# Cognitive flexibility and reasoning patterns in American and German elementary students when sorting addition and subtraction problems

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*This study focuses on sorting and reasoning patterns in second and fourth grade elementary students. Based on a theoretical interpretation of flexibility in mental calculation we argue that sorting and reasoning for addition and subtraction problems provide information about students' cognitive flexibility. For the study reported here, a problem-oriented, guideline-based interview incorporating 12 two-digit addition and subtraction problems was conducted with American and German students. Following the theoretical background and design, this article gives an overview of results concerning reasoning patterns.*

**Keywords**: Reasoning patterns for easy and hard problems, flexibility in mental addition and subtraction, cross cultural study.

## SYNOPSIS OF THEORETICAL BACKGROUND

In recent decades math educators and researchers have taken an increasing interest in students' techniques for performing mental addition and subtraction. Similarly, math educators have emphasized the importance of developing flexibility in performing mental calculations (e.g., Anghileri, 2001; National Council of Teachers of Mathematics, 2000; Selter, 2000). In this context, mental calculation, in the sense of solving multi-digit arithmetic problems mentally without using paper and pencil procedures, has received increasing attention among researchers (e.g., Blőte, Klein, & Beishuizen, 2000; Heirdsfield & Cooper, 2004; Rathgeb-Schnierer, 2006; Threlfall, 2009).

Recent research on flexibility in mental addition and subtraction has revealed the following general patterns: (1) a negative impact on flexibility when addition and subtraction are learned by examples (Heirdsfield & Cooper, 2004); (2) students' mental strategies influenced by multiple factors (Blöte, Klein, & Beishuizen, 2000; Rathgeb-Schnierer, 2006; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009), (3) some approaches more than others support the development of mental math flexibility in elementary students (Heinze, Marschick, & Lipowsky, 2009; Rathgeb-Schnierer, 2006), (4) greater mathematical competence among students who exhibit flexibility in mental calculation (Heirdsfield & Cooper, 2004; Threlfall, 2002), and (5) recognition of number patterns and relationships correlated with computational flexibility (Macintire & Forrester, 2003; Rathgeb-Schnierer, 2010; Threlfall, 2009).

At the same time, inconsistent perspectives appear in the research literature on mental calculation flexibility (Rathgeb-Schnierer & Green, 2013; Star & Newton, 2009). For example, nearly all definitions have the same basic idea of flexibility in mental calculation as an appropriate way of acting when faced with a problem, which is to say that flexible strategies are adapted dynamically to problem situations. Nevertheless, there exist in current research crucial differences concerning the meaning of what constitutes "appropriate" as well as the use of different methods to measure "flexibility" and "appropriate ways of acting" (Rechtsteiner-Merz, 2013). Many researchers define flexibility as the choice of the most appropriate solution to a problem (Star & Newton, 2009; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). While Torbeyns and colleagues (2009) had the same notion, they enhanced their definition to incorporate both computational accuracy and timeliness: "strategy flexibility is conceived as selecting the strategy that brings the child most quickly to an accurate

answer to the problem" (Torbeyns et al., 2009, p. 583). Even contrasting views of "flexibility" focus on mental outcomes, including elements of speed and accuracy.

The project described here refers to a notion of flexibility based on the model of process of calculation (Rathgeb-Schnierer, 2011; Rathgeb-Schnierer & Green, 2013) and current research results which underscore the crucial role of number patterns and relationships for flexibility in mental math (see above). Therefore, the following definition of flexibility has evolved: cognitive actions that match the combination of *strategic means* [1] (Rathgeb-Schnierer & Green, 2013) to the recognized number patterns and relationships of a given problem in the context of processing a problem solution. In this sense, our definition incorporates the capacity to use several strategies and adaptive thinking and is similar to Threlfall's "interaction between noticing and knowledge" (2002, p. 29):

> *When faced with a fresh problem, the child or adult who follows different solution paths depending on the numbers does not do so by thinking about what the alternatives are and trying to decide which one to do. Rather, he or she thinks about the numbers in the problem, noticing their characteristics and what numbers they are close to, and considering possibilities for partitioning or rounding them. (Threlfall, 2002, p. 41)*

Our work seeks to understand the mental processes that underlie problem solutions, the *cognitive elements* [2] that support the solution process. Therefore, we examine directly whether students recognize problem characteristics, number patterns, and relationships, and whether they use this knowledge for solving a problem (Rathgeb-Schnierer & Green, 2013). Our definition of flexibility differentiates between solution processes based on learned procedures (step-by-step mental calculations) and recognized problem characteristics, number patterns, and relationships. Hence, our project is aimed at identifying degrees of flexibility in students' mental arithmetic by identifying the cognitive elements that sustain the solution process (learned procedures or problem characteristics, number patterns, and relationships).

## SYNOPSIS OF THE PROJECT

The project incorporated interview questions directed at students' recognition of problem characteristics, number patterns, and relationships. Students were encouraged to look at the given problems, sort them into either 'easy' or 'hard' categories, and give reasons for the sorting (details in Rathgeb-Schnierer & Green, 2013). In addition, we examined country and classroom-related differences.

Generally, our study was aimed at answering the following empirical questions: Do students recognize problem characteristics, number patterns, and relationships, and what reasoning do they exhibit about these elements? To what extent is cognitive flexibility in mental calculation related to sorting and reasoning patterns? Do students of different grade levels or countries exhibit differences or similarities (or both) in sorting and reasoning? In this paper, we focus predominantly on the first question.

### Design

A qualitative study with guideline-based interviews was conducted with 78 second and fourth grade American (Charlotte, North Carolina) and German (Baden-Württemberg) students from ten different classrooms (three from 2nd grade and two from 4th grade in each country). These classrooms were chosen purposely by observing several math lessons and characterizing them by the degree of directness of instruction and openness of tasks. Typically, seven students per classroom were interviewed, and these were teacher selected as being either average or strong in arithmetic.

The qualitative, problem-oriented and guideline-based interview contained twelve two-digit addition and subtraction problems that were displayed on small cards. Each problem was purposely designed with at least one special numerical pattern or relationship feature: 31-29 (renaming required; range of numbers; 29 is close to thirty), 46-19 (renaming required; 19 is close to twenty), 63-25 (renaming required), 66-33 (no renaming required; double and half relation; double digits; inverse problem to 33+33), 88-34 (no renaming required; double and half relation of the ones), 95-15 (no renaming required; fives at the ones place), 33+33 (no regrouping required; double digits; double facts at the ones place; inverse problem to 66-33), 34+36 (regrouping required; double facts at the tens place; ones add up to ten), 47+28 (regrouping required), 56+29 (regrouping required; 29 close to thirty), 65+35 (regrouping required; fives at the ones place add up to ten), 73+26 (no regrouping required).

Students were interviewed one-on-one for 15 to 30 minutes, with video recording. Normally, each interview had two parts: (1) sorting problems into categories "easy" and "hard" and talking about the reasons for sorting, and (2) solving problems (if not already done during the sorting process). If it was indicated by the situation (for instance by utterance or action of the student), a third part was sometimes added to compare selected problems and direct student's attention to the characteristics of the problem 46-19 ("Is there way to make this problem easier?"). The cards were mixed and laid out on the table, and students were asked to sort cards one-by-one under "easy" or "hard" labels placed on opposite sides of the table. After a card was placed, students were asked, "Why is this problem easy/hard for you?"

Interviews were conducted by one researcher and took part in the last two months of the academic year (Germany 2010 and 2012, USA 2011). All interviews were transcribed in their original language for content-analysis.

Based on theory and student data, a coding system was developed. Two types of reasoning were applied to both easy and hard problems: reasoning by problem characteristics for easy (RCE) and hard (RCH) problems and reasoning by solution procedures for easy (RSE) and hard (RSH) problems. Reasoning by problem characteristics was coded when students referred for instance to features of numbers (e.g. similarity of tens and ones), numerical relations (e.g. double and half, analogies), and relations of tasks (e.g. invers problem). Reasoning by solution procedures was coded when students immediately started to describe a technique of mental, step-by-step computation (e.g., Selter, 2000; Threlfall, 2002).

All interviews were coded using event sampling, the individual sort for each of the 12 cards. An event was defined as one complete statement of reasoning that began with a student's first utterance after sorting a card and ended either when the student stopped talking or was interrupted by the interviewer. Multiple codes in the same category could be assigned to an individual event, but the assignment to one of the four core categories was exclusive. For example, a student's reasoning could either be coded as reasoning by characteristics (RCE/RCH) or reasoning by solution procedure (RSE/RSH), never to both; the deciding factor was always the very first statement. For validation, the entire data set was scored by two independent judges, and all disagreements were resolved through discussion.

## RESULTS

Sixty-nine students were included in our analysis (eight interviews were dropped [3]): 28 fourth graders, 17 American students and 11 German students, as well as 41 second graders, 19 American students and 22 German students.

| Reasoning (31-29) | Coding |
|---|---|
| This one is easy, cause both of the numbers are very close to each other, so – uhm, you can find out that there're only a couple numbers apart and so the answer would be two. | reasoning by problem characteristics – easy (core category)<br>• relation of numbers (code)<br>  • range (sub-code) |
| This is an easy one, because I do 31 – 20 – uhm, and this is 11, and 11 – 9 equals 2. | reasoning by solution procedures – easy (core category)<br>• compose and decompose (code)<br>  • jump method (sub-code) |

**Figure 1:** An example of categorizing

| | Sorting | Reasoning by Problem Characteristics | Reasoning by Solution Procedures | Other |
|---|---|---|---|---|
| **Total** | 902 (100.0%) | 492 (54.54%) | 284 (31.48%) | 127 (14.07%) |
| **Easy** | 644 (71.39%) | 351 (38.91%) | 236 (26.16%) | 58 (6.43%) |
| **Hard** | 258 (28.60%) | 141 (15.63%) | 48 (5.32 %) | 69 (7.64%) |

**Figure 2:** Frequency of sorting and reasoning

### General patterns in sorting and reasoning

In terms of sorting and reasoning, Figure 2 displays frequencies for easy and hard problems sorted by reasoning category.

In total, 902 reasons appeared in the sample, an average of 13.07 per student (minimum 8, maximum 17). More than two-thirds of the problems were judged as easy. Regarding the two types of reasoning, 54.54% belong to reasoning by problem characteristics, 31.48% to reasoning by solution procedure, and 14.07% to other reasons (which were excluded here from further data analyses). Regarding the category "reasoning by problem characteristic" the ratio for easy to hard is 2.48 to 1; within the category "reasoning by solution procedure" the ratio is 5.24 to 1.

The percentage of each reasoning type related to all given arguments in each grade (separated by problem characteristics and solution procedures) is shown in Figure 3. Differences between grades occurred for both problem characteristics and solution procedures. Regarding the fourth graders, 69.53% of the arguments referred to problem characteristics and 30.47% to solution procedures. In contrast, second graders exhibited 10% fewer arguments belonging to problem characteristics and 10% more to solution procedures. Prior research by Selter (2000) had led us to expect fourth graders to argue more using solution procedures, since they are more familiar with the standard algorithms than second graders (in Germany the standard algorithms are introduced in third grade). Interestingly, our fourth graders exhibited less solution procedure reasoning than second graders, and they used nearly 50% more reasoning by problem characteristics than by solution procedures. This finding might be affiliated with the range of numbers used in the interview tasks and is probably explainable by the greater familiarity of fourth graders with two digit numbers.

In order to examine students' recognition of problem characteristics, number patterns, and relationships, we focused on the core categories "reasoning by problem characteristics." Figure 4 shows the percentage of the entire sample for each code in the two core categories "easy" and "hard." Distinct differences between easy and hard problems are readily apparent. With

| | 4th (n=28) | 2nd (n=41) |
|---|---|---|
| **Reasons - total** | **340 100%** | **436 100%** |
| **Characteristics - total** | **234 68.82%** | **258 59.17%** |
| Characteristics - easy | 165 48.52% | 186 42.66% |
| Characteristics - hard | 69 20.29% | 72 16.51% |

| | 4th (n=28) | 2nd (n=41) |
|---|---|---|
| **Reasons – total** | **340 100%** | **436 100%** |
| **Solution - total** | **106 31.17%** | **178 40.82%** |
| Solution - easy | 89 26.17% | 147 33.71% |
| Solution - hard | 14 5% | 31 7.11% |

**Figure 3:** Percentage of reasoning in each grade

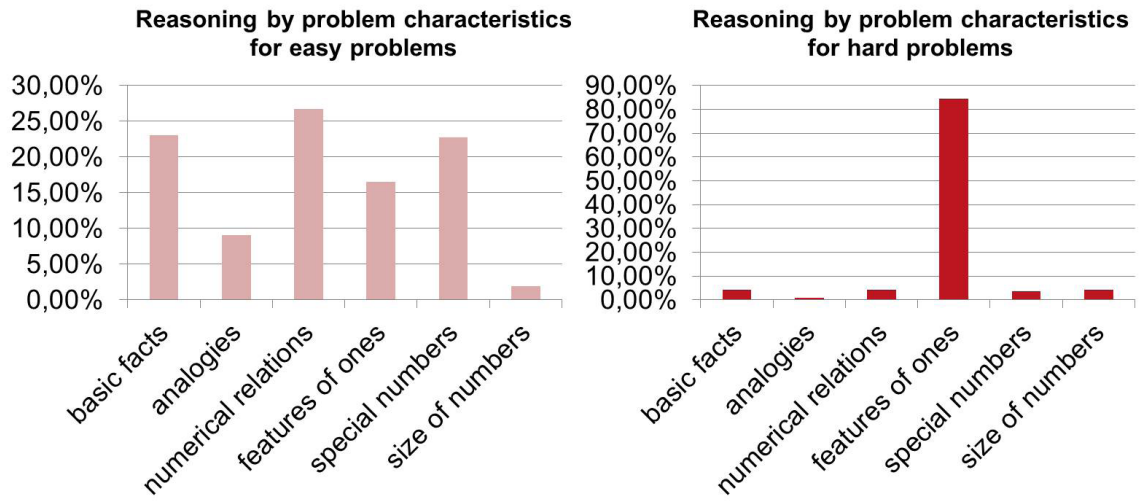


**Figure 4:** Reasoning by characteristics for easy and hard problems

reasoning for easy problems, students consistently referred to a variety of number patterns and problem characteristics. They pointed, for example, to basic facts [4] (23.36%), analogies between tens and ones (8.83%), numerical relations (27.06%, e.g. range of both numbers, double half, inverse problem), special features of numbers at the ones place (16.23%, e.g. two fives at the ones place or numbers that add up to ten), special numbers (22.50%, e.g. double digits or numbers close to the next ten), and also size of numbers (1.99%).

A completely different pattern was exhibited with reasoning by problem characteristics for hard problems. Students referred predominantly to "features of ones" (84.39%), referring directly to the numbers in the ones place. One typical characteristic for hard problems was the specific constellation of numbers in the ones place that leads to the need for renaming.

Student: So *((selects the card 46-19))* ok *((puts it in the "hard" column))* I'll put that one in the hard. So 46 plus, um, minus 19 *((points at the numbers on the card))* I think it's a little harder because first *((points at the tens on the card))* you can do four minus one which is three, and that one *((points at the 6 and the 9))* is kind of hard, because this one *((points at the 9))* is bigger than that one *((points at the 6))*.

Even if problems that required renaming were often considered as hard, we also found opposite judgments especially with those problems that include one number which is close to the next ten (predominantly numbers with a nine) for instance 31-29 (see Figure 2), 46-19 or 56+29. Whenever students discovered problem characteristics other than the requirement of renaming, they considered those problems to be easy.

Example for 56+29:

Student: This one is easy *((points to the problem 56+29))* because 29 is close to 30, I only have to add one.

Interviewer: That means you add one to 29?

Student: Here is 30 *((points to 29))*, and here 55 *((points to 56))*, um, and I do 30 plus 55.

Example for 46-19:

Student: *((points on 46-19))* One could easily subtract 20 *((points on the minuend))*.

Interviewer: Mhm.

Student: So, for 46 minus 20 you need no special trick, you know the answer immediately.

Interviewer: Yes.

Student: - this is 26, and then you need (..) to (..) –moment (..) to add one.

Interviewer: Why do you need to add one? Is this correct?

Student: Yes, because we have one more subtracted, that means we have subtracted one number more.

Summarizing, our data analyses suggest that addition and subtraction problems that require renaming are not regarded by students to be hard in general. In fact, in our sample, students depicted a more differentiated assessment of problems that required renaming: They considered those problems to be easy when additional problem characteristics (e.g., 9 in ones place) were obvious.

### Reasoning in different classrooms

Qualitative analyses suggest classroom-related tendencies concerning reasoning patterns. Figure 5 shows students' reasoning patterns from three different second grade classrooms. Each column represents the arguments of one student, and the size of the dots displays the quantity of arguments in the category. The math lessons in classrooms A (American) and C (German) showed the same high degree of open tasks and self-regulated learning, whereas in classroom B (German) predominantly direct instruction and closed tasks were observed.

Students from classroom A referred almost exclusively to problem characteristics to explain why a problem seems to be easy or hard. Students from classroom B showed a distinct preference for reasoning by solution procedures and exhibited a quite restricted range in their reasoning: Whenever reasoning based on solution procedures appeared, "composing and de-

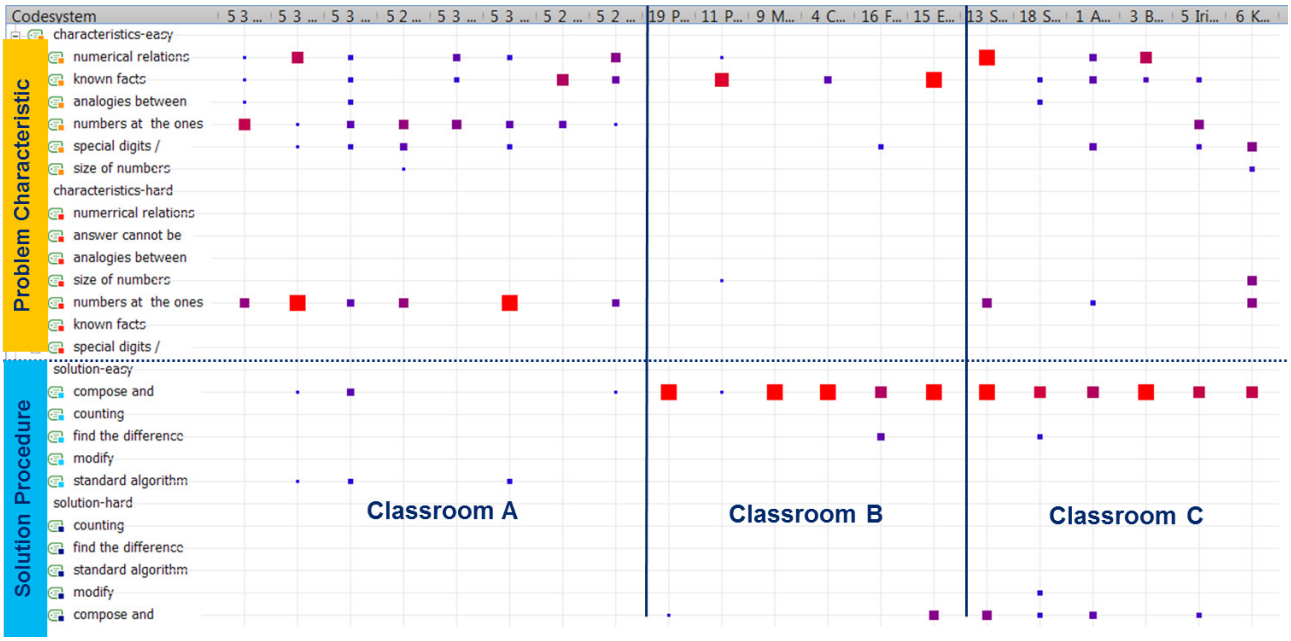


**Figure 5:** Reasoning patterns in classrooms

composing" strategies were exhibited. The few times reasoning based on problem characteristics emerged, students referred predominantly to "known facts" (basic facts). Classroom C students can be described as more comprehensive, since problem-based and procedure-based reasons coexist. In that sense, this group showed a great variety of problem related arguments and no variety at all in the section of procedure related arguments (note that solution procedures for both hard and easy problems were coded "compose and decompose"). This is an important pattern revealed throughout the whole data: Whenever students' reasoning referred to problem characteristic and number patterns, a great variety was noticeable. Therefore, we presume that those students act dynamically and can be considered as mentally flexible. On the other hand, whenever students depicted reasoning based on solution procedures, a quite restricted range of reasons was empirically observable. Hence, those students acted very statically and can be considered as mentally rigid.

## CONCLUSION

In summary, we have reported a variety of patterns in students' reasoning for easy and hard problems that reflected the characteristics we built into the problems. There was obviously a greater variety of arguments for easy problems than for hard ones. Based on individual and classroom related reasoning patterns, we were able to identify three forms of reasoning: flexible (multiple reasons predominantly referring to characteristics), rigid (one reason referring to a solution procedure) and mixed (multiple reasons when referring to characteristics, one reason when referring to a solution procedure). Data analyses suggested that students who exhibited reasoning by characteristics were more cognitively flexible than students who didn't. Our next step will be to identify whether the recognized characteristics (that were visible in students' reasoning) function to sustain the solution process, and therefore can be considered as cognitive elements.

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## ENDNOTES

1. Strategic means are neither holistic strategies nor cognitive menus that complete a solution path; they are distinct devices that can be combined in flexible ways to modify complex problems to make them easier. Strategic means include decomposing and composing, modifying a problem, deriving the solution from a known problem, and using decade analogies.

2. Students' solution processes are based on specific experiences that we designate with the term "cognitive elements." Such cognitive elements that sustain a solution process can be learned *procedures* (such as computing algorithms) or *number characteristics* (such as number patterns and relationships).

3. Eight interviews were dropped from the analysis for inaudibility, early termination or irreversible deviation from the interview guideline.

4. The code "basic facts" was purposely assigned to the core category "reasoning by problem characteristics." In our opinion, referring to basic facts is not a sign of procedure, but an indicator of the recognition that parts of the problem belong to basic facts. In this context, we only assigned an argument to the code "basic facts" when students expressed clearly and consciously that part of a problem or a whole problem is known by heart. The naming of a result by itself was not assigned to the code "basic facts."