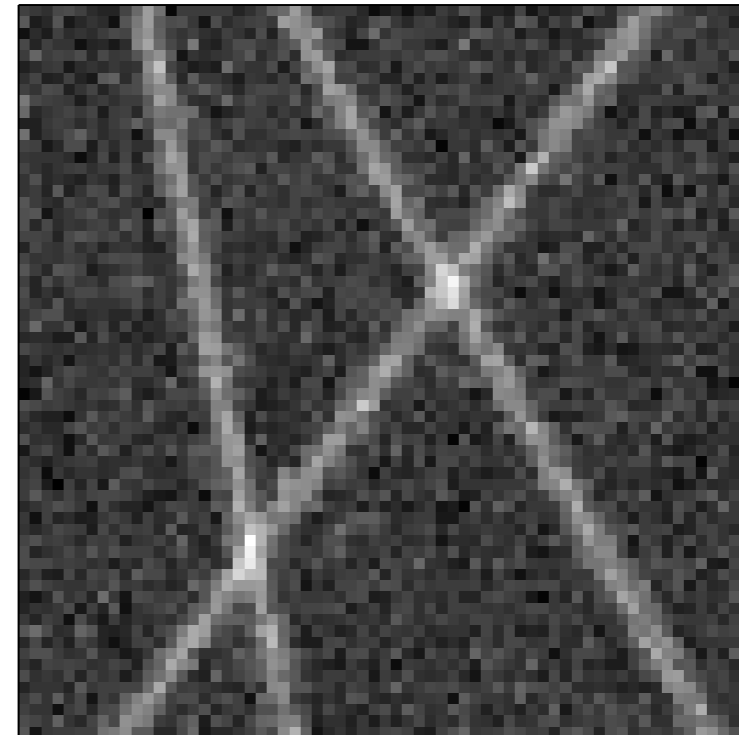
+ Recovering
line parameters

Super-Resolution Detection of Lines

Objective f



Data

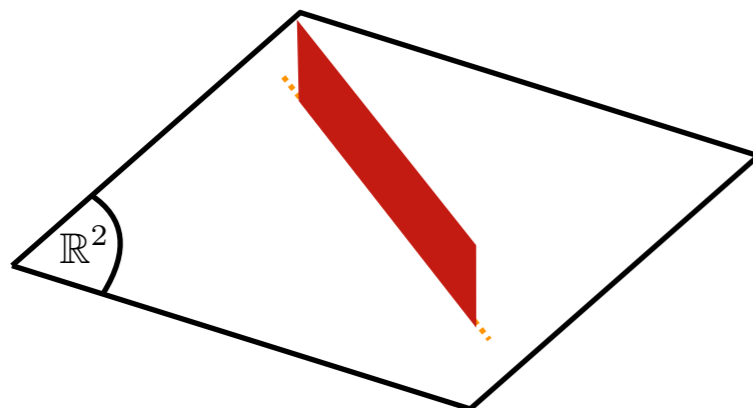
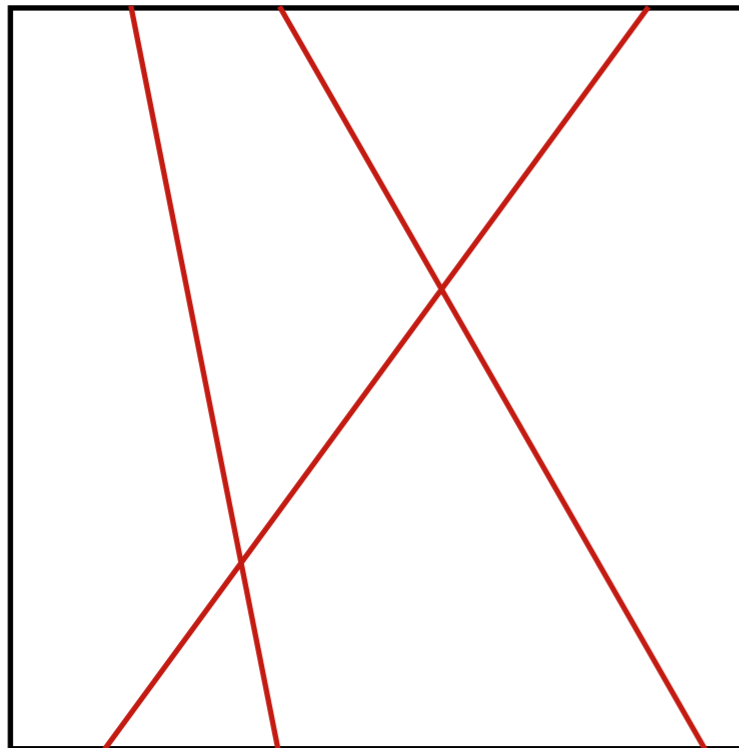


$$f_{\text{obs}} = f * h + \varepsilon$$

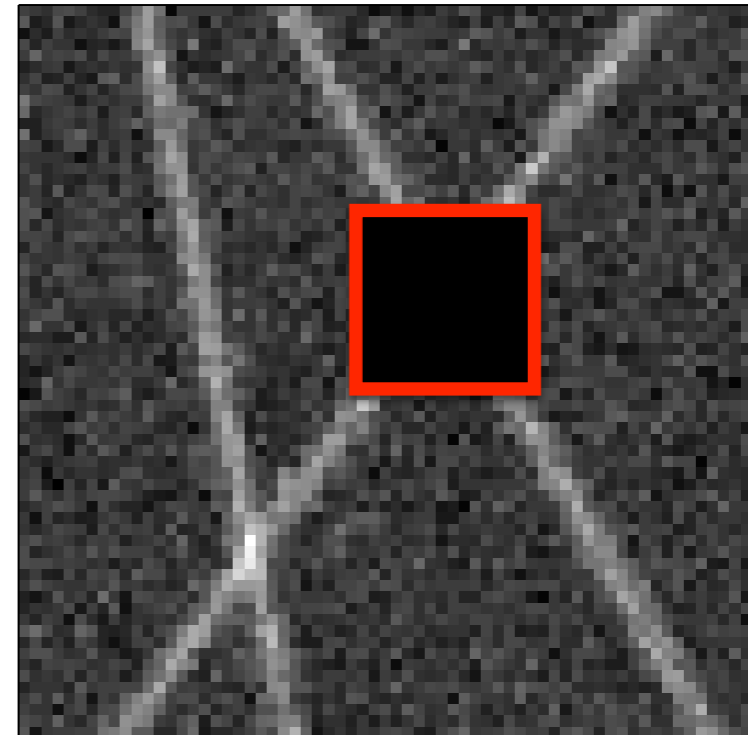
+ Recovering
line parameters

Super-Resolution Detection of Lines

Objective f



Data

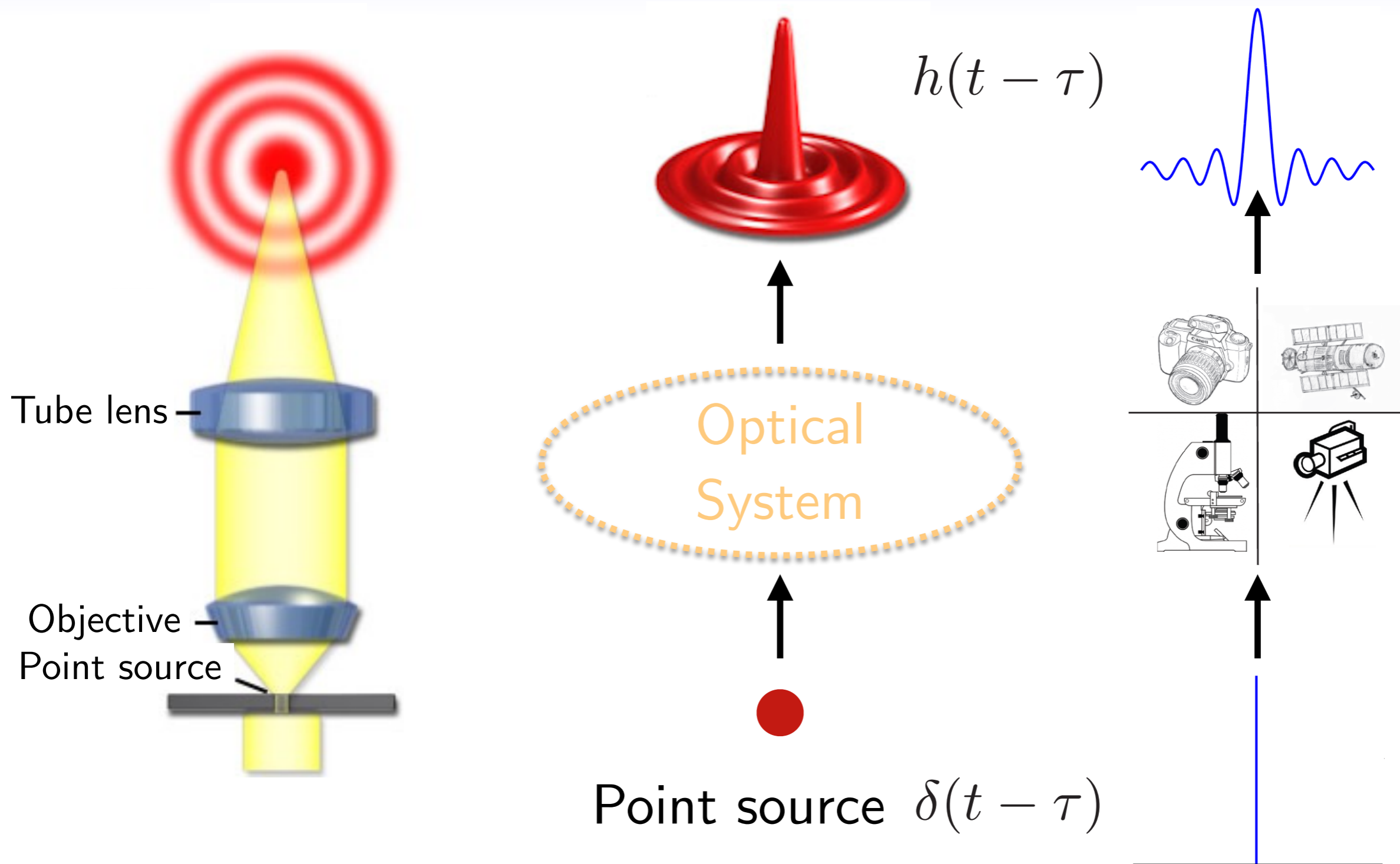


$$f_{\text{obs}} = \mathbf{M}(f * h + \varepsilon)$$

+ Recovering
line parameters

Super-Resolution Principle

The resolving power of lenses, however perfect, is limited (Lord Rayleigh)

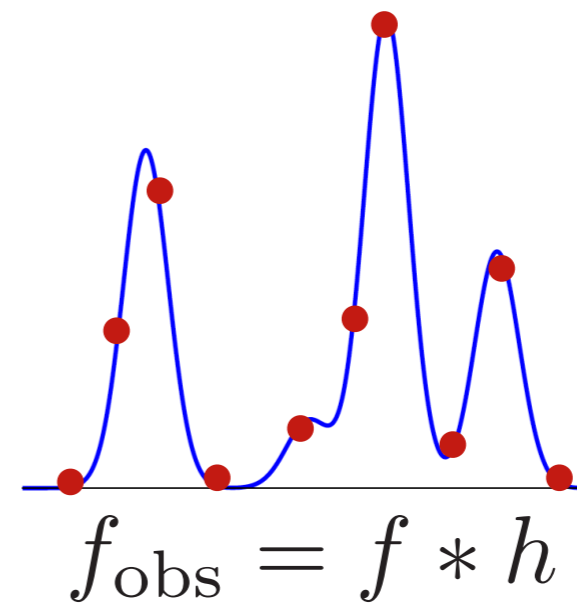
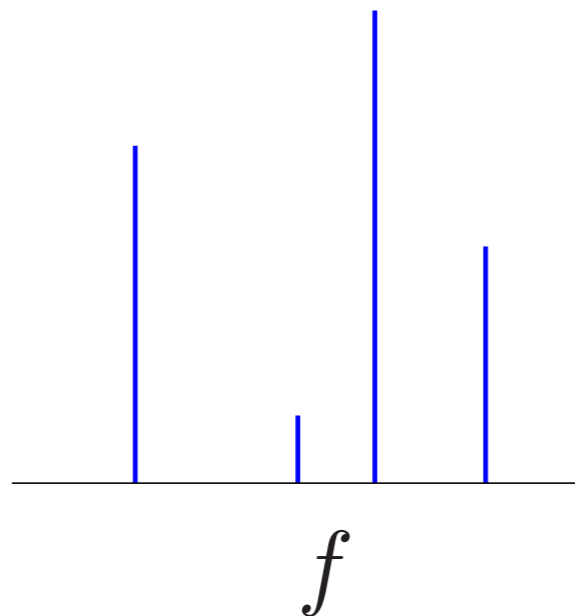


Super-Resolution Principle

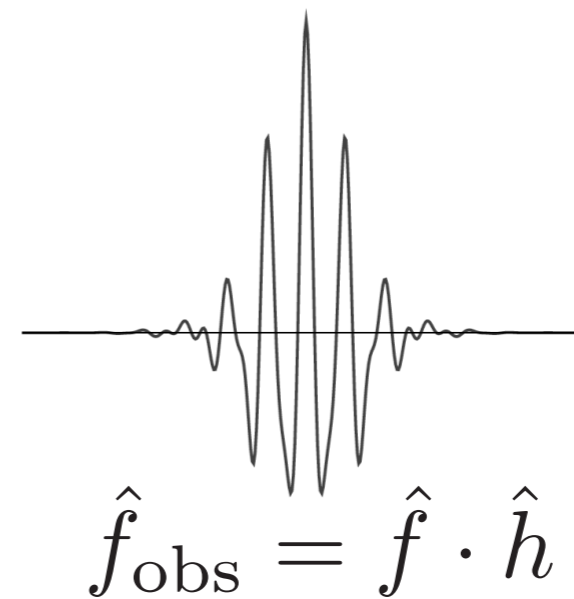
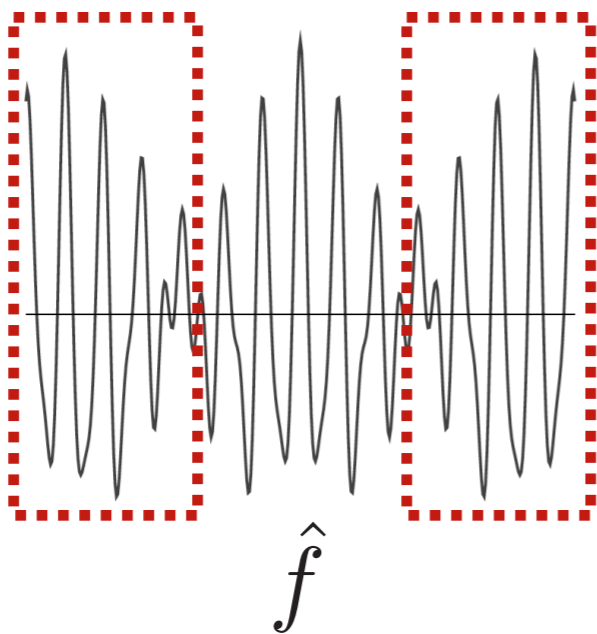
Objective

Data

(*spatial*)



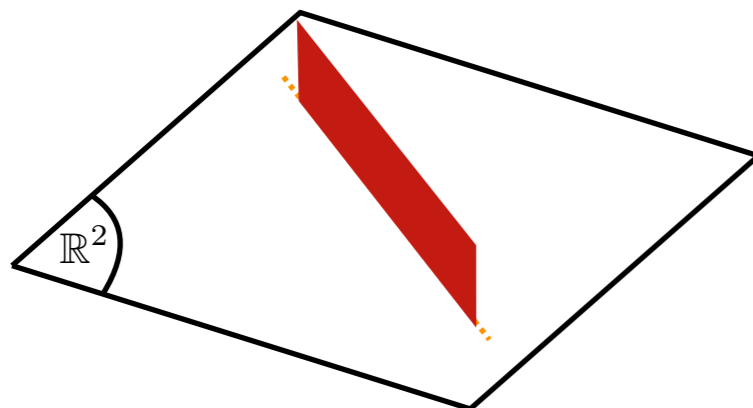
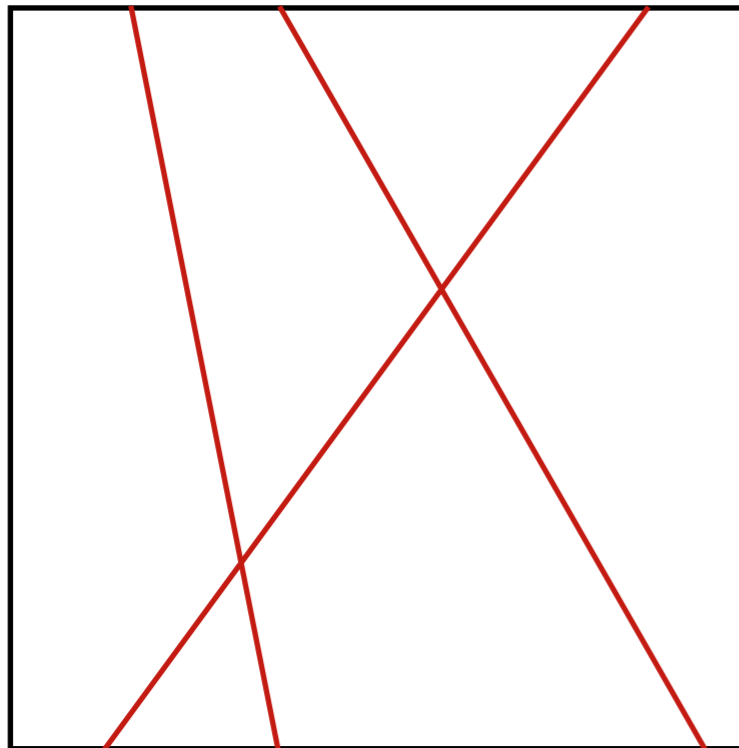
(*Fourier*)



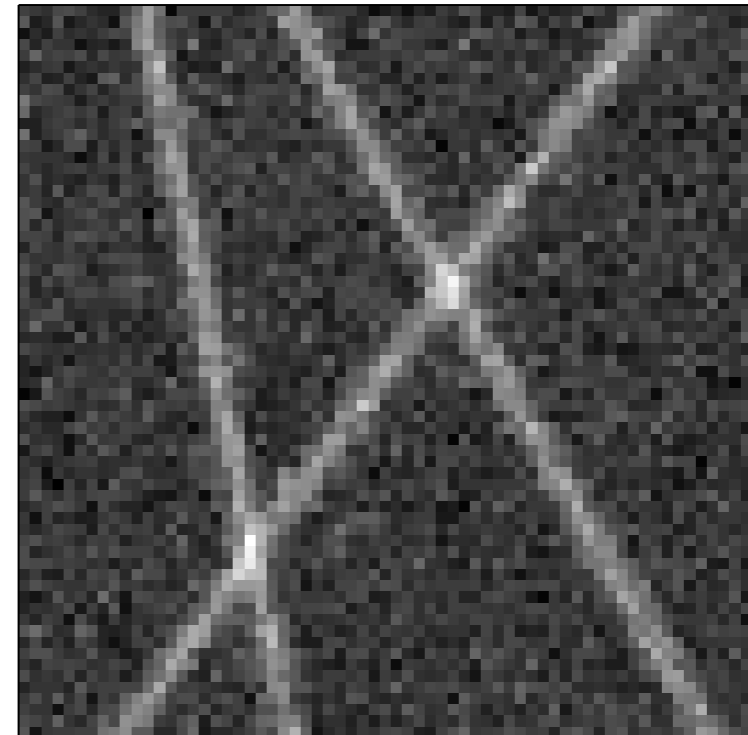
h bandlimited

Super-Resolution Detection of Lines

Objective f



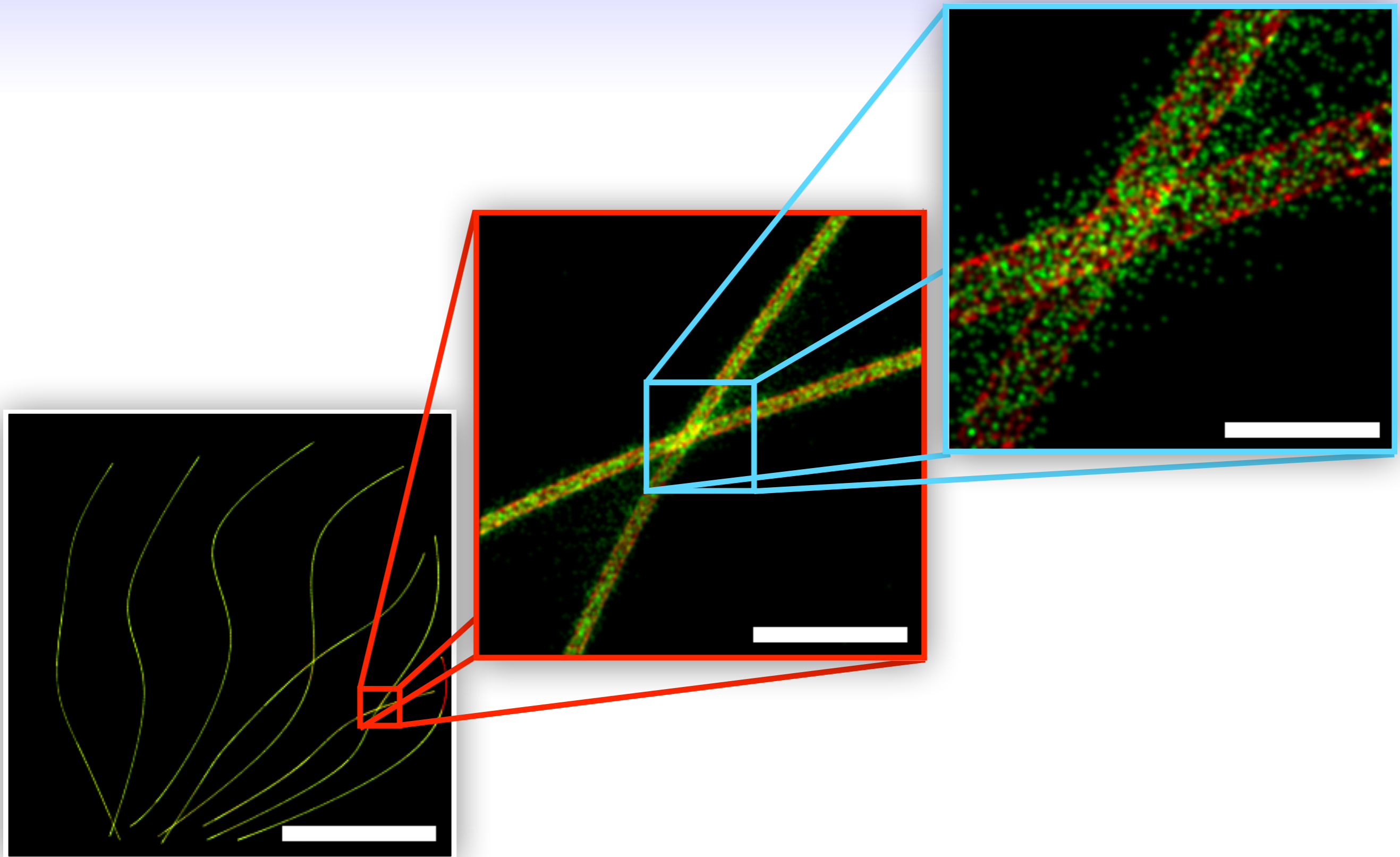
Data



$$f_{\text{obs}} = f * h + \varepsilon$$

+ Recovering
line parameters

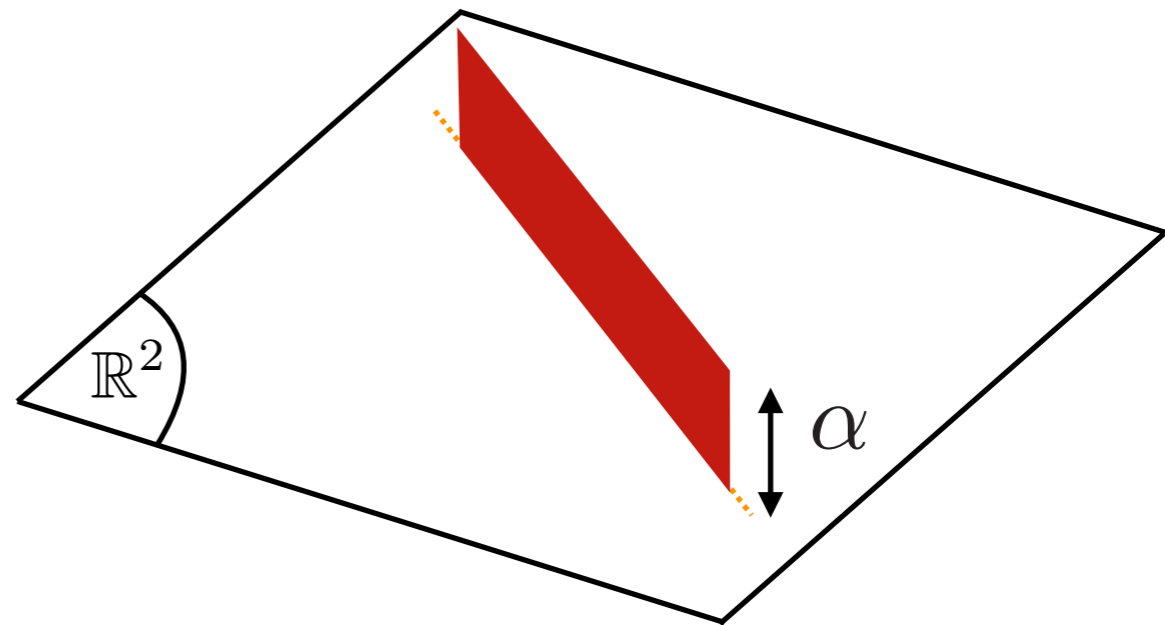
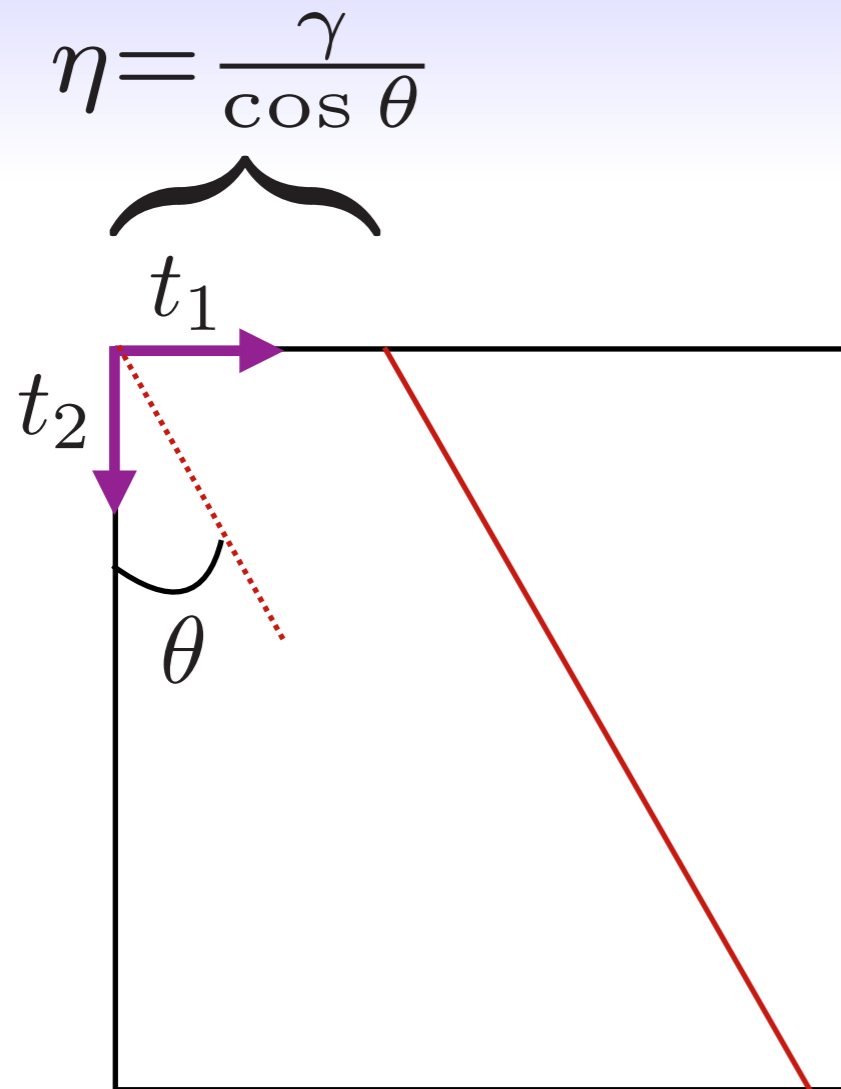
Motivation



Outline

- Introduction
 - Motivation
 - Super-Resolution principle
- **The model: ideal, blurred and sampled lines**
- Reconstruction steps
 - Derive an optimization problem from the framework of atomic norm and solve it with primal-dual algorithm
 - Recovering the line parameters by a Prony-like method
- Numerical experiments

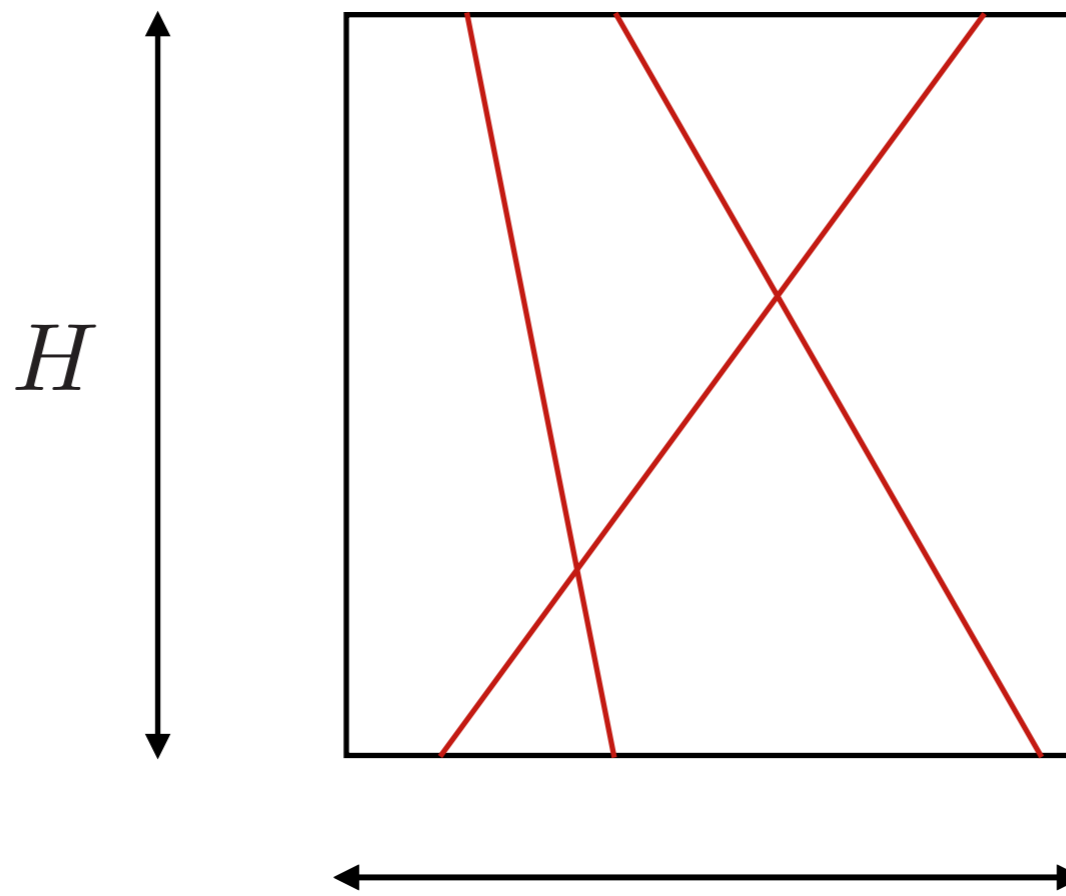
Problem formulation



$$x^\# : (t_1, t_2) \in \mathbb{P} \mapsto \alpha \delta(\cos(\theta)t_1 + \sin(\theta)t_2 - \gamma)$$

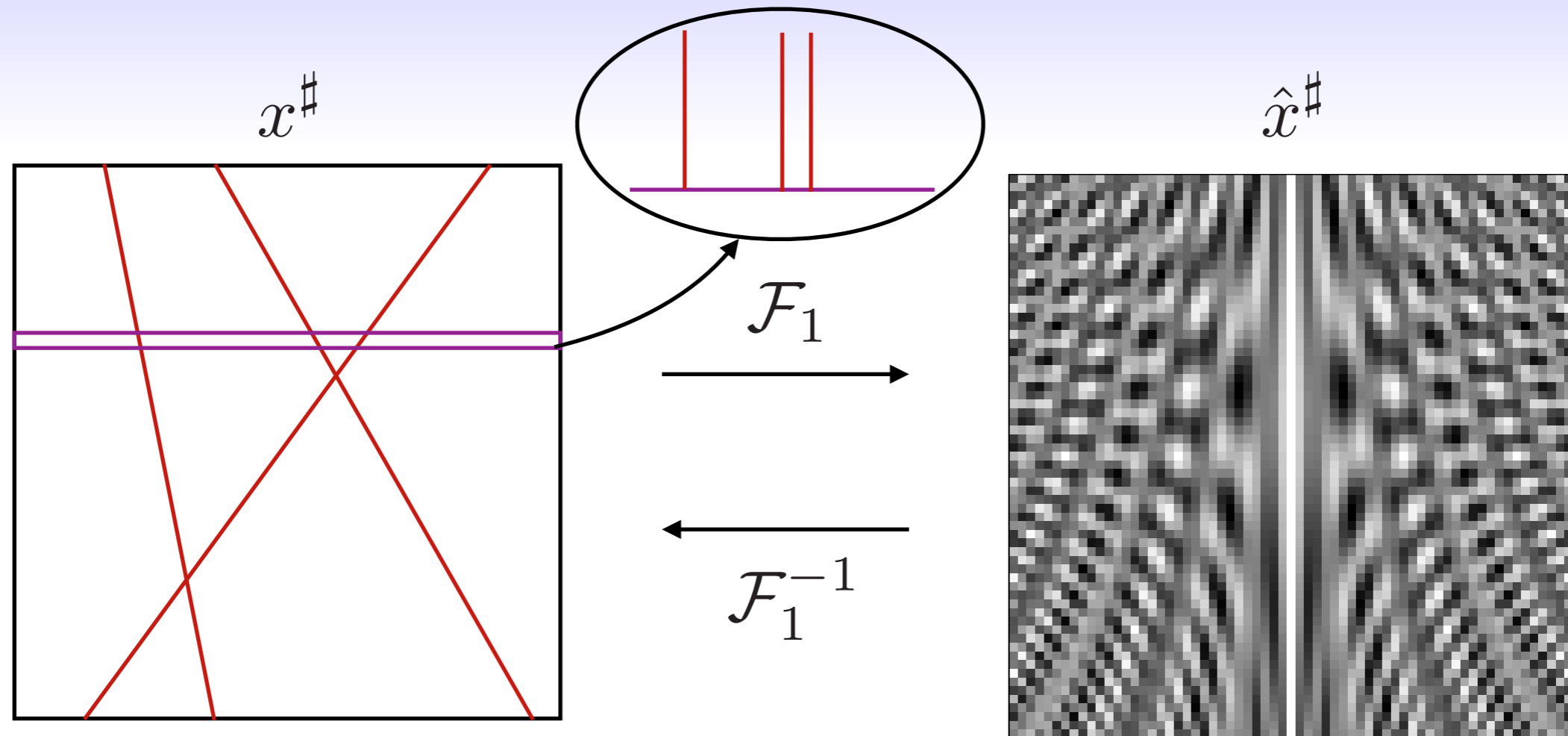
Problem formulation

$$K = 3$$



$$x^\# : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^K \alpha_k \delta(\cos(\theta_k)t_1 + \sin(\theta_k)t_2 - \gamma_k)$$

Spatial vs. Frequency domain



■ $x^\#(t_1, t_2) = \sum_{k=1}^K \alpha_k \delta(\cos(\theta_k)t_1 + \sin(\theta_k)t_2 - \gamma_k)$

■ $\hat{x}^\#[m, n_2] = \sum_{k=1}^K \frac{\alpha_k}{\cos \theta_k} e^{j2\pi(\tan(\theta_k)n_2 - \eta_k)m/W}$

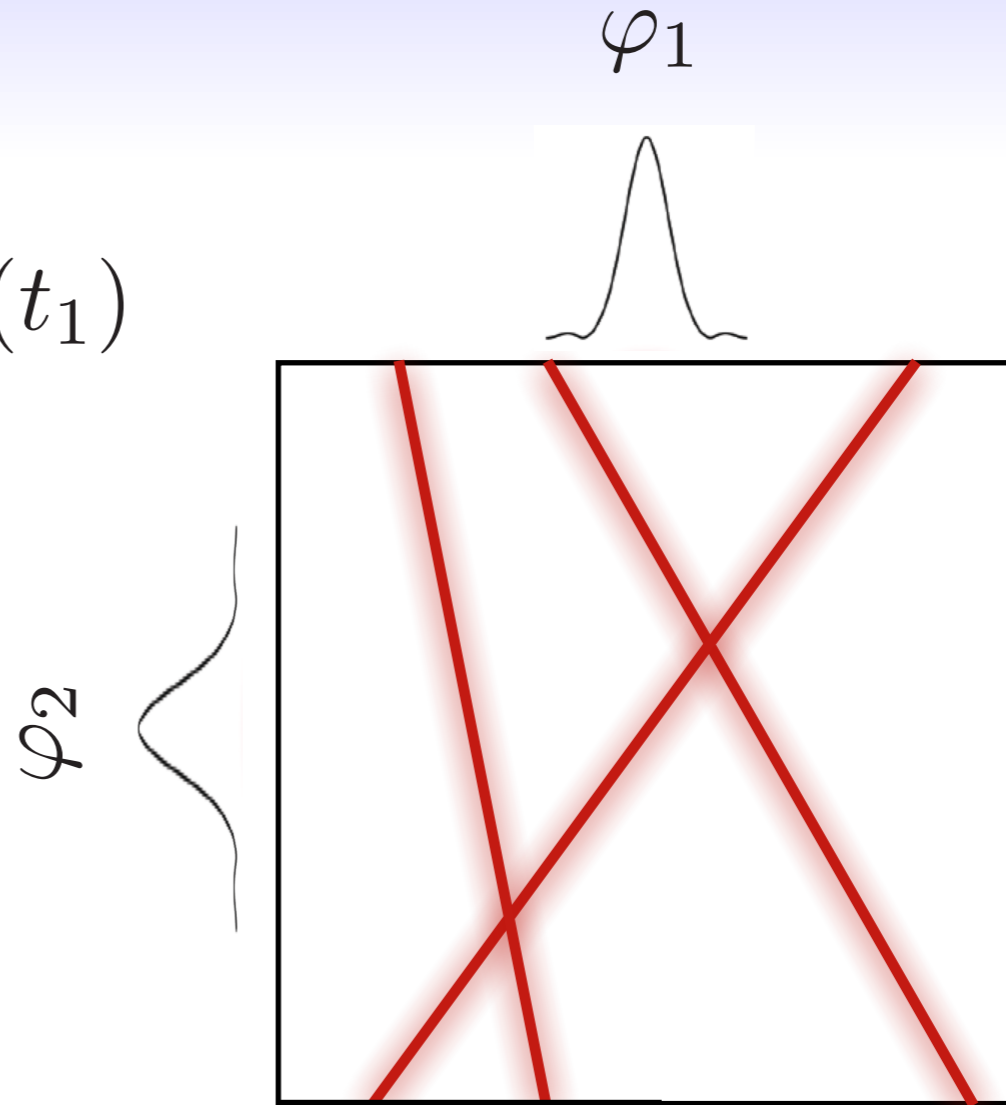
Problem formulation

Separable filter:

$$\phi(t_1, t_2) = \varphi_1(t_1)\delta(t_2) \cdot \varphi_2(t_2)\delta(t_1)$$



- Bandlimited $c_m(\varphi_1) = 0$
for $|m| \geq (W + 1)/2$
- $g[n] = \varphi_1(n)$



$$\psi_k = \left(\frac{1}{\cos(\theta_k)} \varphi_1 \left(\frac{\cdot}{\cos(\theta_k)} \right) \right) * \left(\frac{1}{\sin(\theta_k)} \varphi_2 \left(\frac{\cdot}{\sin(\theta_k)} \right) \right)$$

$$x^\# * \phi : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^K \alpha_k \psi_k (\cos(\theta_k)t_1 + \sin(\theta_k)t_2 + \gamma_k)$$

Problem formulation

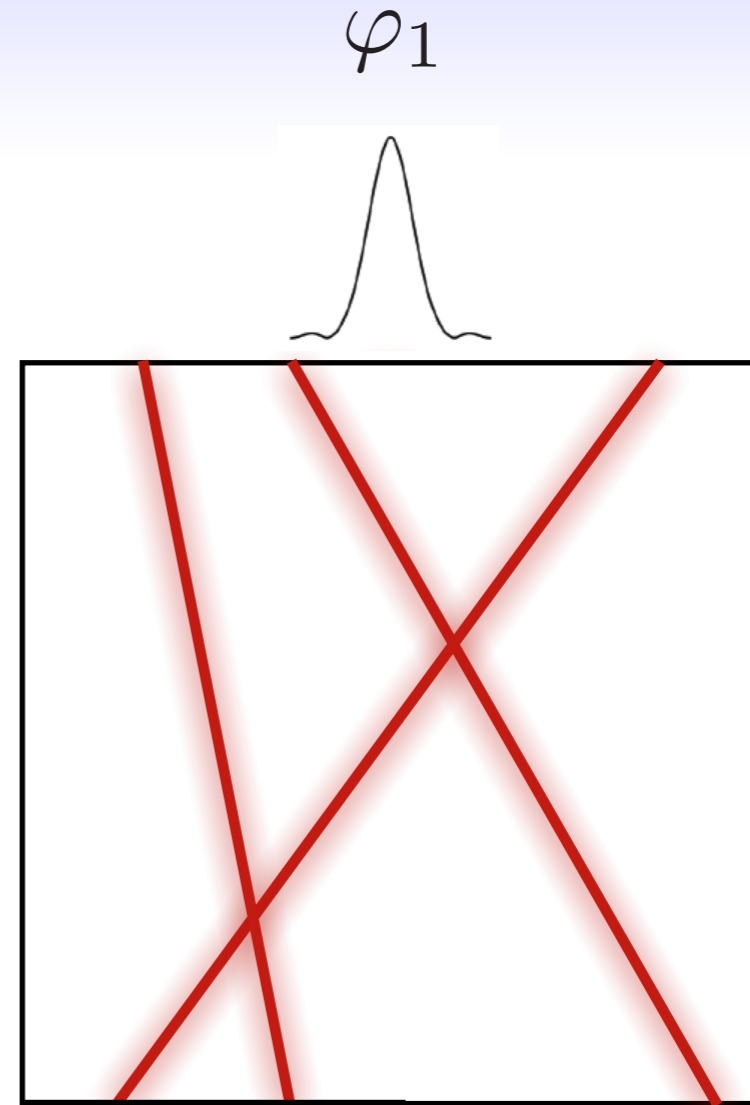
Separable filter:

$$\phi(t_1, t_2) = \varphi_1(t_1)\delta(t_2) \cdot \varphi_2(t_2)\delta(t_1)$$



- $(h[n] = (\varphi_2 * \text{sinc})(n))_{n \in \mathbb{Z}}$
 compact support of length
 $2S + 1$ for $S \in \mathbb{N}$

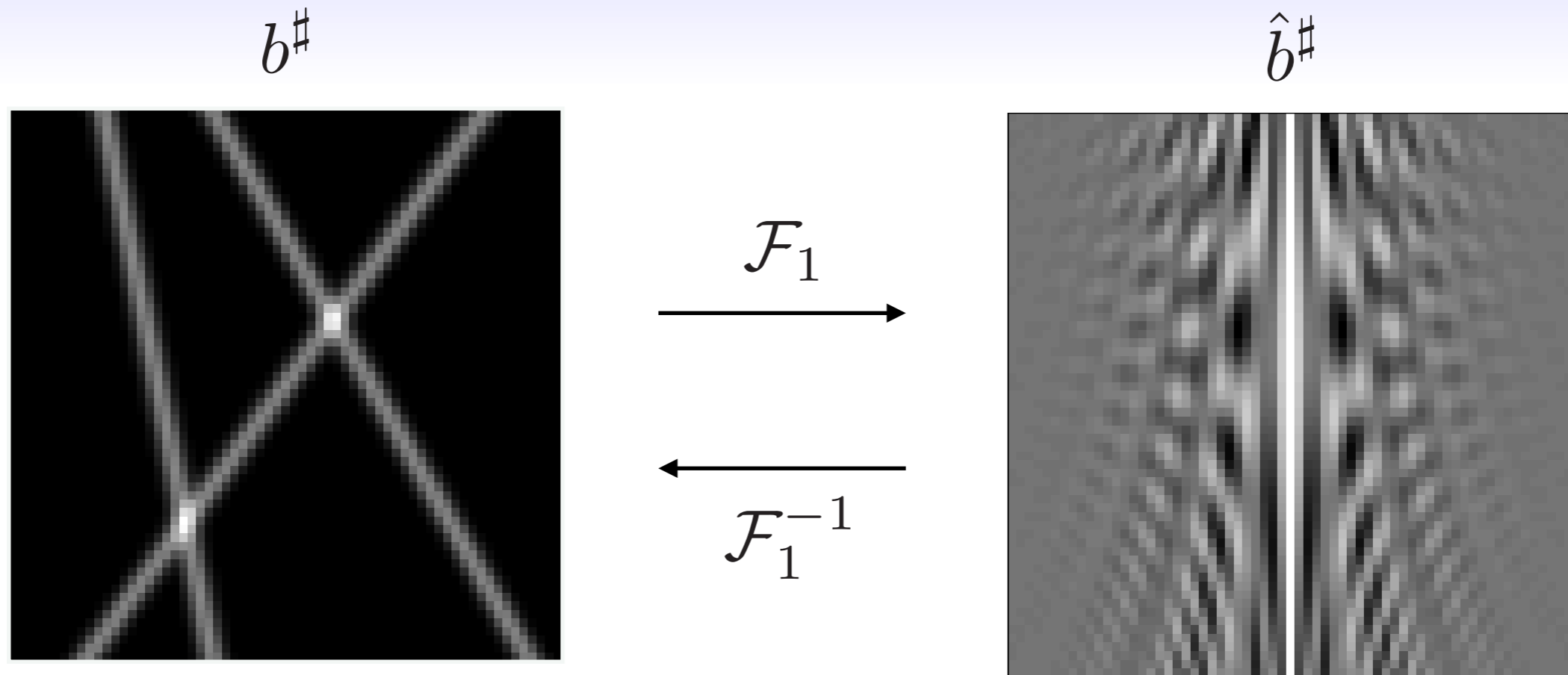
φ_2



$$\psi_k = \left(\frac{1}{\cos(\theta_k)} \varphi_1 \left(\frac{\cdot}{\cos(\theta_k)} \right) \right) * \left(\frac{1}{\sin(\theta_k)} \varphi_2 \left(\frac{\cdot}{\sin(\theta_k)} \right) \right)$$

$$x^\# * \phi : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^K \alpha_k \psi_k (\cos(\theta_k)t_1 + \sin(\theta_k)t_2 + \gamma_k)$$

Spatial vs. Frequency domain

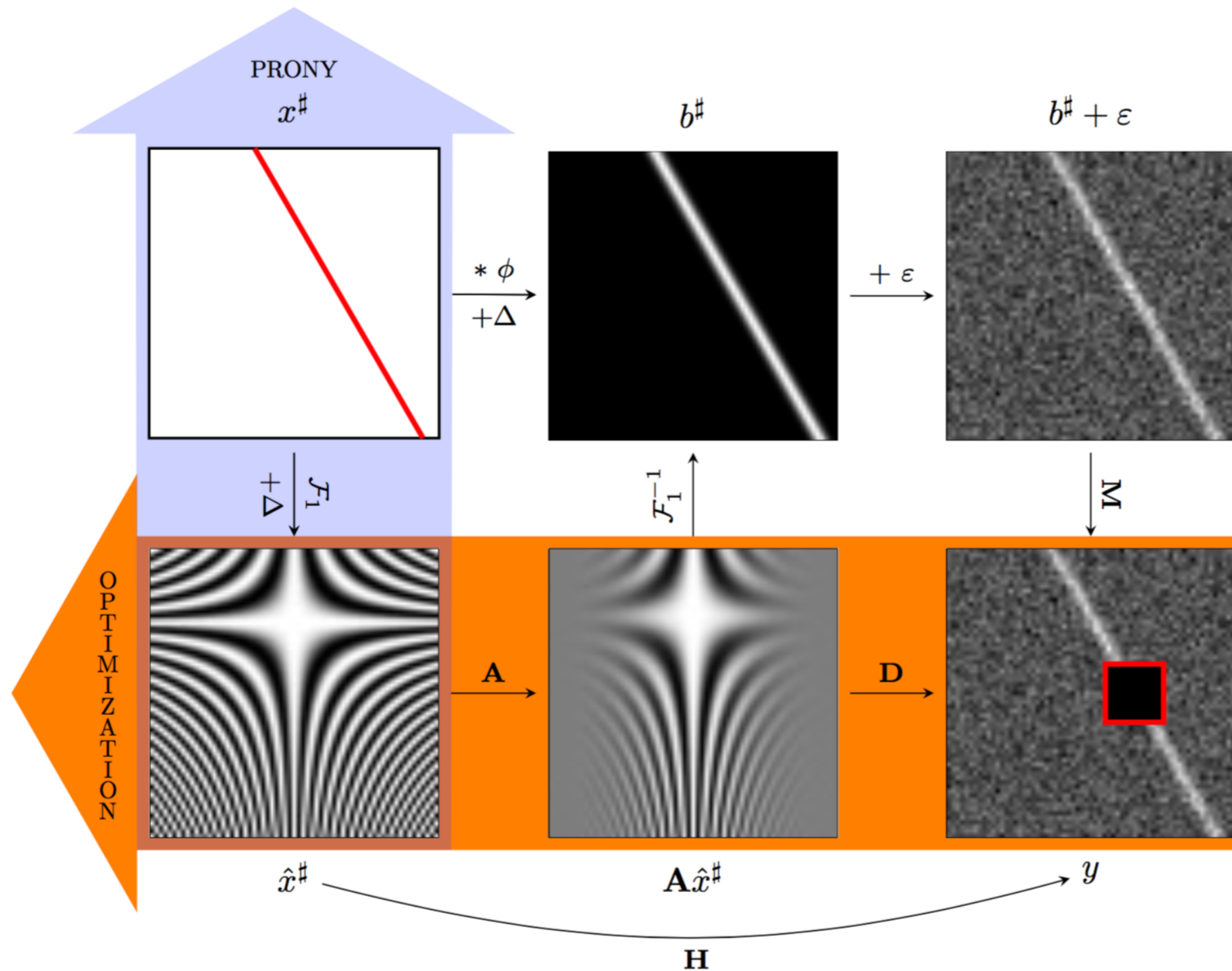


■ $b^\#[n_1, n_2] = (x^\# * \phi)(n_1, n_2)$

$$\mathbf{A}\hat{x}^\# = \hat{b}^\#$$

■ $\hat{b}^\#[m, n_2] = (\hat{g}[m]\hat{x}^\#[m, :]) * h$

Overview of the method



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Framework of atomic norm minimization

■ Simple model from atomic set \mathcal{A}

$$x = \sum_{i=1}^r c_i \mathbf{a}_i$$

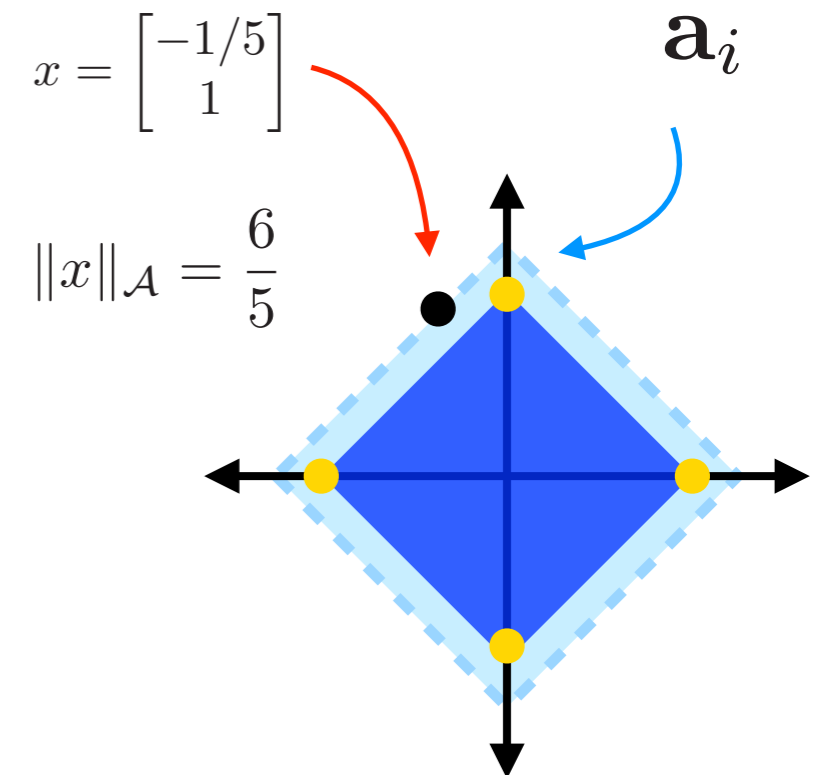
model → x
weights → c_i
atoms → \mathbf{a}_i
rank → r

dictionary
 $\mathbf{a}_i \in \mathcal{A}$
 $c_i \geq 0$

■ $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$

■ $\|x\|_{\mathcal{A}} = \inf \{t > 0 : x \in t\text{conv}(\mathcal{A})\}$

■ $\|x\|_{\mathcal{A}} = \inf_{\substack{c_i \geq 0 \\ \mathbf{a}_i \in \mathcal{A}}} \left\{ \sum_{i=1}^{|\mathcal{A}|} c_i : x = \sum_{i=1}^{|\mathcal{A}|} c_i \mathbf{a}_i \right\}$



Framework of atomic norm minimization

Two examples:

- Sparse vectors $\mathcal{A} = \{\pm e_i\}_{i=1}^N$

$\text{conv}(\mathcal{A}) = \text{cross-polytope}$

$$\|x\|_{\mathcal{A}} = \|x\|_1$$

- Low-rank matrices $\mathcal{A} = \{A : \text{rank}(A) = 1, \|A\|_F = 1\}_{i=1}^N$

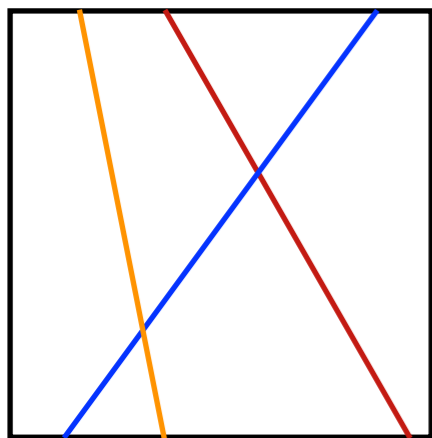
$\text{conv}(\mathcal{A}) = \text{nuclear norm ball}$

$$\|x\|_{\mathcal{A}} = \|x\|_*$$

Formulation as combination of atoms

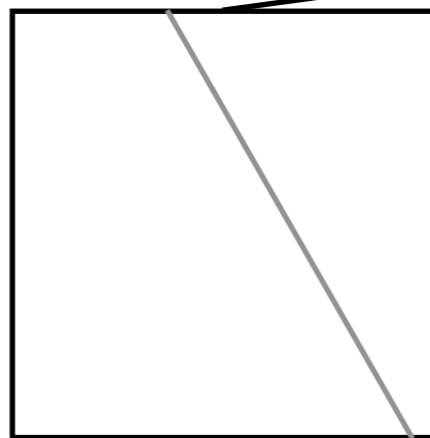
$$a(\theta_k, \eta_k) = \delta(\cos(\theta_k)(t_1 - \eta_k) + \sin(\theta_k)t_2)$$

$x^\#$

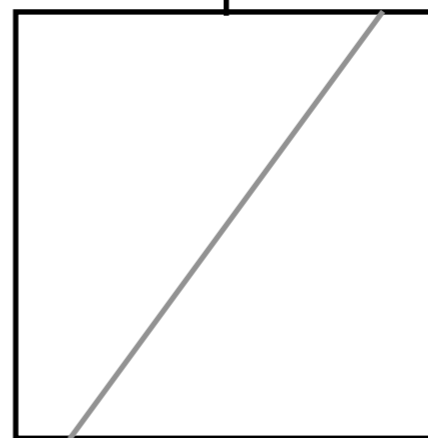


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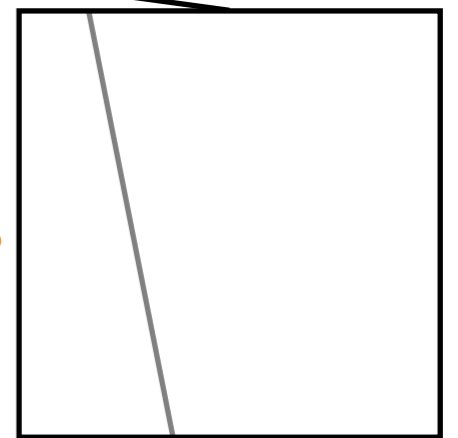
α_1



+ α_2



+ α_3

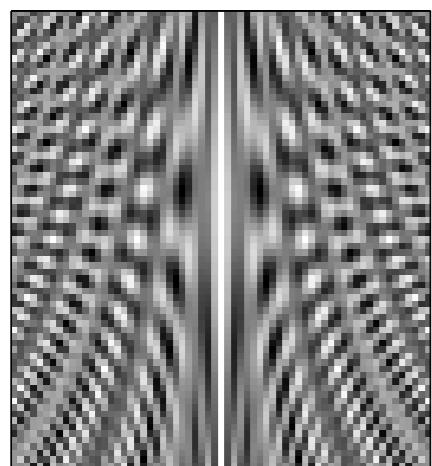


(θ_1, η_1)

(θ_2, η_2)

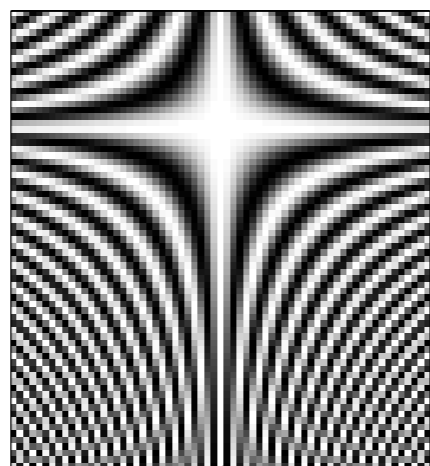
(θ_3, η_3)

$\hat{x}^\#$

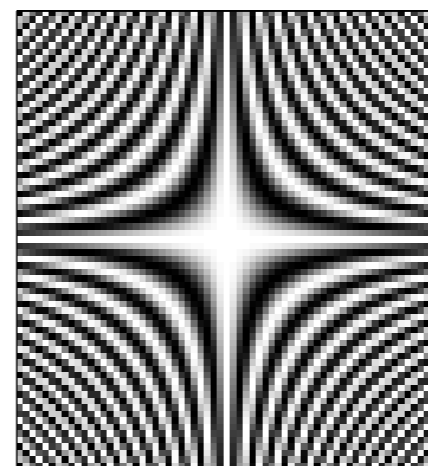


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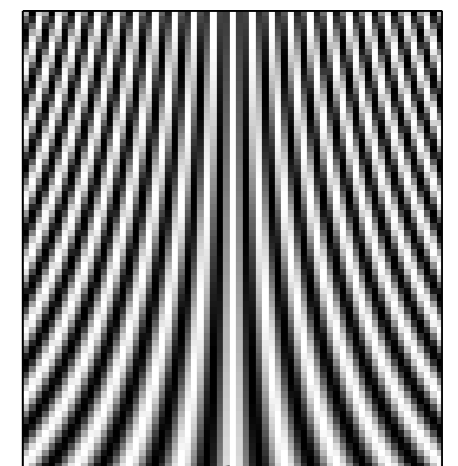
α_1



+ α_2



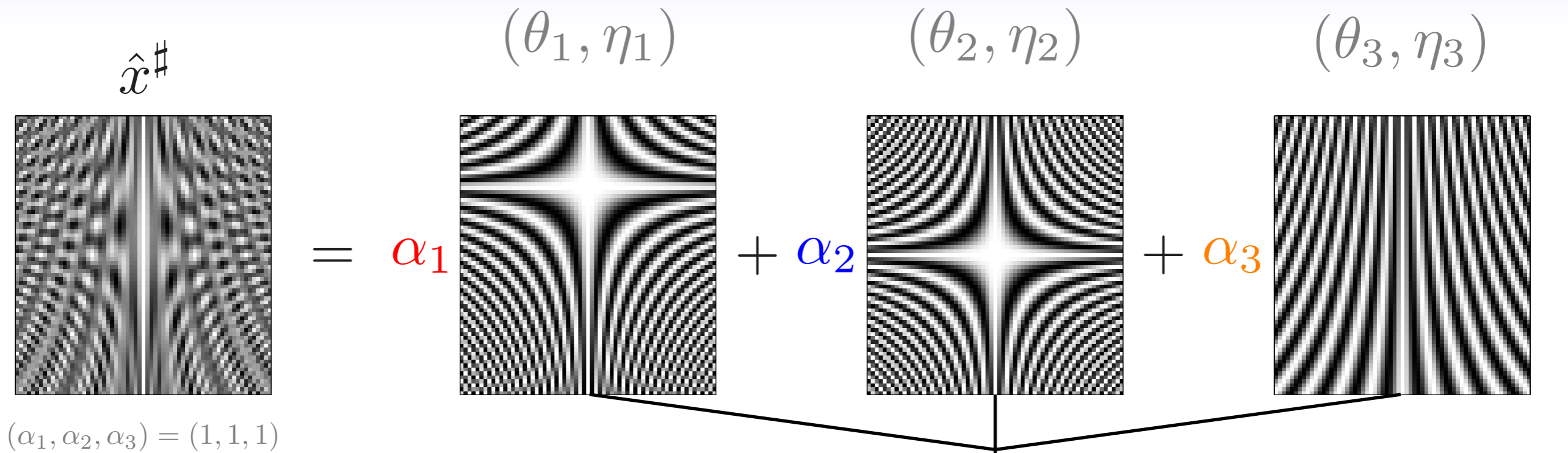
+ α_3



$(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$

$$\hat{a}(\theta_k, \eta_k) = \frac{1}{\cos \theta_k} e^{j2\pi(\tan(\theta_k)\eta_k - \eta_k)m/W}$$

Formulation as combination of atoms



$$\hat{a}(\theta_k, \eta_k) = \frac{1}{\cos \theta_k} e^{j2\pi(\tan(\theta_k)n_2 - \eta_k)m/W}$$

$$\mathcal{A} = \left\{ \hat{a}(\theta_k, \eta_k) : \theta_k \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right], \eta_k \in [-M, M] \right\}$$

$$\|\hat{x}\|_{\mathcal{A}} = \inf_{c'_k, \theta'_k, \eta'_k} \left\{ \sum_k c'_k : \hat{x} = \sum_k c'_k \hat{a}(\theta'_k, \eta'_k) \right\} \leq \sum_{k=1}^3 \alpha_k$$

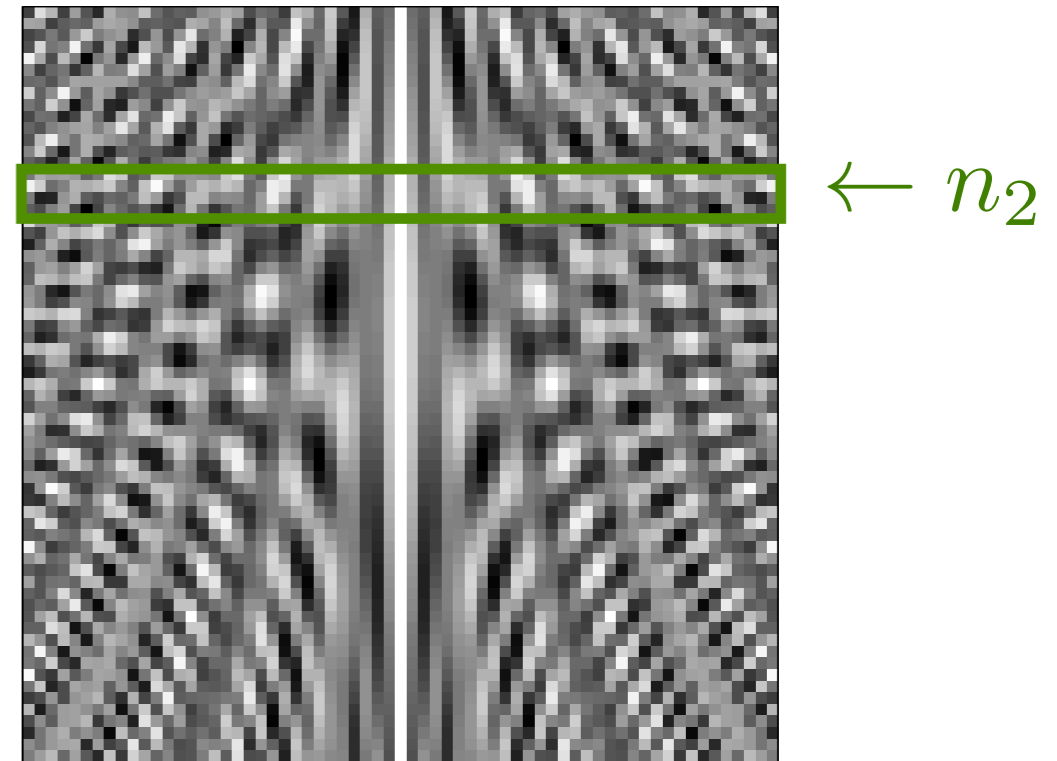
Formulation as combination of atoms

$$\mathcal{A} = \{a(f, \phi) \in \mathbb{C}^{|I|}, f \in [0, 1], \phi \in [0, 2\pi)\} \quad \hat{x}^\#$$

- $[a(f, \phi)]_i = e^{j(2\pi fi + \phi)}, i \in I$

- $[a(f)]_i = e^{j2\pi fi}, i \in I$

- $I = \{-M, \dots, M\}$



- $\hat{x}^\#[m, n_2] = \sum_{k=1}^K \frac{\alpha_k}{\cos \theta_k} e^{j2\pi(\tan(\theta_k)n_2 - \eta_k)m/W}$

- $\ell_{n_2}^\# = \hat{x}^\#[:, n_2] = \sum_{k=1}^K c_k a(f_{n_2, k})$

$$c_k = \frac{\alpha_k}{\cos \theta_k}$$

$$f_{n_2, k} = \frac{\tan(\theta_k)n_2 - \eta_k}{W}$$

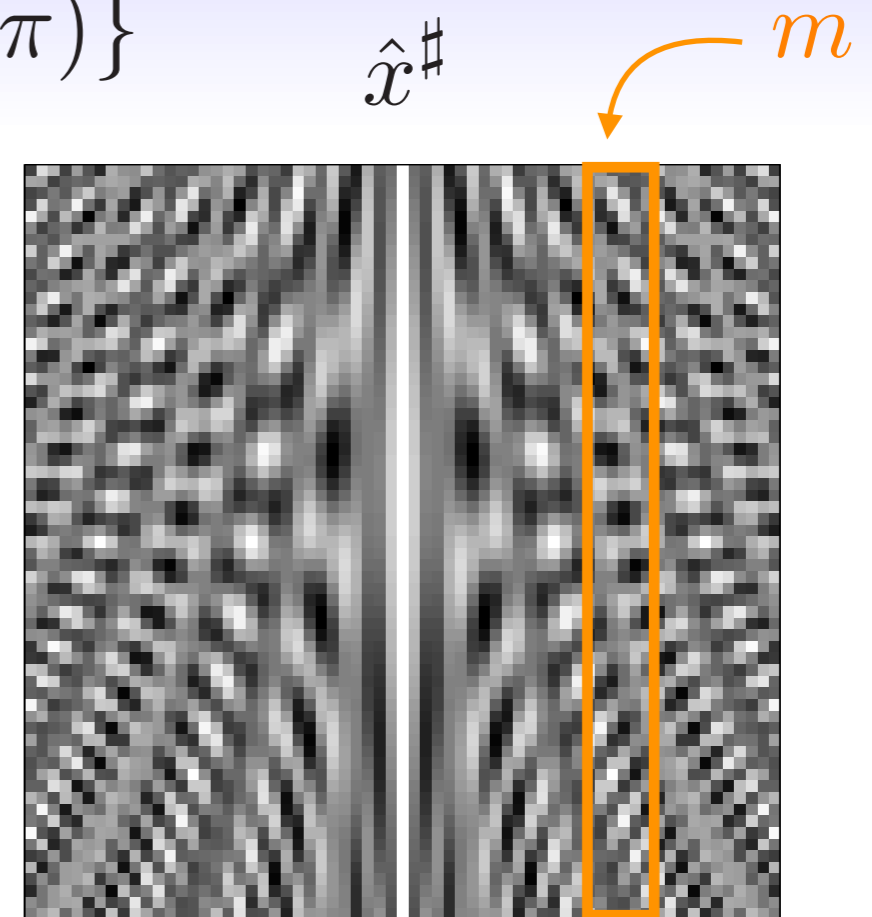
Formulation as combination of atoms

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- $[a(f, \phi)]_i = e^{j(2\pi fi + \phi)}, i \in I$

- $[a(f)]_i = e^{j2\pi fi}, i \in I$

- $I = \{0, \dots, H_S - 1\}$



- $\hat{x}^\# [m, n_2] = \sum_{k=1}^K \frac{\alpha_k}{\cos \theta_k} e^{j2\pi(\tan(\theta_k)n_2 - \eta_k)m/W}$

$$c_k = \frac{\alpha_k}{\cos \theta_k}$$

- $t_m^\# = \hat{x}^\# [m, :] = \sum_{k=1}^K c_k a(f_{m,k}, \phi_{m,k})^T$

$$f_{m,k} = \frac{\tan(\theta_k)m}{W}$$

$$\phi_{m,k} = -\frac{2\pi\eta_k m}{W}$$

Formulation as combination of atoms

$$\|x\|_{\mathcal{A}} = \inf_{\substack{c'_k \geq 0 \\ f'_k \in [0,1] \\ \phi'_k \in [0,2\pi)}} \left\{ \sum_k c'_k : x = \sum_k c'_k a(f'_k, \phi'_k) \right\}.$$

$$\blacksquare \ell_{n_2}^{\#} = \hat{x}^{\#}[:, n_2] = \sum_{k=1}^K c_k a(f_{n_2, k})$$

$$\blacksquare t_m^{\#} = \hat{x}^{\#}[m, :] = \sum_{k=1}^K c_k a(f_{m, k}, \phi_{m, k})^T$$

$$\|\ell_{n_2}^{\#}\|_{\mathcal{A}} ?$$

$$\|t_m^{\#}\|_{\mathcal{A}} ?$$

Caratheodory theorem

Toeplitz Operator:

$$\mathbf{T}_N : (x_0, \dots, x_{N-1}) \mapsto \begin{pmatrix} x_0 & x_1^* & \cdots & x_{N-1}^* \\ x_1 & x_0 & \cdots & x_{N-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ x_{N-1} & x_{N-2} & \cdots & x_0 \end{pmatrix}$$

- **Caratheodory theorem.** A vector x of length N is a positive combination ($c_k > 0$) of $K \leq N$ atoms $a(f_k)$ if and only if $\mathbf{T}_N(x) \succcurlyeq 0$ and of rank K . Also this decomposition is unique if $K < N$.

Atomic norm of the rows

$$\blacksquare \ell_{n_2}^\# = \hat{x}^\#[:, n_2] = \sum_{k=1}^K c_k a(f_{n_2, k})$$

$$\|\ell_{n_2}^\#\|_{\mathcal{A}} = \inf \left\{ \sum_k c'_k : \ell_{n_2}^\# = \sum_k c'_k a(f'_k, \phi'_k) \right\} = \sum_{k=1}^K c_k$$

- Caratheodory theorem.** A vector x of length N is a positive combination ($c_k > 0$) of $K \leq N$ atoms $a(f_k)$ if and only if $\mathbf{T}_N(x) \succcurlyeq 0$ and of rank K . Also this decomposition is **unique** if $K < N$.

Atomic norm of the columns

$$\blacksquare \quad t_m^\# = \hat{x}^\#[m, :] = \sum_{k=1}^K c_k a(f_{m,k}, \phi_{m,k})^T \quad c_k e^{j\phi_{m,k}} \in \mathbb{C}$$

$$\|t_m^\#\|_{\mathcal{A}} = \inf \left\{ \sum_k c'_k : t_m^\# = \sum_k c'_k a(f'_k, \phi'_k) \right\} \leq \sum_{k=1}^K c_k$$

- Caratheodory theorem.** A vector x of length N is a **positive combination** ($c_k > 0$) of $K \leq N$ atoms $a(f_k)$ if and only if $\mathbf{T}_N(x) \succcurlyeq 0$ and of rank K . Also this decomposition is unique if $K < N$.

Atomic norm via a semidefinite program

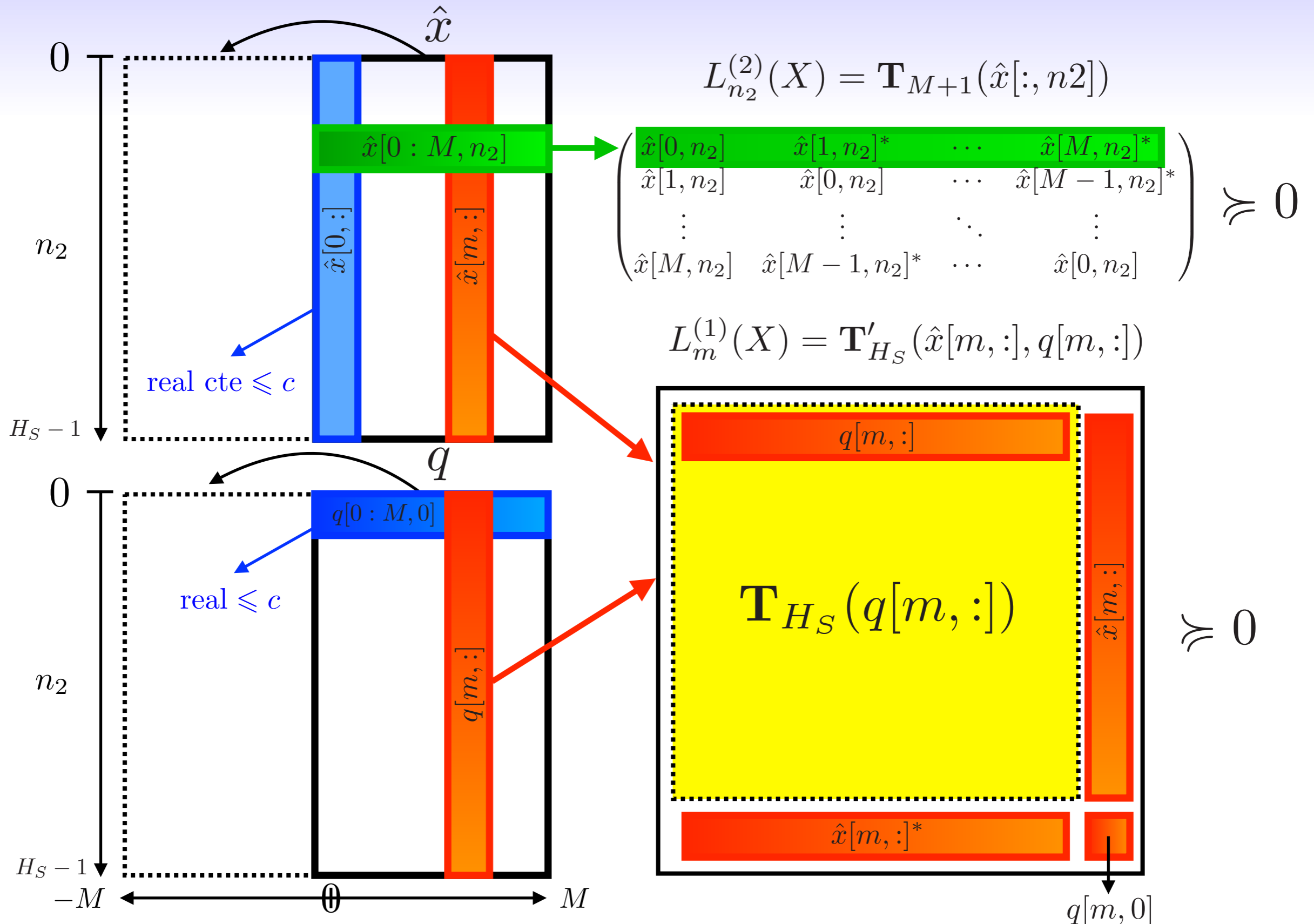
[Bhaskar et al.,2013]

■ **Proposition.** The atomic norm $\|x\|_{\mathcal{A}}$ can be characterized by the following semidefinite program:

$$\|x\|_{\mathcal{A}} = \min_{q \in \mathbb{C}^N} \left\{ q_0 : \underbrace{\begin{bmatrix} \mathbf{T}_N(q) & x \\ x^* & q_0 \end{bmatrix}}_{\mathbf{T}'_N(x,q)} \succcurlyeq 0 \right\}.$$

$$\|t_m^\# \|_{\mathcal{A}} = \min_{q_m \in \mathbb{C}^N} \left\{ q_m[0] : \mathbf{T}'_{H_S}(t_m^\#, q_m) \succcurlyeq 0 \right\}.$$

Constraints



Minimization problem

$$\tilde{x} \in \arg \min_{\hat{x}, q \in \mathcal{X} \times \mathcal{Q}} \frac{1}{2} \|\mathbf{A}\hat{x} - \hat{y}\|^2,$$

Subject to

$$\left\{ \begin{array}{l} \forall n_2 = 0, \dots, H_S - 1, \forall m = 0, \dots, M, \\ \hat{x}[0, n_2] = \hat{x}[0, 0] \leq c, \\ q[m, 0] \leq c, \\ \mathbf{T}'_{H_S}(\hat{x}[m, :], q[m, :]) \succcurlyeq 0, \\ \mathbf{T}_{M+1}(\hat{x}[:, n_2]) \succcurlyeq 0, \end{array} \right.$$

Primal-dual algorithm

[Condat, A primal-dual splitting method

for convex optimization involving Lipschitzian,
proximable and linear composite terms, 2013]

■ **condition:** $\tau\sigma \|\mathbf{L}\|^2 = 1$

Algorithm 1 Primal–dual splitting algorithm

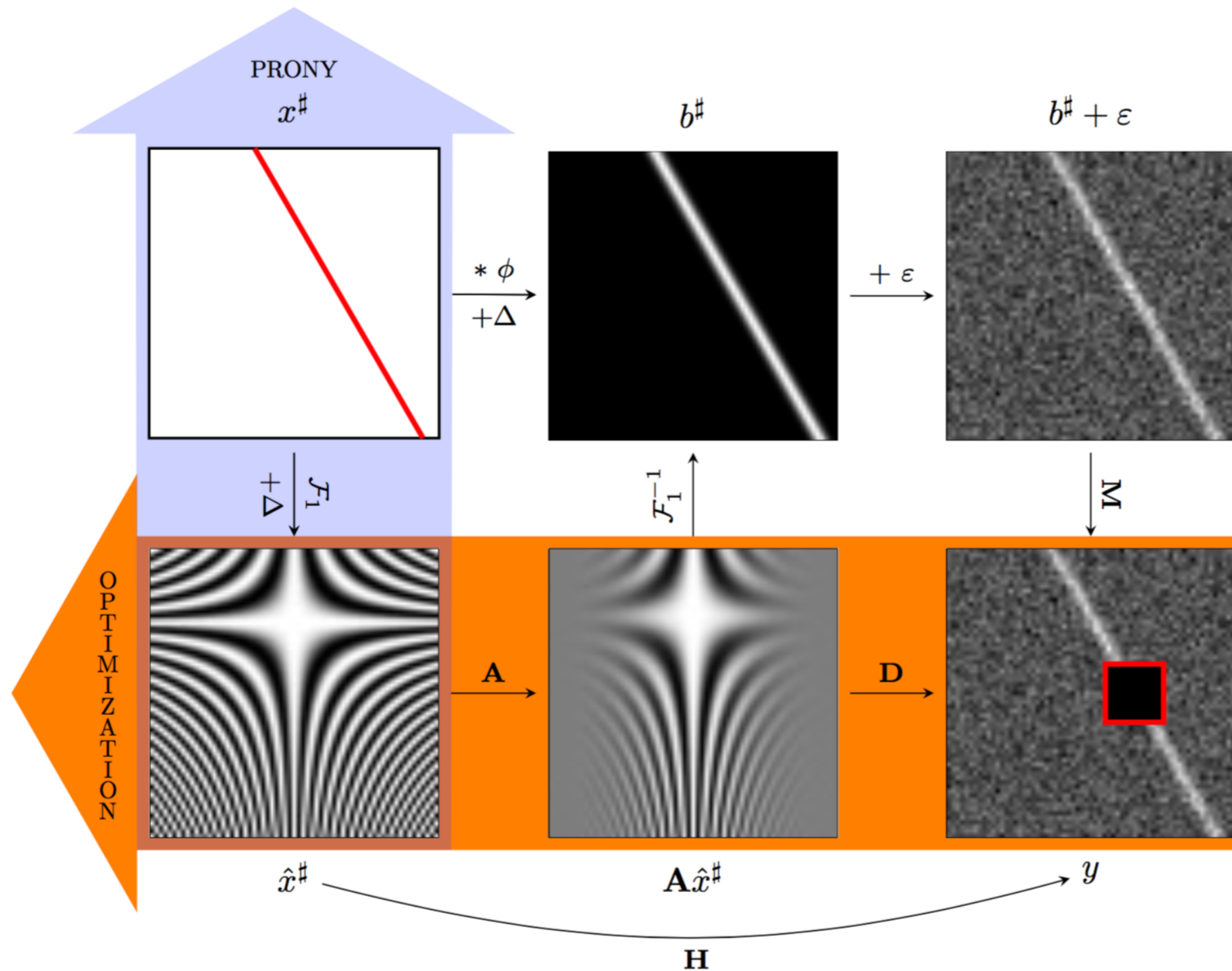
Input: \hat{y} 1D FFT of the blurred and noisy data image y

Output: \tilde{x} solution of the optimization problem

- 1: Initialize all primal and dual variables to zero
- 2: **for** $n = 1$ **to** Number of iterations **do**
- 3: $\tilde{X}_{n+1} = \text{prox}_{\tau G}(X_n - \tau \sum_{i=0}^{N-1} L_i^* \xi_{i,n}),$
- 4: $X_{n+1} = 2\tilde{X}_{n+1} - X_n$
- 5: **for** $i = 0$ **to** $N - 1$ **do**
- 6: $\tilde{\xi}_{i,n+1} = \text{prox}_{\sigma H_i^*}(\xi_{i,n} + \sigma L_i(2X_{n+1} - X_n)),$
- 7: **end for**
- 8: $\xi_{i,n+1} = 2\tilde{\xi}_{i,n+1} - \xi_{i,n}$
- 9: **end for**

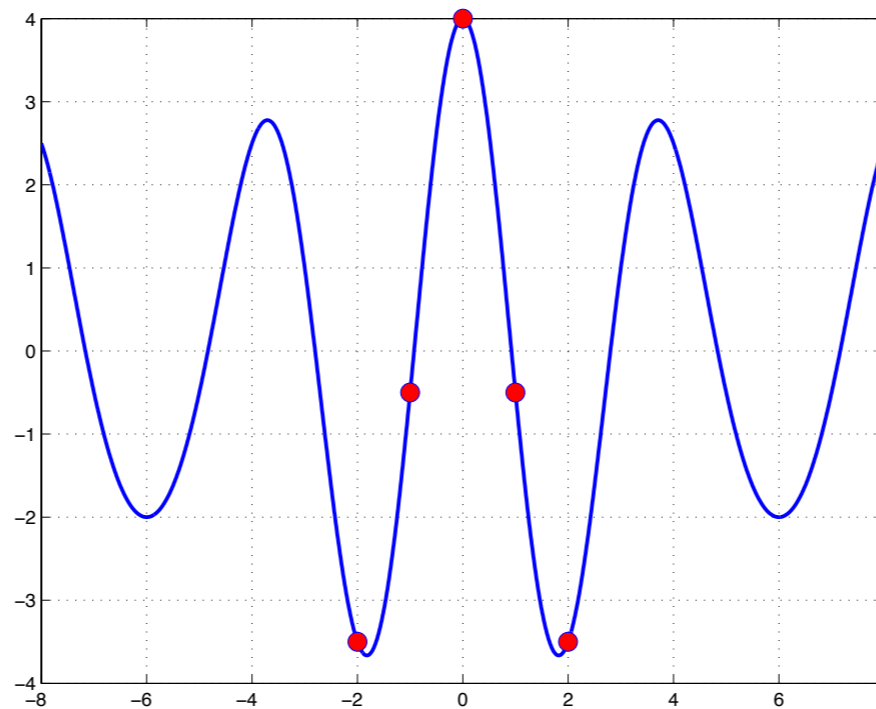
converges toward a solution $(\tilde{X}, \tilde{\xi}_0, \dots, \tilde{\xi}_{N-1})$ of the problem.

Overview of the method



Spectral estimation

$$x(t) = \overbrace{3 \cdot \exp(i2\pi \frac{1}{4}t)}^{x_1(t)} + \overbrace{1 \cdot \exp(i2\pi \frac{1}{3}t)}^{x_2(t)}$$



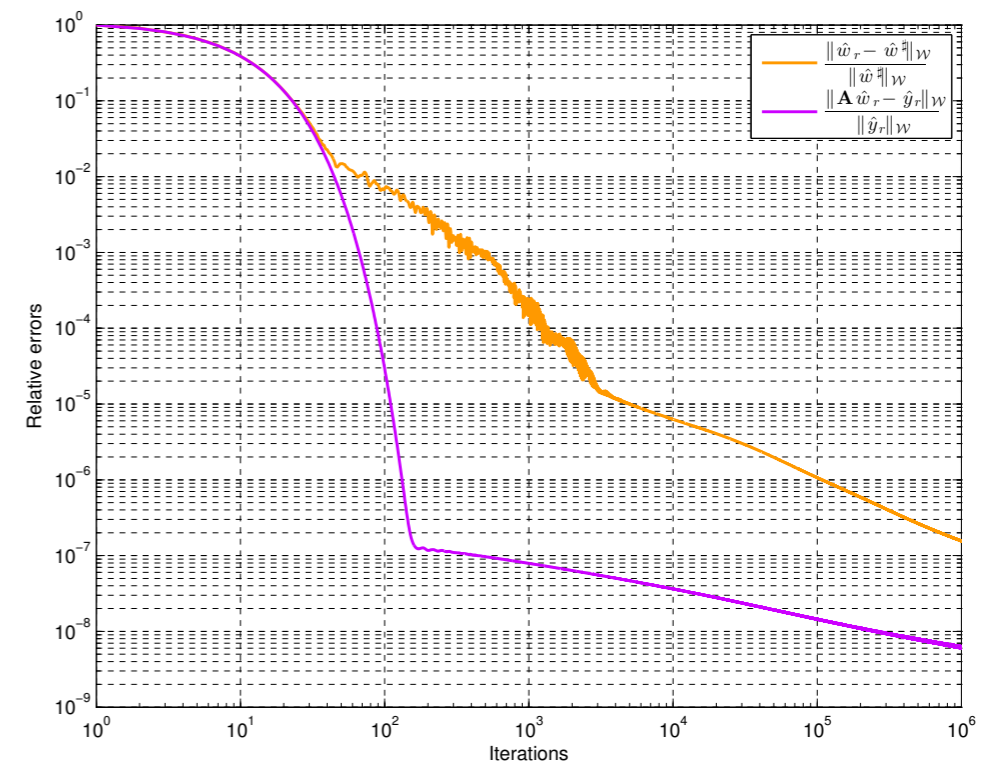
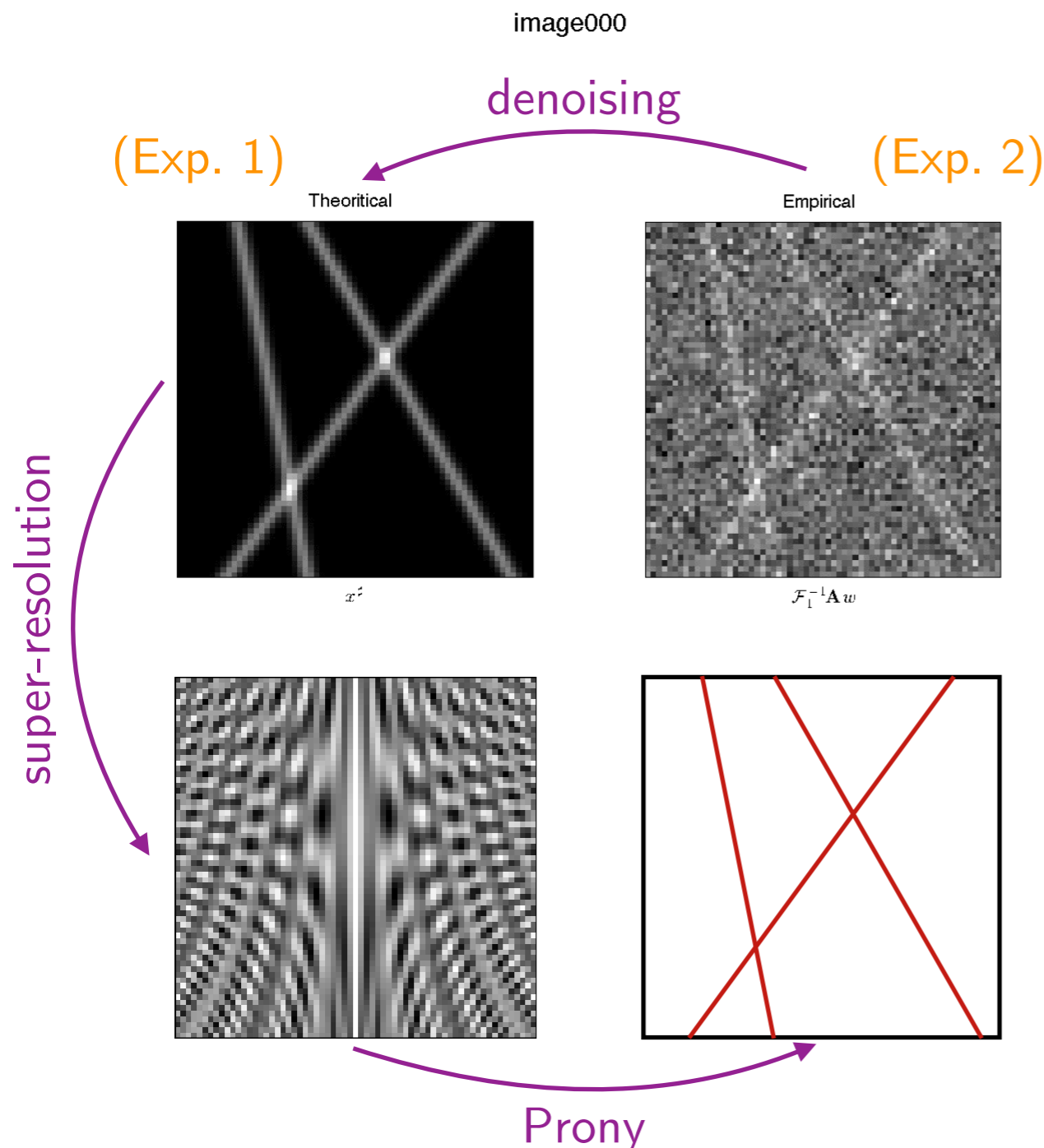
$$\{x_m\}_{m=-M}^M = x(-M : M)$$

$$M = 2$$

Outline

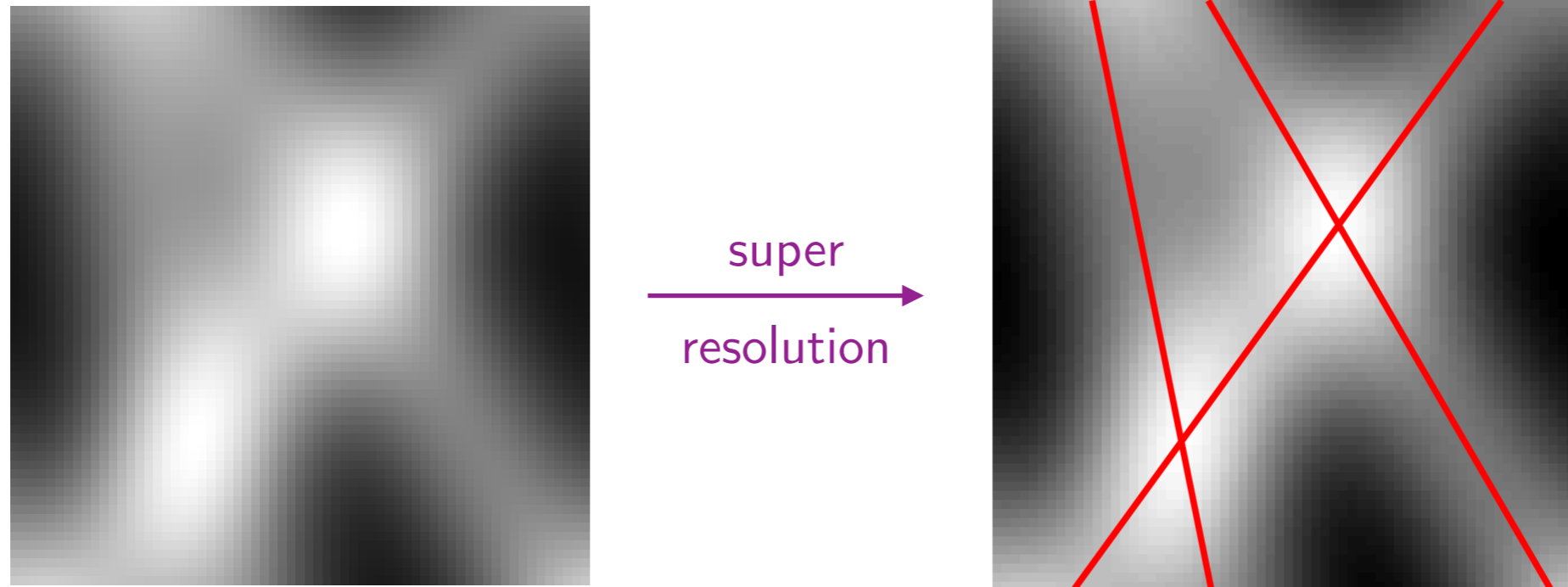
- Introduction
 - Motivation
 - Super-Resolution principle
- The model: ideal, blurred and sampled lines
- Reconstruction steps
 - Derive an optimization problem from the framework of atomic norm and solve it with primal-dual algorithm
 - Recovering the line parameters by a Prony-like method
- **Numerical experiments**

Numerical experiments



	Experiment 1	Experiment 2
Δ_θ/θ	$(10^{-7}, 3.10^{-6}, 7.10^{-7})$	$(10^{-2}, 6.10^{-2}, 9.10^{-2})$
Δ_α/α	$(10^{-7}, 10^{-7}, 10^{-7})$	$(10^{-2}, 9.10^{-2}, 2.10^{-1})$
Δ_η	$(4.10^{-6}, 7.10^{-6}, 7.10^{-6})$	$(5.10^{-2}, 4.10^{-2}, 3.10^{-2})$

Numerical experiments



Experiment 3	
Δ_{θ}/θ	$(6.10^{-7}, 9.10^{-5}, 8.10^{-6})$
Δ_{α}/α	$(4.10^{-5}, 2.10^{-5}, 2.10^{-5})$
Δ_{η}	$(5.10^{-5}, 10^{-4}, 3.10^{-4})$

Conclusion and future work

■ Conclusion

- Model of lines super-resolution based on the atomic norm.
- Solving the optimization problem by primal-dual algorithm.
- Estimating the line parameters by Prony method.

■ Future work

- Application to others inverse problems (inpainting).
- Use local patch to deal with curves super-resolution.

Bibliography

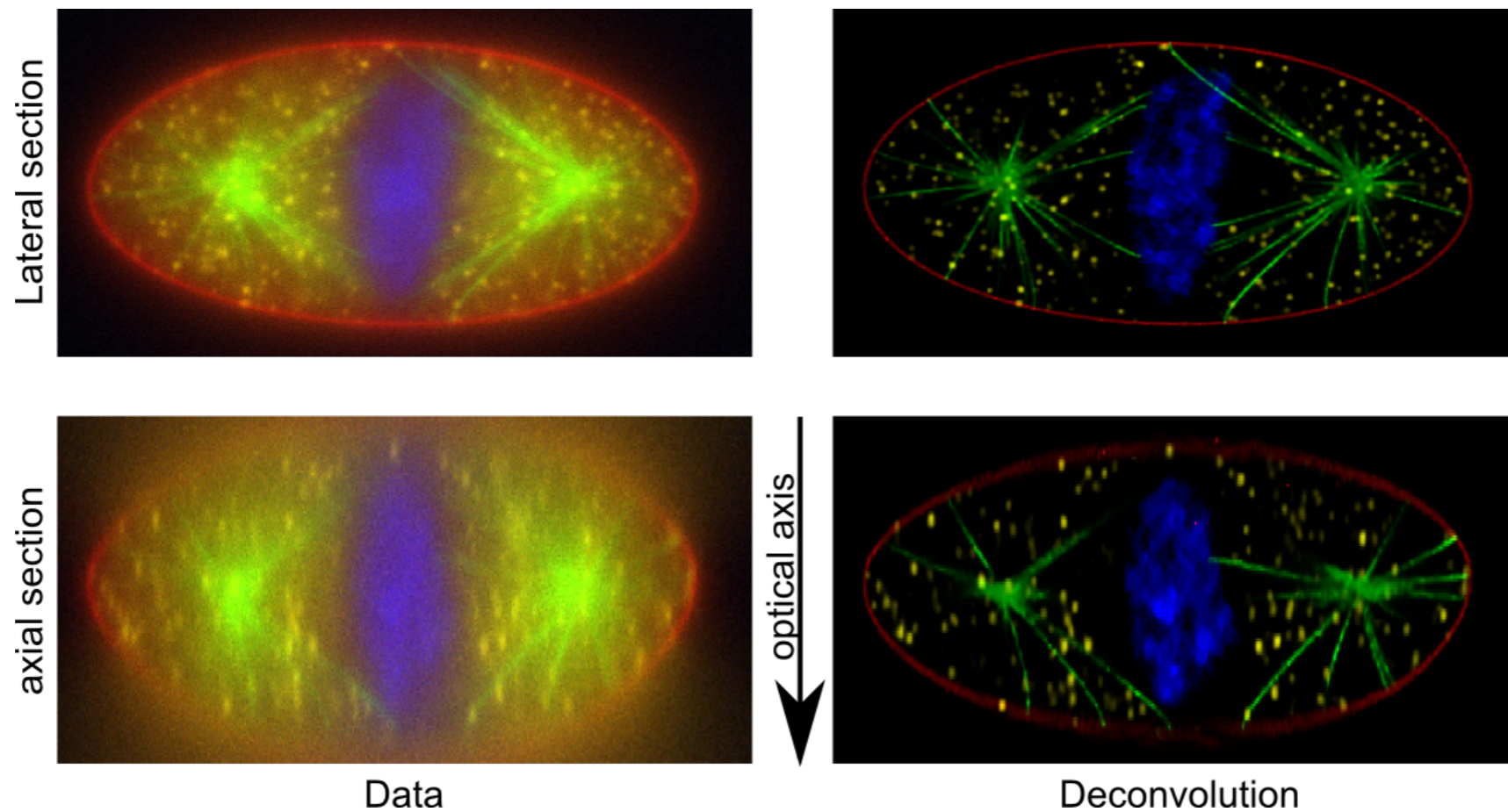
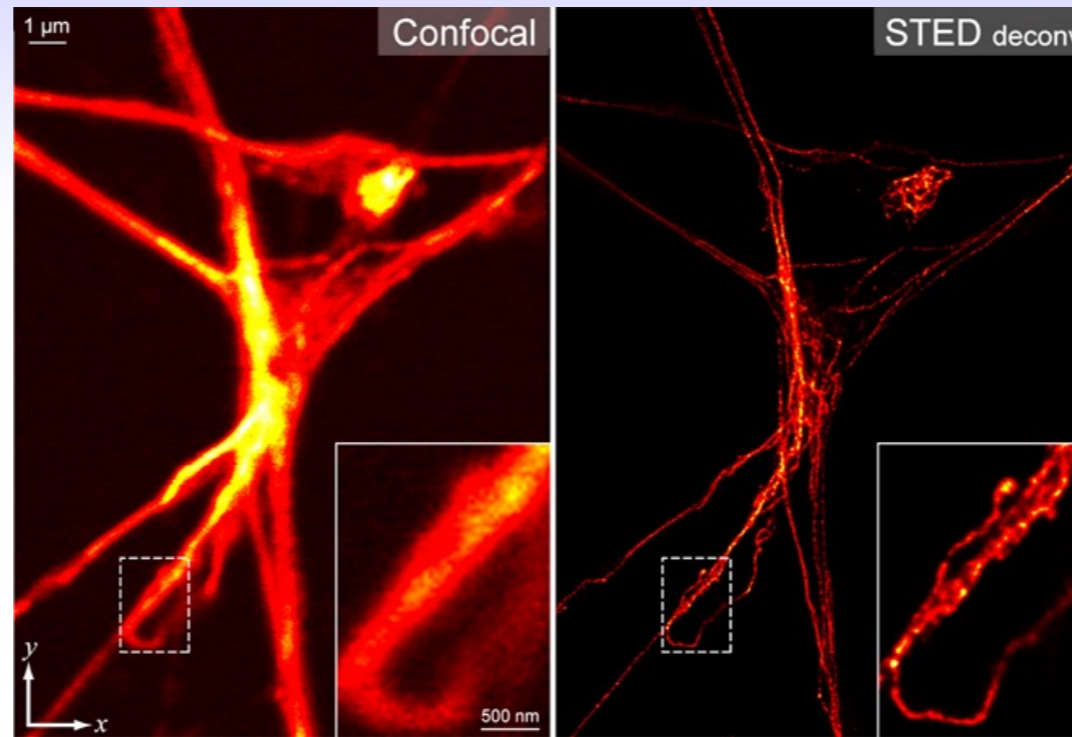
■ Selected papers

- E. J. Candès, and C. Fernandez-Granda, "Toward a mathematical theory of super-resolution", *Communications on Pure and Applied Mathematics*, 2014.
- B. N. Bhaskar, G. Tang, and B. Recht, "Atomic norm denoising with applications to line spectral estimation", *IEEE TIP*, 2013.
- G. Tang, B. N. Bhaskar, P. Shah, and B. Recht, "Compressed sensing off the grid", *Information Theory, IEEE Transactions on*, 2013.
- L. Condat, "A primal-dual splitting method for convex optimization involving lipschitzian, proximal and linear composite terms", *Journal of Optimization Theory and Applications*, 2013.

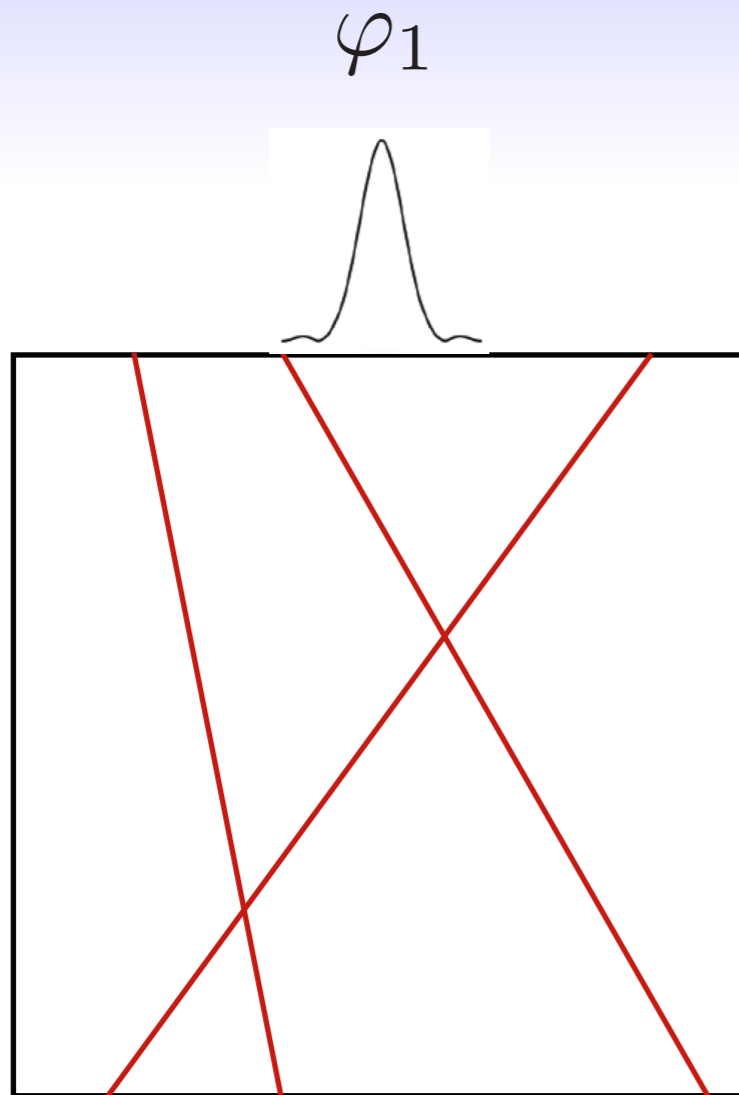
Questions ?

Thank you for your attention.

Motivation

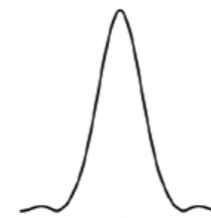


Problem formulation



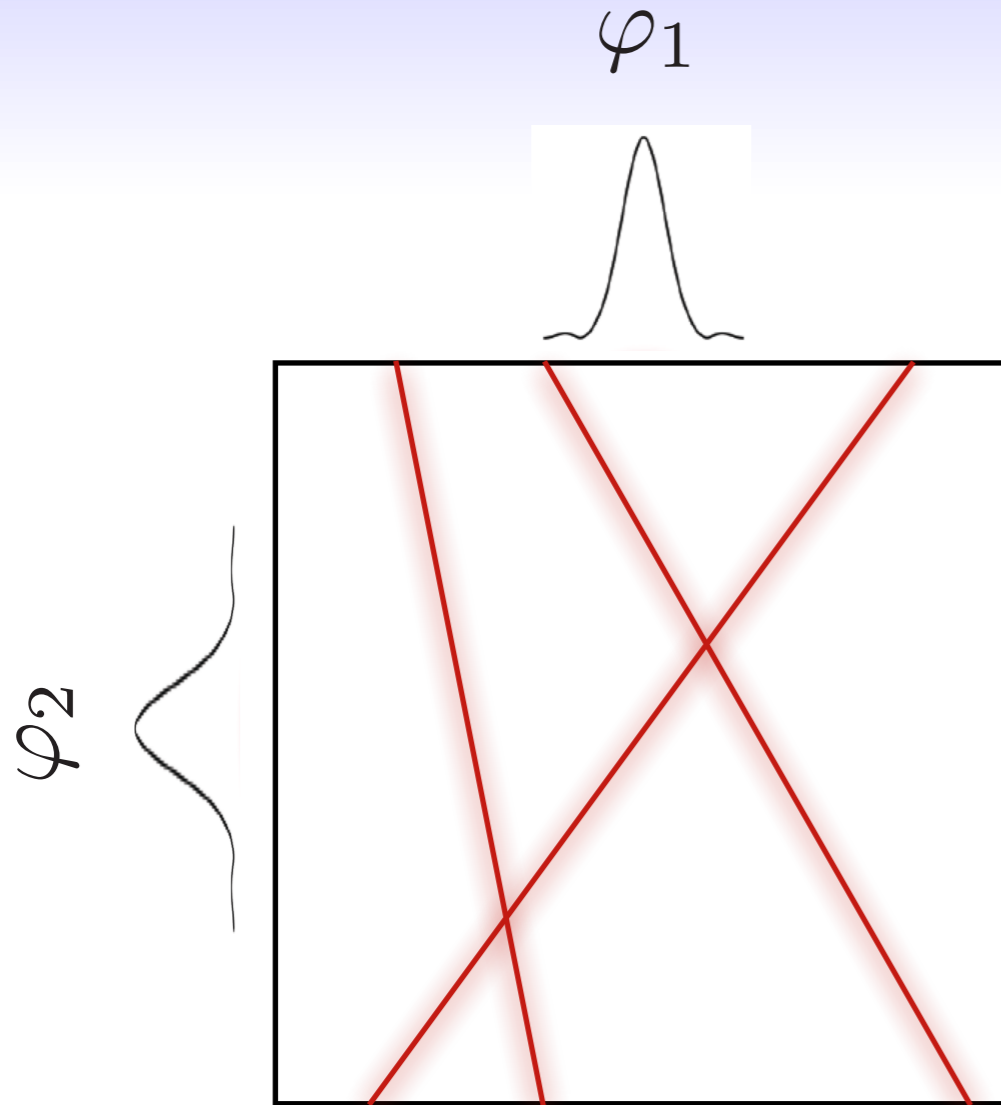
■ Horizontal convolution

$$u^\# = x^\# * \phi_1$$



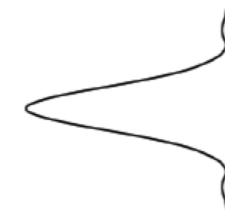
$$x^\# : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^K \alpha_k \delta(\cos(\theta_k)t_1 + \sin(\theta_k)t_2 - \gamma_k)$$

Problem formulation



Vertical convolution

$$x^\# * \phi = u^\# * \phi_2$$



$$u^\# : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^K \frac{\alpha_k}{\cos(\theta_k)} \varphi_1 \left(t_1 + \tan(\theta_k) t_2 + \frac{\gamma_k}{\cos(\theta_k)} \right)$$

Blur model

- Horizontal blur $\varphi_1 \in L^1([0, W))$ Discrete filter
■ W – periodic $g[n] = \varphi_1(n)$
■ Bandlimited $c_m(\varphi_1) = 0$ for $|m| \geq (W + 1)/2$
■ $\int_0^W \varphi_1 = 1$
- Vertical blur $\varphi_2 \in L^1(\mathbb{R})$
■ Discrete filter $(h[n] = (\varphi_2 * \text{sinc})(n))_{n \in \mathbb{Z}}$ has compact support of length $2S + 1$ for $S \in \mathbb{N}$
■ $\int_{\mathbb{R}} \varphi_2 = 1$

Problem formulation

Data generation size: W, H blur: g, h filter size: S

lines: $K, \{\theta_k, \alpha_k, \eta_k\}$ noise: $\varepsilon \sim \mathcal{N}(0, \zeta^2)$

$$\textcircled{1} \quad \varphi_1(t_1) = \sum_{n=0}^{W-1} g[n] \frac{\sin(\pi(t_1 - n))}{W \sin(\pi(t_1 - n)/W)}$$

$$\textcircled{2} \quad u^\# [n_1, n_2] = \sum_{k=1}^K \frac{\alpha_k}{\cos(\theta_k)} \varphi_1 \left(t_1 + \tan(\theta_k) t_2 - \eta_k \right)$$

$$\textcircled{3} \quad b^\# [n_1, n_2] = \sum_{p=-S}^S u^\# [n_1, n_2 - p] h[p]$$

$$y = b^\# + \varepsilon$$

Problem formulation

Data generation size: W, H blur: g, h filter size: S
lines: $K, \{\theta_k, \alpha_k, \eta_k\}$ noise: $\varepsilon \sim \mathcal{N}(0, \zeta^2)$

$$\textcircled{1} \quad \hat{x}^\# [m, n_2] = \sum_{k=1}^K \frac{\alpha_k}{\cos \theta_k} e^{j2\pi(\tan(\theta_k)n_2 - \eta_k)m/W}$$

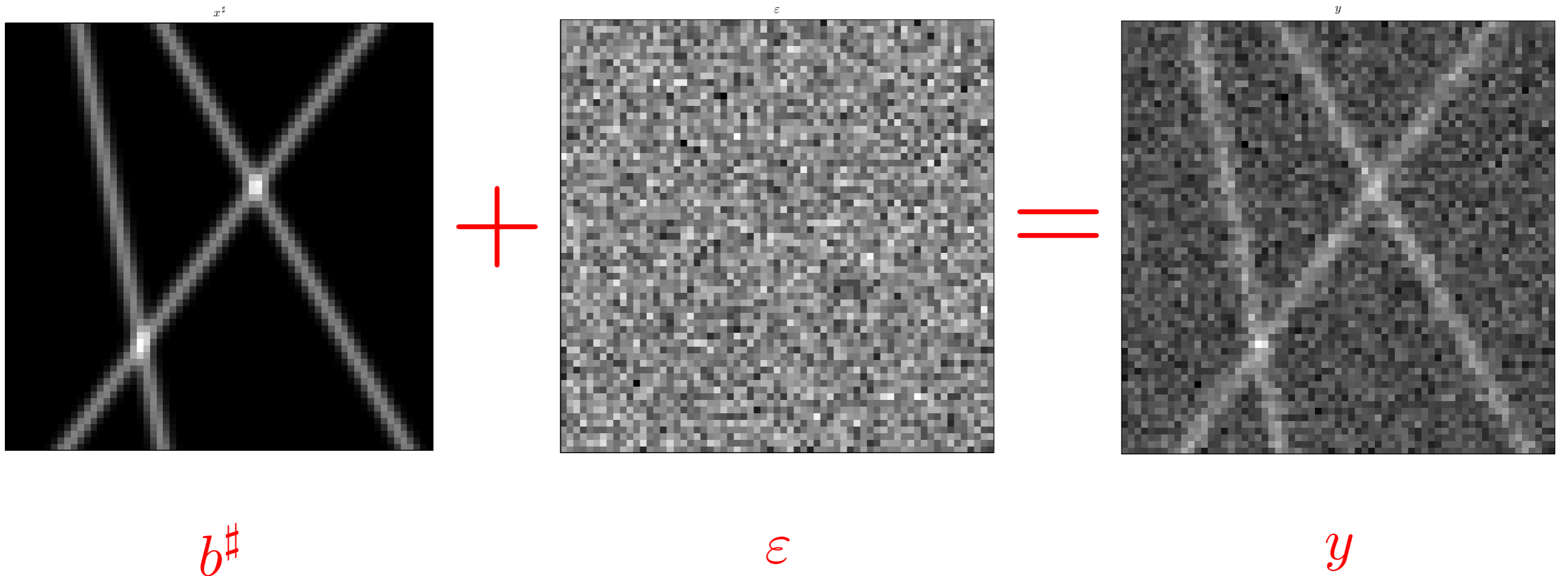
$$\textcircled{2} \quad \hat{b}^\# [m, n_2] = (\hat{g}[m] \hat{x}^\# [m, :]) * h \quad \mathbf{A} \hat{x}^\# = \hat{b}^\#$$

$$\textcircled{3} \quad b^\# = \mathcal{F}_1^{-1} \hat{b}^\#$$

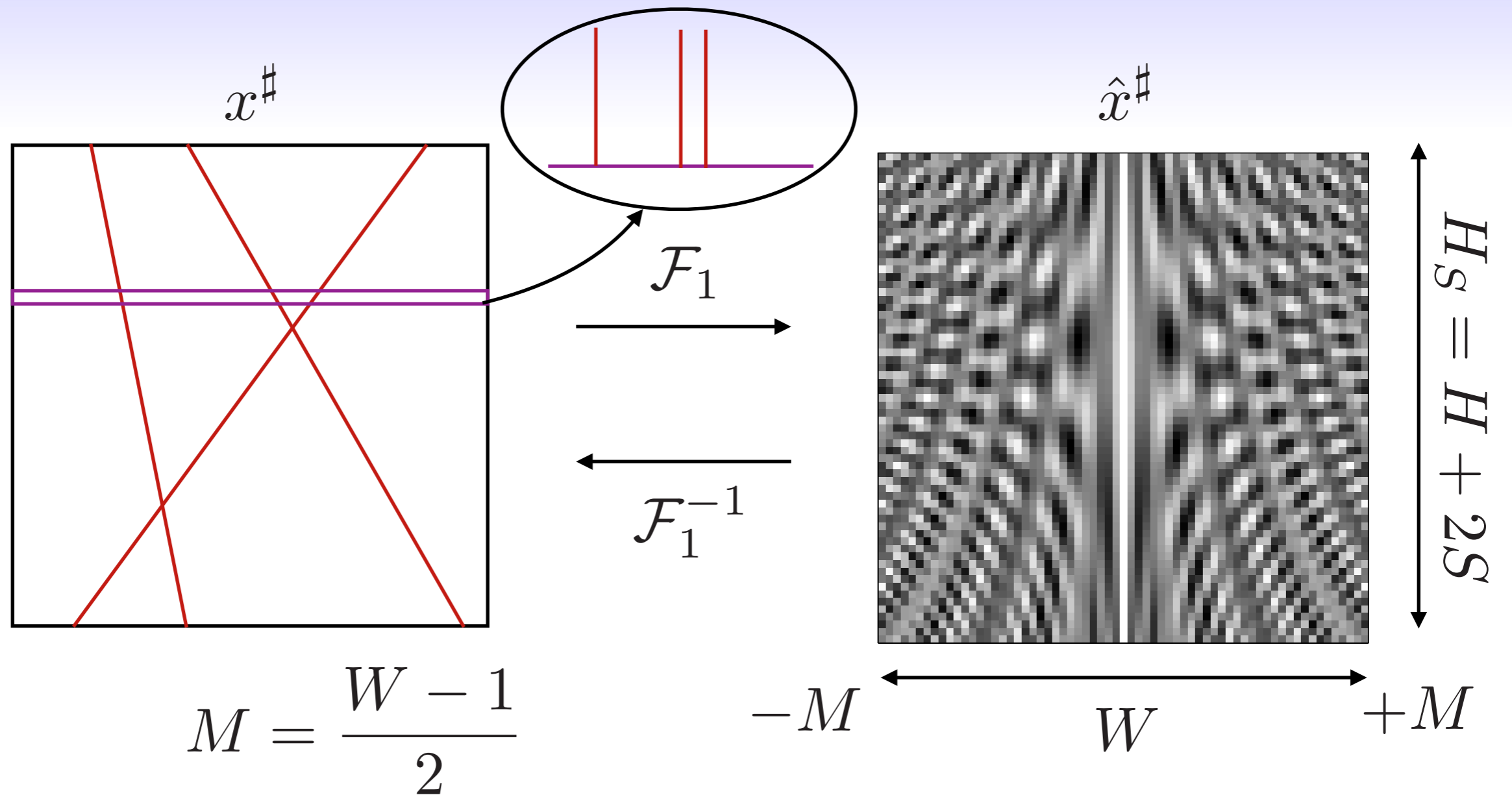
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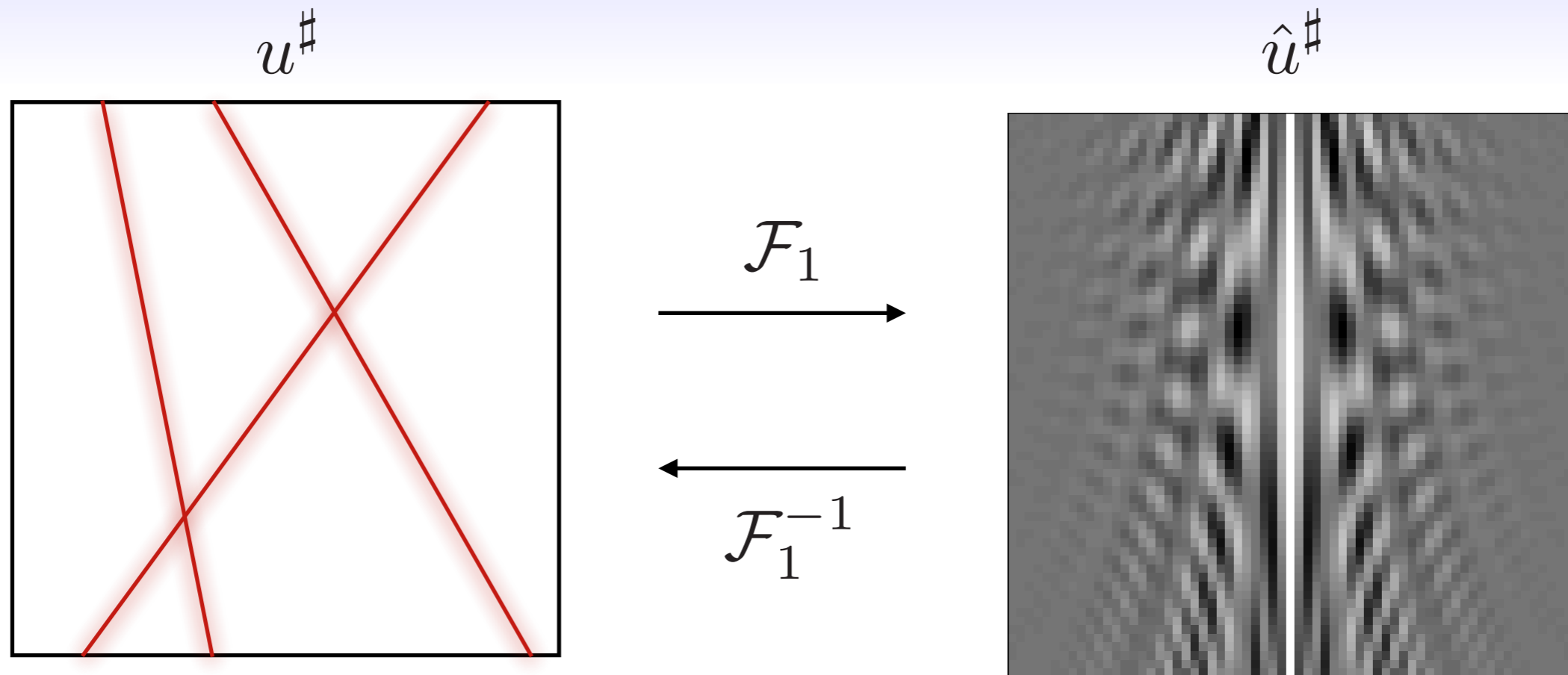


Spatial vs. Frequency domain



■
$$\hat{x}^\#[m, n_2] = \sum_{k=1}^K \frac{\alpha_k}{\cos \theta_k} e^{j2\pi(\tan(\theta_k)n_2 - \eta_k)m/W}$$

Spatial vs. Frequency domain

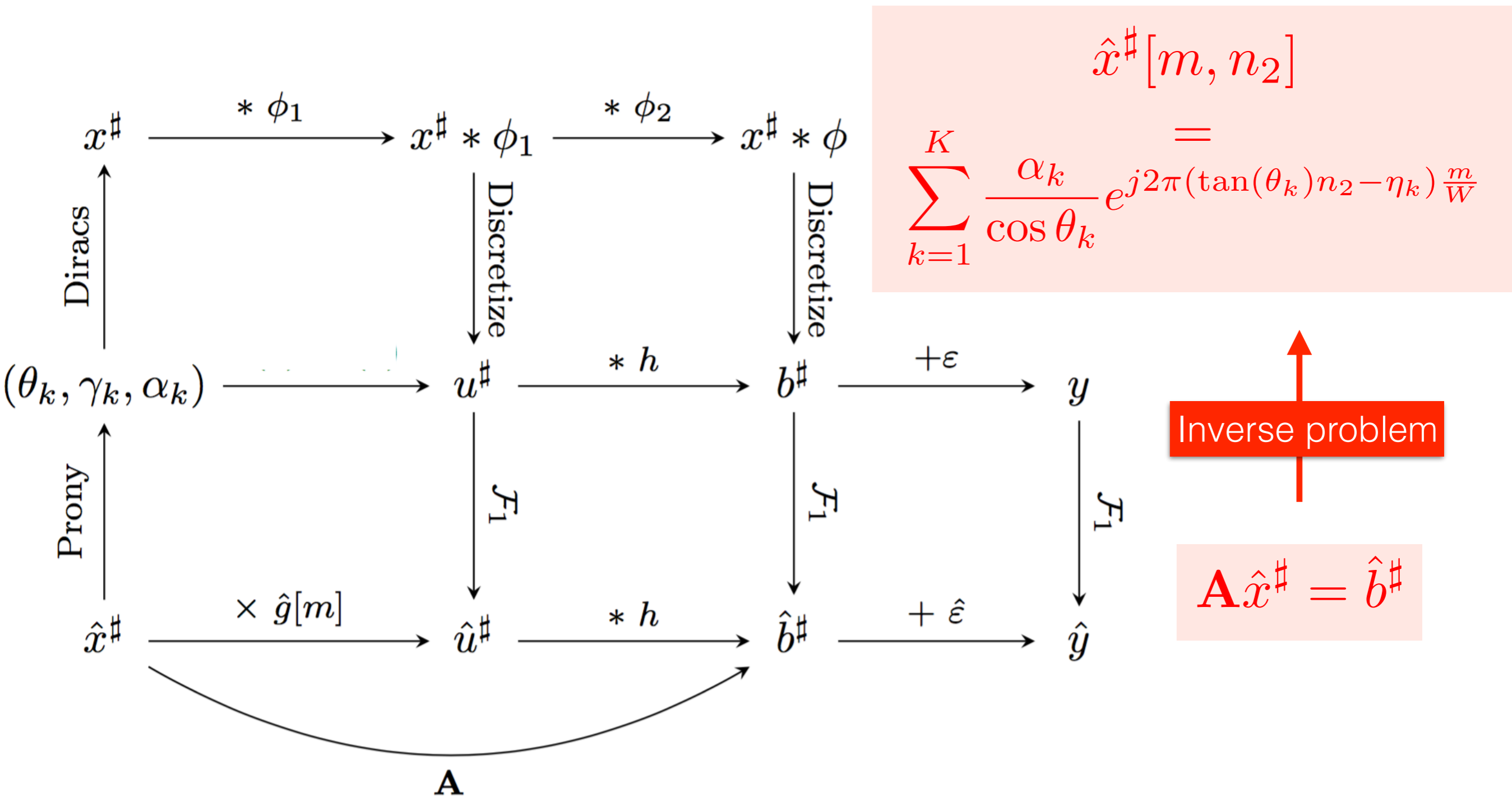


■
$$u^\#[n_1, n_2] = \sum_{k=1}^K \frac{\alpha_k}{\cos(\theta_k)} \varphi_1 \left(n_1 + \tan(\theta_k) n_2 - \eta_k \right)$$

■
$$\hat{u}^\#[m, n_2] = \hat{g}[m] \hat{x}^\#[m, n_2]$$

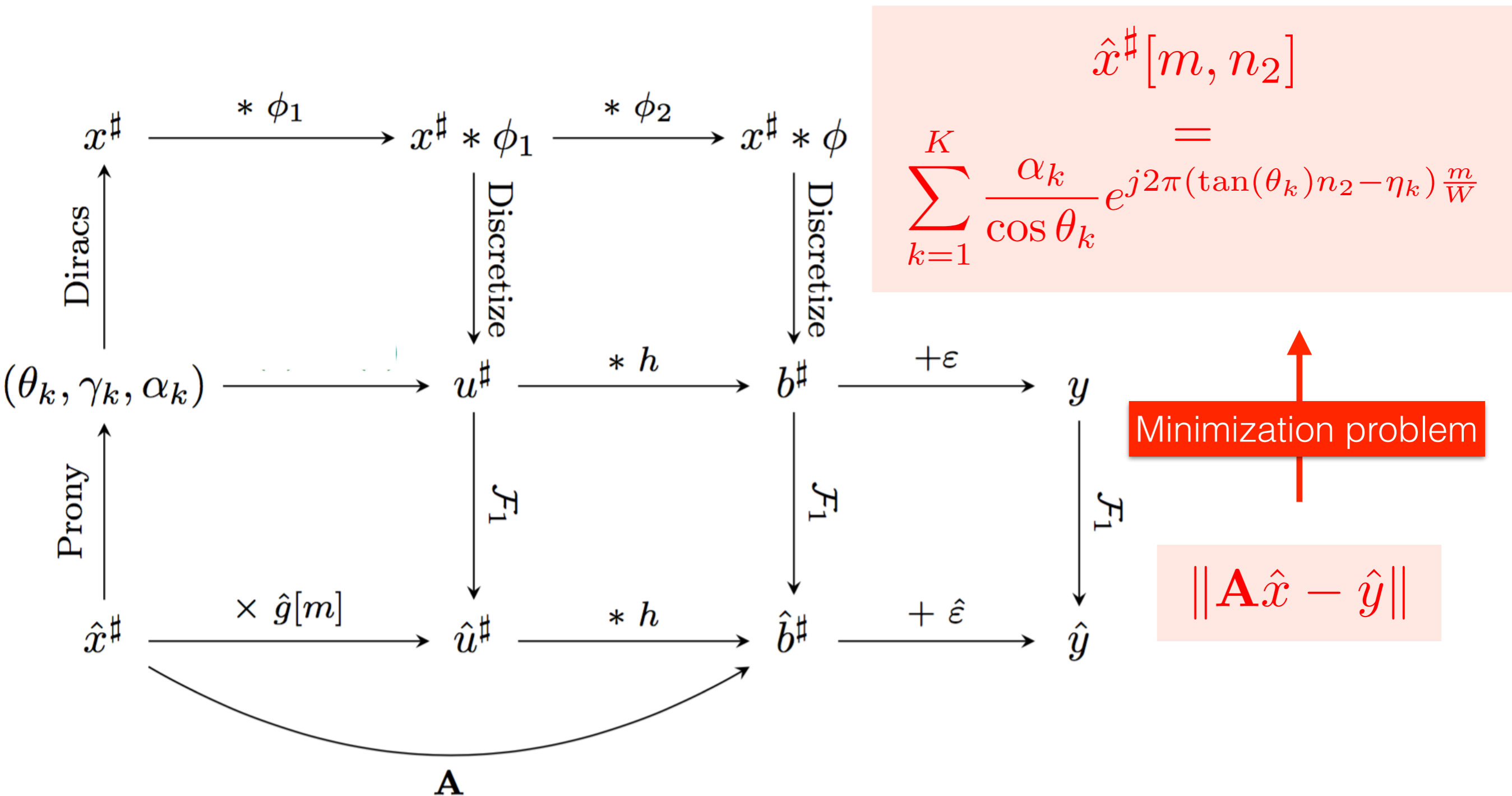
Inverse problem

Goal: From data \hat{y} find the image of the exponentials $\hat{x}^\#$



Inverse problem

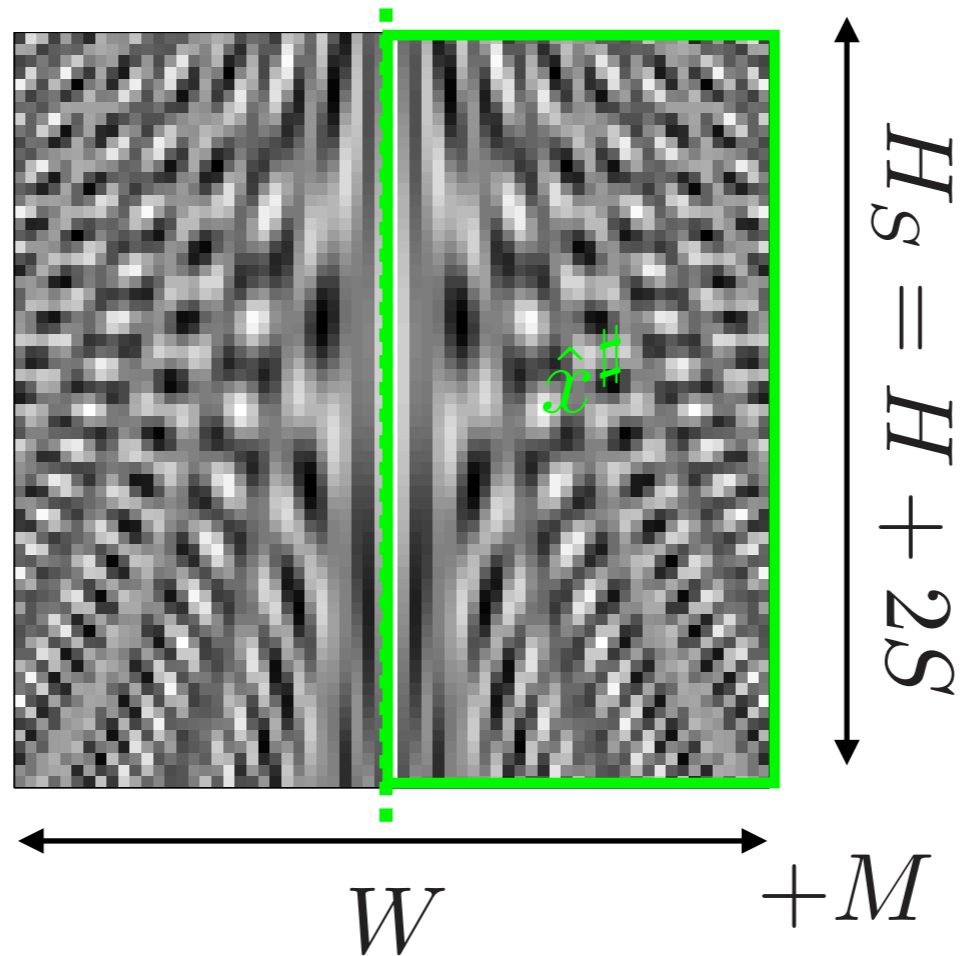
Goal: From data \hat{y} find the image of the exponentials $\hat{x}^\#$



Symmetry

symmetric

$S_1 \hat{x}^\#$



Framework of atomic norm minimization

- Sparse vectors $\mathcal{A} = \{\pm e_i\}_{i=1}^N$

$\text{conv}(\mathcal{A}) = \text{cross-polytope}$

$$\|x\|_{\mathcal{A}} = \|x\|_1$$

- Low-rank matrices $\mathcal{A} = \{A : \text{rank}(A) = 1, \|A\|_F = 1\}_{i=1}^N$

$\text{conv}(\mathcal{A}) = \text{nuclear norm ball}$

$$\|x\|_{\mathcal{A}} = \|x\|_*$$

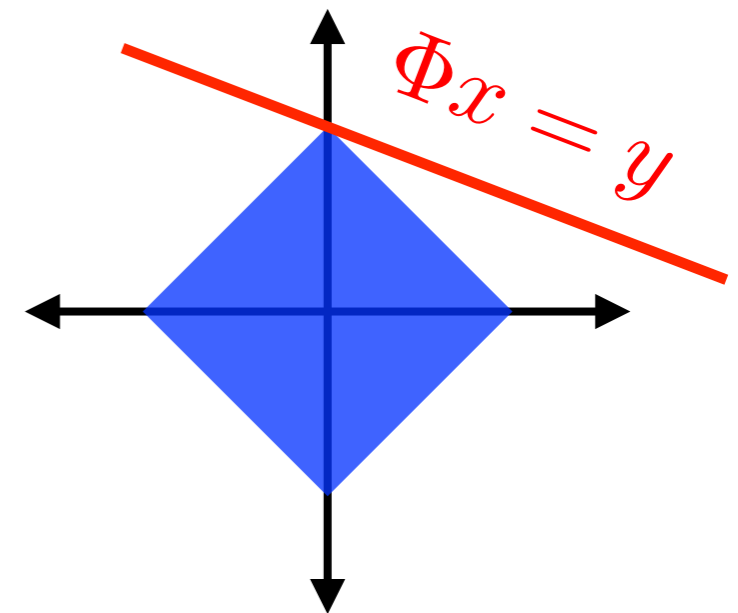
- Binary vectors $\mathcal{A} = \{\pm 1\}_{i=1}^N$

$\text{conv}(\mathcal{A}) = \text{hypercube}$

$$\|x\|_{\mathcal{A}} = \|x\|_{\infty}$$

Minimize $\|x\|_{\mathcal{A}}$

Subject to $\Phi x = y$



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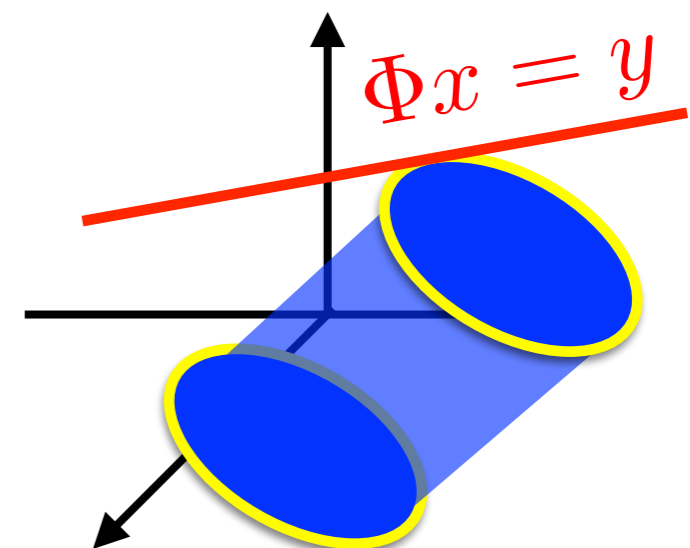
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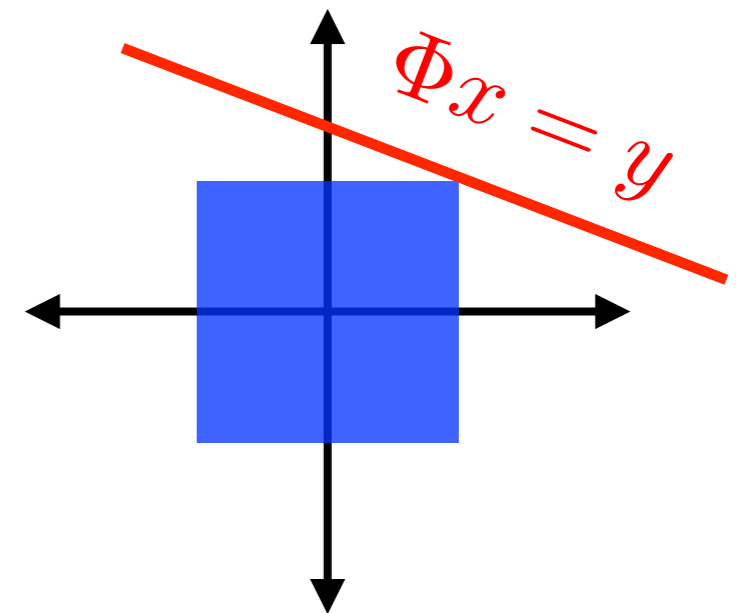
- Binary vectors $\mathcal{A} = \{\pm 1\}_{i=1}^N$

$\text{conv}(\mathcal{A}) = \text{hypercube}$

$$\|x\|_{\mathcal{A}} = \|x\|_{\infty}$$

Minimize $\|x\|_{\mathcal{A}}$

Subject to $\Phi x = y$



A line characterization property

- An image \hat{x} is of the form $\hat{x}[m, n] = c_0 e^{j2\pi((f_1 - f_0)n + f_0)m}$ if and only if its rows ℓ_n and columns t_m satisfy these properties:
For all m and n :
 - $\mathbf{T}_M(\ell_n) \succcurlyeq 0$ and of rank one
 - $\mathbf{P}_1(t_m)$ is of rank one
 - $\hat{x}[0, n] = \hat{x}[0, 0]$

A line characterization property

Relaxed version

- An image \hat{x} is of the form $\hat{x}[m, n] = c_0 e^{j2\pi((f_1 - f_0)n + f_0)m}$ if and only if its rows ℓ_n and columns t_m satisfy these properties:

For all m and n :

- $\mathbf{T}_M(\ell_n) \succcurlyeq 0$ and ~~of rank one~~ **minimizing** $\|\mathbf{T}_M(\ell_n)\|_* = \|\ell_n\|_{\mathcal{A}}$
- ~~$\mathbf{P}_1(t_m)$ is of rank one~~ **minimizing** $\|t_m\|_{\mathcal{A}}$
- $\hat{x}[0, n] = \hat{x}[0, 0] \leq c^*$

$$\|\ell_n\|_{\mathcal{A}} = \sum_{k=1}^K c_k \equiv c^*$$

Minimization problem

■ $X = (\hat{x}, q) \in \mathcal{H}$ $G(X)$

$$\tilde{x} \in \arg \min_{\hat{x}, q \in \mathcal{X} \times \mathcal{Q}} \frac{1}{2} \|\mathbf{A}\hat{x} - \hat{y}\|^2,$$

Subject to $\left\{ \begin{array}{l} \forall n_2 = 0, \dots, H_S - 1, \forall m = 0, \dots, M, \\ \hat{x}[0, n_2] = \hat{x}[0, 0] \leq c, \\ q[m, 0] \leq c, \\ \mathbf{T}'_{H_S}(\hat{x}[m, :], q[m, :]) \succcurlyeq 0, \\ \mathbf{T}_{M+1}(\hat{x}[:, n_2]) \succcurlyeq 0, \end{array} \right.$) boundary constraints \mathcal{B}
) cone of positive matrices \mathcal{C}

■ $L_m^{(1)} : (\hat{x}, q) \mapsto \mathbf{T}'_{H_S}(\hat{x}[m, :], q[m, :])$

■ $L_{n_2}^{(2)} : (\hat{x}, q) \mapsto \mathbf{T}_{M+1}(\hat{x}[:, n_2])$

Minimization problem

$$\tilde{X} = \arg \min_{X=(\hat{x},q) \in \mathcal{H}} \left\{ \frac{1}{2} \|\mathbf{A}\hat{x} - \hat{y}\| + \iota_{\mathcal{B}}(X) + \sum_{m=0}^M \iota_{\mathcal{C}}(L_m^{(1)}(X)) + \sum_{n_2=0}^{H_S-1} \iota_{\mathcal{C}}(L_{n_2}^{(2)}(X)) \right\}$$

$$\tilde{X} = \arg \min_{X \in \mathcal{H}} \{G(X) + \mathbf{H}(\mathbf{L}(X))\}$$

Chambolle-Pock

■ $G = \frac{1}{2} \|\mathbf{A} \cdot - \hat{y}\|$

■ $\mathbf{H}x = \sum_{i=0}^N H_i x_i$

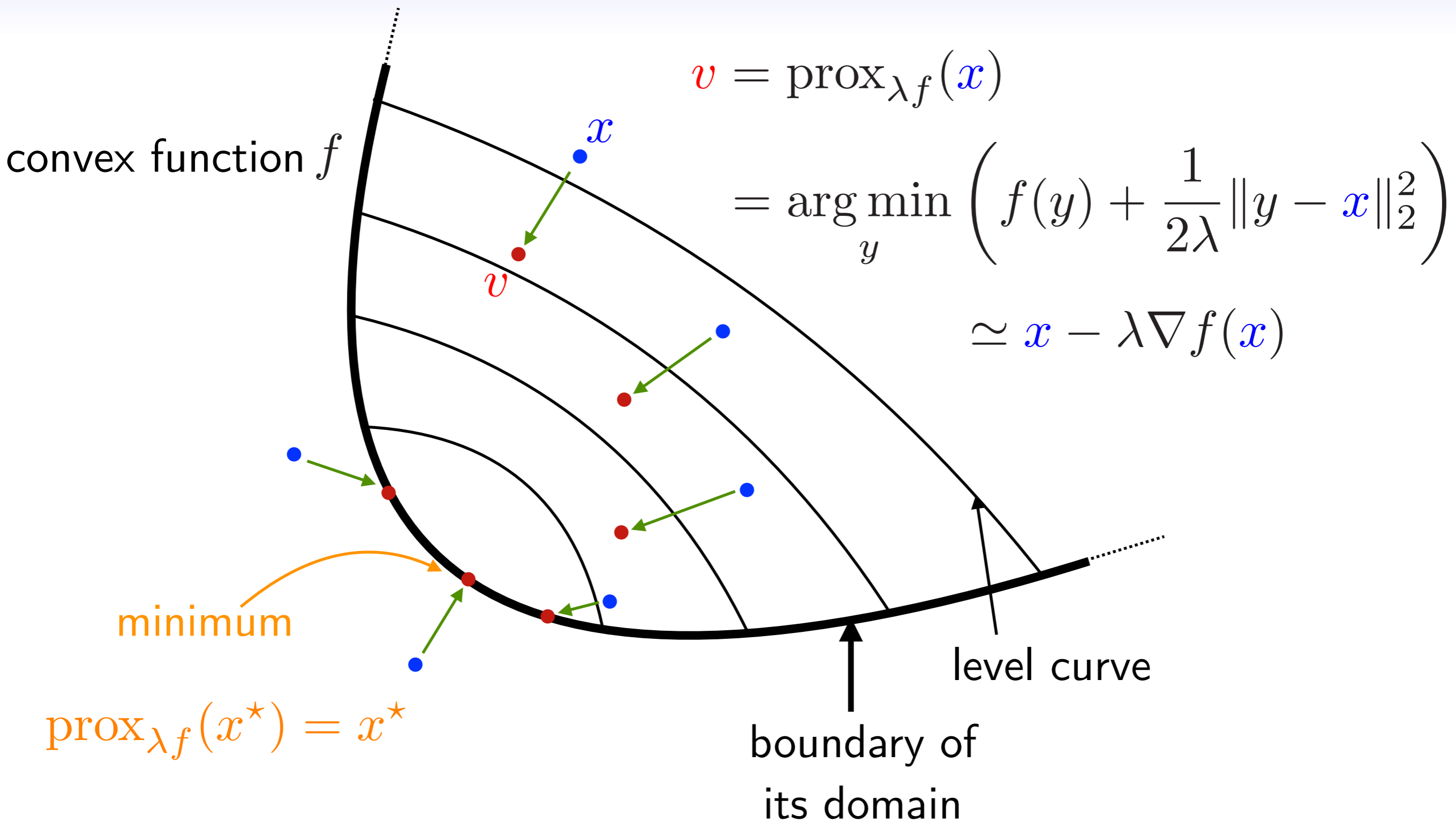
■ $\mathbf{L}x = (L_1, \dots, L_N)$

■ $L_i \in \{L_m^{(1)}, L_{n_2}^{(2)}\} \quad H_i = \iota_{\mathcal{C}} \quad i < N$

■ $L_N = \text{Id} \quad H_N = \iota_{\mathcal{B}} \quad i = N$

$$N = M + H_S + 1$$

Proximal operator



Proximal operator

Example: f the indicator function of a set A

$$\begin{aligned} v &= \text{prox}_{\lambda f}(x) \\ &= \arg \min_y \left(f(y) + \frac{1}{2\lambda} \|y - x\|_2^2 \right) \end{aligned}$$

$$\blacksquare \quad \iota_A(x) = \begin{cases} 0 & x \in A \\ +\infty & x \notin A \end{cases}$$

$$\blacksquare \quad \Pi_A(v) = \arg \min_{y \in A} \|y - v\|_2$$

$\text{prox}_{\lambda \iota}$



Primal-dual algorithm

$$\text{Conditions} \begin{cases} \blacksquare \tau\sigma \|\mathbf{L}\|^2 = 1 \\ \blacksquare \forall n \in \mathbb{N}, \rho_n \in]0, 2[\\ \blacksquare \sum_{n \in \mathbb{N}} \rho_n(2 - \rho_n) = +\infty \end{cases}$$

$$\begin{array}{l} \text{Primal} \\ \text{variables} \end{array} \begin{cases} \blacksquare \tilde{x}_{n+1} = \text{prox}_{\tau G} \left(x_n - \tau \sum_{i=0}^N L_i^* \xi_{i,n} \right) \\ \blacksquare x_{n+1} = \rho_n \tilde{x}_{n+1} + (1 - \rho_n) x_n \end{cases}$$

$$\begin{array}{l} \text{Dual} \\ \text{variables} \end{array} \begin{cases} \blacksquare \forall i = 0, \dots, N \\ \blacksquare \tilde{\xi}_{i,n+1} = \text{prox}_{\sigma H_i^*} (\xi_{i,n} + \sigma L_i (2\tilde{x}_{n+1} - x_n)) \\ \blacksquare \xi_{i,n+1} = \rho_n \tilde{\xi}_{i,n+1} + (1 - \rho_n) \xi_{i,n} \end{cases}$$

converges toward a solution $(x^*, \xi_0^*, \dots, \xi_N^*)$ of the problem.

Proximal operator

■ $H_i = \iota_{\mathcal{C}} \quad i < N \quad \text{prox}_{\lambda H_i} = \text{prox}_{\lambda \iota} = \Pi_{\mathcal{C}}$

↳ the proximal operator of the H_i are the projection onto the cone of positive matrix via a SVD decomposition

■ **Moreau identity**

$$\text{prox}_{\sigma H_i^*}(u) = u - \sigma \text{prox}_{\frac{H_i}{\sigma}} \begin{pmatrix} u \\ \sigma \end{pmatrix}$$

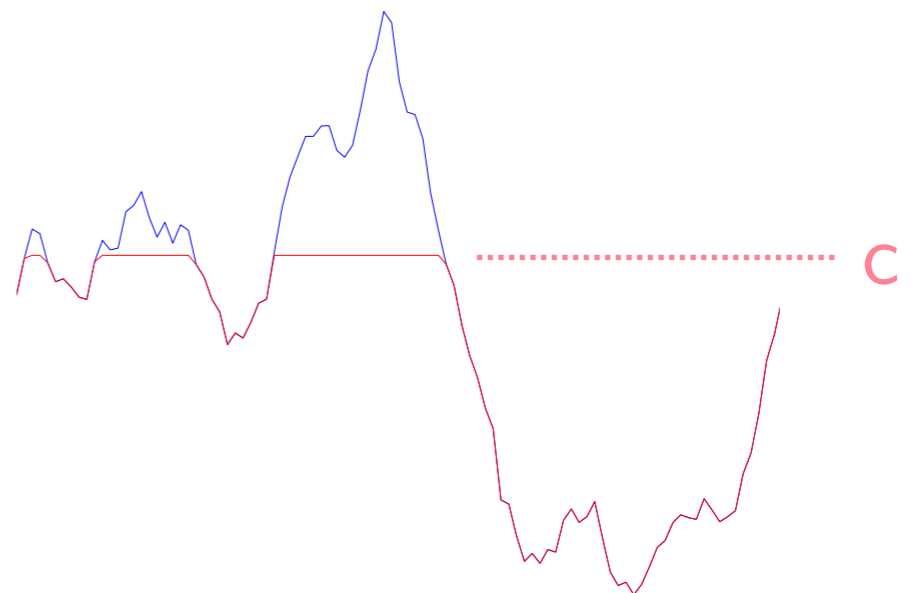
Proximal operator

■ $H_N = \iota_{\mathcal{B}} \quad i = N \quad \text{prox}_{\lambda H_i} = \text{prox}_{\lambda \iota} = \Pi_{\mathcal{B}}$

↳ the proximal operator of H_N is the projection:

$$\begin{cases} \hat{x}[0, n_2] = \bar{x}_0, & \text{if } \bar{x}_0 \leq c, \\ \hat{x}[0, n_2] = c, & \text{otherwise,} \\ q[m, 0] = c, & \text{if } q[m, 0] > c. \end{cases}$$

with $\bar{x}_0 = \frac{1}{H_S} \sum_{n_2=1}^{H_S} \hat{x}[0, n_2]$



Primal-dual algorithm

$$\text{Conditions} \left\{ \begin{array}{l} \blacksquare \tau\sigma \|\mathbf{L}\|^2 = 1 \\ \blacksquare \forall n \in \mathbb{N}, \rho_n \in]0, 2[\\ \blacksquare \sum_{n \in \mathbb{N}} \rho_n(2 - \rho_n) = +\infty \end{array} \right.$$

$$\begin{array}{l} \text{Primal} \\ \text{variables} \end{array} \left\{ \begin{array}{l} \blacksquare \tilde{x}_{n+1} = \text{prox}_{\tau G} \left(x_n - \tau \sum_{i=0}^N L_i^* \xi_{i,n} \right) \\ \blacksquare x_{n+1} = \rho_n \tilde{x}_{n+1} + (1 - \rho_n) x_n \end{array} \right.$$

$$\begin{array}{l} \text{Dual} \\ \text{variables} \end{array} \left\{ \begin{array}{l} \blacksquare \forall i = 0, \dots, N \\ \blacksquare \tilde{\xi}_{i,n+1} = \text{prox}_{\sigma H_i^*} (\xi_{i,n} + \sigma L_i (2\tilde{x}_{n+1} - x_n)) \\ \blacksquare \xi_{i,n+1} = \rho_n \tilde{\xi}_{i,n+1} + (1 - \rho_n) \xi_{i,n} \end{array} \right.$$

converges toward a solution $(x^*, \xi_0^*, \dots, \xi_N^*)$ of the problem.

Proximal operator

- $G = \frac{1}{2} \|\mathbf{A} \cdot -\hat{y}\|$
- $p = \text{prox}_{\tau G}(\hat{x}) \Leftrightarrow \hat{x} - p = \nabla(\tau G)(p)$
 $\Leftrightarrow \hat{x} - p = \tau A^*(Ap - \hat{y})$
 $\Leftrightarrow \hat{x} + \tau A^* \hat{y} = (\text{Id} + \tau A^* A)p$
- $\text{prox}_{\tau G}(\hat{x}) = (\text{Id} + \tau \mathbf{A}^* \mathbf{A})^{-1}(\hat{x} + \tau \mathbf{A}^* \hat{y})$

Prony method

samples

parameters

■ Given

$$x_m = \sum_{k=1}^K \rho_k \underbrace{\left(e^{-j\omega_k} \right)}_{z_k}^m$$

$$\forall m = -M, \dots, M$$

recover

$$\rho_k \in \mathbb{C}$$

$$\omega_k \in [-\pi, \pi]$$

■ **Annihilating filter:** $H(z) = \prod_{k=1}^K (z - \overline{z_k}) = \sum_{k=0}^K h_k z^k$

$$\sum_{j=0}^K h_j x_{m-j} = \sum_{j=0}^K h_j \left(\sum_{k=1}^K \rho_k z_k^{m-j} \right) = \sum_{k=1}^K \rho_k z_k^m \underbrace{\left(\sum_{j=0}^K h_j z_k^{-j} \right)}_{H(\overline{z_k})} = 0$$

$H(\overline{z_k}) = 0$

Prony method

$$\sum_{j=0}^K h_j x_{m-j} = 0, \quad \forall m = -M + K, \dots, M$$

\iff

$$\begin{pmatrix} x_{-M+K} & \cdots & x_{-M} \\ \vdots & \ddots & \vdots \\ x_M & \cdots & x_{M-K} \end{pmatrix} \begin{pmatrix} h_0 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

\mathbf{P}_K h

$$x * h = 0 \iff \mathbf{P}_K h = 0$$

Prony method

$$\omega_k = \arg(\overline{z_k})$$

$$\text{Roots of } H(z) = \prod_{k=1}^K (z - \overline{z_k}) = \sum_{k=0}^K h_k z^k$$

$$\mathbf{P}_K h = 0 \Rightarrow h =$$

Right singular vector associated to the singular value zero

Prony method

$$x_m = \sum_{k=1}^K \rho_k (e^{-j\omega_k})^m$$
$$\forall m = -M, \dots, M$$

 \iff

$$x = U\rho$$

with $U = \begin{pmatrix} e^{jM\omega_1} & \dots & e^{jM\omega_K} \\ \vdots & \ddots & \vdots \\ e^{-jM\omega_1} & \dots & e^{-jM\omega_K} \end{pmatrix}$

size
 $(2M + 1) \times K$

Solve $U^H U \rho = U^H x \iff \rho = (U^H U)^{-1} U^H x$

Application of the Prony method

■ $\theta_k = \mathbb{E}[\theta_{m,k}] \quad \alpha_k = \mathbb{E}[\alpha_{m,k}]$

■ $\alpha_{m,k} = |d_{m,k}| \cos(\theta_{m,k})$

■ $\theta_{m,k} = \arctan(W f_{m,k}/m)$

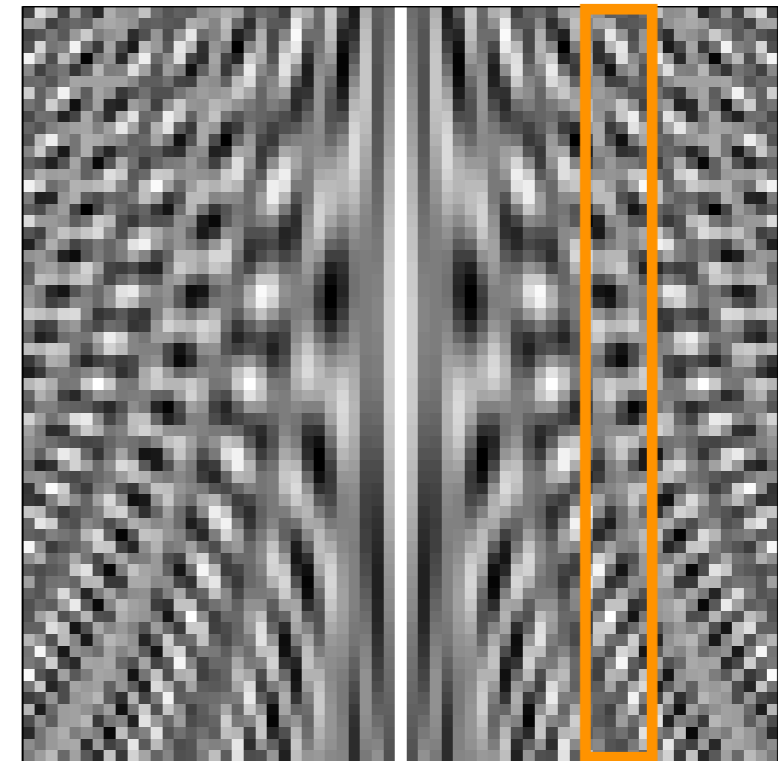
■ **Prony** $\longrightarrow \{d_{m,k}\}_{m,k} \quad \{f_{m,k}\}_{m,k}$

■ $t_m^\# = \hat{x}^\#[m, :] = \sum_{k=1}^K d_{m,k} a(f_{m,k})^T$

$\searrow c_k e^{i\phi_{m,k}}$

■ $t_m^\# = \hat{x}^\#[m, :] = \sum_{k=1}^K c_k a(f_{m,k}, \phi_{m,k})^T$

$\hat{x}^\#$ ↘ m



$$c_k = \frac{\alpha_k}{\cos \theta_k}$$

$$f_{m,k} = \frac{\tan(\theta_k)m}{W}$$

$$\phi_{m,k} = -\frac{2\pi\eta_k m}{W}$$

Application of the Prony method

$$\phi_{m,k} = -\frac{2\pi\eta_k m}{W}$$

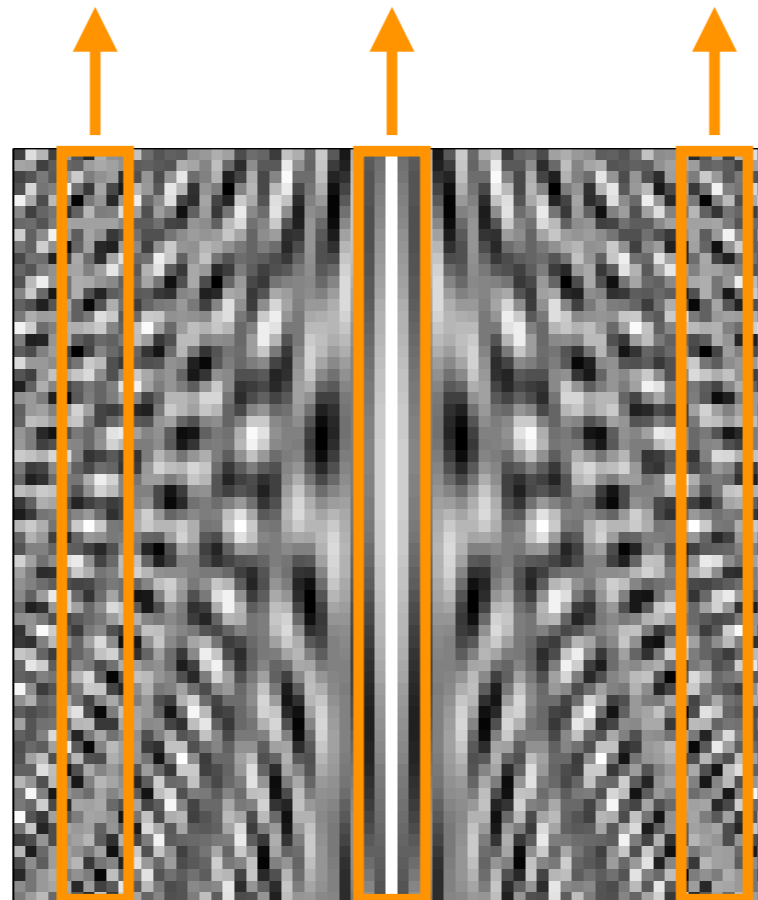
$\div C_k$

$$e^{i\phi_{1,k}} \cdots e^{i\phi_{M,k}} \cdots e^{i\phi_{W,k}}$$

Prony

$\longrightarrow \eta_k$

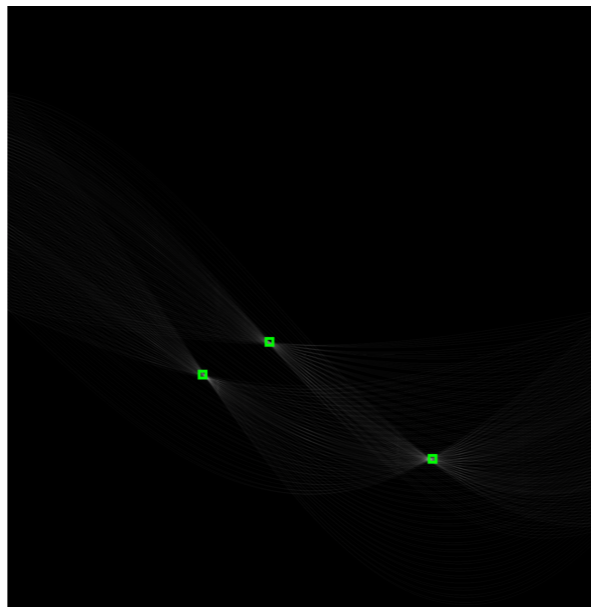
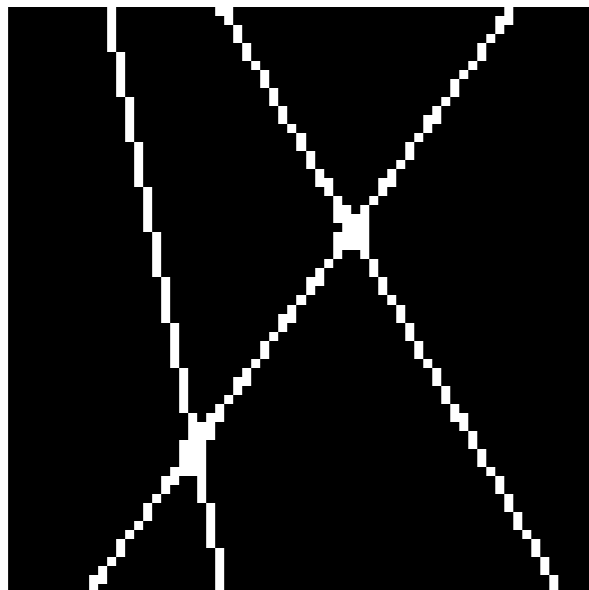
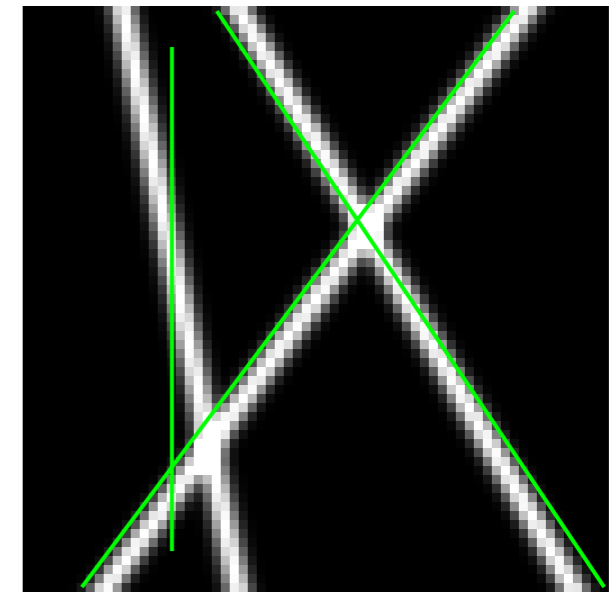
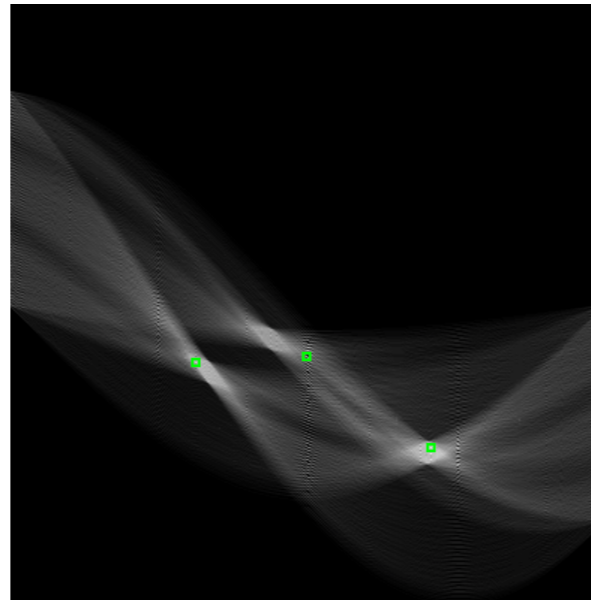
$$\uparrow \quad \uparrow \quad \uparrow$$
$$d_{1,k} \cdots d_{M,k} \cdots d_{W,k}$$



Prony method

- From $m = 1, \dots, M$,
 - 1) Compute $\tilde{f}_{m,k} = \arg(\bar{\xi}_{m,k})/(2\pi)$, where $(\xi_{m,k})_k$ are roots of the polynomial $\sum_{k=0}^K h_{m,k} z^k$ with $\mathbf{h}_m = [h_{m,0}, \dots, h_{m,K}]^T$ being the right singular vector of $\mathbf{T}_K(\hat{w}_m^*)$ with $I = \{0, \dots, H_S - 1\}$. It corresponds to the singular value zero (the smallest value in practise).
 - 2) Compute $\tilde{\theta}_{m,k} = \arctan(W \tilde{f}_{m,k}/m)$ from (15).
 - 3) Form the matrix $\tilde{U}_m = [a(\tilde{f}_{m,1}) \cdots a(\tilde{f}_{m,K})]$, and compute $\tilde{\mathbf{d}}_m = [\tilde{d}_{m,1}, \dots, \tilde{d}_{m,K}]^T$ by solving the least-squares linear system $U_m^H U_m \mathbf{d}_m = U_m^H \hat{w}_m^*$.
 - 4) Compute $\tilde{c}_{m,k} = |\tilde{d}_{m,k}|$ and $\tilde{\alpha}_{m,k} = \tilde{c}_{m,k} \cos(\tilde{\theta}_{m,k})$.
 - 5) Compute $\tilde{e}_{m,k} = \tilde{d}_{m,k}/|\tilde{d}_{m,k}|$.
- For $k = 1, \dots, K$
 - 1) Compute the mean of all estimated angles $\tilde{\theta}_k = \frac{1}{M} \sum_{m=1}^M \tilde{\theta}_{m,k}$ and amplitudes $\tilde{\alpha}_k = \frac{1}{M} \sum_{m=1}^M \tilde{\alpha}_{m,k}$
 - 2) Compute the frequency $\tilde{\nu}_k$ as previously from $\mathbf{T}_K(\tilde{e}_k)$ with $\tilde{e}_k = (\tilde{e}_{m,k})_m$ and $I = \{-M, \dots, M\}$.
 - 3) Compute the horizontal offset $\tilde{\eta}_k = W \tilde{\nu}_k/(2\pi)$

Application of the Hough Transform



Application of the Radon Transform

