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Processes of mathematical reasoning of equations in primary mathematics lessons

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Learning of mathematics in primary classes should not be reduced to learning algorithms and routines or procedures. As we see the learning of mathematics as a process of interpreting mathematical structures and generalizations, it is more important to foster children's thinking and learning of meaningful relations between objects and operations in the context of equations. In this sense children have to learn algebraic relations in primary classes without using algebraic signs. In this paper the results of a design study and a video-based qualitative analysis of teaching/learning situations in the field of reasoning of equations are discussed.

Keywords: Argumentation, equation, algebraic thinking.

INTRODUCTION

This paper is about the well-known problems around the equal-sign in mathematics learning processes: Many pupils understand the equal sign as a clear action-symbol, e.g. the left side of the sign has to be interpreted as a computing-demand, whereas on the right side there can only be found a single number, i.e. the solution of the computing-task (Seo & Ginsburg, 2003; Kieran, 2011; Russel, 2011; Steinweg, 2013). Those pupils don't accept equations like " $8 + 5 = 10 + 3$ " or " $7+9 = 2 \cdot 8$ " because they offend against the clear rule to have the computing term on the left and the result on the right side of the equal sign. For most situations in primary level, this "task-result-interpretation" (Winter, 1982) of equations may be sufficient. But, having in mind the algebra of secondary level curricula, someday the learners will have to overcome this restricted view on equations.

It's more important, I think, to give them many opportunities to use symbols in many situations than simply to tell them, let's say: this is the equals

sign and what's one side of the equals sign should be the same as what's the other side of the equals sign (Seo & Ginsburg, 2003, p. 169).

To realise such *learning opportunities* we need rich learning environments that make algebraic structures accessible to children in primary schools. But, this is not enough. In the following, we give an example to illustrate that the rich learning environment alone does not lead to thinking about equalities in a structural way. Afterwards, we will discuss the necessity of argumentation-processes to motivate *fundamental learning processes*, i.e. to support the children on their way to an algebraic view on equalities. At least, we will give an example from our design study of how to initiate such *collective argumentations*.

An example of a fourth grade class

The example took place in a learning unit that was on the structure of "Rechendreiecke" (for an example see Figure 1), a substantial learning environment based on the well-known "arithmogons" (Wittmann, 2001): Every two numbers in the inner fields of the triangle are added, and the sum is noted in the appropriate field outside of the figure.

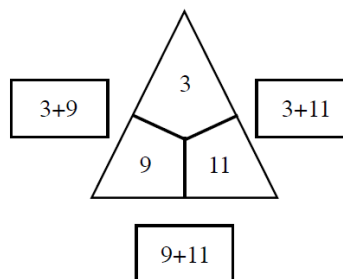


Figure 1

At the beginning of the unit the teacher gave the children some ordinary tasks in computing, i.e. the

numbers for the inner fields were given and the children had to calculate the sums for the fields outside of the triangle. After some of these standard-exercises the children made the first discovery on the essential structures of the triangles: The sum of the outer numbers is twice the sum of the numbers in the inner fields, because every inner number appears in two outer fields.

In the present lesson, the teacher's goal was to discuss the most difficult tasks within this learning environment: She wanted the children to find the inner numbers for a triangle with given numbers in the outer fields. So, she wrote an example for this kind of task on the blackboard (Figure 2), and asked the whole class for first ideas about how to solve this problem. The teacher clearly expected that the children would use the discovery mentioned above to find out the inner numbers: Firstly, they could calculate *the sum* of the inner numbers (by halving the sum of the given outer numbers). Then, they could find a fitting partition of this sum for the inner numbers by trying a systematic way.

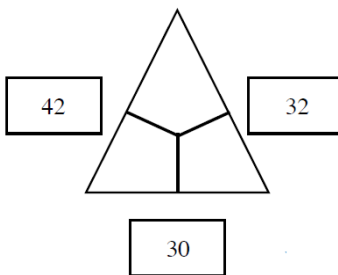


Figure 2

Having in mind this problem-solving approach, the teacher asked Robert for his idea:

Robert I think that this is the same problem as we had with the number-walls, that 30 plus 32 take away 42 divided by two, the result must be noted in the bottom field (while talking, the pupil arrives at the blackboard).

Children (make some noise to demonstrate their incomprehension)

Robert Then we will have the ten there (notes 10 in the bottom field on the left) and twenty there (notes 20 in the bottom field on the right, then notes 12 in the field on the top, Figure 3)

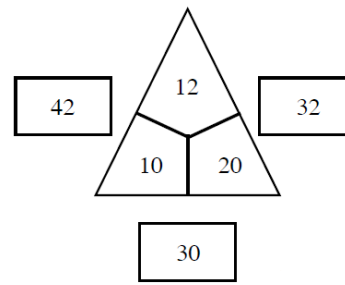


Figure 3

Teacher Okay, now we have a ten here and a twenty over there (points to the numbers in the figure on the blackboard). How can we go on? The idea of Robert is not bad!

Clarissa We could change the places of the twenty and the ten. Because, I think it would fit better. Then one would note twenty-two at the top and we had the result. (Clarissa modifies the numbers, Figure 4)

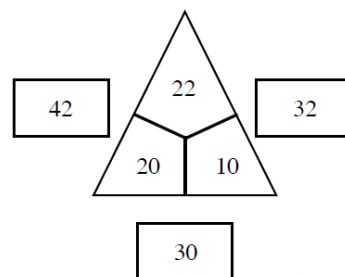


Figure 4

After this episode, the teacher and the pupils seemed to be satisfied, having found the fitting inner numbers. Neither the pupils nor the teacher not even Robert himself had the need to analyse whether it is coincidence or not that Clarissa only had to change the places of Robert's numbers in order to find the right solution of the triangle. Hence, even though the class found a solution for the special task, the underlying mathematical structure of the general problem remained covered. In other words: The *learning opportunity* given by the idea of Robert did not unfold its structural potential, we name it a "missed substantial learning opportunity".

Briefly, two questions remain, resulted from our observation of this episode:

- 1) Why is Robert's idea so hard to understand?

- 2) Why is there obviously no need to analyse the proposal of Robert, and no need to question whether the idea may lead to a general way of problem solving?

Analysis: The mathematical core of the missed learning opportunity

Even if Robert's proposal does not lead to the correct solution, the underlying idea is correct: Given the inner numbers a , b , c , Robert firstly builds the sum of two outer numbers (i.e. $a+b+b+c$). Afterwards, he takes away the third outer number ($a+c$), the result is the double of one of the inner numbers (here $2b$).

Although we do not want to shift the mathematics curriculum of the secondary level into the primary school, from our point of view, it should be possible to discuss arithmetical structures like the one above in a fourth grades class. For example, one could organise the learning process within a reflective exercise (s. Figure 5), forcing the children to calculate with the outer numbers in the way Robert does: "Build the sum of two outer numbers and take away the third one – what do you observe?" Having in mind that every outer number is built by the sum of two inner numbers, the children might be able to explain the result of their calculations, and then to use this new knowledge as an effective solving strategy for problems like the one above.

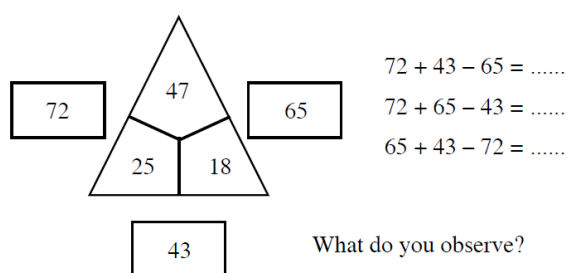


Figure 5

The main difficulty on the way to find an explanation like this is the necessary flexibility in interpreting the inner and outer numbers: Sometimes, they are computing numbers, sometimes they are the result of a computation and the problem-solver has to decide which number would be used to calculate and which one must be understood as a computing-result.

From our point of view, we should help the children to construct a flexible understanding of equations already in primary level. But, again, we do not want

to shift the mathematics curriculum of the secondary level into the primary school. Hence, we propose to promote the development of a content-related, flexible *understanding of equality* rather than teaching the children *how to handle equations* in an adequate form to prepare algebraic notations. According to Steinbring (2005), the children in primary school should work on a flexible concept of mathematical equality mainly by constructing several adequate reference contexts rather than learning how to use the standard mathematical signs within the solution of equations.

THE PRESENT PROJECT

Within our project we started a variety of different teaching-learning experiments, like whole class instruction, group working and peer interviews. The experiments were planned on the basis of already existing substantial learning environments, mainly taken from the project "mathe 2000" (Wittmann, 2001). Our goal is to strengthen the *content-related concept* of equality. Hence, the equal sign and its algebraic correct and formal rules to use do not play the leading role within the learning environments. In some of them (like arithmogons, number walls, etc.; Wittmann, 2001), this special sign does not even appear in the tasks. But nevertheless, the learning environments focus on equality. For example, the children have to find different number walls with equal numbers in the top stones.

Our main research interest is to understand the micro-processes of teaching and learning mathematics rather than to measure the success of a learning environment. Hence, our analysis of the teaching-learning-experiments follows the interpretative paradigm, mainly using approaches of symbolic interactionism and ethnomethodology (Bauersfeld, Krummheuer, & Voigt, 1998; Voigt, 1994; Yackel & Cobb, 1996), epistemological theories (Steinbring, 2005) and theories of argumentation (Schwarzkopf, 2003). In the following sections we discuss some aspects of the first results of the study.

Theoretical embedding: Different types of learning processes

What does it mean to modify a restricted "task-result" interpretation of the equal-sign to a more sophisticated, flexible and structurally sustainable concept of equality? Following Steinbring (2005), the construc-

tion of an adequate equality-concept moves between two epistemological poles:

On the one hand, there is an *empirical, situated description* of mathematical knowledge like finding the result for a computing task in the sense of Clarissa's offer for the solution of the problem above. This kind of knowledge is easy to handle in communication, one simply has to offer some empirical facts, and everybody will know what they mean. But, in conclusion, the pupils can only learn new facts; the associated learning opportunities will not help them to construct a better structural understanding of the mathematical background.

On the other hand, the understanding of Robert's idea for example, requires a *relational generality* of mathematical knowledge. Learning opportunities that concentrate on this kind of knowledge have to offer the need for the children to create a new interpretation of the thematic mathematics. At the same time one has to consider that the pupils cannot create interpretations that are completely detached from their experienced points of view.

These fundamental learning processes (cf. Miller, 1986, 2002) can only be realised in situations of *collective argumentations*, i.e. the children must be confronted with a problem that makes it somehow impossible for them to go on by routine and they first have to solve that problem in an argumentative way (Miller, 1986; Schwarzkopf, 2003). But, according to Miller, argumentation is a very stressful kind of interaction, and, normally, people try hard to solve their problems in a non-argumentative way.

Even within mathematics classrooms, where argumentation is one of the learning goals, there are many opportunities to avoid a content related argumentation – at least one can always ask the teacher as an expert in mathematics. Remember the example above: There is no need for an argumentation around the idea of Robert because Clarissa found an easier way to solve the problem by changing numbers, and presenting directly the correct solution. In the sense of Steinbring (2005, p. 194–213), the offers of Clarissa and Robert stand for the two poles of communication between which the interactive constructions of knowledge move: Whereas Clarissa's gives a simple "mediation of facts" (correct calculations), an adequate understanding of Robert's idea needs a "construction

of a new interpretation" that is obviously not accessible to the communication partners.

Hence, collective argumentations do not emerge in mathematics lessons in a somehow natural way. On the contrary, the teacher has to *initiate* the concerning processes in both, a careful and persistent way (Schwarzkopf, 2000). For this, it is important to think about social requirements and mathematical learning goals that are necessary to initiate substantial learning opportunities by cooperative argumentation.

To enforce the emergence of substantial learning opportunities, we develop tasks that initiate mathematical needs for collective argumentation. Our intention is to confront the children with a "productive irritation" (Nührenbörger & Schwarzkopf, 2013), concerning their social experiences in classroom discussions. The tasks or problems, for example, provide phenomena that were not expected by the children so that they have to reflect on the given structures and see the need to re-interpret the experienced mathematics behind the problem.

This approach bases on Piaget's (1985) work on cognitive conflicts. Roughly speaking, Piaget points out that a child develops new ideas when it is confronted with facts or beliefs that contradict their expectations, depending on their individual experiences. If the concerning cognitive schemas resist the child's possibilities of assimilation, there is a need for the child to generate a cognitive consensus, i.e. a learning-process emerges.

To initiate productive learning opportunities in this sense, one of the main difficulties is that an observation of a pattern or a surprising discovery is not enough to create the need of argumentation – we gave an example at the beginning of this paper. Moreover, it is necessary that the observation becomes an amazing phenomenon for the pupils. For example, the children discover a pattern in a series of tasks. Then, they are confronted with another task that does not exactly fit to the previously solved series and they are asked to make a prognosis: Will the result of the next task fit to the pattern or not? By this we try to force the children to create an expectation on the result of the next task. The initiation of a "productive irritation" is successful, when this expectation fails while computing the next result – by this, there is an interactive need to find explanations for the failure of the prognosis. In this

sense, a productive irritation can widen the implicit working, often sense-restricting “sociomathematical norms” (Yackel & Cobb, 1996) that influence the interpretation of arithmetical terms in routinized (cf. Voigt 1994) classroom discussions.

**Initiating productive irritations:
Arguing for amazing equalities**

In the following part we give an example for a short learning unit with the goal to initiate an argumentation, initiated by a productive irritation within a primary class (3rd or 4th class). The content comes from the well-known substantial learning environment “number-walls” (see Figure 6), where you have to note the sum of the numbers in every stone from the two stones under it. Typically, children discover different types of terms to describe the top stone of the wall (e.g. as a result of a calculation or as a relation between the bottom stones: $a+b + b+c = a+2b+c$).

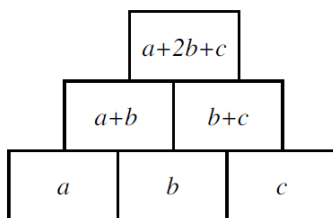


Figure 6

In the first part of the present learning unit, the children calculate a series of number-walls and discover a pattern, that is very familiar to them: Increasing the number in the right bottom-stone and decreasing the number in the left bottom-stone by the same difference will leave the number at the top of the wall constant (keeping the same number in the middle of the bottom, of course) in this example 650 (Figure 7).

The children can calculate the number of the top as well as they can argue with the relations between the stones of a number wall. To point out the equalities between the number walls the children can use so called “term walls” (see Figure 8), noting calculations instead of their results in the stones. These term-walls might build bridges for the children to change their interpretations of the numbers as results (e.g. $650 = 460 + 190$ and $650 = 450 + 200$) and to see them as computing numbers ($650 = 380 + 2 \times 80 + 110 = 370 \times 2x80 + 120$).

However, having calculated some of these tasks, the children are confronted with another number-wall that does not exactly fit to the previously discovered pattern (e.g., the one in Figure 9), and they are asked about their expectations on the result: Will the number at the top of the wall change or not?

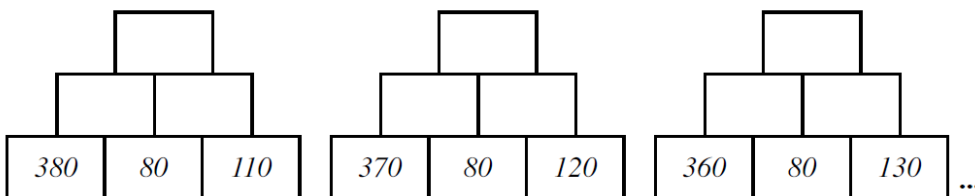


Figure 7

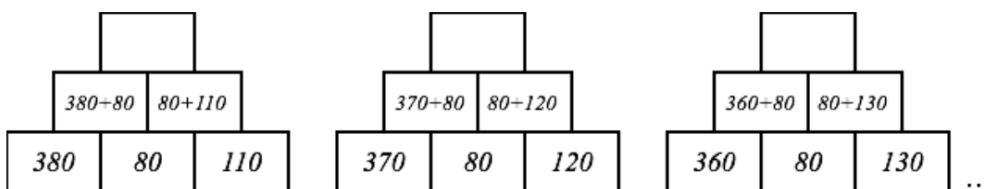


Figure 8a

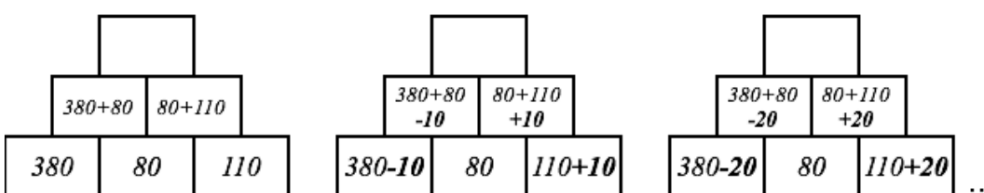


Figure 8b

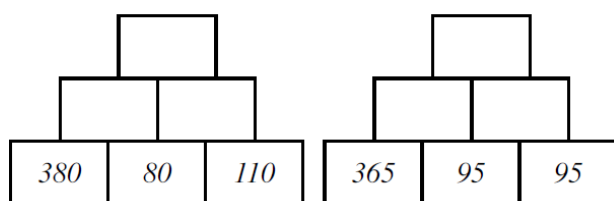


Figure 9

The following episode takes place in an interview-situation, where a teacher-student works together with two children on the above given number walls. The children have already discovered that the top numbers of the number-walls in Figure 7 are all the same, namely 650. This pattern seemed to be clear for the children, having in mind that from wall to wall one of the bottom numbers increases and one of the others decreases by the same difference. Afterwards, the teacher showed the children some number walls like the one in Figure 9, asking for a prognosis about the number of the top-stone. As expected, both of the children gave the prognosis that the top stone must change, because two of the bottom-numbers increase by fifteen and only one of them decreases by fifteen.

Teacher Do you think that in this top stone (pointing to the right number-wall in Figure 9) there will be the 650 again?

Moritz no.

Luisa no.

Teacher Why do you say no?

Moritz Well, here are fifteen more than there (points to 380 and 365), and here are fifteen more than there (points to 110 and 95 in the right bottom stone), but here are fifteen more than there (points to 95 in the middle bottom stone and 80), and fifteen plus fifteen are together thirty and not fifteen (.), exactly.

Moritz reasons by comparing the changes of the two increasing bottom numbers with the decreasing number. In this way, he activates his experiences with the constancy-law of additions that was successfully used in the first part of the unit. The teacher moderates the interaction, so that also Luisa has to reason for her prognosis:

Teacher Luisa, do you have the same opinion?

Luisa Yes, the result here above is the same.

Teacher How do you mean it?

Luisa Well, if you add the bottom stones (points to 365, 80, 110) then they must have the same result as there (points to 380, 95, 95) for getting the same result on the top, and this is another sum.

Luisa points out a hypothesis for a general rule: Two number walls with the same numbers in the top stones must be equal in the sum of the bottom numbers.

Even if the two given arguments are different in detail, from an epistemological point of view, both of them can be characterised as an *empirical, situated description of knowing* (Steinbring, 2005; Schwarzkopf, 2003): The comparisons of the (unknown) top numbers are based on empirical observations of the given examples without leading to a structural deeper understanding of the mathematical structures.

After the children have verified their hypothesis and found out that their observation offended against their expectation, they rethink the arithmetical structures between the stones of the number walls and find new arguments:

Luisa Mmm, they are the same because this (points to the left and right bottom stone, Figure 10) is coming to the other stones, and this (points two times to the middle bottom stone) meets twice. So we have to calculate them together.

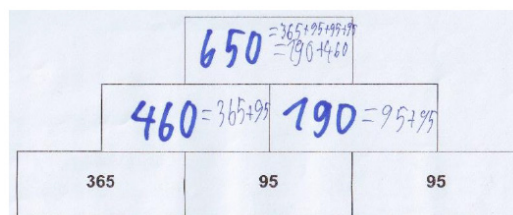


Figure 10

Moritz Mmm, because in the middle we have 460 is equal 365 plus 95 (.) that is 460 (.), and because there are not four bottom stones you have to calculate once again 95 plus 95 is equal 190 (.). You have to take once again the 95.

Teacher Ok, and why do we get in both number walls the same number 650 in the top?

Moritz Because you need the 95 for both sides. However, for example, here you have to

take 15 more and here exactly the same (points to both number walls).

In their arguments, both children become aware of the special function of the number in the middle bottom stone. On the first view, Luisa argues in a somehow dynamic, still empirical way – as if the numbers would crawl through the stones to the top. But regarding her offer in detail, the argument already shows qualities of a more theoretical approach: The middle stone “meets twice”, although the number exists only once. This discovery builds the bridge from the empirical view to the structural understanding of number walls: The meaning of the numbers result of their position in the wall. Moritz modulates this approach (Krummheuer, 1992) of Luisa to a more static view, and by this he strengthens the theoretical character of the argumentation: The number in the middle bottom stone is not a concrete, single object, but it is part of two calculations. His remark “there are not four bottom stones” builds the bridge between the empirical understanding of the numbers as objects (that crawl through the wall) and the numbers as part of calculations that depend only on their position in the wall: An empirical version of the number wall would have four bottom stones (see Figure 11), providing every number as often as it will be needed in the wall.

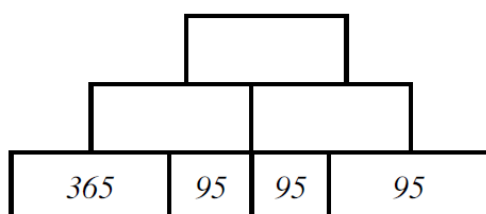


Figure 11

Finally, Moritz is able to interpret the new view on the number walls on his hypothesis of the beginning of the episode: Changing the numbers in the bottom stones means changing the operators in the calculation terms.

The main aspect within this argumentation seems to be that the numbers of the task have changed their computational functions: Previously, the pupils interpreted the number wall in the sense of a task-result-view on equations: At the bottom there are three numbers, and at the top there is one result. Within the argumentation for the unexpected equality, they changed this view to a more flexible, theoretical one (Figure 12): No longer the numbers, but the comput-

ing-terms are the main objects of the number-wall (Steinweg, 2013). In this sense, the children have to construct this as a common object a new term concerning twice the decrease of 15 and the increase of 15: $(380+80)+(80+110)=(380-15+80+15)+(80+15+110-15)=(365+95)+(95+95)$.

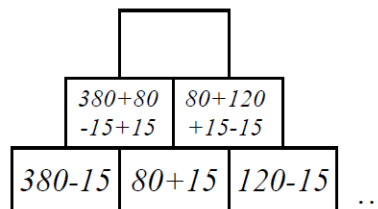


Figure 12

In conclusion, the pupils construct a new interpretation of the arithmetical relations which is related to their old knowledge. According to Steinbring (2005), the children construct a *relational generality* of mathematical knowledge.

FINAL REMARKS

By initiating substantial learning opportunities we try to promote the development of a flexible and structural sustainable concept of mathematical equality. In our work we mean by “understanding equalities in primary classes” that the children operate with the structures of computing-terms rather than only focussing on pure numbers as if they were concrete objects. To understand equality between two terms means to find one theoretical interpretation that fits for two different looking terms (Winter, 1982). In this paper, we discussed a somehow reflective approach to the promotion of accompanying activities, concerning the initiation of collective argumentation through productive irritations.

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