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What is a better buy? Rationale and empirical analysis of unequal ratios tasks in commercial offers contexts

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In this work, the comparison of unequal ratios tasks in commercial offers contexts is studied. A rational and empirical analysis of tasks helps to identify the critical components and students responses to each task. The results confirm the deficits in the relative thought of pre-service teachers, and also that their difficulties are not in the algorithmic aspects and “norming” techniques, but in conceptual aspects and ratio referents.

Keywords: Ratio and proportion, relatively, norming, didactic phenomenology.

INTRODUCTION AND RESEARCH OBJECTIVES

“The box of Bites (net weight 16 oz.) costs \$3.36 and the box of Bits (net weight 12 oz.) costs \$2.64. Which cereal is the better buy?”. This example is used by Lamon (2012, p. 106) to encourage thinking flexibly in unitizing. Examples of this type involve the comparison of relative quantities, which are ratios, though the explicit formulation of the problem needs a relational term, the word “relatively”, to specify the price comparison must be in relation to the weight of the product. The ability to compare the large amounts in this way widens the range of applicability of certain words, such as the word “more” which has two meanings, one absolute or additive and one relative or multiplicative, both of them are correct.

As Streefland (1985, p. 75) says: “In mathematics programs for elementary instruction as far as ratio is concerned, one is often struck by the poverty and brevity of the approach chosen by their authors”. The poverty of this approach to ratio can be more generally characterized as follows: the concept building is exercised with mathematical objects unrelated to reality; the

lack of real applications, and isolation of the subject “ratio”, which is not connected with any other subject.

In this way, Freudenthal (1983) in his didactic phenomenology, highlights the importance of considering ratios in situations in which the idea of “relatively” (or comparatively) and the complex of techniques designated by *norming* are required.

Understanding “relatively” in the sense of “in relation to...” involves the use of the term ratio as Smith (2002, p.14) proposes: “I will use the term ratio to describe a relational number that has two properties: (1) it relates two quantities in one situation, and (2) it projects that relationship onto a second situation in which the relative amounts of the two quantities remain the same”. This use of ratio is in accordance with the very meaning of ratio: “to speak about equality (and inequality) of ratios, without knowing how large the ratio is” (Freudenthal, 1983, p. 180).

All of this situates our interest in problems of quantitative comparison of ratios. In particular, in commercial offers comparison, which offer discounts that are given as relative amounts. As a standard norming percentages are usually used to express discounts.

In order to provide relevant tasks for a didactic phenomenology of ratio involving these ideas, a test has been designed. The tasks, which are typical of commercial offers, have been analyzed in a rational and empirical way. This allowed for a better understanding of critical components and their relationship with the response patterns of students.

According to Cramer, Post & Currier (1993, p. 2), “the critical component of proportional situations is the

multiplicative relationship that exists among quantities that represent the situation". In the quantitative comparison of ratios problems, the multiplicative relationship that exists between the quantities represented in the situation can be equal or unequal. These multiplicative relationships express relative quantities, that is, quantities put in multiplicative relationship with other quantity of reference. This is usually called "the referent". So, we consider that the critical components (c.c.) in these situations are: not only the multiplicative relationships, but their equality or inequality and their referents.

Note that tasks that have been experimented can be used not only with the intention of assessing knowledge, but also in teaching situations and for teacher training. It allows them to promote the metacognitive reflection about their own cognitive processes and the didactic task complexity and the mathematical contents involved. So, the research questions are: which are the critical components of tasks?, what strategies do the students use?, and what difficulties do the students show?

CONCEPTUAL FRAMEWORK

As we have just said before, comparing relatively is to put something in relation to, and norming is a process of reconceptualization of a system in relation to some fixed unit or standard (Lamon, 1994, p.94). One of the common forms of norming is the unification of the antecedent (numerator) or consequent (denominator) of the ratios to favor the comparison. This can be done by an algorithmic process that links them to the unit (e.g. the unit rate obtained by quotient), the decimal numbering system (percentage or decimal) or equivalent fractions. These techniques connect the various forms of ratio: fraction, decimal, percentage or quotient, and are linked to the flexibility of thinking in order to choose convenience.

The norming techniques are used with the intention to make more visible the comparison of pairs of phenomena that Freudenthal (1983) calls "expositions" or "compositions". When two distinct defined expositions are compared on the same set, e.g., Ω is a set of countries, each with its assigned inhabitants and its area by the ω_1 and ω_2 functions. Then ratio ω_1/ω_2 expresses the population density. Comparison of density couples allows to state whether a country has in proportion to its area the same number of inhabitants

or a higher or lower number than another country. In the comparison of two compositions on the same set, e.g., Ω is an alloy composed by copper and zinc to form bronze and each component is assigned a different mass in each alloy by the ω_1 and ω_2 functions. Comparison of the pairs of internal ratios, copper mass/zinc mass, allows one to know which alloy has relatively more, less or the same amount of copper to zinc (Freudenthal, 1983, p. 186).

The "best buy problems" can be interpreted as a pair of expositions or compositions. Under this interpretation, we define the objectives of this study. The first one is to determine, through the rational analysis, the critical components of the tasks such as: the multiplicative relationships, the equality or inequality of ratios and their referents. The second one is to determine, through the empirical analysis, the relationship between these components and students' performance.

METHODOLOGY

The work is based on the methodology of the empirical and rational analysis of tasks. According to Lamon (2007, p. 641), the distinction between empirical and rational analysis is adopted to distinguish between children's mathematics (children's actual performance on tasks); and the exam of content from a mature mathematical perspective, making assumptions about the ways of thinking that are necessary to solve problems. The rational analysis begins at the theoretical level, in order to identify the critical components of tasks and their procedural cognitive and conceptual objectives, to support theoretical inferences from the data obtained in the empirical analysis. The empirical analysis begins with the implementation of tasks given to students in order to interpret their responses. This is taken as criteria for analyzing the critical components identified in the rational analysis.

We choose 4 tasks for a pencil and paper test. They are called Pizza, Beer, Softener and Mosquito repellent. They are realistic tasks taken from offers in current commercial brochures. Due to their typology they are characterized as quantitative comparison of ratios tasks, where one has to judge which of two ratios is higher, lower, or perhaps the same, so you can do it roughly or precisely. Moreover, they may be characterized by their phenomenology as pairs of expositions or compositions involving comparing relatively

and applying norming techniques, as Freudenthal says. So, we choose these tasks because they are real applications of ratio and one has to judge equality or inequality of ratios, as Streefland and Freudenthal require. The test was implemented using individual worksheets given to 9 groups of students working in a normal mathematics class time in their second year of the teaching degree, at the beginning of the course (341 students). The study was conducted during 2013–2014 at the University of Valencia.

We select these participants because, according to Ben-Chaim, Ilany & Keret (2002), we think that the pre-service teachers need to improve their knowledge and their attitudes toward mathematics, in general, and all of the components and aspects of ratio and proportion, in particular. And we think that realistic tasks, such as the commercial offers, are suitable for teaching the topic of ratio and proportion in pre-service elementary teacher education.

TASKS

In Pizza, one asks: What is better, two regular pizzas 30cm in diameter for 14.95€ each or a large pizza 50cm diameter for 27.95€? Justify your answer. In

Softener, the question is: Which of the two options is more expensive, the concentrate on the left or the non concentrate on the right? Justify your answer.

In Beer, the text says: Usually beer cans are 1/3 liter or what is the same, 33.3cc. One option offers a 15% discount on the price and the other option has 14% more beer. What is more expensive? And, in Mosquito repellent: Fogo and Bloom sell for the same price and have the same volume without any promotion. What discount is better? Explain your answer. Softener and Mosquito are easier than the other two because the text provides all the data needed for doing a correct comparison.

Rational analysis and critical components of tasks

In Pizza and Softener the critical component is the equality or inequality relationship of the two relative quantities. These quantities are given by composed ratios formed by the pair (€, washes) or (€, cm²). It is required to apply norming by quotient for making the comparison visible. In Pizza, the solution process requires finding the areas and the cost of the pizzas, and comparing prices with areas or vice versa. Alternatively, one can compare areas, prices, and then

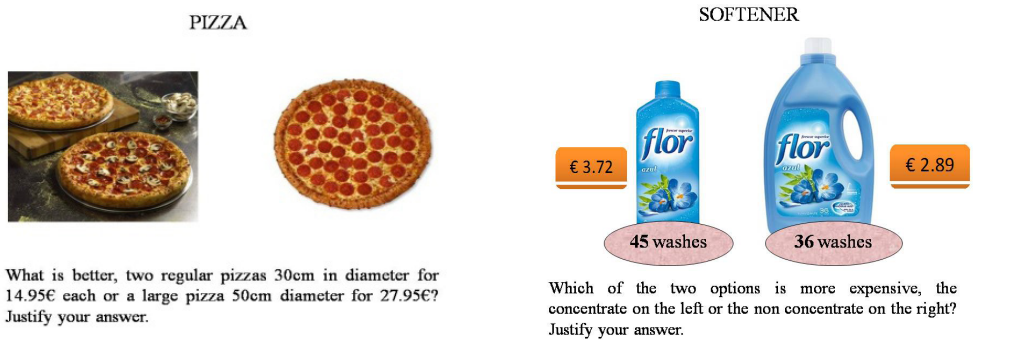


Figure 1: Pizza and Softener

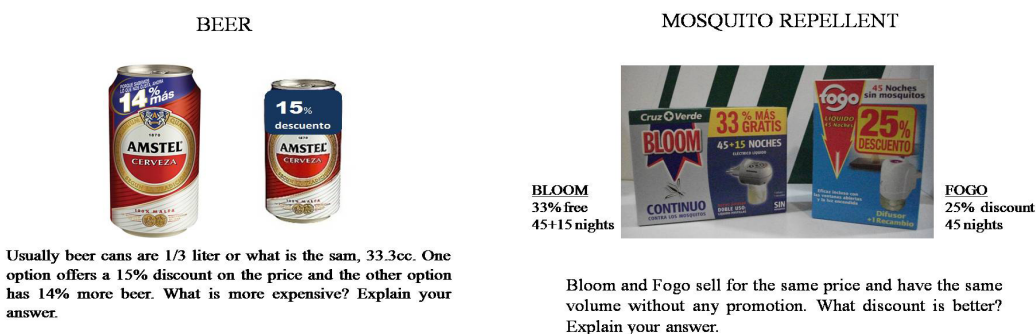


Figure 2: Beer and Mosquito repellent

compare both comparisons. In Softener, the process is reduced to compare the relative quantities €/washes. In Beer and Mosquito, it is needed to convert the gift in a discount or vice versa, i.e., to change one of the two referents. Beer task requires finding the volume of the larger can and calculating what percentage discount is equivalent to the extra volume. Once found, we can compare two discounts. Likewise, the mosquito repellent task requires finding what discount percentage is equivalent to the 33% extra free nights. Once found, you can compare the discount percentage in Bloom with the 25% discounts in Fogo. In both tasks, the reciprocal process (to transform a discount percentage into a gift percentage) is similar.

The processes that account for these transformations, calculations and norming techniques, are displayed in tables from 1 to 6, as pairs of expositions or compositions. In the case of pair of expositions, Ω is formed by the set of offers; and, in the case of pair of compositions, Ω is composed by the parts that form each offer.

As we can see in tables 1 and 2, in comparing norming ratios by quotient, $\omega_1(\Omega)/\omega_2(\Omega)=\text{cost}/\text{cm}^2$ or $\text{cost}/\text{washes}$, the unit rate (u.r.) or its reciprocal (r.u.r.) are obtained. These comparisons are the main c.c., because they show which offer pays more, less or the same compared to what is acquired (comparing how many € per cm^2 or per wash). Alternatively, in Pizza, while comparing the internal norming ratios, $\omega_1(\Omega)/\omega_1(\Omega)=\text{cost}$

PIZZA \ Ω	Large Pizza	Regular pizzas	Internal comp. \rightarrow
$\omega_1: \Omega \rightarrow \text{Cost}$	$C_L = 27.95\text{€}$	$C_R = 2 \cdot 14,95 = 29.90\text{€}$	$\frac{C_L}{C_R} = \frac{27.95}{2 \cdot 14.95} = 0.94$
$\omega_2: \Omega \rightarrow \text{Area}$	$A_L = \pi \cdot 25^2 \text{cm}^2$	$A_R = 2 \cdot \pi \cdot 15^2 \text{cm}^2$	$\frac{A_L}{A_R} = \frac{\pi \cdot (50/2)^2}{2 \cdot \pi \cdot (30/2)^2} = 1.39$
External comp. \downarrow	$\frac{C_L}{A_L} = \frac{27,95}{\pi \cdot 25^2} = 0,014 \frac{\text{€}}{\text{cm}^2}$	$\frac{C_R}{A_R} = \frac{2 \cdot 14,95}{2 \cdot \pi \cdot 15^2} = 0,021 \frac{\text{€}}{\text{cm}^2}$	

Table 1: Couple of expositions. Pizza task

SOFTENER \ Ω	Concentrate softener	Non concentrate softener
$\omega_1: \Omega \rightarrow \text{Cost}$	$C_C = 3.72 \text{€}$	$C_{NC} = 2.89 \text{€}$
$\omega_2: \Omega \rightarrow \text{Washes}$	$W_C = 45$	$W_{NC} = 36$
External comp. \downarrow	$\frac{C_C}{W_C} = \frac{3.72}{45} = 0.082 \frac{\text{€}}{\text{wash}}$	$\frac{C_{NC}}{W_{NC}} = \frac{2.89}{36} = 0,080 \frac{\text{€}}{\text{wash}}$

Table 2: Couple of expositions. Softener task

BEER \ Ω	Part free	Part paid	Internal comp. \rightarrow (increase percentage)
$\omega_1: \Omega \rightarrow C_L$ (large can)	$F_L = 14$	$P_L = 100$	$\frac{F_L}{P_L} = \frac{14}{100} = 14\%$
$\omega_2: \Omega \rightarrow C_S$ (small can)	$F_S = 15$	$P_S = 85$	$\frac{F_S}{P_S} = \frac{15}{85} = 0,176 = 17\%$

Table 3: Couple of compositions. Beer task

MOSQUITO \ Ω	Part free	Part paid	Internal comp. \rightarrow (increase percentage)
$\omega_1: \Omega \rightarrow \text{Bloom}$	$F_{\text{Bloom}} \cong 15$	$P_{\text{Bloom}} = 45$	$\frac{F_{\text{Bloom}}}{P_{\text{Bloom}}} = \frac{15}{45} = 33\%$
$\omega_2: \Omega \rightarrow \text{Fogo}$	$F_{\text{Fogo}} \cong 11 (45 - 25\% \cdot 45)$	$P_{\text{Fogo}} = 34 (45 - 11)$	$\frac{F_{\text{Fogo}}}{P_{\text{Fogo}}} = \frac{11}{34} = 33\%$

Table 4: Couple of compositions. Mosquito repellent task

BEER \ Ω	Large can	Small can
ω ₁ : Ω → free part	F _L = 14	F _S = 15
ω ₂ : Ω → total product	T _L = 114cc.	T _S = 100cc.
External comp. ↓ (discounts)	$\frac{F_L}{T_L} = \frac{14}{114} = 12\%$	$\frac{F_S}{T_S} = \frac{15}{100} = 15\%$

Table 5: Couple of expositions. Beer task

MOSQUITO \ Ω	Bloom	Fogo
ω ₁ : Ω → free part	F _{Bloom} = 15	F _{Fogo} = 11
ω ₂ : Ω → total product	T _{Bloom} = 60	T _{Fogo} = 45
External comp. ↓ (discounts)	$\frac{R_{Bloom}}{A_{Bloom}} = \frac{15}{60} = 25\%$	$\frac{R_{Fogo}}{A_{Fogo}} = \frac{11}{45} = 25\%$

Table 6: Couple of expositions. Mosquito repellent task

regular pizza/cost large pizza, or regular pizzas area/ large pizza area, we see that for almost the same price, the large pizza has got more area.

When comparing the norming ratios by quotient, ω₁(Ω)/ω₂(Ω)=free part/total product or ω₁(Ω)/ω₁(Ω)=free part/part paid, after homogenizing the referent with respect to the part paid (h.p., tables 3 and 4) or with respect to the product acquired (h.a., tables 5 and 6) decimals or percentages are obtained. All of these processes are the c.c., because they show which discount or increase percentage are higher, lower or the same.

EMPIRICAL ANALYSIS OF TASKS

The empirical analysis takes into account the response patterns of students in relation to their strategies and their difficulties. Apart from the difficulties related to the c.c. other conceptual difficulties have been observed such as: the linearity (specific of Pizza) and the misinterpretation of the r.u.r. We highlight also two strategies that have appeared in the tasks and they are different from the strategies pointed in tables from 1 to 6.

Most significant alternative strategies and difficulties: Examples

Difficulty in linearity of the unit rate in Pizza task. Students adopt a relative approach, comparing external ratios and applying norming by quotient to obtain a u.r. Rather than comparing prices with pizza areas, they compare prices with pizza diameters (example

1). In this case, the difficulty is in the referent of the relative quantities that they have to compare.

$(14.95/30) \cdot 2 = 0.996$; $27.95/50 = 0.559$. A cm of the large pizza is cheaper. That is, the large pizza is more economical.

Example 1: Student's response to Pizza task

The student compares the relative amounts given by the ratios between the diameters and the prices, i.e. the u.r. of each offer. This strategy could be valid if there was only one item in each offer because more diameter implies more area. Note that the student multiplies the u.r. of a regular pizza by 2 [Regular: $(14.95/30) \cdot 2 = 0.996$]. This calculation suggests us that he is not aware of the invariance of the ratio.

Difficulty in interpreting the reciprocal of the unit rate. Students adopt a relative approach, comparing external ratios and applying norming to obtain a unit rate, but they interpret the unit rate in the reverse way that corresponds to the stated ratio. In this answer, the difficulty is the loss of meaning of the referent when they apply norming techniques.

b) $100\% - 15\% = 85\%$ price → 33cl; a) 14% of 33.3 = 4.662 cl more. $33.3 + 4.662 = 37.962$ cl in can A; $33/85 = 0.388$; $37.962/100 = 0.3796$. In offer B you pay 0.388€ per cl, while in offer A you pay 0.3796€ per cl so that, offer A is cheaper.

Example 2: Student's response to Beer task

At first glance, it is obvious that the concentrate is more expensive, as it costs 3.72€, which is 0.082cents per cup. On the other hand, the 36 wash product has an added cost of 0.080 for each of the 9 cup difference and results in a price of 3.61€, making it cheaper than the concentrate and therefore, less expensive than the one on the right. $2.89/36 = 0.080$ $2.89 + 0.72 = 3.61$; $0.080 \cdot 9 = 0.72$ → cup difference from the concentrate.

Left: 45 cups = 3.72. Right: 45 cups = 3.61.

Example 7: Student's response to Softener task

This student finds the difference between the 2 softeners: $9=45-36$; calculating both unit rates: $3.72/45=0.082\text{€/wash}$ the concentrate softener, and $2.89/36=0.080\text{€/wash}$ the non concentrate softener, and uses one of them to calculate the total cost that the non concentrate softener would have if there were a 9 wash increase. If he had $36+9$ washes, it would cost $2.89+9 \cdot 0.080=3.60\text{€}$, which is cheaper than the 3.72€ concentrate.

RESULTS

The absolute frequencies are displayed in Table 7. The columns (from left to right) show: the number of students that use the strategy of u.r., those who have difficulties in this strategy, students who homogenized the referent, the participants who compare the quantities of the same nature, those who use the strategy of cost comparison after matching amounts, and the group of qualitative answers, random, blank, etc (others).

Regarding the critical components of the tasks, we observed that in Beer and Mosquito there are more students who compare absolute quantities instead of relative quantities. Nevertheless, in Pizza and Softener tasks the contrary occurs. Moreover, there

are very few students who homogenize the referents (4 in Beer, 2 in Mosquito). They may not interpret the gift like a discount or vice versa. It highlights how students are inclined to calculate the u.r. although it is not needed in this case. If we focus on the strategies and difficulties identified, in Pizza there are few students who calculate the u.r. or its reciprocal without difficulties because of the linearity. In fact, 175 students calculate the u.r. with the diameter instead of the area. In contrast, in Softener the u.r. strategy dominates as we expected. Only 23 students show difficulties calculating the r.u.r. Finally, there are students who give blank, random, qualitative or incomplete responses, especially in Beer (82 students) and Mosquito (44 students).

CONCLUSIONS

Predominance of the unit rate strategy in Beer and Mosquito tasks can be due to the lack of flexibility and the application of a mechanical rule learnt at school. The use of this strategy implies to assign an arbitrary price although it is not needed. It does not imply that their responses are wrong, but it is an indicative of a price-dependence attitude. Note that, there are other more efficient strategies. In addition, resistance is also observed in accepting that an increase percentage can be interpreted as a discount and vice versa. Moreover, there are students who misunderstand the r.u.r., suggesting a mechanical knowledge of the rule. Finally, the use of linearity is widespread in Pizza, and very few students perceive the invariance of the ratio calculating the u.r. of two regular pizzas instead of the u.r. of one regular pizza. They do not realize that both of them are equivalent.

These results confirm the deficits in the relative thought of pre-service teachers, and that, according to Ben-Chaim, Ilany & Keret (2002, p. 81), their knowledge is frequently technical, unrelated and incoherent. Moreover, the difficulties shown by the

Strat. Task	Unit rate		Difficulties with u.r.		h.p. o h.a.	Same nature		Matching amounts	Others
	u.r.	r.u.r.	Recip.	Linearity		Absolute	Relative		
Pizza	13	1	2	175	-	87	1(lin.)	42	20
Softener	215	5	23	-	-	41	1	33	23
Beer	103	3	16	-	4	113	-	20	82
Mosquito	107	2	11	-	2	168	-	7	44

Table 7: Tasks' results

students are not the algorithmic aspects and norming techniques, but in conceptual aspects, ratio referents and the “price-dependence” when they are comparing discounts in commercial offers. The next step is to design a teaching sequence that helps the students to widen their knowledge of ratio and, according to Lamon (2012, p. 107), to help them to develop flexibility in situations like the best buy problems, to encourage multiple correct strategies and to discuss which strategies are easier, faster or more reasonable.

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