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Student strategies and errors in mental computation with rational numbers in open number sentences

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Mental computation is an important skill for mathematics learning. This paper analyzes mental computation strategies and errors in grade 6 students with positive rational numbers in a teaching experiment based on tasks. The tasks involve the four basic operations and are done in conjunction with a discussion of strategies with the entire class. We focussed on student relational thinking and our findings suggest that open number sentences highlight the meaning of the equal sign and the importance of using properties of operations in mental computation. This is reflected in student's strategies when they use numerical relationships to compute mentally. Mental representations such as propositional representations are strongly linked to strategies based on numerical relationships.

Keywords: Mental computation, rational numbers, student errors, student strategies, relational thinking.

INTRODUCTION

Carrying out mental computation tasks in the classroom helps in the development of important mathematics skills. As Taton (1969) indicates, mental computation develops notions of order, logic, reflection, concentration and memory. It also prepares students for daily life, providing them with tools to perform simple computations without the help of external resources. Former studies (e.g., Caney & Watson, 2003; McIntosh, 2006) highlight the importance of understanding student strategies in mental computation, particularly the numerical relationships they use in solving problems involving rational numbers. This understanding may support teacher planning and teaching activity. This paper is drawn from a teaching experiment based on mental computation tasks involving open number sentences with four operations

and the discussion of strategies analyzing mental computation strategies and errors in grade 6 student, focusing on how they use relational thinking.

MENTAL COMPUTATION WITH RATIONAL NUMBERS

Reys, Reys, Nohda and Emori (1995) refer to mental computation as the process of computing exact arithmetic results without external support. In line with this perspective this paper takes mental computation to be the process of computing arithmetic results in a quick and efficient way using number facts (e.g., multiplication tables), memorized rules (e.g., how to multiply by powers of ten) and numerical relationships (e.g., converting between different representations) supported by mental representations (e.g., mental images of fractions or relational thinking based on propositional representations).

The main elements of our framework are the use of (i) number facts; (ii) memorized rules, (iii) numerical relationships and (iv) mental representations (Figure 1).

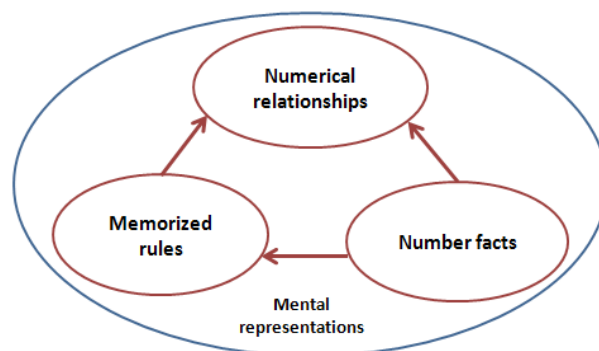


Figure 1: Conceptual framework

To develop mental computation, Heirdsfield (2011) indicates that students need to know number facts but also to understand the size and the value of numbers, the effect of an operation on a number, and to be able to make estimates to check if the solution is reasonable. As computing mentally with rational numbers involves more complex reasoning than computing with whole numbers (Barnett-Clarke, Fisher, Marks, & Ross, 2010), we assume that the use of memorized rules may also support student computation and relational thinking (Empson, Levi, & Carpenter, 2010).

The use of mathematical relationships is evidence of relational thinking. This is related to the capacity to use fundamental properties of operations and the notion of equality to analyse and solve problems. As Carpenter, Franke and Levi (2003) highlight, “many fundamental mathematical ideas involve relations between different representations of numbers and operations on them” (p. 38). From their perspective, the use of open number sentences (e.g., $1/2 - _ = 1/4$) may be a viable way to draw a student’s attention to relations between numbers and operations and, if students are experienced in looking at these kind of relations, they will be able to use them to compute quickly. For example, if students understand that $3/4$ may be split into $3 \times 1/4$ they quickly compute $3/4 \div _ = 1/4$. Also, if they understand 10% as a benchmark and know how to multiply/divide by powers of ten, they easily compute 5% by halving or 20% by doubling.

Another important idea to consider in mental computation is the fact that we construct mental representations from the world that surrounds us, which we use in sense making and making inferences. Understanding student mental representations may help us to understand their relational thinking because they use mental representations to relate numbers and operations to their real world knowledge, including knowledge about mathematics. In the theory of mental models (Johnson-Laird, 1990), these mental representations may be mental models, mental images or propositional representations: (i) mental models are generalized perceptions of the world (e.g., using a generalized context of sales to compute 20% of 25); (ii) images involve a more specific perception of a real world where some characteristics are considered (e.g., relating the symbolic representations $1/2$ to one part of a pizza divided in two parts); (iii) propositional representations represent true/false propositions that play an important role in the inference process (e.g.,

to compute 40% of ? = 48, using a sequence of propositions leading to the solution: ‘if 48 are 40%, $48 \div 4$ are 10% which is 12. So, $10\% \times 10$ is 100% and 12×10 is 120 which is the missing value’).

When students compute mentally, they often make errors that arise from a lack of understanding of rational numbers. This is incorrect usage of the elements presented in Figure 1. Carpenter and colleagues (2003) suggest that such errors may be based on generalizing properties in other situations. For example, knowing that multiplication and addition are commutative may lead students to apply this knowledge to subtraction and division. In adding fractions students often regard them not as a single number but as two separate numbers (Lamon, 2006), thus adding numerators and denominators. As McIntosh (2006) indicates students make conceptual and procedural errors in mental computation. A conceptual error arises when the student fails to understand the nature of the numbers or of the operation involved and a procedural error takes place when the student knows what strategy to use but makes a computation error when putting it into practice.

RESEARCH METHODOLOGY

The study is qualitative and interpretative with a design research approach (Cobb, Confrey, diSessa, Lehere, & Schauble, 2003). This is a developmental study that aims to solve problems identified from practice—the difficulties in learning rational numbers and near absence of mental computation with this number set. It is based on a teaching experiment with mental computation tasks that provide opportunities for discussing student strategies and errors. It was carried out in three phases: preparation; experimentation and analysis. In the preparation phase (2010) a preliminary study in grade 5 was undertaken (conducted by the first author in their classes) in order to collect information on strategies that students use and errors that they make when computing mentally with rational numbers. This showed that practical aspects of students’ mental computation practice were potentially important for planning the teaching experiment. Such planning also took into account research on rational numbers and mental computation with rational numbers. In the second phase (2012–13), two experimental cycles were implemented involving two teachers and two grade 6 classes (39 students) from two different schools with the first author as

a participant observer. During this phase some refinements were made in the teaching experiment (e.g., changing the sequence of tasks). Data was collected through video and audio recordings of classwork with mental computation and researcher notes. Finally, in the analysis phase, audio and video records of student strategies and errors were transcribed. In this paper we focus on the collective discussions in order to understand student strategies and errors, particularly student relational thinking in questions involving open number sentences. The analysis of student mental computation strategies is carried out using categories taken from previous studies (e.g., Caney & Watson, 2003) and from the concept of mental representations (mental models, images and propositional representations), number facts, memorized rules, and numerical relationships. From these concepts and taking into account the data, three main categories of strategies were defined (numerical relationships, number facts and memorized rules) and in each of these, several subcategories (Table 1). The categorization was made according to the strongest concept involved in student strategy. For example, if there is strong usage of numerical relationships, for example, the part-whole relationship a strategy from the category “numerical relationships” and subcategory “comparison part-whole” was named Mental representations were not considered as a fourth category since they are common to all categories. Thus mental representations

were analyzed as a support to student strategies. To analyze student errors McIntosh’s (2006) notion of conceptual and procedural error was used. In each of these categories some subcategories are identified as shown in Table 2.

THE TEACHING EXPERIMENT

The teaching experiment relies on the conjecture that a systematic work with mental computation tasks with rational numbers represented as fractions, decimals and percentages, and whole class discussions may contribute to the development of students’ mental computation strategies and understanding of their errors. Before the teaching experiment the students had already worked with rational numbers in different representations and operations, with an emphasis on algorithms. The teaching experiment involves ten mental computation tasks (with several number sentences or word problems each) with rational numbers to carry out weekly, all prepared by the first author and discussed in detail with the participating teachers. These tasks (Table 3) were presented at the beginning of a mathematics class, using a timed PowerPoint. The students had 15 seconds to solve each number sentence and 20 seconds to solve each word problem individually, in both cases recording the results on paper.

		Categories		
		Numerical relationships	Number facts	Memorized rules
Subcategories	Equivalence		Two halves make a unit	Rule to multiply/divide by powers of ten
	Part-part comparison		A half of a half is a quarter	Rule to add/subtract fractions with different denominators
	Part-whole comparison		Two quarters is a half	
	Inverse operation		10% of...	

Table 1: Categories of strategies in mental computation with rational numbers

		Categories	
		Conceptual error	Procedural error
Subcategories	Strong use of additive reasoning in multiplicative problems		Calculation error
	Ignore the place value in decimal representation		Solve a part of an expression when a valid strategy is possible to identify
	Add/subtract numerators/denominator when operating with fractions		

Table 2: Categories of errors in mental computation with rational numbers

Each task has two parts, with 5 numbers sentences or 4 word problems in each part. On finishing each part, there was a collective discussion of student strategies and errors. These mental computation lessons lasted between 30 and 90 minutes. Some examples of the content of the tasks used in the teaching experiment are presented in Table 3. The timed tasks were seen as a way to challenge students to compute mentally. The discussion moments were regarded as the most important part of the lesson, and allowed students to show how they think, the strategies that they use, and the errors that they make. They were important for students to think, reflect, analyze, make connections, share, and extend mental computation strategies, as well as to identify skills that they could develop in numbers and operations.

In the teaching experiment the students began to compute mentally with rational numbers represented as fractions (addition/subtraction in task 1 and multiplication/division in task 2), then with decimal and fraction representations with the four basic operations (task 3), and then solely with decimal representation (addition/subtraction in task 4 and multiplication/division in task 5). Then they solved word problems in measurement and comparison contexts involving fractions and decimals (task 6). The percent representation was used in task 7, as the teacher began working with statistics. Then, students used the three representations (decimal, fractions and percent) in tasks 8, 9 and 10. In task 10, they solved word problems.

The design of tasks is based on three principles, taking into account previous research on mental computation and rational numbers.

Principle 1. Use contexts to help students to give meaning to numbers. Structured knowledge is associated with the context in which it was learned and most of the time it is difficult for a student to bridge this knowledge to new situations. Therefore, number sentences and word problems were used to provide

students with a variety of contexts and help them to establish connections between them.

Principle 2. Use multiple representations of rational numbers. Fractions, decimals, and percent representations were used in the same task and in several tasks across the teaching experiment (e.g., question from task 3). We used even numbers and multiples of 5 and 10, benchmarks such as 25% or 1/2 to facilitate equivalence between decimals, fractions and percent, and to stress numerical relationships (e.g., multiplying by 1/5 is the same as dividing by 5) and part-whole relationships (e.g., if 10% corresponds to 5, then 100% is 10 times bigger). The rational number representation used was related to the topics that students were working on in class with their teacher because the tasks had been integrated into the teachers overall planning. This option provided students with a further opportunity to learn rational numbers on different mathematical topics throughout the mathematics curriculum.

Principle 3. Use tasks with different cognitive demands. For example, taking into account mental computation levels (Callingham & Watson, 2004) tasks in which the students have to use the concept of half to compute (e.g. 50% of 20 or $1/2 + 1/2 = _$) or need to use more complex numerical relationships (e.g., 20% of $_ = 8$) to do the computation were designed.

In planning the lessons, we tried to anticipate possible student strategies and errors, to prepare better collective discussions. All classroom activities were led by the teachers with the first author making occasional interventions to clarify aspects related to student strategy presentation.

STUDENT MENTAL COMPUTATION STRATEGIES AND ERRORS

Student relational thinking through their strategies and errors, supported by mental representations in

Think fast! What is the exact value?	
Task 1:	$3/4 - 1/2 = _ ; 1/2 - _ = 1/4$
Task 6:	The area of the base of a cylinder is 4.2 m^2 and the volume 12.6 m^3 Calculate the height of the cylinder.
Task 7:	90% of 30 = $_$
Task 8:	$_ \%$ of 20 = 18

Table 3: Examples of tasks

open number sentences was analysed. Examples of related questions were chosen (regarding numbers or the numerical relations between them).

In task 1, students were challenged to mentally compute “ $3/4 - 1/2 = _$ ” and “ $1/2 - _ = 1/4$ ”. These sentences involve two possibilities to reach $1/4$. To solve “ $3/4 - 1/2 = _$ ” Ivo explained: “I made the rule. 2 multiply by 2 gives 4 and then following this I multiplied by 2 also. Then 3 minus 2 is 1. $1/4$.” Ivo used a strategy based on memorized rules where he applied the procedure to subtract two fractions with different denominators, which may have been supported by a mental image of a written algorithm. Concerning “ $1/2 - _ = 1/4$ ”, we present a dialogue where it is possible to understand Marta’s and Rogério’s relational thinking and Eva’s error:

- Eva: I put two quarters.
 Teacher: First, Eva will explain to us why she put two quarters.
 Eva: Because if we multiply by 2 ... We need to have the same denominator and if we multiply the 2 by 2 it gives us 4. It gives us 4 and then 2 multiplied by 1 gives 2. One quarter ... gives two quarters.
 Teacher: Now, you are going to listen to your colleagues and then I will talk to you.
 Ivo: Eh, I had a quarter and I crossed it out!
 Teacher: Do you think that a quarter is correct Ivo?
 Ivo: Yes.
 Teacher: Why?
 Ivo: Because 2 minus 2 is not one ... Because following this in the numerator, 2 minus 2 is not 1. If you use two quarters you have to multiply the numerator by 2 and the denominator also, and it is not 2 minus 2.
 Teacher: Marta.
 Marta: It gives a quarter. I saw that $1/4 + 1/4$ is $1/2$ then, if we take a quarter from a half, it gives us a quarter.
 Teacher: Rogério, did you hear Marta?
 Rogério: I put a quarter.
 Teacher: Explain your reasoning. Let’s see if it’s the same as Marta or not.
 Rogério: We have a cake. A half of a cake. We ate a half [from this cake] and left only a quarter. So, it is half less a quarter.

Marta and Rogério presented a number fact strategy, but supported by different mental representations. Marta used the number fact “ $1/4 + 1/4 = 1/2$ ” and the propositional representation: (if $1/4 + 1/4$ is $1/2$ then, $1/2 - 1/4$ is $1/4$) to relate this knowledge with the open number sentences that she needed to solve. She applies a property of subtraction (to get the subtractive, we take the difference from the additive). Rogério also used a number fact (a half of a half is a quarter) but he used a mental image (“A half of a cake”; “We ate a half”; “and left only a quarter”). Eva had a strategy to solve the question but she only focused on the fraction equivalent to $1/2$ and did not make the subtraction between $2/4$ and $1/4$ so she made a procedural error. It seems that she was trying to apply the same rule that Ivo used above and a similar mental representation (mental image of a written algorithm). The explanations of Marta and Rogério show that a strategy based on number facts may be supported by different mental representations, probably according to the student’s strongest experience. Marta used mathematical knowledge in her propositional representation and Rogério used a real world experience in his mental image, which shows their relational thinking. A student’s experience is crucial in defining the mental image that supports their strategies. Rogério used a different mental image from those used by Ivo and Eva. A strong mental image of a written algorithm (like Ivo and Eva used) does not emphasize student relational thinking and may lead them to make some errors (like Eva did) because they are focused on the procedure and not on the relationships between numbers and operations.

In the second part of task 6 students were challenged to compute word problems mentally. To solve the problem shown in Table 3 involving the concept of volume the students could use open number sentences like “ $12.6 = 4.2 _$ ”. To solve the problem Pedro, Maria and Acácio used different strategies supported by propositional representations. They explained: “It gives 3. I divided 12.6 by 4.2” (Pedro), “I didn’t divide. I multiplied. I tried to find a number. 4.2 times a number to get 12.6 and I found 3” (Maria) and “It gives 3.3. I divide 12.6 by 4.2. 4 plus 4 is 8, plus 4 is 12. 2 plus 2 is 4, plus 2 is 6” (Acácio). They probably had in mind the open number sentence that we had indicated. Pedro and Maria used relational thinking, taking advantage of the properties of operations (Pedro) and relations between the numbers on each side of the equal sign (Maria). Pedro used the inverse operation strategy

and a propositional representation based on the proposition “if 4.2, then ,” showing knowledge about the relation between multiplication and division. Maria used an equivalence strategy supported by a propositional representation based on a sequences of true/false propositions where she “tried to find a number” that could make the equivalence true ($4.21 = 12.6$ (false); $4.22 = 12.6$ (false) and $4.23 = 12.6$ (true)). Acácio made a conceptual error because he was not able to understand how to give meaning to his result. He identified the correct operation ($12.6 \div 4.2$ like Pedro did) and used a splitting strategy (separating the integer and the decimal points of the number in order to operate/compare them) but was not able to understand that the relation concerning the whole number must be seen and not each part separately. His strategy may be supported by a propositional representation based on a false proposition: 12 is and 6 is. If 12.6 is “a twelve plus a six” and 4.2 is “a four plus a two” then the result is 3.3, “a three plus a three”). The origin of this error could lie in the additive reasoning used by Acácio. The three students used different strategies but the same type of mental representation in their relational thinking. This suggests that propositional representations play an important role in this process, where different mathematical knowledge must be analyzed and related. Acácio could have achieved success in his relational thinking if his reasoning were more multiplicative and not as strongly additive.

In task 7, students had to compute “90% of 30 = __”, which is related to the open number sentence “__% of 20 = 18” from task 8. To solve these sentences, Dina and João used different strategies based on propositional representations. Dina explained: “100% of 30 is 30. 10% of 30 is 3. Then, from 100% to 10% this gives 27”. She used a part-whole comparison strategy supported by a propositional representation: (if 100% is 30 and 10% is 3, then 100% - 10% is 90% and 30 - 3 is 27). She used 10% as a benchmark, compared 90% with the whole and removed 10% from 100%. To solve “__% of 20 = 18”, João used a part-part comparison strategy and explained: “I saw that 10% was 2. So I divided 18 by 2 which is the same as dividing the total [result 18] by 10% which gives me 9. Then I multiplied 9 by 10 to get 90.” He used 10% as a benchmark (like Dina did) and probably, 10% of 20 as a number fact (he did not explain how he computed 10% of 20) to relate part to part based on a propositional representation: (If 10% of 20 is 2 and % of 20 is 18, then $\% / 10\% = 18 / 2$). He divided 18 by 2 not to “divide the total by 10%” but to relate

part to part. He gets 9 and multiplies it by 10 to get 90 because the whole is 10 times bigger than 10%. Dina’s and João’s strategies show that they understood that the equal sign expresses a relation, an important aspect in relational thinking. Once more, propositional representations seem to play an important role in student relational thinking. The part-whole (Dina) or the part-part strategy (João) used by students requires a systematic analysis of the relation between the whole and its parts which is supported by true propositions. These propositions help students to make inferences about the relationships between numbers and operations on each side of the equal sign.

CONCLUSION

Students’ mental computation strategies are mostly based on using equivalence, number facts, operation properties, and part-part or part-whole comparison, supported by mental representations. Their relational thinking is more evident when they use strategies based on numerical relationships. We also identified conceptual and procedural errors (McIntosh, 2006). The source of some of these errors may be a lack of some mental representations as well as a lack of understanding of multiplicative structures, especially in operating with rational numbers. The theory of mental models (Johnson-Laird, 1990) helps us to understand and interpret the mental representations used by students as a support of their mental computation strategies. For example, a strong use by students of mental images of written algorithms suggests a lack of knowledge of relationships between numbers and operations and a poor repertoire of mental representations, obstacles to relational thinking and conceptual understanding. In contrast, as has been shown, the use of propositional representations based on true propositions is evidence of student relational thinking. It highlights student understanding of properties of operations and how they use them to establish numerical relationships and use the equal sign to express relationships between numbers and operations. This was more explicit in sentences like “__% of 20 = 18”. Such open number sentences lead students to analyze numbers and operations in a systematic way on both sides of the equal sign as they did in the part-part comparison strategy.

We understand that in analyzing procedural errors students have a viable strategy to compute mentally but a computation error has led them to an incorrect

solution. An emphasis on applying memorized rules and procedures based on mental images of written algorithms may also contribute to such an error. Students that make strong use of additive reasoning (as Acácio) make conceptual errors easily as they fail to understand the multiplicative relation that exists in the rational numbers set.

This study provides suggestions for the teachers practice in the classroom in developing student mental computation and relational thinking. It provides information to teachers about the kinds of knowledge that students need and use to compute mentally such as number facts, memorized rules, numerical relationships, and mental representations. It also shows the kind of errors that need to be clarified and discussed in the classroom to improve student mathematics learning.

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