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Sixth grade students' explanations and justifications of distributivity

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Equal groups and rectangular arrays are examples of multiplicative situations that have different qualities related to students' understanding of the distributive and the commutative properties. These properties are, inter alia, important for flexible mental calculations. In order to design effective instruction we need to investigate how students construct understanding of these properties. In this study sixth grade students were invited to reason with a peer about calculation strategies for multiplication with the goal of explaining and justifying distributivity. Their discussions demonstrate that the representation of multiplication as equal groups helps them to explain and justify distributivity. At the same time this representation hinders their efficient use of commutativity.

Keywords: Multiplicative reasoning, distributivity, commutativity, equal groups.

INTRODUCTION

Three fundamental properties of arithmetic, the distributive, the commutative and the associative property all apply to multiplication. These properties underpin flexible mental calculations and later algebraic understanding (Carpenter, Levi, Franke, & Koehler, 2005; Ding & Li, 2010; Lampert, 1986; Young-Loveridge, 2005). Although the significance of these properties is well known, researchers have only recently “begun to discuss ways to teach these ideas in the elementary grades” (Ding & Li, 2010, p. 147). To design effective teaching of the arithmetical properties more knowledge about students' understanding of the properties is needed. This study's aim is to investigate how students make sense of the arithmetical properties in multiplicative calculations. The distributive property (DP) is the main focus, but the commutative property (CP) for multiplication is also investigated since students need to manage the CP when they undertake

calculations involving the DP. The associative property is not discussed here, which is not a reflection on its importance but this paper's focus. In the next section some general concepts central to this study are presented followed by a review of findings concerning students' understanding of the DP and the CP before, finally, the aims for this study are clarified.

BACKGROUND

The DP, which states that $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$, underpins mental multiplication by splitting one factor to make two multiplications which are then summed. For example one might solve $7 \cdot 14$ as $7 \cdot (10+4) = (7 \cdot 10) + (7 \cdot 4) = 70+28$. Students can develop this mental strategy when they view multiplication as repeated addition and focus on the invariance of the total (Schifter, Monk, Russel, & Bastable, 2008). When this implicit use of the DP is transferred to problems where both factors are multi-digit numbers, a common error is to solve multiplications such as $26 \cdot 19$ by only multiplying the first terms and the second terms with each other; $(20+6) \cdot (10+9) = 20 \cdot 10 + 6 \cdot 9$ (Lo, Grant, & Flowers, 2008). Ding and Li (2014) suggest that the difficulties students have learning arithmetical properties, leading to calls for more concretisation, stem from their abstractness and “lack [of] close relevance to learners' lives” (p. 103). Concretisation by contextual and visual representations, in order to build a mental image of the operation and its properties, is argued to help students to structure and “organize their thinking and reasoning” (Yackel, 2001, p. 27). Both contextual and visual representations can reflect different multiplicative situations such as equal groups and rectangular arrays. A simple equal group situation is 4 bags of 8 apples in each bag, while a simple rectangular array can be a chocolate bar with 4 rows of 8 squares. In asymmetrical situations, such as equal groups, one factor is the multiplier (number of bags) and the other the multiplicand (number of apples). In

symmetrical situations, such as rectangular arrays, the two factors have the same role.

A contextual representation suitable for illustrating the DP is the total cost for 4 cups of coffee and 4 cakes, where the answer would be the same whether you first multiply the cost for one coffee by 4, then the cost for one cake by 4 and then add the products or if you first add the cost for one coffee and one cake and then multiply that sum by 4. A visual representation for equal groups can be used for discussing the ways the total number of objects could be calculated, by means of the DP (Lampert, 1986). A rectangular array of dots or squares can also illustrate how 7 rows, each with 14 objects, can be partitioned flexibly by use of the DP into for example $2 \cdot 14 + 5 \cdot 14$ rectangles (Young-Loveridge, 2005).

The CP, $a \cdot b = b \cdot a$, allows numbers to change places in multiplication and addition. Young students seem to discover and understand the CP for addition without instruction but not for multiplication (Squire, Davies, & Bryant, 2004). The rectangular array model is well suited to concretise the CP, as its rotational quality makes it perceptually self-explanatory, for example a box of 18 eggs in 3 rows of 6, which, when rotated 90° is simply perceived as 6 rows of 3 eggs. With an equal groups situation, it is not perceptually transparent that 7 bags of 14 coins is equivalent to 14 bags of 7 (Lo et al., 2008). Carpenter and colleagues (2003), however, found young students justifying an equal groups approach to the CP by rearranging the objects in the group, as exemplified by a student who said: "I would always get new groups that are the same size as the number of groups I started with and the number of new groups I would get would be the same as the number I had in each of the old groups" (p. 95). Some researchers argue that rectangular arrays should be introduced to enrich students' images for multiplication and to illustrate the DP (Carpenter et al., 2003; Young-Loveridge, 2005). Indeed, the illustration of integer multiplication as a rectangular array can explain standard algorithms and illustrate the extension of multiplication to rational numbers.

Squire and colleagues (2004) investigated 9–10 year old students' ability to use the DP and the CP in varied contextual situations. They gave a multiple-choice test whereby a multiplicative situation was given as a cue, and the problem was to solve a word problem of the same situation by means of either the DP or the CP. The

DP problems presented the total number of objects in a groups of b objects (an equal groups problem) and asked for the total number of objects in $a+1$ groups of b objects. For example the students were given a multiplicative word problem incorporating the assertion $26 \cdot 21 = 546$ and invited to solve a similar problem involving $27 \cdot 21$. The CP problems were constructed analogously; if the cue stated the total number of objects in a groups of b objects the task objective was to find total amount of objects in b groups of a objects. They concluded that 9–10 years old English students could manage the CP in all situations but not the DP. DP problem reflecting equal groups were more often solved correctly than any other situations, leading Squire and colleagues to suggest that the representation of equal groups might be a natural way for young students to imagine what happens when the multiplier is changed by one. They conclude that equal groups should be employed when introducing students to the DP. This is in line with Lampert's (1986) findings that fourth grade students (about 10 years old) made sense of the DP by means of stories in combination with drawings illustrating equal groups. She argues that equal groups is more intuitive for young students than array models as that is how they model multiplication. This is also confirmed by literature in the field of early algebra (Schifter et al., 2008).

In short, we know that younger students (9–10 years old) do not invoke the DP as easily as the CP; that equal groups are more intuitive for the understanding of the DP than rectangular arrays, even though rectangular array is proposed to enrich students' understanding of the DP. But for middle grade students (12–13 years old) there is a lack of research of how they understand the DP and what representations they employ when reasoning about multiplication. Given the importance of arithmetical properties for the understanding of algebra as well as flexibility in calculation, it is appropriate to pose the question: how do sixth grade students understand distributivity?

METHOD

Students from two sixth grade classes already enrolled in a research project were invited to take part in this study. The 19 students who agreed to participate do not form a random or representative sample but a typical mix of Swedish students; some have diagnoses concerning concentration or dyslexia, some struggle with mathematics while others excel. In order to en-

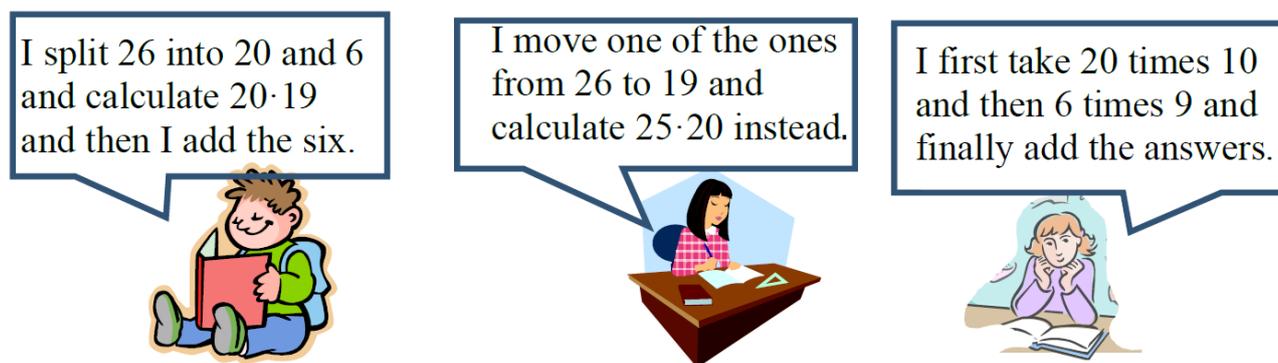


Figure 1: Problem 1 (P1), Problem 2 (P2) and Problem 3 (P3)

hance the possibility of a rich discussion, students were placed in homogeneous pairs (one worked alone) based on the evidence of earlier tests and interviews concerning the forms of multiplicative reasoning they had previously shown. They were presented with three problems written on separate cards, see Figure 1, each comprising a strategy from a fictitious student for the calculation of $26 \cdot 19$. The students' tasks were to discuss each problem with their peer and a) evaluate the validity of the suggested strategy and b) reason why the suggested strategy was valid or not. They were not informed that the suggested strategies were incorrect. A separate card with the multiplication problem ($26 \cdot 19$) was visible throughout the discussions.

Problem 1, (P1), reflects an incomplete use of the DP, where one factor is partitioned and multiplied by the other factor but the last part of the partitioned factor is added to the product without any multiplication.

Problem 2, (P2), is analogous to a common method for addition, where a suitable part is moved from one term to the other, in order to make an easier calculation.

Problem 3, (P3), reflects a well known error (Lo et al., 2008). However, all three strategies derived from (incorrect) strategies exploited by this group of students the previous year when they were tested on multiplication of two-digit numbers.

Each of the three problems is an example of how students have partitioned the numbers in order to simplify calculation. When using the DP correctly partitioning is the starting point, but to demonstrate understanding of the DP involves explaining what to do with the parts as well as why. By inviting students to evaluate and explain an incorrect use of the DP it was

possible to draw conclusions from their reasoning about how they understand the DP.

The students' discussions, which took place in a room adjacent to their normal classroom, were video and audio recorded and all written material collected. The tasks were explained to the students, who were explicitly told that they did not need to do any calculations themselves. Each card was read aloud and left on the table. The discussions for all three problems lasted between 10 and 25 minutes including the oral information about the tasks. The transcribed student discussions were repeatedly read and their answers categorised according to the arguments they employed to explain their decisions about the validity of the strategy. Some students used multiple arguments for each problem and some arguments were used in all problems while other arguments were used for one or two of the problems.

RESULTS

In this section the categories of arguments that emerged from the data are presented and exemplified by excerpts from the students' discussions. These are followed by a discussion about the different types of reasoning in respect to the DP.

General justification reflects the arguments of four students who not only solved P2 by justifying why the suggested strategy was invalid but also investigated the strategy further in order to find out under what conditions it would work and when the answer would be bigger or smaller. Their arguments clearly reflect a discussion *about* multiplication and its properties.

Emil If you increase the smaller number and decrease the larger number, then it always gets bigger.

Marcus: It might not work with zero point, but with ordinary numbers, whole numbers, then it always...

Here, Emil concluded that the product gets bigger if you move one from the larger factor to the other, and Marcus' statement reflected a discussion-derived argument concerning which numbers Emil's claim is valid for. In this case, "zero point" refers to decimal numbers.

Equal groups is a category where students made contextual references and represented $26 \cdot 19$ as 26 sets of 19 objects or 19 sets of 26. This argument was used in all three problems and reflected an awareness of why the DP is valid in multiplication.

Lucas If you take away one pile, then you take away nineteen. And that should be put in twenty-five piles. That doesn't work.

Here Lucas showed why the strategy was invalid, that one in the number 26 represents a group of 19, which is construed as understanding the DP.

Counterexample was used to evaluate the calculation strategy by use of numerical examples. Hence, it reflected multiplicative reasoning by implicit use of the DP. Students gave examples of moves of one from one factor to the other would not yield the same product, as exemplified by Ida.

Ida It is not the same, if it is eight times nine or seven times ten.

Here, Ida showed an awareness that if she used the same strategy as suggested by the fictitious student in P2, the calculation would yield an incorrect result, since it transformed the problem. She transferred the strategy into easier numbers to make arguments of why the strategy did not work. There were also students who gave counterexamples to the suggested strategy in P1, which reflected knowledge of the DP, stating that 6 needs to be multiplied by 19 and then the two partial products can be added.

Check the answer was used to assess the result rather than the strategy, and drew on a calculation of the answers for both the suggested strategy and for $26 \cdot 19$; if the answers were not the same the strategy must be invalid. This reflected a result-oriented view on multi-

plication without argumentation as to why the results differ; hence, such an argument does not demonstrate understanding of the DP. It was used for P2 and P3 and is exemplified by Hanna.

Hanna I will calculate that $[25 \cdot 20]$ and then this $[26 \cdot 19]$ and see if it is the same.

Experience was when students knew that the strategy was invalid or "knew" that it was valid from their experience of calculations. This argument offered no evidence of understanding of the DP since there was no reasoning as to why the strategy does not work. Alice used it to justify the falseness of P3 and Wilma its truth.

Alice That does not work. [...] I thought it worked before, but it doesn't.

Wilma It works. That is how I calculate.

Additive reasoning reflects the students' incorrect additive reasoning when calculating. This type of reasoning occurred in relation to all three problems.

Matilda If you first take twenty-six, and split it, that's the same. You can take nineteen and then take [one] to first make a twenty, that is, you can take one from the six to the nineteen, so it becomes twenty times twenty, and then just add five. Then you will get the same answer, just that you split it in different [parts].

Here, Matilda described an additive way of handling big numbers to partition the numbers into parts that are easier to handle. She suggested that 26 is split into 20 and 6, and then add 1 from the six (splitting 6 into 5 and 1) to make 20, take $20 \cdot 20$. Then finally add 5, $((19+1) \cdot (26-1-5) + 5)$.

The category *other* consists of vague and unclear arguments as well as no answer. Felicia gives an example of a vague argument and Alva of unclear reasoning to P1.

Felicia Then the six isn't timesed [multiplied].

Alva It is hard to explain, but it is just wrong.

Some of the students who reasoned like Felicia meant to take $20 \cdot 19 + 6 \cdot 19$, others meant $20 \cdot 19 - 6$, while others never explained further how the six should be mul-

plied. Alva's statement is an example from which it is impossible to draw any conclusions about the student's understanding of the DP, which is this study's aim.

When the categories of arguments were analysed further, and in relation to the DP, different types of reasoning were found. Four students investigated the suggested strategy in P2 on a meta-level; for example, under what premises their arguments were valid, which can be described as an *investigative reasoning* on a *meta-level* about multiplication. These students not only considered the structure of multiplication but also the generality of their claims. The arguments, which drew on 'equal groups' and 'counterexamples', were construed as *multiplicative reasoning by the DP* since arguments in both these categories reflected an implicit understanding of the DP as a theorem-in-action (Vergnaud, 2009). The difference between the two categories of argument was that in 'equal groups' all arguments were validated by contextual examples, e.g., Lucas' piles (see above), while the 'counterexamples' were validated by numerical examples without references to objects in groups. In contrast to the category 'check the answer', 'counterexample' was focussed on explaining why the strategy was invalid while 'check the answer' was focussed on calculating answers, and hence labelled as *procedural reasoning*. To use 'experience' reflects a *descriptive reasoning* by stating that the suggested strategy was invalid, but not why. The descriptions were focussed on the calculations as a procedure to get the right answer, implying that the student did not understand the DP. Finally, there were students who gave arguments *not showing multiplicative reasoning*, some by 'additive reasoning' and some categorised as "other". Vague arguments and unclear statements are not necessarily indicative of their not being able to use multiplicative reasoning; they might have had problems verbalising their understanding. All arguments are presented in Table 1.

Since many students gave multiple arguments for the same problem, the sum of arguments for P2 and P3 exceed the number of participating students. The category "other" is only presented when students did not provide any other argument, hence reflecting the number of students unable to give any clear argument to each problem.

The distribution of arguments for each problem varies. For P1 six students employed multiplicative reasoning when they explained why the suggested strategy was invalid and thirteen used arguments showing no multiplicative reasoning. For P2 the distribution of arguments spread over all categories except 'experience', and the 19 students used 33 arguments, demonstrating students' use of multiple arguments. This problem also engaged students in general justifications, which did not occur for the other problems. For P3 six students drew on 'experience', three correct and three incorrect, in the evaluation of the erroneous strategy. The incorrect evaluations drawing on experience are the only arguments, besides additive reasoning, which led students to make incorrect conclusions about the validity of strategies.

In the rich discussions where students reasoned about P2 it was possible to infer understanding of both the DP and the CP. The following utterance from Emil shows his distinguishing the multiplier from the multiplicand. The transcript also demonstrates that he was aware of the CP when speaking about calculating "the other way around", a common way for Swedish students to talk about the CP.

Emil But if you take less there [pointing at 19] then it has to be fewer times multiplied. Or it depends if you do it the other way around, so if I calculate $19 \cdot 26$, then it is anyway twenty-six times multiplied...

Type of reasoning	Category of arguments	P1	P2	P3
Investigative reasoning on meta-level	General justification	0	4	0
Multiplicative reasoning by the DP	Equal groups	2	6	7
	Counterexample	4	8	0
Procedural reasoning	Check the answer	0	9	8
Descriptive reasoning	Experience	0	0	3+3
Not showing multiplicative reasoning	Additive reasoning	1	4	4
	Other/no answer	12	2	3

Table 1: Number of students using different arguments for each problem

When students reasoned about equal groups they distinguished the multiplier from the multiplicand. Some of them interpreted $26 \cdot 19$ with 19 as the multiplier, others interpreted 26 as the multiplier. In some pairs this different interpretation of a fixed factor as the multiplier caused some confusion. However, all students were aware of the CP being valid for multiplication and overcame confusion by stating, that it did not matter which factor was multiplier, as exemplified by Johanna and Ida.

Johanna Ida, it is ten times nineteen. Not ten times twenty-six.

Ida What? You did it like this before, and then I thought twenty-six times ... but it doesn't matter what way around you do it.

During the discussions about all three problems some students offered other suggestions as to how to calculate $26 \cdot 19$. These suggestions clarified how they understand the DP, as with Hugo's statement where he gives a suggestion how to proceed with the strategy in P2 to get a correct answer.

Hugo She has multiplied twenty times, and then she must take away what stands for one time, that is twenty-five. She has to take away twenty-five. [...] Then she gets that one times nineteen, so she has plus nineteen.

Here, Hugo demonstrates his understanding of the DP as he proposes compensating for the erroneous strategy of taking $25 \cdot 20$ instead of $26 \cdot 19$ by subtracting 25 and adding 19 to the product of $25 \cdot 20$; $26 \cdot 19 = (26 - 1) \cdot (19 + 1) - (1 \cdot 25) + (1 \cdot 19)$.

In summary, by engaging in the evaluation of incorrect strategies for two-digit multiplication, students with an implicit understanding of the DP demonstrated their understanding, mainly by reasoning about multiplication as equal groups. In equal groups the multiplier is distinguished from the multiplicand and this representation helped students to offer valid justifications about the incorrect strategies and to suggest other valid strategies employing the DP, but it also contributed to miscommunication connected to the CP. The arguments to explain and justify strategies demonstrated different types of reasoning in-

volving both the DP and the CP as building blocks for understanding of multiplication. Students showing additive reasoning did not show knowledge about the arithmetical properties.

DISCUSSION AND CONCLUSIONS

The different arguments that students gave for the invalidity of the three strategies reflected different degrees of understanding the DP. When students cannot explain why or how a multiplication strategy works or not works it might be due to difficulties in expressing what they mean. It might also reflect shallow understanding of multiplication or explanatory difficulties due to perceptions of self-evidence. From my readings of students' explanations to P1, I infer that it was too easy for the majority of students to explain why you need to multiply the six as well. Still, one student, Matilda, who showed additive reasoning to all three problems, was convinced that P1 was a valid strategy, which she had trouble reconciling with the fact that the answer was wrong.

Matilda It should work, but it doesn't. It might work if you take $20 \cdot 19$ and then multiply by 6. (After checking the calculation.)

Even though students like Matilda can be exposed as additive reasoners, P1 was not very productive since most students' answers and arguments were vague. This is in contrast to the other two problems, especially P2. The suggested strategy in P2, to move one from one factor to the other, caused long and elaborate discussions among most of the pairs. One reason might be that this strategy was new to most students. The novelty, and the analogous strategy for addition, might have evoked students' curiosity to investigate and engage in discussion about the strategy on a more general level than the other problems. In contrast to novelty, six students drew on experience to P3, the common mistake to only multiply tens by tens and ones by ones (Lo et al., 2008). Experience might have decreased the interest to engage in discussions about the strategy; students just knew that it "worked" or did not work.

Students who represented multiplication as equal groups in order to make sense of the DP and calculation strategies were successful, see for example Lucas' explanation why the suggested strategy in P2

did not work. The representation of equal groups as piles of objects served as a thinking tool to sort out the multiplication. To use the representation of equal groups as a thinking tool was demonstrated both for invalid strategies in the problems and for valid strategies employing the DP that the students offered as an alternative calculation. These Swedish middle school students seemed to prefer thinking about multiplication as equal groups just as the younger students from other studies (Lampert, 1986; Squire et al., 2004) and prospective teachers (Lo et al., 2008). However, the successful representation of multiplication as equal groups in respect to the DP proved to have a drawback concerning the CP. Even though students knew the validity of the CP for multiplication, there were instances where their view of one of the factors as the multiplier hindered their understanding of their peer's reasoning. This may be due to the fact that they represented the expression $26 \cdot 19$ differently, either as 26 groups of 19 or as 19 groups of 26. The students' explicit statements about the CP being valid can be construed as if the students did not take the CP as something self-evident when they were engaged in discussions drawing on equal groups. Interestingly, there were no utterances at all where students drew on rectangular array to make sense of calculation strategies or the CP.

The results of this study suggest that if we want students to learn and understand the DP we might better introduce the DP by equal groups and discuss the limits of its validity as well as how it can be used. On the other hand, the rectangular array, also an important representation of multiplication, highlights the CP by making it self-evident that $a \cdot b = b \cdot a$ (Carpenter et al., 2003) and can also be used to illustrate the DP (Carpenter et al., 2003; Young-Loveridge, 2005). If the underlying reason for illustrating the abstract properties of multiplication by contextual and visual representations is to build mental representations that can enhance understanding of the concepts (Yackel, 2001), it would be of interest to introduce multiple representations of multiplication. The findings from this study also suggest that more effort might be needed to incorporate representation of multiplication by rectangular arrays (and areas) in the instruction as complimentary representation to the equal groups, not as a substitute. Further research might give suggestions to how instruction can enhance the possibilities for students to build solid and useful mental representations of multiplication and its properties.

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