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► **To cite this version:**

Isabelle Chalendar, William T. Ross. COMPACT OPERATORS ON MODEL SPACES. 2016. hal-01281967

HAL Id: hal-01281967

<https://hal.science/hal-01281967>

Preprint submitted on 3 Mar 2016

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COMPACT OPERATORS ON MODEL SPACES

ISABELLE CHALENDAR AND WILLIAM T. ROSS

ABSTRACT. We give a characterization of the compact operators on a model space in terms of asymptotic Toeplitz operators.

1. INTRODUCTION

If H^2 denotes the classical Hardy space of the open unit disk \mathbb{D} [4, 7], a theorem of Brown and Halmos [3] says that a bounded linear operator T on H^2 is a Toeplitz operator if and only if

$$S^*TS = T,$$

where $Sf = zf$ is the well-known unilateral shift on H^2 . By a *Toeplitz operator* [2], we mean, for a given symbol $\varphi \in L^\infty(\mathbb{T}, m)$ (\mathbb{T} is the unit circle and m is normalized Lebesgue measure on \mathbb{T}), the operator

$$T_\varphi : H^2 \rightarrow H^2, \quad T_\varphi f = P(\varphi f),$$

where P is the orthogonal projection of L^2 onto H^2 .

This notion of “Toeplitzness” was extended in various ways. Barria and Halmos [1] examined the so-called *asymptotically Toeplitz operators* operators T on H^2 for which the sequence of operators

$$\{S^{*n}TS^n\}_{n \geq 1}$$

converges strongly. This class certainly includes the Toeplitz operators but also includes other operators such as those in the Hankel algebra. Feintuch [5] discovered that one need not restrict to strong convergence of $\{S^{*n}TS^n\}_{n \geq 1}$ and worthwhile classes of operators arise from the weak and uniform (or norm) limits of this sequence. Indeed, an operator T on H^2 is uniformly asymptotically Toeplitz, i.e., $S^{*n}TS^n$ converges in operator norm, if and only if

$$(1.1) \quad T = T_1 + K,$$

2010 *Mathematics Subject Classification.* 30J05, 30H10, 46E22.

Key words and phrases. Hardy spaces, inner functions, model spaces, compact operators, Toeplitz operators.

where T_1 is a Toeplitz operator, i.e., $S^*T_1S = T_1$, and K is a compact operator on H^2 . Nazarov and Shapiro [8] examined other associated notions of “Toeplitzness” with regards to certain composition operators on H^2 .

In this paper we explore a model space setting for this “Toeplitzness” discussion. For an inner function Θ on \mathbb{D} (i.e., a bounded analytic function on \mathbb{D} whose radial boundary values are unimodular almost everywhere on \mathbb{T}), one can define the *model space* [6, 9]

$$\mathcal{K}_\Theta = H^2 \ominus \Theta H^2.$$

Beurling’s theorem [4] says that these spaces are the generic invariant subspaces for the backward shift operator

$$S^*f = \frac{f - f(0)}{z}$$

on H^2 . By model theory for contractions [9], certain types of Hilbert space contractions are unitarily equivalent to compressed shifts

$$S_\Theta = P_\Theta S|_{\mathcal{K}_\Theta},$$

where P_Θ is the orthogonal projection of L^2 onto \mathcal{K}_Θ .

In this model spaces setting, we examine, for a bounded operator A on \mathcal{K}_Θ , the sequence

$$\{S_\Theta^{*n} A S_\Theta^n\}_{n \geq 1}.$$

Here we have a similar result as before (see Lemma 2.6 below) in that $S_\Theta^{*n} A S_\Theta^n$ converges in operator norm if and only if

$$A = A_1 + K,$$

where K is a compact operator on \mathcal{K}_Θ and A_1 satisfies $S_\Theta^* A_1 S_\Theta = A_1$. In the analogous H^2 setting, the operator T_1 from (1.1) is a Toeplitz operator. In the model space setting, the corresponding operator A_1 is severely restricted. Indeed,

$$A_1 \equiv 0.$$

Thus, as the main theorem of this paper, we have the following characterization of the compact operators on \mathcal{K}_Θ .

Theorem 1.2. *For an inner function Θ and a bounded linear operator A on \mathcal{K}_Θ , the following are equivalent:*

- (i) *The sequence $S_\Theta^{*n} A S_\Theta^n$ converges in operator norm;*
- (ii) *$S_\Theta^{*n} A S_\Theta^n \rightarrow 0$ in operator norm;*
- (iii) *A is a compact operator.*

One can also explore the convergence of the sequence $S_\Theta^{*n} A S_\Theta^n$ in other topologies, such as the strong/weak operator topologies. Surprisingly what happens is entirely different from what happens in H^2 .

Proposition 1.3. *For any inner function Θ and any bounded linear operator A on \mathcal{K}_Θ , the sequence $S_\Theta^{*n} A S_\Theta^n$ converges to zero strongly.*

In other words, the convergence of $S_\Theta^{*n} A S_\Theta^n$ in the strong or weak topology is always true (and to the same operator) and provides no information about A .

2. CHARACTERIZATION OF THE COMPACT OPERATORS

The following lemma proves the implication (iii) \implies (ii) of Theorem 1.2.

Lemma 2.1. *If K is a compact operator on \mathcal{K}_Θ then*

$$\lim_{n \rightarrow \infty} \|S_\Theta^{*n} K S_\Theta^n\| = 0.$$

Proof. Let $\mathcal{B}_\Theta = \{f \in \mathcal{K}_\Theta : \|f\| \leq 1\}$ denote the closed unit ball in \mathcal{K}_Θ . First observe that

$$\|S_\Theta^n\| \leq \|S_\Theta\|^n \leq \|P_\Theta S|_{\mathcal{K}_\Theta}\|^n \leq \|S\|^n = 1.$$

From here we see that

$$\begin{aligned} \|S_\Theta^{*n} K S_\Theta^n\| &= \sup_{f \in \mathcal{B}_\Theta} \|S_\Theta^{*n} K S_\Theta^n f\| \\ &\leq \sup_{g \in \mathcal{B}_\Theta} \|S_\Theta^{*n} K g\| \\ (2.2) \quad &\leq \sup_{h \in \overline{K(\mathcal{B}_\Theta)}} \|S_\Theta^{*n} h\|. \end{aligned}$$

Second, note that $S^{*n} \rightarrow 0$ strongly. Indeed, if $f = \sum_{k \geq 0} a_k z^k \in H^2$, then

$$\|S^{*n} f\|^2 = \sum_{k \geq n+1} |a_k|^2 \rightarrow 0 \quad n \rightarrow \infty.$$

Thus since $S_\Theta^{*n} = S^{*n}|_{\mathcal{K}_\Theta}$ (since \mathcal{K}_Θ is S^* -invariant), we see that

$$(2.3) \quad S_\Theta^{*n} \rightarrow 0 \text{ strongly.}$$

Let $\epsilon > 0$ be given and let $h \in \overline{K(\mathcal{B}_\Theta)}$. Since $S^{*n} \rightarrow 0$ strongly, there exists an $n_{h,\epsilon}$ such that $\|S_\Theta^{*n} h\| < \epsilon/2$ for all $n > n_{h,\epsilon}$. The continuity

of the operator $S^{*n_{h,\epsilon}}$ implies that there exists a $r_{h,\epsilon}$ such that for all q belonging to

$$B(h, r_{h,\epsilon}) = \{q \in \mathcal{K}_\Theta : \|q - h\| < r_{h,\epsilon}\}$$

we have $\|S^{*n_{h,\epsilon}}q\| < \epsilon$.

Again using the fact that $\|S_\Theta^*\| \leq 1$, we see that for all $q \in B(h, r_{h,\epsilon})$ and all $n > n_{h,\epsilon}$ we have

$$(2.4) \quad \|S_\Theta^{*n}q\| = \|S_\Theta^{*(n-n_{h,\epsilon})}S_\Theta^{*n_{h,\epsilon}}q\| \leq \|S_\Theta^{*n_{h,\epsilon}}q\| < \epsilon.$$

Moreover, we have

$$\overline{K(\mathcal{B}_\Theta)} \subset \bigcup_{h \in \overline{K(\mathcal{B}_\Theta)}} B(h, r_{h,\epsilon}).$$

The compactness of $\overline{K(\mathcal{B}_\Theta)}$ implies that there exists h_1, \dots, h_N ($N = N_\epsilon$) belonging to $\overline{K(\mathcal{B}_\Theta)}$ such that

$$\overline{K(\mathcal{B}_\Theta)} \subset \bigcup_{k=1}^N B(h_k, r_{h_k,\epsilon}).$$

For all $n > \max\{n_{h_1,\epsilon}, \dots, n_{h_N,\epsilon}\}$ we use (2.4) along with (2.2) to see that

$$\|S_\Theta^{*n}h\| < \epsilon \quad \forall h \in \overline{K(\mathcal{B}_\Theta)}.$$

This proves the lemma. \square

Remark 2.5. Important to the proof above was the fact that $S_\Theta^{*n} \rightarrow 0$ strongly (see (2.3)). One can show that S_Θ is unitarily equivalent to S_Ψ^* , where Ψ is the inner function defined by $\Psi(z) = \overline{\Theta(\bar{z})}$ [6, p. 303]. From here we see that $S_\Theta^n \rightarrow 0$ strongly. This detail will be important at the end of the paper in the proof of Theorem 1.3.

Lemma 2.6. *Suppose T is a bounded operator on \mathcal{K}_Θ such that $S_\Theta^{*n}TS_\Theta^n$ converges in norm. Then $T = T_1 + K$, where K is a compact operator on \mathcal{K}_Θ and T_1 is a bounded operator on \mathcal{K}_Θ satisfying $S_\Theta^*T_1S_\Theta = T_1$.*

Proof. Let A be a bounded operator on \mathcal{K}_Θ such that

$$\|S_\Theta^{*n}TS_\Theta^n - A\| \rightarrow 0.$$

Then

$$\begin{aligned} \|S_\Theta^{*(n+1)}TS_\Theta^{n+1} - S_\Theta^*AS_\Theta\| &= \|S_\Theta^*(S_\Theta^{*n}TS_\Theta^n - A)S_\Theta\| \\ &\leq \|S_\Theta^{*n}TS_\Theta^n - A\| \rightarrow 0. \end{aligned}$$

This implies that

$$(2.7) \quad S_\Theta^*AS_\Theta = A.$$

From here it follows that

$$(2.8) \quad S_{\Theta}^{*n} T S_{\Theta}^n - A = S_{\Theta}^{*n} (T - A) S_{\Theta}^n, \quad n \geq 0.$$

Define

$$P_n := S_{\Theta}^n S_{\Theta}^{*n} \quad \text{and} \quad Q_n := I - P_n = I - S_{\Theta}^n S_{\Theta}^{*n}$$

and observe that

$$(2.9) \quad \begin{aligned} P_n(T - A)P_n &= (T - A) \\ &- Q_n(T - A) + Q_n(T - A)Q_n - (T - A)Q_n. \end{aligned}$$

Furthermore by (2.8) we have

$$\begin{aligned} \|P_n(T - A)P_n\| &= \|S_{\Theta}^n S_{\Theta}^{*n} (T - A) S_{\Theta}^n S_{\Theta}^{*n}\| \\ &\leq \|S_{\Theta}^{*n} (T - A) S_{\Theta}^n\| \\ &= \|S_{\Theta}^{*n} T S_{\Theta}^n - A\| \rightarrow 0. \end{aligned}$$

If

$$k_{\lambda}(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda}z}, \quad \lambda, z \in \mathbb{D},$$

is the reproducing kernel for \mathcal{K}_{Θ} , then [10, p. 497] gives us the well-known operator identity

$$S_{\Theta} S_{\Theta}^* = I - k_0 \otimes k_0.$$

Iterating the above n times we get

$$S_{\Theta}^n S_{\Theta}^{*n} = I - \sum_{j=0}^{n-1} S_{\Theta}^j k_0 \otimes S_{\Theta}^{*j} k_0.$$

In other words,

$$Q_n = I - P_n = \sum_{j=0}^{n-1} S_{\Theta}^j k_0 \otimes S_{\Theta}^{*j} k_0$$

is a finite rank operator.

By (2.9) this means that

$$F_n := -Q_n(T - A) + Q_n(T - A)Q_n - (T - A)Q_n$$

is a finite rank operator which converges in norm to $A - T$. Hence $A - T$ a compact operator and, by (2.7), A satisfies $S_{\Theta}^* A S_{\Theta} = A$. \square

So far we know from Lemma 2.1 that every compact operator K on \mathcal{K}_Θ satisfies

$$\lim_{n \rightarrow \infty} \|S_\Theta^{*n} K S_\Theta^n\| = 0.$$

Furthermore, from Lemma 2.6 we see that an operator A for which $S_\Theta^{*n} A S_\Theta^n$ converges in operator norm can be written as $A = A_1 + K$ where K is compact and A_1 satisfies $S_\Theta^* A_1 S_\Theta = A_1$. To complete the proof of Theorem 1.2, we need to show that

$$S_\Theta^* A S_\Theta = A \iff A \equiv 0.$$

This is done with the following result.

Proposition 2.10. *Suppose A is a bounded operator on \mathcal{K}_Θ . Then $S_\Theta^* A S_\Theta = A$ if and only if $A \equiv 0$.*

Proof. Recall that

$$k_\lambda(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar{\lambda}z}$$

is the kernel function for \mathcal{K}_Θ . There is also the “conjugate kernel”

$$\tilde{k}_\lambda(z) = \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}$$

which also belongs to \mathcal{K}_Θ [10, p. 495]. The proof depends on the following kernel function identities from [10, p. 496]:

$$\begin{aligned} S_\Theta \tilde{k}_\lambda &= \lambda \tilde{k}_\lambda - \Theta(\lambda) k_0, \\ S_\Theta k_\lambda &= \frac{1}{\lambda} k_\lambda - \frac{1}{\lambda} k_0. \end{aligned}$$

This gives us

$$\begin{aligned} (A\tilde{k}_\lambda)(z) &= \langle S_\Theta^* A S_\Theta \tilde{k}_\lambda, k_z \rangle \\ &= \langle A S_\Theta \tilde{k}_\lambda, S_\Theta k_z \rangle \\ &= \langle A(\lambda \tilde{k}_\lambda - \Theta(\lambda) k_0), \frac{1}{z} k_z - \frac{1}{z} k_0 \rangle \\ &= \frac{\lambda}{z} (A\tilde{k}_\lambda)(z) - \frac{\Theta(\lambda)}{z} (A k_0)(z) - \frac{\lambda}{z} (A\tilde{k}_\lambda)(0) + \frac{\Theta(\lambda)}{z} (A k_0)(0). \end{aligned}$$

Re-arrange the above identity:

$$(A\tilde{k}_\lambda)(z) \left(1 - \frac{\lambda}{z}\right) = -\frac{\Theta(\lambda)}{z} (A k_0)(z) - \frac{\lambda}{z} (A\tilde{k}_\lambda)(0) + \frac{\Theta(\lambda)}{z} (A k_0)(0).$$

Multiply through by z :

$$(z - \lambda)(A\tilde{k}_\lambda)(z) = -\Theta(\lambda)(A k_0)(z) - \lambda(A\tilde{k}_\lambda)(0) + \Theta(\lambda)(A k_0)(0).$$

Divide by $(z - \lambda)$ and re-arrange:

$$(2.11) \quad (A\tilde{k}_\lambda)(z) = -\Theta(\lambda) \left(\frac{(Ak_0)(z) - (Ak_0)(\lambda)}{z - \lambda} \right) - \lambda \frac{(A\tilde{k}_\lambda)(0)}{z - \lambda}.$$

Observe that the functions

$$(A\tilde{k}_\lambda)(z) \quad \text{and} \quad \frac{(Ak_0)(z) - (Ak_0)(\lambda)}{z - \lambda}$$

belong to \mathcal{K}_Θ for all $\lambda \in \mathbb{D}$. This means that

$$\lambda \frac{(A\tilde{k}_\lambda)(0)}{z - \lambda}$$

must also belong to \mathcal{K}_Θ for all $\lambda \in \mathbb{D}$ which means (since there is an obvious pole at $z = \lambda$) that

$$(2.12) \quad (A\tilde{k}_\lambda)(0) = 0.$$

The identity in (2.11) can now be written as

$$(2.13) \quad (A\tilde{k}_\lambda)(z) = -\Theta(\lambda) \left(\frac{(Ak_0)(z) - (Ak_0)(\lambda)}{z - \lambda} \right).$$

Plug in $z = 0$ into the previous identity and use (2.12) to see that

$$0 = (A\tilde{k}_\lambda)(0) = \frac{\Theta(\lambda)}{\lambda} ((Ak_0)(0) - (Ak_0)(\lambda)), \quad \lambda \in \mathbb{D}.$$

Since Θ is not the zero function, we get

$$(2.14) \quad (Ak_0)(\lambda) = (Ak_0)(0), \quad \lambda \in \mathbb{D}.$$

Plus this into (2.13) to get that

$$A\tilde{k}_\lambda = 0 \quad \forall \lambda \in \mathbb{D}.$$

But since the linear span of these conjugate kernels form a dense subset in \mathcal{K}_Θ (the conjugation operator $f \mapsto \tilde{f}$ is isometric and involutive [10, p. 495]), we see that A must be the zero operator. \square

Proof of Proposition 1.3. For any $f, g \in \mathcal{K}_\Theta$ and $n \geq 0$ we have

$$(2.15) \quad |\langle S^{*n} A S^n f, g \rangle| = |\langle S_\Theta^n f, A^* S_\Theta^n g \rangle| \leq \|S_\Theta^n f\| \|A^* S_\Theta^n g\|.$$

Taking the supremum in (2.15) over $g \in \mathcal{K}_\Theta$ with $\|g\| \leq 1$, and using the fact that $\|S_\Theta\| \leq 1$, we get

$$(2.16) \quad \|S^{*n} A S^n f\| \leq \|S_\Theta^n f\| \|A^*\|.$$

From Remark 2.5, we conclude that the right hand side of (2.15) goes to zero as $n \rightarrow \infty$. Thus $S_\Theta^{*n} A S_\Theta^n \rightarrow 0$ strongly. \square

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