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# The rules for the order of operations: The case of an inservice teacher

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*In this paper a two-stage project is presented concerning the rules for the order of operations. During the first stage the mal-rules used by an experienced teacher as he evaluated arithmetical expressions were recorded and a session for repairing these misinterpretations followed. During the second stage the influence of the teacher's teaching on his sixth graders was examined. The findings showed that the initial understanding of the teacher was so persistent that almost all his students and in order to evaluate the same arithmetical expressions used exactly the same mal-rules.*

**Keywords:** Mal-rules, order of operations.

## INTRODUCTION

The major issue in mathematics teaching and learning is whether students really know mathematics or whether they just memorize rules or conventions. The rules for the order of operations that are introduced during the fifth and sixth grades constitute a representative example. Even though, one can argue that the precedence rules are a matter of procedural knowledge. Lampert (1986) claims that there are contexts that order of operations matters, and according to Merlin (2008) the order of precedence is not simply the sort of convention adopted without careful consideration; rather, it reflects something essential and deep about the operations themselves. Students are taught that symbols of inclusion (i.e.,  $()$ ,  $[\ ]$ ) can be used to show which operation is to be performed first in an expression. If there are more than three operations then there would be so many parentheses and brackets that the expression would look extremely complex. Therefore, in order to avoid this, mathematicians have agreed on an order for performing the operations and the parentheses are used only to change this order (Foerster, 1994). The rules for the order of operations are:

- i. Brackets first
- ii. Evaluate expressions with exponents
- iii. Carry out multiplications or divisions from left to right
- iv. Carry out additions or subtractions from left to right.

For young students these rules might appear random and therefore meaningless. So, according to Wu (2007) the situation is like this: students are encouraged to memorize things without understanding their use or meaning and teachers are also becoming part of the game as they know that exam points can be surely achieved by this useless memorization. This way of teaching order of operations results in some overgeneralizations that are used equally by primary school students (Linchevski & Herscovics, 1994), middle school students (Blando, Kelly, Schneider, & Sleeman, 1989), university students (Pappanastos, Hall, & Honan, 2002), and prospective elementary teachers (Glidden, 2008). However, there is no research on the teaching practice of in-service teachers in this topic. This leaves out in-service teachers. Therefore, in this study we try to find out how such misconceptions of an experienced in-service primary school teacher may influence the way his students conceive the topic of the order of operations.

## BRIEF LITERATURE REVIEW

Perhaps, the whole story starts during the early years of schooling when teachers use problems of the type  $5 + 2 \times 3 + 10 - 5 = ?$ . In these problems each operation is followed by "and then" (i.e., 5 plus 2 and then what you found times 3 and then that answer plus 10, etc.). Indeed, very often this kind of problems is considered as a proper one since the students are encouraged to

think and carry out mental operations. However, this may – unintentionally- contribute to students’ difficulties related to the order of operations convention (Schrock & Morrow, 1993). They disregard the order of operations practice and therefore learn that operations are simply worked from left to right (Kieran, 1989). Linchevski and Livneh (1999), working with 53 sixth graders, found that possibly students have generalized incorrectly the rule for the order of operations and believe that addition takes precedence over subtraction and multiplication over division. This is not unrelated to the fact that in many textbooks the order of operations is given via the known mnemonic BOMDAS (Brackets first then “Of”, Multiplication, Division, Addition and Subtraction) (Herscovics & Linchevski, 1994). PEMDAS (Please Excuse My Dear Aunt Sally, which stands for Parentheses, Exponents, Multiplication and Division (from left to right), Addition and Subtraction (from left to right)) is another example of students’ reliance on mnemonics. Pappanastos, Hals, and Honan (2002) surveyed over 300 business school students at two universities and the results showed that (a) they recalled the above mentioned acronym, and (b) one third of the respondents applied incorrectly the order of operations. Rambhia (2002) states that as a result of PEMDAS focused teaching many students are convinced that multiplication has to be done before division and that addition is more important than subtraction. Thus, PEMDAS and similar mnemonic devices either hide or assist the learning of operations. But what is especially interesting in the case of these mnemonics is that their use is a symptom of a lack of attention to structure. Finally, Glidden (2008) who investigated how well 381 prospective elementary teachers solved four arithmetic problems that required using the order of operations found that fewer than half of them answered more than two questions correctly.

All these findings are important since (a) teachers must know the correct order of operations to teach the concept correctly, and (b) light might be shed on whether students do in fact understand the order of operations completely or merely interpret the mnemonics literally (Pappanastos et al., 2002). However, despite the importance of these findings there is an aspect that is not represented in the body of the research literature. The above mentioned presented studies concern how very young students overgeneralize the order of operations (Linchevski & Herscovics, 1994), and then how older students tend to perform

the operations sequentially from left to right (Kieran, 1979), and then how college students use mnemonics devices to remember order of operations since they do not understand the proper order of precedence that should be applied to mathematical operations (Pappanastos et al., 2002), and then how prospective elementary school teachers give incorrect answers attributable to misunderstanding the order of operations (Glidden, 2008). All this incorrectly learned information perhaps continues to handicap learning into their next math courses and beyond for several years. So, what if an in-service teacher did not manage to “repair” this incorrect knowledge acquired years ago? How this influences the way he teaches the specific concept to his young students? How easily these errors are transferred as an incorrect knowledge to them? Actually, these are the research questions of this paper.

## THE SETTING OF THE STUDY

This is a two-stage study. During the first stage a collection of tasks was designed to uncover errors with precedence rules. The collection of the tasks was administered to an in-service primary school teacher before teaching the unit of order of operations to his 6<sup>th</sup> graders. Four of the tasks are presented in Table 1.

Item 1:	$14 : 2 \cdot 14 - 12 =$
Item 2:	$18 + 19 + 14 \cdot (11 + 22) =$
Item 3:	$9 + 23 + (11 - 3) \cdot (4 : 2) =$
Item 4:	$7 \cdot (6 \cdot 6 + 14) \cdot 5 + 6 =$

**Table 1:** Tasks of the study

The answers of the teacher were examined and the incorrect answers were coded on the basis of the work of Blando and colleagues (1989). In their work, errors are mentioned as *mal-rules* which stand for violations of legal mathematics rules. Their theoretical model relies on *repair theory* which states that errors occur when a student is faced with a difficult or unfamiliar feature of a task. In this case the student may react by modifying a known procedure and applying it (incorrectly) to the task. Blando and his colleagues (1989) described the students’ incorrect solutions in terms of mal-rules. In relation to Precedence Errors they listed six mal-rules using acronyms (see examples in Table 2).

PAM	Add before multiplying	Example: $4 + 2 \times 3 \rightarrow 6 \times 3$
PAD	Add before dividing	Example: $10 / 2 + 3 \rightarrow 10 / 5$
PSM	Subtract before multiplying	Example: $9 - 2 \times 3 \rightarrow 7 \times 3$
PSD	Subtract before dividing	Example: $8 - 6 / 2 \rightarrow 2 / 2$
PAS	Add before subtracting	Example: $6 - 4 + 3 \rightarrow 6 - 7$
PIP	Ignore Parenthesis	Example: $8 - (2 + 4) \rightarrow 6 + 4$

**Table 2:** List of mal-rules for precedence errors (from Blando et al. (1989))

Thus, in our study, the teacher’s errors were coded according to the above mentioned list.

In Item-1 the solution given by the teacher was:

$$14 : 2 \cdot 14 - 12 = (14 : 2) \cdot (14 - 12) = 7 \cdot 2 = 14$$

The general format of this task is  $a : b \cdot c - d$ , the correct rule is  $[(a : b) \cdot c] - d$  and one of the possible mal-rules is  $(a : b) \cdot (c - d)$ . This could be regarded as “subtract before multiplying” (PSM) mal-rule. The teacher correctly made the division  $14 : 2$ . But then he ignored the multiplication and chose to subtract  $14 - 12$  before multiplying.

In Item-2 the solution of the teacher was:

$$18 + 19 + 14 \cdot (11 + 22) = (18 + 19 + 14) \cdot (11 + 22) = 51 \cdot 33 = 1683$$

The general format of the task is  $a + b + c \cdot (d + e)$ . The correct solution is  $a + b + [c \cdot (d + e)]$  and the mal-rule applied by the teacher was  $(a + b + c) \cdot (d + e)$ , which means “add before multiplying” (i.e., PAM). The teacher correctly evaluated the part inside the parentheses but then he violated the precedence rule and carried out addition prior to multiplication.

In Item-3 the answer given by the teacher was:

$$9 + 23 + (11 - 3) \cdot (4 : 2) = [(9 + 23) + (11 - 3)] \cdot (4 : 2) = (32 + 8) \cdot 2 = 40 \cdot 2 = 80$$

Even though the general format seems to be more complex than the previous one, the core idea of the teacher’s error was the same. He did not ignore the parentheses ( $11 - 3 = 8$ ,  $4 : 2 = 2$ ), however, he added before multiplying (PAM) instead of following the correct rule (i.e.,  $9 + 23 + [(11 - 3) \cdot (4 : 2)]$ ).

Finally, in Item-4, the PAM mal-rule was again prevalent:

$$7 \cdot (6 \cdot 6 + 14) \cdot 5 + 6 = [7 \cdot (6 \cdot 6 + 14)] \cdot (5 + 6) = [7 \cdot (36 + 14)] \cdot 11 = 7 \cdot 50 \cdot 11 = 350 \cdot 11 = 3850$$

The correct solution was  $[7 \cdot (36 + 14) \cdot 5] + 6 = (7 \cdot 50 \cdot 5) + 6 = 1750 + 6 = 1756$ . For one more time what actually the teacher did was to add ( $5 + 6 = 11$ ) before multiplying (PAM). Once more he respected the parentheses but then he incorrectly evaluated the expression  $5 + 6$  before multiplying. These answers signalled the need for an intervention session aiming to let the teacher become aware of his errors. Research has suggested that interventions focused on developing teachers’ subject-matter knowledge can positively impact teachers’ knowledge (Swafford et al., 1997) and makes them able to improve their ability to make sense of and evaluate students’ thinking strategies in a variety of mathematical context (Tyminski et al., 2014)

The session was decided to take place at school and lasted about 90 minutes. During the session a lot of examples were discussed offering the opportunity for the teacher to internalize (a) the order of operations as an introduction to the necessity for structure and rules, and (b) the need to have a unique answer in tasks and also to have rules required to achieve it. For each example it was clarified that there was a unique answer. The application of various mal-rules result to different outcomes but only one answer is the correct and certain rules are required to achieve it. When multiple operations are included in an expression, different numerical results are obtained according to the precedence given to some operations over others. Visual representations were used to illustrate the notion of operation precedence. More specifically, the *tree diagrams* were used (Kirshner & Awtry, 2004). Their central feature is an iterative procedure for analysing the syntactic structure of a mathematical expression, and representing it as a partially ordered hierarchical structure. In a tree diagram the operation that is more precedent appears lower on the tree than an operation less precedent. In this tree notation it is easy to see the independence of

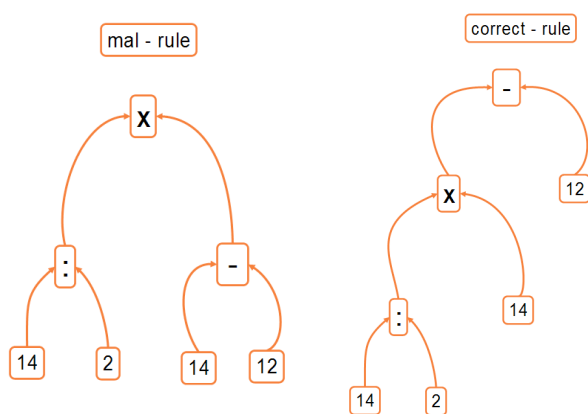


Figure 1: Tree diagrams for Item-1

the operations in the separateness of the tree’s main “branches”. In Figure 1 two tree diagrams illustrating the precedence decision for Item-1 are presented.

Following Ernest (1987), the teacher’s errors might be attributed to a failure to discern the hierarchical syntactical structure or orders of precedence within mathematical expressions. So the tree model was used exactly to exhibit this structure explicitly to the teacher. He was ‘trained’ to analyse expressions into written tree forms.

It is worth mentioning that the teacher was an experienced one in the sense that he had been teaching for at least 25 years. This means that during this period he was carrying this incorrect perspective about the order of operations which may influence the learning of his students. Consequently, the question is: To what extent an intervention can sufficiently help the teacher to correct his stable errors relevant to order of operations?

This is why the teacher was left to teach the specific unit to his students which lasted for about a week. The second stage of the study took place almost a month after completing the unit and at that moment the teacher was not aware of our intention to come back again and

work with his 6<sup>th</sup> graders. This decision was made on purpose since it was important to let the teacher work on his own terms and not under the feeling that he has to be cautious due to our future return into his classroom. We requested permission to work with the students on this topic. He gladly agreed and confessed that in the meantime he was wondering about the possible impact of this experience on him and/or on his students.

So, the students were invited to cope with the same collection of the tasks that was initially administered to their teacher. Twenty two 6<sup>th</sup> graders participated. The study took place a month after the beginning of the school year. The students had been taught comparison between natural and decimal numbers as well as the four operations among these numbers. Their worksheets were collected and their incorrect answers were coded according to two criteria: (a) whether their incorrect answers corresponded to the mal-rule list of Blando and colleagues (1989), and (b) whether their answers reflected their teacher’s incorrect ones.

## RESULTS AND DISCUSSION

The students’ answers were classified into three categories:

- (i) Incorrect answers that were identical to their teacher’s ones (i.e., the ones during the first stage of the study).
- (ii) Answers that were correct, and
- (iii) Answers that did not fit to any of these two categories (i.e., the students did not answer the item at all or the students answered but there was not a clear way of showing their thinking for the computations).

	Incorrect answers identical to the teacher’s ones (students followed mal-rules)	Correct solutions (correct application of the rules for the order of operation)	Other
Item-1 (PSM)	6 (27.27%)	12 (54.54%)	4 (18.19%)
Item-2 (PAM)	<b>19 (86.36%)</b>	3 (13.64%)	-
Item-3 (PAM)	<b>18 (81.81%)</b>	3 (13.64%)	1 (4.55%)
Item-4 (PAM)	<b>12 (54.54%)</b>	4 (18.19%)	6 (27.27%)

Table 3: Summarized view of the student’s answers

A summarized view of the students' answers is presented below in Table 3.

It is interesting that many students used symbols of inclusion that facilitated their erroneous way of evaluating the arithmetical expressions. Even though they followed mal-rules for the order of operations they respected the symbols of inclusion they used. Some examples of the students' usage of inclusion symbols that supported their incorrect generalization for the order of operations can be seen in Figure 2.

It is evident that almost the whole class reacted exactly in the same way their teacher did a month before. The percentages for Items 2–4 are extremely high. More than half the students repeated the mal-rule of their teacher for Item-4 (12/22), all but three students for Item-2 (19/22), and all but four students for Item-3 (18/22). It seems that this did not happen for Item-1 since only 6 students out of 22 repeated the mal-rule of “subtract before multiplying” (PSM). A potential explanation for this might be that the students were accustomed to work sequentially from left to right. During their early grades the students were given exercises which disregarded the order of operations and therefore many of them learned incorrectly that operations are simply worked from left to right. The fact is that in Item-1 following the operations from left to right happened to be the same as following the rules for the order of operations and this possibly explains the small percentage of the PSM mal-rule.

The results were also equally surprising for the teacher since he realized how similarly incorrect were his own errors to his students' ones. So, the question is: Given that during the intervening session the rules for the order of operations were examined and became clear, then what would be a possible explanation for having almost all the students repeating their teacher's initial errors?

$$[(14 : 2)(14 - 12)] \cdot 7 \cdot 2 = 14$$

$$[(9 + 23) + (11 - 3)] \cdot (4 : 2)(32 + 8) \cdot 2 = 40 \cdot 2 = \underline{80}$$

$$(18 + 19 + 14) \cdot (11 + 22) = 51 \cdot 33 = \underline{1683}$$

$$[7 \cdot (6 \cdot 6 + 14)] \cdot 5 + 6 = [7 \cdot (36 + 14)] \cdot 5 + 6 = [(7 \cdot 50) + 5] + 6 = (350 + 5) + 6 = \underline{361}$$

Figure 2: Students' usage of inclusion symbols

Before presenting our thesis it has to be acknowledged that teachers' content knowledge in the subject area does not suffice for good learning. However, it is also true that the knowledge of mathematics obviously influences the teachers' teaching of mathematics and subsequently they cannot help children learn things they themselves do not understand. This could explain the impact of the specific teacher in his students' performance for the time period before this study. The difference now was that the teacher was led to face his weak mathematical background concerning order of operations and moreover he participated in a session that made him to see why the rules he applied were mal-rules as well as to get practice on a series of tasks that challenged him to apply now the correct rules for the precedence of operations. He declared that he understood the violation of the rules for the order of operations he used to follow. However, the findings of the study did show that the teaching that took place after the session was dominated by his persistent misinterpretation on the order of operations. This contradiction may be explained by accepting that the session that took place was not sufficient to confront the teacher's erroneous long-time way of teaching. Thus, a stronger intervention might be needed to establish a more compact knowledge on order of operations accompanied with guidance concerning instructional strategies for the unit. Moreover, it can be said that learners generalize in a way that they are initially taught and this can lead to the construction of schemata at an early stage that have a strong inherent robustness (Waren, 2003). Linchevski and Livneh (2002) claim that occasionally these old schemata become tacit models of comprehension and this could mean that –as in our case– despite the intervening session, initial understanding persists.

## CONCLUSIONS

The deep content knowledge of mathematics is – among others- necessary for teaching successfully mathematics. Until a few years ago, the subject matter knowledge of teachers was largely taken for granted in teacher education. But recent research focused on the ways in which teachers and prospective teachers understand the subjects they teach, reveals that they often have misconceptions or gaps in knowledge (Ball & McDiarmid, 1990). In the same paper Ball and McDiarmid also argue that as teachers are themselves products of elementary and secondary schools in which pupils rarely develop deep understanding of the subject matter they encounter, we should not be surprised by teachers' inadequate subject matter preparation. This was clearly presented in our study. An experienced teacher, teaching more than 25 years, introduced a specific mathematical topic (i.e., order of operations) based on certain mal-rules that probably influenced the quality of learning of his students.

It also seems that these misinterpretations of the correct rules for the order of operations made it difficult for the teacher to have control over his own teaching under the light of the new information. Despite a session that aimed to support the teacher to confront the situation the findings showed that he continued to be based on the same mal-rules. It seems that the session was not really efficient. This is verified by the fact that his students showed similar behaviour to his towards the same collection of tasks. Their responses followed the same incorrect formats that were used by their teacher. This is very helpful to depict clearly the situation in our country. And the situation is like this:

The mathematical content knowledge of primary school teachers takes place only during their university courses. Later on, for pre-service and in-service teachers there are training programs but all of them are focused on the new curriculum, on the new teaching methods, on the usage of technology in mathematics classrooms, etc. Implicitly, all these training programs take for granted that these people know mathematics and this is why they give emphasis to pedagogical aspects of mathematics teaching. So, in this paper we tried to show that this is a rather false image of the real situation. Even though the work of only one teacher was presented and this could be considered as a case study (and therefore we could not generalize) we still believe that this is evidence that

cannot be ignored. Part of the training programs must give emphasis on the subject matter knowledge of the persons who are responsible for teaching mathematics in young students and influence by their teaching the mathematical thinking of their students.

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