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Introduction to the papers of TWG02: Arithmetic and number systems

Sebastian Rezat¹, Lisser Rye Ejersbo², Darina Jirotkova³ and Elisabeth Rathgeb-Schnierer⁴

1 University of Paderborn, Institute for Mathematics, Paderborn, Germany, srezat@math.upb.de

2 Aarhus University, Department of Education, Aarhus, Denmark

3 Charles University in Prague, Faculty of Education, Prague, Czech Republic

4 University of Education Weingarten, Weingarten, Germany

THE FOCUS AND SCOPE OF TWG02

Thematic Working Group 2 focused on six questions related to teaching and learning of arithmetic and number systems from grades 1 to 12:

- 1) What is the interplay between conceptual understanding and procedural skills for number operations, and how should these two aspects be balanced in the design of learning environments?
- 2) What does it mean to operate flexibly with numbers, and what knowledge and skills are required therefore?
- 3) What are the roles of models in teaching and learning of arithmetic and number systems, and how do they support flexibility and conceptual understanding?
- 4) What are effective approaches for teaching and learning of arithmetic and number systems in inclusive education? How can this content area be taught for students with special needs?
- 5) What aspects of number theory (including specific reasoning) are supposed to be taught in primary and secondary school, and how can this be done?
- 6) How is it possible to support and analyze long-term learning processes from grade 1 to grade 12? How can different transitions that come with a long-term learning process be taken into account, especially the primary-secondary transition, but also the secondary-tertiary transition.

NUMBER SENSE, CONCEPTUAL UNDERSTANDING AND FLEXIBILITY

Although it is not mentioned explicitly in the focus of TWG02, the notion of number sense included in most of these questions.

In their paper, Sayers and Andrews remind us that there are very different definitions of number sense. Not only different notions of number sense are used in different domains, but in fact, there are no two researchers who use the same definition of number sense. Sayers and Andrews distinguish three distinct perspectives on number sense, which they label *pre-verbal*, *applied* and *foundational*. In their paper, they aim to offer a concise conceptualisation of what they call foundational number sense in a way that would support a range of activities, including developments in curriculum, teacher education or assessment, as well as cross-cultural classroom analyses.

An important issue related to the first question is the role of counting strategies in the development of conceptual understanding of number and operations with numbers. Counting strategies are usually regarded as an intermediate stage in the development of conceptual understanding of number. On the one hand, it is assumed that counting strategies are necessary, but on the other hand, they need to be substituted by other strategies that rely on knowledge about number facts and relationships between numbers. At a certain point counting strategies are even regarded as an indication for mathematics disabilities.

Gaidoschik, Fellmann and Guggenbichler challenge the view that counting strategies are actually a necessary intermediate stage in the development. They in-

investigate how teaching of derivation strategies affects strategy use by first graders. They are able to provide empirical support for the claim that counting-on is not a necessary strategy in the development of conceptual understanding of number and operations with numbers.

Häsel-Weide is also interested in possible ways of substituting counting strategies. Her study focuses on the role of peer interaction in cooperative learning situations that encourage children to pay attention and use number relations as well as relations between tasks and problems. She concludes that it is rewarding for both children in heterogeneous pairs to work together. Additionally, she observed that children rather focus on number relations than on relations between tasks and problems.

The first question also addresses the design of learning environments as an important issue. Nührenböcker and Schwarzkopf tackle this issue related to the conceptual understanding of equations. They investigate the development of a rich learning environment that enable students to access the algebraic meaning of equalities and the equal sign. Furthermore, they argue that a rich learning environment itself does not suffice, and highlight the importance of argumentation in order to foster the learning process.

Veldhuis and van den Heuvel-Panhuizen also emphasize the fact that it is not only the design of learning environments and related activities that foster students' learning of mathematics, but also a matter of classroom assessment. In a large-scale experimental study they are able to provide evidence that teachers' use of classroom assessment techniques in mathematics has a positive effect on students' achievement.

The notion of flexibility in terms of operating flexibly and adaptively with numbers is taken up in the second question. Explicitly or implicitly flexibility is seen as an indicator of what Sayers and Andrews call applied number sense. The papers in this TWG focus mainly on two aspects of flexibility:

- conceptualizing flexibility and adaptivity in mental calculation,
- promoting flexibility in mental calculation.

Rathgeb-Schnierer and Green target to identify degrees of flexibility in students' mental arithmetic by revealing the cognitive elements that sustain the solution process (learned procedures or problem characteristics, number patterns, and relationships). They are able to identify three forms of reasoning: flexible (multiple reasons predominantly referring to characteristics), rigid (one reason referring to a solution procedure) and mixed (multiple reasons when referring to characteristics, one reason when referring to a solution procedure). Furthermore, the results show the tendency that students who refer to problem characteristics in their reasoning seem to be more cognitively flexible.

Serrazina and Rodrigues are also concerned with the notion of flexibility in terms of adaptive thinking. However, their approach is different since they do not regard strategies as the unit of analysis, but quantitative reasoning. They argue that quantitative reasoning underlies the development of flexible calculation because it focuses on the description and modeling of situations and the involved comparative relationships. In a qualitative approach they attempt to understand how children establish a network of connections through their reasoning about different representations of the numbers, and about relationships between numbers and quantities.

Given the importance of the fundamental arithmetic properties for flexible mental calculations on the one hand, and for algebraic thinking on the other hand, Larson aims to better understand how students make sense of arithmetical properties in particular of the distributive property in multiplicative calculations. She analyses students' arguments when they have to evaluate the validity of (wrong) strategies to carry out a multiplication of two two-digit numbers. Finally, she makes inferences from students' arguments to students' understanding of the distributive property.

Besides developing a better theoretical understanding of the notion of flexibility, the issue of fostering the development of flexibility is another important issue related to the second question. Rechtsteiner-Merz and Rathgeb-Schnierer pay attention to the development of flexible mental calculation in less advanced students. In this regard, they investigate the contribution of a specific approach (called "Zahlenblickschulung") to foster the recognition of problem characteristics, number patterns and numerical relationships.

Within a qualitative design they identify different types based on three dimensions: (1) the amount of correct solutions, (2) solution procedures, and (3) related reasoning. They conclude that knowledge of basic facts and strategies seem to be insufficient for the development of a deep understanding of calculation that goes beyond counting. Therefore, the focus on numerical relationships and structures is essential in order to develop flexibility in mental calculation. Their approach of “Zahlenblickschulung” seems to be promising in this regard.

Lübke tackles the issue of estimation as another aspect related to number sense. She investigates fourth graders’ conceptual understanding of computational estimation using indirect estimation questions. She argues that the interrelation between an estimate and the exact calculation is not only a very important aspect of understanding computational estimation, but also proved to be a useful tool to analyse students’ concept of computational estimation. Her findings indicate that even students who are able to carry out estimations do not necessarily understand the concept of estimation.

Carvalho and da Ponte as well as Almeida and Bruno address aspects of number sense in the domain of rational numbers. Carvalho and da Ponte analyze mental computation strategies and errors of 6th grade students particularly focusing on how they use relational thinking. Based on the theory of mental models they are able to better understand the relation between mental representations and mental calculation strategies and errors respectively.

Almeida and Bruno analyze the effects of an intervention on three abilities of grade 8 students related to number sense in the domain of fractions: the use of benchmarks, the use of graphical representations of numbers and operations, and the recognition of the reasonableness of a result.

THE ROLE OF MODELS

The third question relates to the role of models in teaching and learning arithmetic. The notion of model consciously has been left open. The papers in the TWG mainly address four different kinds of models: models in the sense of manipulatives, models in the sense of visual representations, models in the sense of mental models, and finally, models in the sense of role models.

Hejny, Jirotková and Slezáková offer a theoretical approach to the role of models in the learning process. A process that they call “desemantization” lies at the heart of their Theory of generic models. It describes stages of learning processes in mathematics in terms of two different kinds of mental models, which differ in their semantic embedding: isolated models denote the set of pupils’ experiences related to a certain mathematical concept, relationship, or situation, and generic models refer to generalizations of this previous experience. Through an abstraction process the generic model is transferred into abstract knowledge, which does not include the semantic embedding. The ability of pupils to work with models is thus an indicator of their level of understanding of mathematical phenomena.

Finesilver analyzes strategies and errors of S.E.N. students when asked to determine the number of cubes in cuboid blocks made up of multilink cubes. It was intended to evoke the use of different multiplicative strategies by highlighting different structures using different colors. The tasks proved to be a useful tool to understand students’ multiplicative thinking in terms of structuring, enumeration and errors.

Hattermann and vom Hofe analyze the effects of a game in the domain of negative numbers on students’ performance in tasks with addition and subtraction of negative numbers and students’ argumentation schemes. The intention of the game is to foster metaphorical reasoning, which is regarded as crucial for the understanding of negative numbers and the construction of mental models by the authors. First results that are based on a small number of students indicate positive effects on both, performance and metaphorical reasoning.

Pöhler, Prediger and Weinert investigate the influence of different representations (numerical, visual, verbal) in percent problems on students’ performance in relation to their language proficiency. Their findings indicate that the difficulties with verbal test items cannot be explained by students’ restricted reading proficiency, but rather seem to be a consequence of their lack in conceptual understanding of percentages.

Papadopoulos focuses on the teacher as a role model. He analyses the influence of a teacher’s persistent misconceptions on students’ performance. The re-

sults highlight once more the importance of content knowledge in pre- and in-service teacher education.

SUBJECT-MATTER ANALYSIS

Three papers take the subject-matter and related content knowledge as their starting point. Real and Figueras present a framework of notions, concepts and processes for fractions and rational numbers, which was developed based on Freudenthal's Didactical Phenomenology. The framework is organized according to five classes of phenomena/processes related to fractions: describing, comparing, dividing, distributing, and measuring. It has proved to be a useful analytical tool.

Nicolaou and Pitta-Pantazi evaluate the impact of an intervention, which was based on a framework comprising seven abilities that are supposed to be important for conceptual understanding of fractions in elementary school. They conclude that the intervention had a positive impact on students' understanding of fractions.

Gómez and García carry out a rational analysis of problems with unequal ratios. In the first step, their analysis aims to work out critical components of the problems in order to evaluate them empirically in the second step. Their findings indicate that students do not apply different strategies flexibly when solving the problems, but rather stick to a standard strategy.

OPEN QUESTIONS AND FUTURE DIRECTIONS

Many of the papers in TWG relate to the notion of number sense. Consequently, number sense can be regarded as one of the focal concepts of this TWG. The papers in this TWG have contributed significantly to improve the understanding and applicability of this concept. However, many questions are still open.

First of all, very different conceptualizations of number sense are used in different domains. Whereas number sense in cognitive psychology refers to the innate arithmetic of the human brain, in mathematics didactics it relates to the ability to perceive number relations and to make use of them when solving problems. It seems promising to relate these different perspectives and strive for a comprehensive definition of number sense. Main questions in this context are:

- How is number sense in the didactical meaning related to the innate number sense of the brain?
- How can insights into the innate number sense contribute to the development of number sense in the didactical meaning of the term?

Secondly, the relation between number sense on the one hand and flexibility and adaptivity on the other hand needs to be further clarified. Is number sense conceptualized via flexibility and adaptivity or is it a prerequisite for flexibility and adaptivity?

Finally, it is tempting to use the term number sense related to other number domains than the natural numbers – and some papers in this TWG actually do so. However, it is not clear yet, what number sense means related to these number domains and how it relates to the number sense in the domain of natural numbers.

Important questions related to number sense in every number domain are:

- What influences the development of number sense?
- What is the role of metaphors, manipulatives, models, and mental models in the development of (didactical) number sense?
- What is the role of reasoning and argumentation in the development of (didactical) number sense?
- What is the role of counting in the development of (didactical) number sense?

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