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A network of notions, concepts and processes for fractions and rational numbers as an interpretation of Didactical Phenomenology

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One of the main purposes of the research project that is being carried out in Mexico consists on building, from Filloy's point of view, a Local Theoretical Model (LTM) for fractions and rational numbers. For this purpose, four components of the LTM have to be constructed: models of formal competence, teaching models, models for the cognitive processes and models of communication. In this paper, a network of notions, concepts and processes for fractions and rational numbers that constitutes a fundamental part of the models of formal competence is described. This network was structured taking into account five mathematical processes grounded on Freudenthal's specimen of Didactical Phenomenology of fractions.

Keywords: Fractions, rational numbers, mathematical processes, teaching models.

INTRODUCTION

In the sixties, prestigious university educators discussed two approaches for the teaching of fractions in basic education. One group argued that decimal numbers could replace common fractions while the other one asserted that fractions should be taught, but only those that are more frequently used in everyday life (Rappaport, 1962). Those different positions generated discussions about the teaching and learning of fractions. Rappaport (1962) argued that fractions should be taught, since it is a necessary mathematical concept in order to understand rates and ratios. Usiskin (1979) claimed that a fraction is a representation of many numbers, and identified different uses of fractions involved in situations of splitting up, rates, proportions, mathematical formulae and equations.

Kieren's research (1976 & 1988) showed the complexity of the rational number concept. He stated that in order to construct a permanent knowledge of rational numbers, a person has to understand several constructs: measure, quotient, operator and ratio. For Kieren, part-whole relationship serves as a basis for connecting those constructs, which supports a construction of more formal knowledge that includes equivalence, and additive and multiplicative structures of a quotient field.

Lesh, Landau and Hamilton (1980) pointed out that the concept of rational number is a group of sub-constructs and integrated processes related to a wide range of basic concepts, appearing in a variety of problem situations.

Aware of the constructs identified by Kieren (1976); Behr, Lesh, Post and Silver (1983) proposed a model by grouping them in view of their relationships and links to equivalence, operations and problem solving. They also considered that the part-whole relationship is a fundamental meaning in order to understand the other meanings.

Although the part-whole relationship is an important meaning of fractions, Freudenthal (1983, p. 144) argued that the teaching based only on this relationship is restricted not only phenomenologically, but also mathematically, for it yields only proper fractions.

The research findings influenced worldwide educational reforms in the 90s, in order to improve the teaching and learning of fractions and rational numbers. In Mexico, in 1993, three different fraction meanings (quotient, measure and ratio) were included explicitly in the study programmes for primary

and secondary school. However, nowadays secondary school students go on obtaining low scores on questions about fractions in national tests. For instance, a comparative study (data from 2005 to 2008 provided by the *Instituto Nacional para la Evaluación de la Educación* [INEE], 2012) showed that the correct answers obtained by third grade secondary school students in the Excale test, applied in Mexico, were lower than 40%.

For many years – at least since 1962, as mentioned above – it has been thought that knowledge of fraction concept is important in understanding other mathematical concepts. Siegler and colleagues (2012) found that “elementary school students’ knowledge of fractions and division uniquely predicts students’ knowledge of algebra and overall mathematics achievement in high school” (p. 691). Siegler and collaborators’ findings imply that almost 60% of Mexican students would have a poor performance in mathematics in high school. Therefore, the teaching and learning of fractions and rational numbers pose still a problematic area for mathematics education.

As a consequence, it was decided to design a research project so as to characterise the structure and organization of the current teaching model of fractions; to analyse the interpretation made by authors of textbooks of the study programmes, and to characterise the effectiveness of the overall design and the understanding of those authors. This information will provide knowledge for designing an alternative teaching model of fractions for secondary school, and carrying out its experimentation in the classroom.

In this paper the most important result of the building up of the models of formal competence of the LTM for fractions and rational numbers is described. This outcome is a network of notions, concepts and processes for fractions and rational numbers, which is used as a methodological tool for analysing and designing teaching models.

THEORETICAL FRAMEWORK

Filloy’s Local Theoretical Models theory (see Filloy, Rojano, & Puig, 2008) is used as a theoretical and methodological framework to organize the research bearing in mind its four components: 1) models of formal competence, 2) teaching models, 3) models for the cognitive processes, and 4) models of communication.

Models of formal competence are related to fractions and rational numbers as mathematical objects; the teaching models and models of communication are associated to those mathematical objects as teaching objects, and models for the cognitive processes deal with fractions and rational numbers as learning objects.

METHODOLOGY

To construct the LTM, a specific methodology has been defined for each component. The models of formal competence were built mainly considering Freudenthal’s Didactical Phenomenology of fractions (1983). The teaching models comprise analyses of the Mexican current study programme for secondary school and mathematics textbooks used by students. The models for the cognitive processes and communication include: 1) Kieren’s explicative model (1988) which provides a global overview of individual construction of rational numbers knowledge; 2) a review of the results found in empirical studies carried out among secondary school students; 3) an analysis of other theoretical perspectives; 4) a characterisation of Mathematical System of Signs (MSS) (Filloy et al., 2008) of fractions and rational numbers, and 5) an analysis of items designed to assess these kinds of knowledge.

MODELS OF FORMAL COMPETENCE

This component of the LTM is related to mathematical knowledge, emphasising the structure and properties of fractions and rational numbers as mathematical objects. It is an important component due to: 1) the observer of educational experimentation should be capable to interpret the messages and texts generated by students and their teacher in the classroom. Therefore, the researcher needs more abstract MSS that encompasses all the MSSs used in the observed processes (Filloy et al., 2008, p. 36); 2) the design of a teaching model requires the knowledge of the characteristics of a competent user of fractions and rational numbers, and 3) the task designer must have examples of diverse situations in which these numbers are used readily available.

For the authors of this paper, the fulfilment of those requirements is possible through analysis of Didactical Phenomenology of fractions done by Freudenthal (1983). In these types of analyses, those phenomena

organized by fraction and rational number concepts are described; consequently they provide diverse examples of how those numbers are used in various situations. In the processes of identifying different phenomena, the characteristics of the necessary competencies for using fractions and rational numbers for solving problems emerge.

Freudenthal starts the characterisation of phenomena from the first ideas of fractions embedded in everyday language through to the complex concepts included in the mathematical theory of rational numbers.

The specimen of Didactical Phenomenology of fractions was reinterpreted by the authors, identifying classes of phenomena: description and comparison of quantities, magnitude values or objects, division of substances measured by magnitudes, distribution of quantities, measurement, and numbers as part of a numerical system. For each one of the first five classes, different notions and concepts of fractions and rational numbers were identified, organized, schematized and linked with five mathematical processes: describing, comparing, dividing, distributing and measuring. The sixth class refers to the mathematical constructions of rational numbers.

In the following sections of this paper a brief description of these processes/classes of phenomena associated with fractions and rational numbers is made.

Describing. In everyday language there are expressions in which fractions are used to describe: i) a quantity or magnitude value through another quantity or magnitude value, ii) a measure expressed by numbers, iii) cyclic or periodic processes, and iv) ratios. These expressions are related to the process of ‘describing’ (see Figure 1) and are linked with four kinds of phenomena in different contexts in which *the fraction is acting as a descriptor*.

Expressions such as: ‘half of a way’, ‘three quarters of an hour’ and ‘one third of the world’s population will suffer water shortages by 2015’ are examples in which a description of a quantity or magnitude value through total distance of a way, of a number of minutes of an hour, and of the number of people that live in the world is made. Measures can be described using phrases as ‘½ litre of milk’ or ‘¼ of a pound’. Different cyclic or periodic processes are described using fractions and mixed numbers, for example: ‘a driver only could run ½ times the track due to a mechanical failure’ and ‘a Chinese competitor presented an almost perfect dive in which he rotated 2 ½ times’. Likewise, fractions are employed to describe ratios relating two quantities or magnitude values. For instance, ‘3 out of 5 parts’ and ‘six of every one hundred Mexican people speak an indigenous language’.

Comparing. Three types of phenomena associated to the process of ‘comparing’ in which *fractions represent comparers* were identified (see Figure 2). Fractions are

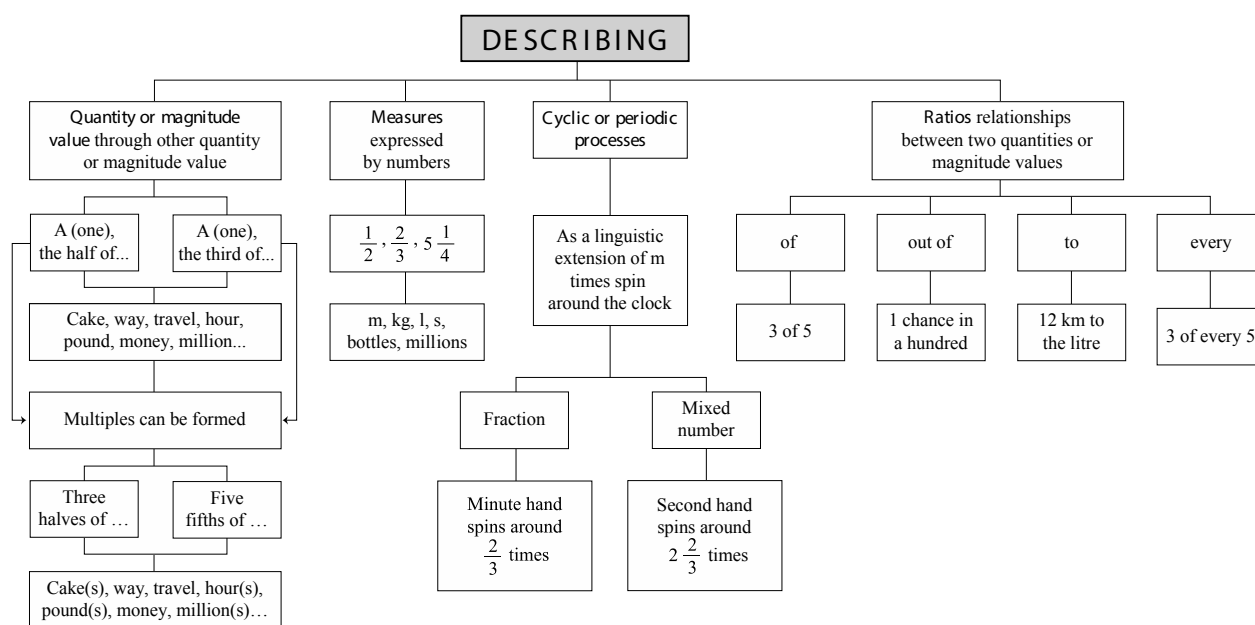


Figure 1: Schematic representation of the ‘describing’ process as part of the network

also used in everyday language to express the result of a comparison between two quantities or magnitude values; this comparison is related to a first abstraction level. In a comparison of the salary of two persons, one that earns 1 800 € and the other 1 200 € it is possible to say: 'one person earns $\frac{3}{2}$ times than the other'.

In a second abstraction level, diverse aspects of the fraction were incorporated: fracturing operator, fracturing relation, ratio relation, ratio operator and transformer.

Two objects that are brought close together, or are in some other way considered, as though the smaller were part of the bigger, can be compared using the fracturing operator or fracturing relation. For example, if we compare the height of two bookcases, one has six shelves and the other has seven shelves, the fractions $\frac{6}{7}$ and $\frac{7}{6}$ represent these two possible comparisons.

Fraction as a ratio relation is the result of comparing two objects that are separated. This comparison can be

made with respect to a number or a magnitude value of each object. To compare the height of two of the tallest towers in the world: the Tokyo Skytree and the Canton Tower, having in mind that one of them measures 634 m and the other 610 m, the fractions $\frac{634}{610}$, $\frac{317}{305}$, $\frac{610}{634}$ and $\frac{305}{317}$ are ratio relations that represent the comparisons of the towers' heights.

The aspect of fraction as ratio operator acts on a quantity or magnitude value, transforming it into another quantity or magnitude value. For example, to obtain the amount that each type of heir gets from an inheritance of 60 000 €, knowing that $\frac{2}{3}$ are for the sons and $\frac{1}{3}$ for the widow, the fractions $\frac{2}{3}$ and $\frac{1}{3}$ are ratio operators that transform 60 000 € in 40 000 € and 20 000 € respectively.

In its intermediate stage as transformer, the fraction performs on dimensions of objects by deforming or mapping to a scale $\frac{a}{b}$. To determine a scale of the transformation of an image whose size is 1280 x 960 pixels to another size of 1600 x 1200 pixels, the fraction $\frac{5}{4}$ represents both a scale and a transformer.

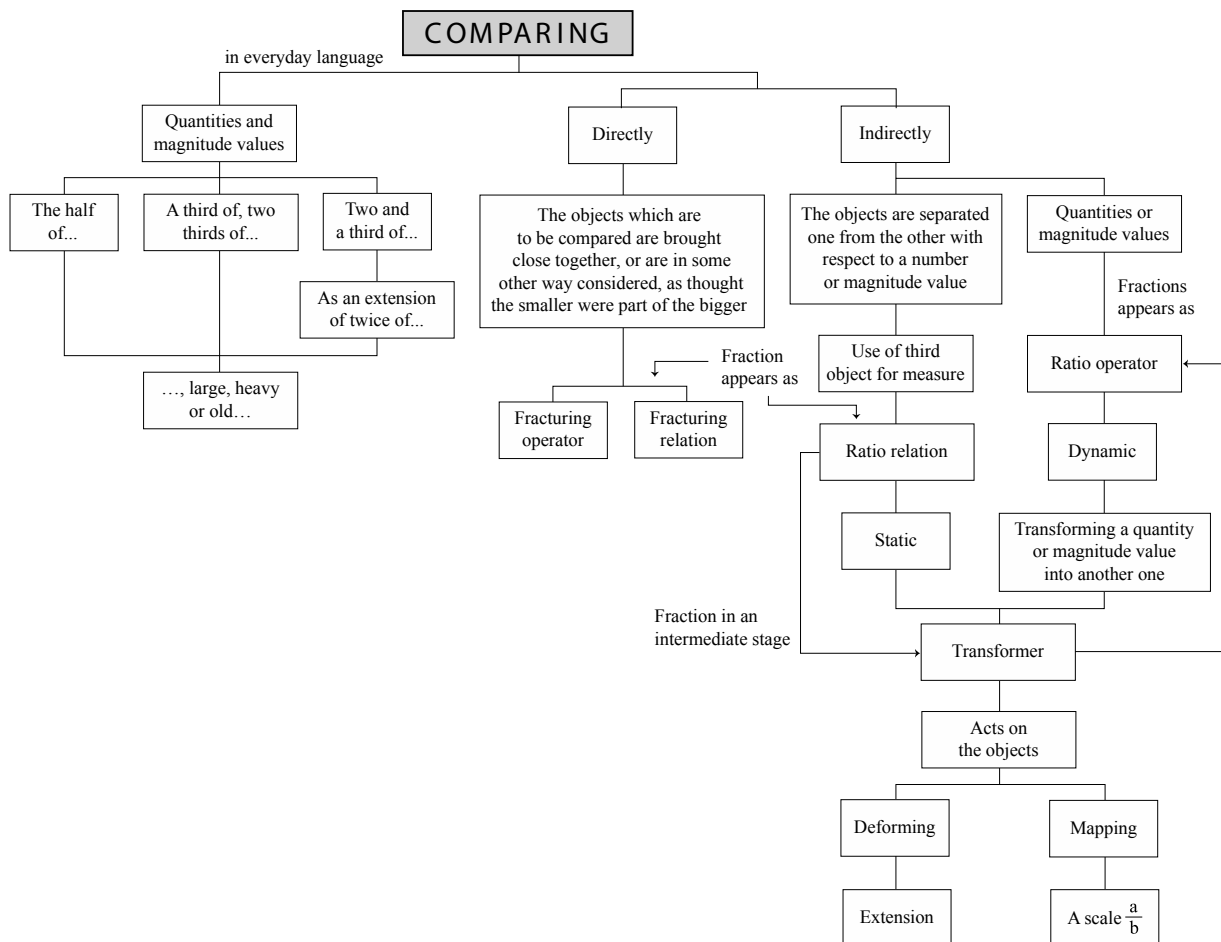


Figure 2: Schematic representation of the 'comparing' process as part of the network

Dividing. To divide substances measured by magnitudes, the whole is partitioned in equal parts, relating it to one or more of those parts (see Figure 3). In that way, *the fractions represent part-whole relationships and are acting as fracturers.* For instance, 18/27 parts of the roulette are black or 11/25 of the numbers of a bingo card are even numbers. The expression numerator/denominator represents the part-whole relationship.

There are different methods of fracturing in equal parts: in an irreversible, reversible or symbolic way. The estimating of equality of the parts can be made at sight, by feel or by others methods like the use of congruences or symmetries or employing different measuring instruments. Regarding the whole it can be discrete, continuous, definite, indefinite, structured or lacking structure. There are transitions among those types of wholes. The parts of the whole can be connected or disconnected.

Distributing. A distribution process of small or large quantities can be related to a finite set model and/or to a magnitude model (see Figure 4); *the fraction represents the result of this process.* On the one hand, when 10 leaf packs are distributed among 5 persons, each person gets 1/5 of the total of leaf packs ($10 \text{ leaf packs} \div 5 \text{ persons} = 2 \text{ leaf packs per person} = 1/5 \text{ of } 10 \text{ leaf packs}$

for each person). In this case, the process has finished and is associated with the finite set model.

On the other hand, if 3 sacks of rice whose mass is 50 kg each are distributed between 2 persons, each person would get 1 whole sack of rice and it would be necessary to distribute the remaining sack considering its mass, thus each person would get $\frac{1}{2}$ of a sack more ($50 \text{ kg} \div 2 \text{ persons} = 25 \text{ kg per person} = \frac{1}{2} \text{ of a sack per person}$). In this case, the process is related to a magnitude model because the use of the mass unit has made possible the distribution.

When the magnitude model is introduced to distribute the remainder, it is necessary to constitute a magnitude in a system of quantities that include different requirements ranging from an equivalent relation to the division of an object in an arbitrary number of partial objects. The expression numerator/denominator represents, in this case, the quotient of a division; in other words, it is the result of the distribution process.

Measuring. Fractions also represent measures of magnitudes (see Figure 5). Magnitudes can be measured with unconventional units or using metric and non-metric systems. The metric system is related to decimal fractions. The non-metric systems are the an-

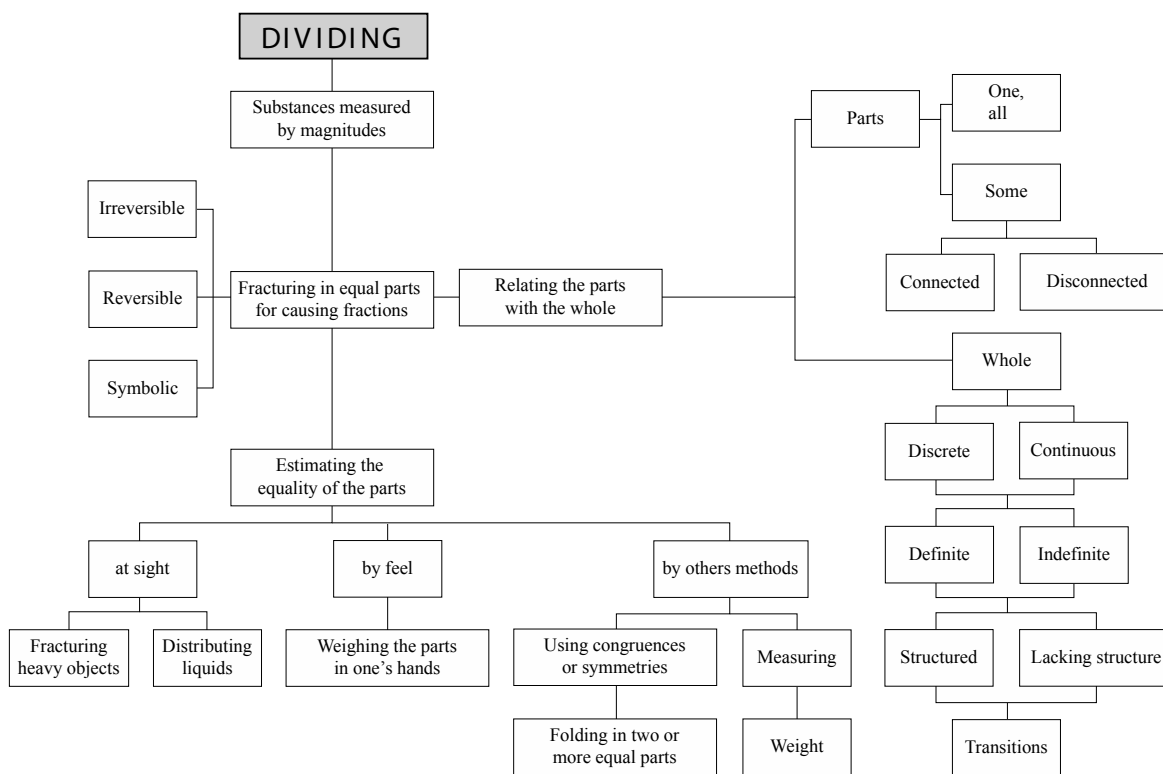


Figure 3: Schematic representation of the 'dividing' process as part of the network

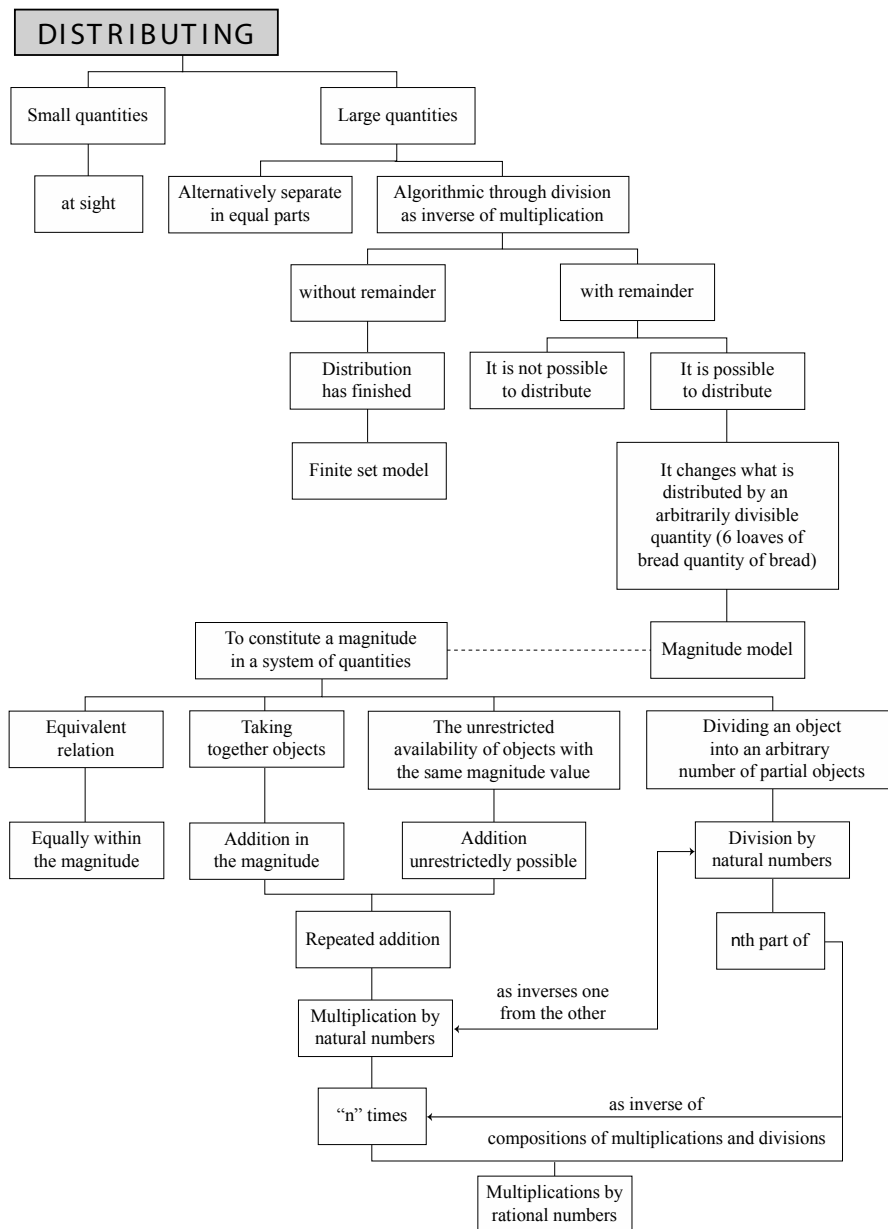


Figure 4: Schematic representation of the 'distributing' process as part of the network

cient systems of measurement and the English system; some of the former have various decimal relations; for example, in the Egyptian system 1 khet = 100 sq cubits.

Decimal fractions can be thought of as elements of ever finer nets. There are connections among these elements in the same net through addition, subtraction and comparison, or between elements from two different nets using multiplication.

To measure magnitudes it is also possible to use the number line, selecting an arbitrary length to represent a unit of measurement to be used for building different scales of length, time, temperature and mass measurements among other magnitudes. *The fraction*

represents the result of a measurement process and acts as a measurer.

In the last part of the network of notions, concepts and processes, the phenomena that are associated with the rational numbers as a number system are included. Two ways of constructing this kind of numbers outlined by Freudenthal – an *a posteriori* axiomatic way (as equivalence classes of fractions) connected with the algebraic nature of these numbers, and an *a priori* genetic way (as a commutative semigroup) linked with the ratio operator – were considered as the sixth class of phenomena. These constructions of rational numbers correspond to a more abstract level. In the latter construction, an arbitrary length was selected

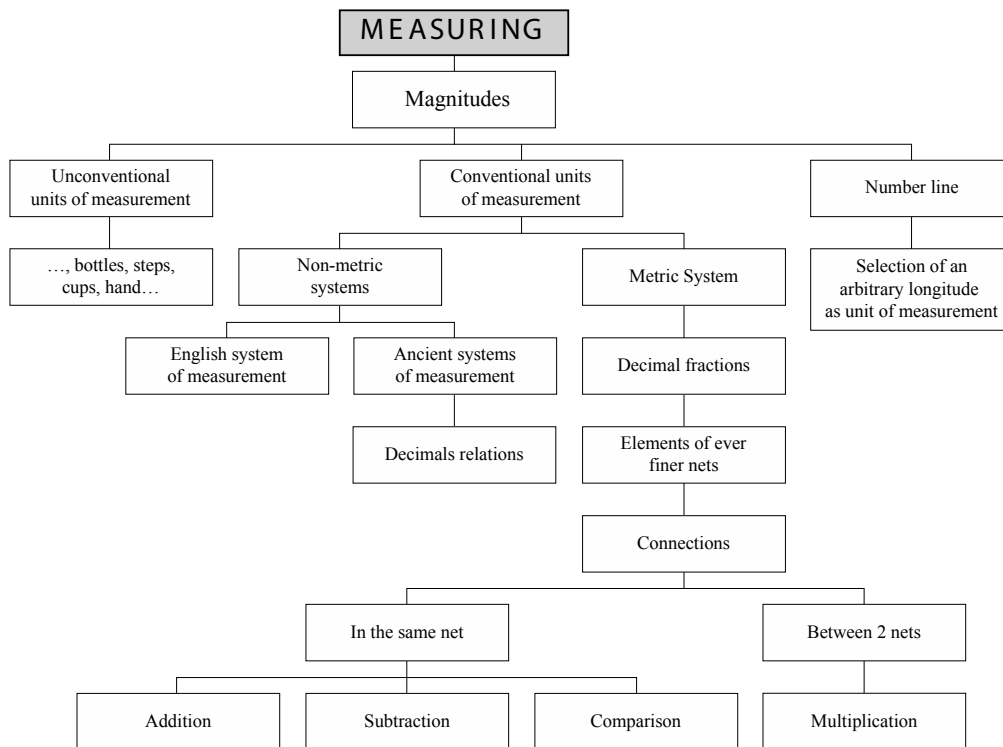


Figure 5: Schematic representation of the 'measuring' process as part of the network

in order to discover rational numbers as measures of a segment or points on the number line.

The network encompasses from the use of first ideas of fractions embedded in everyday language through to the mathematical constructions of rational numbers. Due to the extension of the network, it cannot be included in this paper, but the authors expect that readers could get a general idea from the above paragraphs.

THE USE OF NETWORK AS METHODOLOGICAL TOOL

In order to analyse a teaching model of fractions, the network was used as a methodological tool through examination of the activities included in mathematics textbooks, which were used by secondary school students in Valencia, Spain (Real, Figueras, & Gómez, 2013). The results of that analysis showed that the aspects of fractions in such teaching model are: descriptor, fracturer, ratio operator, and measurer (this last aspect appears in an implicit way). For representing a part-whole relationship the fraction as descriptor is used. The fraction as comparer is utilised for comparing two quantities or magnitude values. For transforming a quantity or magnitude value into another one, the fraction as ratio operator is employed. The

fraction as a measurer is a means for solving some activities, but its use to compare different fractions in the number line is not included. In an overall view, some phenomena within the network are included in this teaching model of fractions, even though the authors state that it is based on problem solving.

FINAL REMARKS

All these mathematical processes related to different notions and concepts of fractions and rational numbers, included in the network, describe an overall perspective about teaching and learning of these mathematical objects. The network and its description constitute a potential theoretical element to analyse and to design an activity or a teaching model. Its use as a methodological tool has been proved through the analysis of two different teaching models: one from Spain (Real et al., 2013) mentioned above, and the other from Mexico (Real, 2014). Additionally its validation to design a teaching model for secondary school students in Mexico is being carried out.

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