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► **To cite this version:**

Milan Hejný, Darina Jirotková, Jana Slezáková. Reversible and irreversible desemantization. CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education, Charles University in Prague, Faculty of Education; ERME, Feb 2015, Prague, Czech Republic. pp.288-294. hal-01281846

HAL Id: hal-01281846

<https://hal.science/hal-01281846>

Submitted on 2 Mar 2016

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Reversible and irreversible desemantization

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A child's first experience with mathematics comes from their everyday life. The child does not know what three is but knows what three fingers, apples or candies are. Later, in consequence to abstraction, this semantic anchoring of mathematical ideas is expanded by ideas and concepts that are not directly dependent on semantics. We call this process "desemantization". If the process of desemantization ousts semantic links from a pupil's mind as a result of too fast a drill of additive and later multiplicative structures, we speak of irreversible desemantization whose consequence is mechanical knowledge of a child. The paper develops this concept.

Keywords: Desemantization, cognitive process, mental schema, generic model.

INTRODUCTION

Czech pupils' and students' negative attitudes to mathematics which can be observed e.g. in international surveys TIMSS and PISA and their low level of understanding of mathematics are a prevailing problem of school mathematics in the Czech Republic and a challenge for mathematics educators. Our experience and research imply that responsible for inauspicious situation is not the content of mathematics but the way it is taught at school. We are convinced that in most cases the teacher presents and explains new subject matter and their pupils only imitate the teacher, they reproduce what has been said and by repetition try to store it in their memory. A pupil is not expected to use their natural will to discover, he/she is reduced to the role of a consumer of knowledge transmitted by the teacher. Knowledge of mathematics enters a pupil's mind from the outside and makes a mosaic of more or less isolated items of knowledge. The chance that this knowledge will not be forgotten and that it will be linked to other knowledge is very low. It has been known for many years that if transmissive model of teaching is replaced by constructivist approach,

where new knowledge is born in a pupil's mind as a result of the pupil's intellectual activity, the situation changes (Noddings, 1990; Pehkonen, 1997; Gruszczyk-Kolczyńska, 2012). The here presented study casts the light on the possible causes of why key knowledge of arithmetic is often stored as mechanical knowledge in a pupil's mind and thus becomes unusable in the future when the pupil meets new topics and solves new problems. It also looks for a solution of the first three main questions formulated for TWG02 both on CERME 8 and CERME 9.

The stories used in the paper as illustration of our points come from several different research projects from different periods of time.

THEORETICAL BACKGROUND

We understand the cognitive process in mathematics as a pentad of stages in the sense of the Theory of Generic Models (for details see Hejný, 2012):

1. motivation, it guides the cognitive process and provides energy;
2. creation of items of initially disconnected experience – isolated models;
3. discovery of generalized knowledge – generic model;
4. discovery of abstract knowledge;¹
5. crystallization when the new item of knowledge becomes organic part of mathematical knowledge of the individual. This stage in fact underlies all the first four stages.

1 The difference between generalized and abstract knowledge is illustrated in Story 4.

This learning process was introduced to CERME audience in (Hejný & Kratochvílová, 2005). Hejný (2012) states that unlike a number of other theoretical studies describing the cognitive process, the Theory of Generic Models (TGM) is well comprehensible both to researchers and teachers. The division of the cognitive process into stages allows a comprehensible application of the theory both into primary and lower secondary education. TGM explains what the sources of development pupils' mechanical knowledge are and how this can be prevented and reeducated. A complete set of textbooks covering all areas of primary mathematics were developed based on this theory. Over the past 8 years the set of textbooks has spread into 20% of all elementary schools in Czech Republic and currently it is being piloted at several schools in Poland and Slovakia.

The Theory of Generic Models has been also used as a tool for analysis of experiments related to cognitive processes as well as when conceiving textbooks for pre-service teacher education.

Sets of generic models in a pupil's mind create mathematical mental *schemas* that are the bearers of an individual's mathematical knowledge (Hejný, 2012).

Attention should be paid to two abstraction transfers:

isolated models
→ generic models
→ abstract knowledge, (*)

in which a pupil's semantic experience changes into abstract cognition.

Let us remark that the process (*) as a tool for discovery of mathematics happens not only in ontogeny, but also in phylogeny. Ernst Haeckel's biogenetic law, which states that ontogeny recapitulates phylogeny, is inspiring also for didactics of mathematics. P. M. Erdnjev (1978, p. 197) formulated the idea as follows: "The growth of the tree of mathematics knowledge in an individual's mind will be successful only if we recapitulate to a certain degree the history of development of mathematics." We also work with this idea. When exploring the issue of desemantization we build on our former studies as well as works (Křepel & Zemanová, 2011) and (Zemanová, 2014).

THE CONCEPT OF DESEMANTIZATION

The term *desemantization* refers to the process (*). It emphasizes the fact that once abstract knowledge is formed, it is no longer dependent on the initial semantic ideas and exists independently.

Story 1

I ask a five-year-old Adam how much two plus three is. The boy looks at apples on the table and asks: "Two apples and three apples?" and when I agree he first takes two apples, then three apples, puts them together, counts them and says: "five apples". Then I point at a bowl with candies and ask: "How much is two candies and three candies?" The boy proceeds analogically and says: "five candies". He does not realize that in both cases it is the same calculation. This is surprising for Adam's father who was observing us. The father was convinced I should have told the boy that he could use the first calculation in the second case. The father was disappointed that his son failed to see the analogy.

Half a year later, when asked how much two and three is, the boy used his fingers to find out the result is "five". Another year later the boy answers "five" was without counting. When he later mastered numbers in the language of higher abstraction, he was able to record this knowledge in the abstract form: $2 + 3 = 5$.

What has taken place in the boy's mind is *desemantization*. If the symbol 2^* describes a semantically anchored number 2 and similarly symbols 3^* and 5^* semantically anchored numbers 3 and 5, then the described desemantization can be described as the transfer $(2^* + 3^* = 5^*) \rightarrow (2 + 3 = 5)$. When practicing addition and subtraction at school, the link gets automated. However, when his younger sister asks him how much two plus three is, his advice is to count it on fingers. This implies that the abstract knowledge $2 + 3 = 5$ is still connected to semantics in Adam's mind. The boy is able to make the abstract item of knowledge $2 + 3 = 5$ comprehensible to his sister using the relation $2^* + 3^* = 5^*$. The desemantization that Adam went through has not broken the individual stages of the process (*). The boy naturally goes back to the stage of isolated models. That is why we speak in this case of *reversible desemantization*.

Story 2

Adam is in the second grade and is one of the fastest arithmeticians in his class. He can add, subtract, multi-

ply and partially also divide very quickly and reliably. He is not so good at word problems. The class is solving the following problem:

Problem 1. Mum paid 163 CZK for her shopping. She has 509 CZK after this purchase. How many crowns did mum have before she went shopping?

Adam sees the signal word “spent” but is not sure whether to subtract, as he finds the text a bit strange. He prefers to ask his teacher: “Miss, is it plus or minus?” The teacher answers plus. Adam then quickly answers six hundred and seventy two. And the teacher commends him for his answer.

The teacher overvalues calculation skills and fails to realize that Adam fails to understand mathematics. She does not realize that calculation in word problems is only secondary, what is of primary importance is understanding the assignment, the pupil’s ability to grasp in their mind what the problem asks and requires. When the teacher commends the boy, she deforms his metacognitive belief that mathematics is about being fast in calculations rather than about thinking. The boy’s mathematical cognition is no longer supported in his semantic ideas. If we ask him to pose a problem with addition $163 + 509$, he will use the standard addition of two sets of data. If we ask him to use the word “spent” in the assignment, he will not manage to do so. Desemantization in the boy’s mind is in this case *irreversible*.

Story 3

Adam is in the seventh grade. He can now add fractions. This knowledge entered his mind not by the process (*) but through structural deduction. First the pupils were introduced to reduction and raising of fractions and based on this knowledge the teacher deduced the relation: $a/b + c/d = ad/bd + bc/bd = (ad + bc)/bd$. Adam does not understand the presented deduction and thus has no idea of what is actually happening as there are very few real-life problems based on this concept of fractions. Adam has only learnt a rule and knows that when adding fractions he must follow this specific rule.

When his younger sister asks him how to add one half and one third, he shows her the rule. When she asks him for an explanation of this rule, he says it cannot be explained, it must be learnt.

In this case Adam’s knowledge of addition of fractions is not supported by semantic ideas as it entered the boy’s mind from the outside with very little semantic support which is, moreover, not further developed in the subsequent lessons. On the contrary, the little semantic support that existed was forced out through the subsequent drill. Desemantization is very weak in this case and the support in semantic ideas ceases to exist. In this case we also speak of *irreversible* desemantization. More precisely we speak of mechanical knowledge that does not enter pupils’ minds by (*) but by direct transmission.

ISOLATED MODELS

Adam’s story shows how a pupil with clear ideas about mathematics turns into a pupil without any ideas. The story also illustrates the important stages of isolated models. Once we master abstract thinking and cognition, we often believe it is a waste of time to be learning isolated models. That is the case of Adam’s father from Story 1 and the teacher who was teaching Adam to add fractions. This common mistake can be avoided if the teacher carefully monitors how their pupils understand mathematics and if they look for ideas in history of mathematics. Let us present here one historical illustration which is underlain by phylogenetic parallel to Adam’s ontogenetic activity in Story 1.

Phylogenetic parallel

We can find evidence of the fact that addition can be dependent on the objects we calculate with also in phylogeny. For example the Japanese numeral “two” is “ni” and is recorded by the character (二). However, this concept is not on the level of abstract cognition as in the semantic context the Japanese supplement this numeral by a particle called numerative. This means that in our transcription they do not work with number 2 but with anchored number 2*. Thus two people are futari (二人), two small animals are nihiki (二匹), two large animals are nitó (二頭), two elongated cylindrical objects are nihon (本), two glasses are nihai (二杯) etc.

The number of Japanese numeratives shows that creation of a generic model may require a considerable number of isolated models. It is almost impossible to find out how many different calculations $2^* + 3^* = 5^*$ had been carried out by Adam in Story 1 before he grasped the relation $2 + 3 = 5$.

Both isolated models of the relation $2^* + 3^* = 5^*$ presented in Story 1 were of the same type:

- A) number + number = number.

But the relation $2 + 3 = 5$ has more semantic contexts. For example two more types:

- B) address + operator of comparison = address (I live on the 2nd floor, Michal lives 3 floors higher. Which floor does Michal live on?)
- C) operator of change + operator of change = operator of change (I make 2 steps, then 3 steps. How many steps have I made?)

The spectrum of different contexts of isolated models is crucially important for the quality of future desemantization. First of all it brings a variety of ideas into the nascent schema of “addition and subtraction” and enriches semantic background of arithmetic. Moreover it prepares the grounds for new concepts and relations and also type

- A) prepares the grounds for the concept of fraction as we can add $1/2$ of an apple + $1/3$ of an apple;
- B) prepares for grasping the concept of a number line on which there will be all numbers;
- C) prepares for work with vanishing models of numbers (e.g. three steps, three winks, they vanish once they have been carried out) and negative numbers.

The presented stories, their analysis and further considerations show that a well-built rich system of isolated models is prerequisite to good and successful desemantization. Although we discuss here only introduction to arithmetic, the conclusion is universally valid. It applies also to fractions, decimal numbers, equations, perimeters and areas of plane figures, combinatorics. Combinatorics will be the setting of the story in the following chapter.

FIRST ABSTRACTION TRANSFER

Based on the analysis of several dozens of video recordings of the cognitive processes, the stage of isolated models was divided into four sub-stages:

1) First experience enters our mind – the seed of future knowledge.

2) Gradual entry of more isolated models that are not interlinked yet. It may happen that we accept quasi-models and refuse surprising models.

3) Some of the models start pointing at each other, we group them together and separate them from other models. We get the feeling that these models are somehow alike.

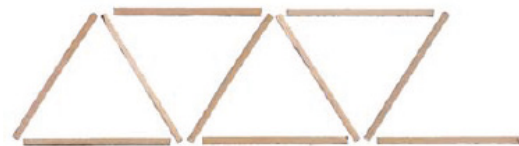
4) Discovery of this alikeness results in creation of a community of at least a subset of all isolated models.

As soon as the pupil has reached stage four, he/she chooses one of the isolated models and says “and this is how it always works”. This isolated model becomes a generic model, which shows how to solve problems of this type. In Adam’s case from Story 1 this generic model was the addition 2 fingers + 3 fingers = 5 fingers.

Story 4

In the second grade pupils were solving problems with sticks. The following is one of the problems.

Problem 2. Continue making other triangular windows and write down into the table how many sticks you need.



cells	1	2	3	4	5	6	7
sticks							
cells	8	9	10	11	12	13	14
sticks							

Figure 1: How many sticks?

The problem was solved about 10 days ago and the pupils had found out that for creation of 3 (4, 5) windows they needed 7 (9, 11) sticks. None of the pupils had noticed the number of sticks was increasing by two. It means that the pupils reached the 1st and 2nd sub-stage of isolated models.

Later, Ben was solving the problem at home on his own. When he put numbers 3, 5, 7 and 9 in the table, he saw the numbers were increasing by two. (Ben reached the 3rd sub-stage.) Therefore he added number 11 beneath number 5 and checked the result. He was happy to see that his conjecture worked. (Ben reached the 4th sub-stage.) He filled in the whole table and ran to his father to show. The father commended his son and asked how many sticks would be needed for 50 windows. The boy realizes that this will require a lot of writing and calculations. He takes his things and goes to work in his room. He makes Table 1.

While filling in the table he notices that beneath numbers 19 and 29 there are numbers 39 and 59. Then he realizes that beneath numbers 20 and 30 there are numbers 41 and 61. He is convinced that beneath numbers 40 and 50 there will be numbers 81 and 101. He runs to his father to show him this discovery. He commends his son and asks him what number there will be beneath number 57. Ben creates table for numbers 50, 51 to 57 and beneath numbers 101, 103 and 115. The father asks what number there will be beneath number 100. Ben writes 100 and beneath immediately 201. The father asks what there will be beneath number 113. Ben writes 110, 111, 112 and 113. Beneath 221, 223 and 225. The father applauds.

The father resisted the temptation to disclose to his son that the rule works for all numbers, not just tens. Ben discovered this rule later together with his other two classmates and recorded it in a very simple way: $sticks = 2 \cdot windows + 1$. Later when Ben was explaining it to his friend, he wrote briefly $s = 2 \cdot w + 1$.

Ben's solving procedure contains all 4 sub-stages of isolated models. The first two took place in the classroom when pupils were solving the problem for 3, 4 and 5 windows. The third sub-stage was supported by the use of the table thanks to which the first discovery was made: the numbers increase by two. Ben discovered the instruction on how the process continues. This was the 4th sub-stage that immediately transcended into a generic model. Generic models of this type are therefore called *processual*.

The father, by asking about number 50, guided the boy to search for the rule how the number of sticks can be derived from the number of windows immediately, without having to make a long table. The boy first discovered the answer for numbers 10, 20, 30, ... This is a discovery of the model that we will call *partially conceptual*. Finally the third discovery made with classmates is a fully *conceptual model*.

The story shows a complex, several day long process of discovery of a generic model. Each of the three AHA-effects accompanying the process was a very exhilarating experience. And this experience guarantees that the knowledge about the relation of windows and sticks given by the abstract formula $s = 2 \cdot w + 1$ is the consequence of reversible desemantization. Ben is able to recapitulate the whole process even a year later.

ISOMORPHISM OF GENERIC MODELS

In the paragraph on Isolated models we presented several different semantic types of anchoring of the knowledge $2 + 3 = 5$. The rich spectrum of semantic anchoring of most knowledge in mathematics helps reversibility of desemantization. Let us illustrate this on an example of knowledge of combinatorics: $() = 10$.

Problem 3. Alice has birthday. She invited her friends Betty, Cecil, Dee and Elis to her party. Each who came kissed each other girl who had already come to the party. How many kisses were there?

Problem 4. Find out how many matches will take place in a football tournament if there are 5 teams, each of them playing each other once.

Both problems serve as a semantic illustration of combinatorial number $()$.

Problem 3 is of processual, problem 4 of conceptual nature.

Story 5

Problem 3 was assigned to 5th graders. Jana was solving it by a simulated dramatization and then she drew 5 points A, B, C, D and E on a sheet of paper. She joined

15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61						

Table 1: How many cells and sticks?

A to points B, C, D and E and wrote 4. Then she joined B with C, D and E and wrote + 3 next to 4. Thus she continued with C and D and finally gained the result $4 + 3 + 2 + 1 = 10$. Jana finished very fast so the teacher asked her to try to solve the problem for 6 and even 10 girls. Jana started to work. She got engulfed and was working only on this problem until the end of the lesson. Ivan found the problem difficult but together with his friend managed to get the result.

A month later the same class was working on problem 4. Hynek wrote down the names of 5 teams: Sparta, Ostrava, Brno, Hradec, Jihlava. Then he wrote down all the matches: Sparta – Ostrava, Sparta – Brno, ... In the end he stated that there were 11 matches because he had calculated Brno – Hradec and Hradec – Brno.

Jana gave up looking for the solution. She said she did not understand football. The teacher knew Jana liked playing chess and so she advised the girl to solve the same problem for a chess tournament. Jana then started to solve the problem and later, when the problem was discussed by the whole class, she contributed with a very powerful idea.

The first to speak was Hynek (deliberately, because his solution 11 was wrong). When he wrote Sparta – Ostrava, Mirek suggested he should write only S – O to speed up the process. So Hynek changed it and discovered on his own that if there is B – H, he cannot add H – B. There were more pupils who had

	A	B	C	D	E
A		X	X	X	X
B			X	X	X
C				X	X
D					X
E					

Table 2: How many tournament matches?

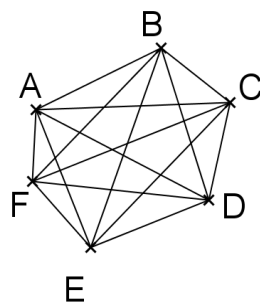


Figure 2: Jana's drawing

been solving the problem analogically to Hynek. Each such solution is an isolated model of the result of problem 4. The most popular solution (especially with boys) was Ivan's solution. He created a table of the tournament and showed there would be 10 matches. This table (Table 2) is closer to a generic model than to an isolated model.

Lada was waving her hand to get the attention of the class to explain that this was the same as kisses on Alice's party. Some pupils agreed but most did not understand. Then Jana suggested it was pretty clear: before each match the captains shake hands, which is the same as when the girls kiss. She ran to the board, drew 5 points and joined each point with all the other points (Figure 2). She said this was how it could be drawn – both Alice and the matches. Everybody could understand this explanation. Jana's drawing became a generic model and Hynek's and Ivan's solution were in this perspective merely isolated models.

When they reached 7th grade, these pupils learnt that the combinatorial number $\binom{n}{2}$ is

the number of all 2-element subsets of an n -element set. The first to understand this difficult definition was Jana. She pointed out that for $n = 5$ it is the same as with Alice or with the matches. Understanding of the abstract concept $\binom{n}{2}$ is now for some pupils based on the generic model "Alice's party for n girls", for other pupils on the generic model "tournament for n teams" but for more pupils on isomorphism of both these models represented by an n -gon, its all sides and diagonals. Obviously the understanding of the last group of pupils is the deepest.

CONCLUSION

The above presented stories were used to cast light on the process of desemantization and to point out the phenomenon of reversibility and irreversibility of this process, which is of paramount importance with respect to understanding arithmetical phenomena. We showed that desemantization is impossible where the abstract idea enters a pupil's mind from the outside, without previous semantic preparation. Story 5 illustrated the case where semantic preparation took place three years before the abstract concept was introduced from the outside.

The scope of this paper does not allow us to discuss the issue of desemantization in detail. Some of the phenomena had to be omitted. For example the importance of *counting backwards* for reversible desemantization, *amalgamation process-concept* (Gray & Tall, 1994), or *grasping, coding and transformative power of language* for second abstraction transfer (Kvasz, 2010).

Let us conclude this study by a summary of the main findings of our research:

- Desemantization is a long-term mental process whose mechanism can be described by two abstraction transfers (*).
- Desemantization is reversible if the pupil even after having achieved the abstract level is able to project abstract ideas into generic models.
- Reversibility of desemantization has positive influence on the richness of generic models and links between them.
- Knowledge that enters a pupil's mind from the outside in its final form (i.e. mechanical knowledge) has no semantic anchoring. In this case we cannot speak of desemantization. However, semantic anchoring can be developed later.

ACKNOWLEDGEMENT

The research was supported by the research project No. 15 (PRVOUK) School and the teaching profession in the context of increasing demands on education.

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