



HAL
open science

When Does Channel-Output Feedback Enlarge the Capacity Region of the Two-User Linear Deterministic Interference Channel?

Victor Quintero, Samir M. Perlaza, Iñaki Esnaola, Jean-Marie Gorce

► **To cite this version:**

Victor Quintero, Samir M. Perlaza, Iñaki Esnaola, Jean-Marie Gorce. When Does Channel-Output Feedback Enlarge the Capacity Region of the Two-User Linear Deterministic Interference Channel?. [Research Report] RT-8862, INRIA Grenoble - Rhône-Alpes. 2016, pp.30. hal-01281717v2

HAL Id: hal-01281717

<https://hal.science/hal-01281717v2>

Submitted on 1 Apr 2016 (v2), last revised 23 Jun 2017 (v6)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



When Does Channel-Output Feedback Enlarge the Capacity Region of the Two-User Linear Deterministic Interference Channel?

Victor Quintero, Samir M. Perlaza, Iñaki Esnaola, Jean-Marie Gorce

**TECHNICAL
REPORT**

N° 8862

March 2016

Project-Team Socrate



When Does Channel-Output Feedback Enlarge the Capacity Region of the Two-User Linear Deterministic Interference Channel?

Victor Quintero, Samir M. Perlaza, Iñaki Esnaola, Jean-Marie
Gorce

Project-Team Socrate

Technical Report n° 8862 — March 2016 — 32 pages

Victor Quintero, Samir M. Perlaza and Jean-Marie Gorce are with the CITI Laboratory, a joint lab between the Institut National de Recherche en Informatique et en Automatique (INRIA), Université de Lyon and Institut National de Sciences Appliquées (INSA) de Lyon. 6 Av. des Arts 69621 Villeurbanne, France. ({victor.quintero-florez, samir.perlaza, jean-marie.gorce}@inria.fr).

Iñaki Esnaola is with the University of Sheffield, Dep. of Automatic Control and Systems Engineering, Mappin Street, Sheffield, S1 3JD, United Kingdom. (esnaola@sheffield.ac.uk).

Victor Quintero is also with Universidad del Cauca, Popayán, Colombia.

This research was supported in part by the European Commission under Marie Skłodowska-Curie Individual Fellowship No. 659316 (CYBERNETS) and the Administrative Department of Science, Technology and Innovation of Colombia (Colciencias), fellowship No. 617-2013.

**RESEARCH CENTRE
GRENOBLE – RHÔNE-ALPES**

Inovallée
655 avenue de l'Europe Montbonnot
38334 Saint Ismier Cedex

Abstract: The two-user linear deterministic interference channel (LD-IC) with noisy channel-output feedback is fully described by six parameters that correspond to the number of bit-pipes between each transmitter and its corresponding intended receiver, i.e., \vec{n}_{11} and \vec{n}_{22} ; between each transmitter and its corresponding non-intended receiver i.e., n_{12} and n_{21} ; and between each receiver and its corresponding transmitter, i.e., \overleftarrow{n}_{11} and \overleftarrow{n}_{22} . An LD-IC without feedback corresponds to the case in which $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = 0$ and the capacity region is denoted by $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$. In the case in which feedback is available at both transmitters, $\overleftarrow{n}_{11} > 0$ and $\overleftarrow{n}_{22} > 0$, the capacity is denoted by $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$. This technical report presents the exact conditions on \overleftarrow{n}_{11} (resp. \overleftarrow{n}_{22}) for observing an improvement in the capacity region $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ (resp. $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$) with respect to $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$, for any 4-tuple $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$. Specifically, it is shown that there exists a threshold for the number of bit-pipes in the feedback link of transmitter-receiver pair 1 (resp. 2), denoted by \overleftarrow{n}_{11}^* (resp. \overleftarrow{n}_{22}^*) for which any $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^*$ (resp. $\overleftarrow{n}_{22} > \overleftarrow{n}_{22}^*$) enlarges the capacity region, i.e., $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ (resp. $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$). The exact conditions on \overleftarrow{n}_{11} (resp. \overleftarrow{n}_{22}) to observe an improvement on a single rate or the sum-rate capacity, for any 4-tuple $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ are also presented in this technical report. Finally, a closed-form expression for the number of generalized degrees of freedom (GDoF) for the symmetric two-user LD-IC-NOF is presented.

Key-words: Linear Deterministic Interference Channel, Noisy Channel-Output Feedback, Capacity Region.

Résumé : Le canal linéaire déterministe à interférences avec rétroalimentation dégradée est entièrement décrit par six paramètres qui correspondent au nombre de niveaux de signal entre chaque émetteur et le récepteur destinataire correspondant, i.e., \vec{n}_{11} et \vec{n}_{22} ; entre chaque émetteur et le récepteur non-destinataire correspondant i.e., n_{12} et n_{21} ; et entre chaque récepteur et son émetteur correspondant, i.e., \overleftarrow{n}_{11} et \overleftarrow{n}_{22} . Un canal linéaire déterministe à interférences sans rétroalimentation correspond au cas pour lequel $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = 0$ et pour lequel la région de capacité est notée $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$. Dans le cas où la rétroalimentation est disponible aux deux émetteurs, $\overleftarrow{n}_{11} > 0$ and $\overleftarrow{n}_{22} > 0$, la région de capacité est notée $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$. Ce rapport présente les conditions exactes sur \overleftarrow{n}_{11} (resp. \overleftarrow{n}_{22}) permettant d'observer une amélioration de la région de capacité $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ (resp. $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$) comparée à $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$, pour tout quadruplet $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$. Plus précisément, ce rapport montre l'existence d'un seuil pour le nombre de niveaux de signal dans le lien de rétroalimentation de la paire émetteur-récepteur 1 (resp. 2), noté \overleftarrow{n}_{11}^* (resp. \overleftarrow{n}_{22}^*) pour lequel tout $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^*$ (resp. $\overleftarrow{n}_{22} > \overleftarrow{n}_{22}^*$) améliore la région de capacité, i.e., $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ (resp. $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$). Les conditions exactes sur \overleftarrow{n}_{11} (resp. \overleftarrow{n}_{22}) permettant d'observer une amélioration sur le taux de transmission individuel ou sur la somme des taux de transmission, pour tout quadruplet $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ sont également présentées dans ce rapport. Enfin, une expression généralisée du nombre de degrés de liberté pour le modèle linéaire déterministe du canal à interférences symétrique avec rétroalimentation dégradée est établie.

Mots-clés : Modèle linéaire déterministe, canal à interférences, rétroalimentation dégradée, Région de Capacité.

Contents

1	Introduction	5
2	Linear Deterministic Interference Channel with Noisy-Channel Output Feedback	6
3	Preliminaries	8
3.1	Definitions	8
3.2	Rate Improvement Metrics	10
4	Main Results	11
4.1	Enlargement of the Capacity Region	11
4.2	Improvement of the Individual Rate R_i by Using Feedback in Link i	15
4.3	Improvement of the Individual Rate R_j by Using Feedback in Link i	16
4.4	Improvement of the Sum-Capacity	18
5	Generalized Degrees of Freedom	18
6	Conclusions	20
	Appendices	21
A	Proof of Theorem 1: Enlargement of the Capacity Region by Using Feedback in one Transmitter-Receiver Pair	21
B	Proof of Theorem 2: Improvement of the Individual Rate R_1 by Using Feedback in Link 1	28
C	Proof of Theorem 3: Improvement of the Individual Rate R_2 by Using Feedback in Link 1	29
D	Proof of Theorem 4: Improvement of the Sum-Rate Capacity by Using Feedback in one Transmitter-Receiver Pair	29
E	Proof of Theorem 5: Generalized Degrees of Freedom	30

1 Introduction

Channel-output feedback is an interference management technique that significantly increases the number of degrees of freedom (DoF) for the two-user Gaussian interference channel (IC) in most of the interference regimes [1]. Essentially, in the very strong interference regime, the DoF gain provided by perfect-channel output feedback can be arbitrarily large when the interference to noise ratios (INRs) and signal to noise ratios (SNRs) grow to infinity. One of the reasons why feedback provides such a surprising benefit stems from the fact that it uses interference to create alternative paths to the existing point-to-point paths. For instance, in the two-user IC, feedback creates a path from transmitter 1 (resp. transmitter 2) to receiver 1 (resp. receiver 2) in which symbols that are received at receiver 2 (resp. receiver 1) are fed back to transmitter 2 (resp. transmitter 1), which decodes the messages and retransmits them to receiver 1 (resp. receiver 2). This significant improvement of the capacity region is also observed in the decentralized IC, i.e., the case in which the transmitter-receiver pairs autonomously choose their own transmit-receive configurations. More specifically, the Nash equilibrium region of the Gaussian IC is significantly enlarged with respect to the case in which feedback is not available [2]. Another metric to determine the benefits of feedback is the number of generalized DoF (GDoF), see [3] for a discussion on DoF and GDoF, as well as other metrics. The GDoF gain due to feedback in the IC depends on the topology of the network and the number of transmitter-receiver pairs in the network. For instance, in the symmetric K -user cyclic Z -interference channel, the DoF gain does not increase with K [4]. In particular, in the very strong interference regime, the DoF gain is shown to be monotonically decreasing with K . Alternatively, in the fully connected symmetric K -user IC with perfect feedback, the number of GDoF per user is shown to be identical to the one in the two-user case, with an exception in a particular singularity, and totally independent of the exact number of transmitter-receiver pairs [5]. It is important to highlight that the network topology, the number of transmitter-receiver pairs and the interference regime are not the only parameters determining the effect of feedback. Indeed, the presence of noise in the feedback links turns out to be another relevant factor. As shown later in this technical report, in the case in which one transmitter-receiver pair is in a high interference regime (the interfering signal is stronger than the intended signal) and the other is in a low interference regime (the interfering signal is weaker than the intended signal), the use of feedback in the former does not enlarge the capacity region, even in the case of perfect output feedback. Conversely, using feedback in the latter might enlarge the capacity region depending on the SNR of the feedback link. The exact values of the feedback SNRs beyond which the capacity region is enlarged depend on all the other channel parameters: two forward SNRs and two forward INRs. In [6], the capacity region of the two-user Gaussian IC (GIC) with noisy channel output feedback is approximated to within a constant number of bits for the symmetric case. These results are generalized in [7] for non-symmetric cases. However, from the available descriptions of the capacity regions with and without feedback, identifying whether or not the existence of a feedback link with a given SNR enlarges the capacity region is not a trivial task.

An alternative for dealing with the challenges described above is to study the GIC via its linear deterministic IC (LD-IC) approximation [8], for which the capacity region is perfectly known [9, 10]. The two-user LD-IC with noisy channel output feedback (LD-IC-NOF) is fully described by six parameters: $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) \in \mathbb{N}^6$. There exists a mapping between the parameters describing the two-user LD-IC and the parameters describing the GIC. More specifically, there are two forward SNRs ($\overrightarrow{\text{SNR}}_i \geq 1$); two forward INRs ($\text{INR}_{ij} \geq 1$); and two backward SNRs ($\overleftarrow{\text{SNR}}_i \geq 1$), with $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$. In the LD-IC, the parameters of the GIC are mapped into the number of bit-pipes between each transmitter and its corresponding intended receiver, i.e., $\vec{n}_{ii} = \lfloor \frac{1}{2} \log_2(\overrightarrow{\text{SNR}}_i) \rfloor$; between transmitter j and receiver

i i.e., $n_{ij} = \lfloor \frac{1}{2} \log_2(\text{INR}_{ij}) \rfloor$; and between each receiver and its corresponding transmitter, i.e., $\overleftarrow{n}_{ii} = \lfloor \frac{1}{2} \log_2(\overleftarrow{\text{SNR}}_i) \rfloor$. An LD-IC without feedback corresponds to the case in which $\overleftarrow{n}_{11} = \overleftarrow{n}_{22} = 0$ and the capacity region is denoted by $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$. In the case in which feedback is available at both transmitters, $\overleftarrow{n}_{11} > 0$ and $\overleftarrow{n}_{22} > 0$, the capacity is denoted by $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$.

This technical report presents the exact conditions on \overleftarrow{n}_{11} (resp. \overleftarrow{n}_{22}) for observing an improvement in the capacity region $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ (resp. $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$) with respect to $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$, for any 4-tuple $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$. More specifically, it is shown that there exists a threshold for the number of bit-pipes in the feedback link of transmitter-receiver pair 1 (resp. 2), beyond which the capacity region of the two-user LD-IC-NOF can be enlarged, i.e., $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ (resp. $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$). The exact conditions on \overleftarrow{n}_{11} (resp. \overleftarrow{n}_{22}) to observe an improvement on a single rate or the sum-rate capacity, for any 4-tuple $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ are also presented in this technical report. Surprisingly, these values can be expressed in closed-form using relatively simple expressions that depend on some of the parameters $\vec{n}_{11}, \vec{n}_{22}, n_{12}$ and n_{21} .

Based on these results, several relevant engineering questions arise in this setting. For instance, in which of the two transmitter-receiver pairs must the feedback link be implemented if the objective is to improve: (a) the individual rate of the transmitter-receiver pair in which feedback is implemented; (b) the individual rate of the other transmitter-receiver pair; or (c) the sum-rate of both transmitter-receiver pairs. In each of these scenarios, the feedback SNR, either \overleftarrow{n}_{11} or \overleftarrow{n}_{22} , must be bigger than a given threshold for the improvement to be observed. Interestingly, for each of these scenarios there exists a complete different answer. As a by-product of the results described above, the exact values of \overleftarrow{n}_{11} or \overleftarrow{n}_{22} for which feedback does not enlarge the capacity region are also identified.

2 Linear Deterministic Interference Channel with Noisy-Channel Output Feedback

Consider the two-user LD-IC-NOF, with parameters $\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}$ and \overleftarrow{n}_{22} described in Fig. 1. The parameters \vec{n}_{ii}, n_{ij} and \overleftarrow{n}_{ii} with $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, are non-negative integers. Parameter \vec{n}_{ii} represents the number of bit-pipes between transmitter i and receiver i ; parameter n_{ij} represents the number of bit-pipes between transmitter j and receiver i ; and parameter \overleftarrow{n}_{ii} represents the number of bit-pipes between receiver i and transmitter i (feedback). At transmitter i , with $i \in \{1, 2\}$, the channel-input $\mathbf{X}_i^{(n)}$ at channel use n , with $n \in \{1, \dots, N\}$, is a q -dimensional binary vector $\mathbf{X}_i^{(n)} = (X_{i,1}^{(n)}, \dots, X_{i,q}^{(n)})^\top$, with

$$q = \max(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}), \quad (1)$$

and N the block length. At receiver i , the channel-output $\vec{\mathbf{Y}}_i^{(n)}$ at channel use n is also a q -dimensional binary vector $\vec{\mathbf{Y}}_i^{(n)} = (\vec{Y}_{i,1}^{(n)}, \dots, \vec{Y}_{i,q}^{(n)})^\top$. The input-output relation during channel use n is given as follows

$$\vec{\mathbf{Y}}_i^{(n)} = \mathbf{S}^{q-\vec{n}_{ii}} \mathbf{X}_i^{(n)} + \mathbf{S}^{q-n_{ij}} \mathbf{X}_j^{(n)}, \quad (2)$$

and the feedback signal available at transmitter i at the end of channel use n is:

$$\overleftarrow{\mathbf{Y}}_i^{(n)} = \mathbf{S}^{(q-\overleftarrow{n}_{ii})^+} \vec{\mathbf{Y}}_i^{(n-d)}, \quad (3)$$

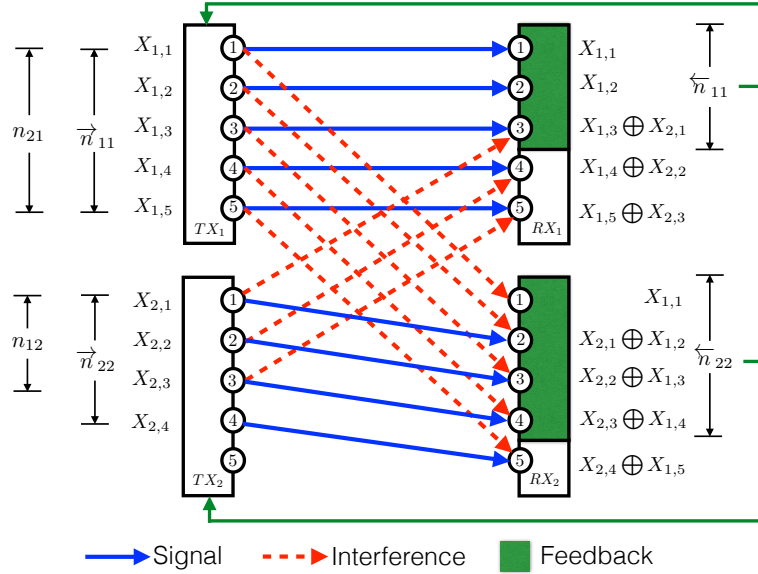


Figure 1: Two-user linear deterministic interference channel with noisy channel-output feedback. The bit-pipe line number 1 represents the most significant bit.

where d is a finite feedback delay, additions and multiplications are defined over the binary field, \mathbf{S} is a $q \times q$ lower shift matrix of the form

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

and $(\cdot)^+$ is the positive part operator.

Transmitter i sends M_i information bits $b_{i,1}, \dots, b_{i,M_i}$ by sending the codeword $(\mathbf{X}_i^{(1)}, \dots, \mathbf{X}_i^{(N)})$. The encoder of transmitter i can be modeled as a set of deterministic mappings $f_i^{(1)}, \dots, f_i^{(N)}$, with $f_i^{(1)} : \{0, 1\}^{M_i} \rightarrow \{0, 1\}^q$ and $\forall n \in \{2, \dots, N\}$, $f_i^{(n)} : \{0, 1\}^{M_i} \times \{0, 1\}^{q(n-1)} \rightarrow \{0, 1\}^q$, such that

$$\mathbf{X}_i^{(1)} = f_i^{(1)}(b_{i,1}, \dots, b_{i,M_i}) \text{ and} \quad (4)$$

$$\mathbf{X}_i^{(n)} = f_i^{(n)}(b_{i,1}, \dots, b_{i,M_i}, \hat{\mathbf{Y}}_i^{(1)}, \dots, \hat{\mathbf{Y}}_i^{(n-1)}). \quad (5)$$

At the end of the block, receiver i uses the sequence $\mathbf{Y}_i^{(1)}, \dots, \mathbf{Y}_i^{(N)}$ to generate the estimates $\hat{b}_{i,1}, \dots, \hat{b}_{i,M_i}$. The average bit error probability at receiver i , denoted by p_i , is calculated as follows

$$p_i = \frac{1}{M_i} \sum_{\ell=1}^{M_i} \mathbb{1}_{\{\hat{b}_{i,\ell} \neq b_{i,\ell}\}}. \quad (6)$$

A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is said to be achievable if it satisfies the following definition.

Definition 1 (Achievable Rate Pairs) The rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable if there exists at least one pair of codebooks \mathcal{X}_1^N and \mathcal{X}_2^N with codewords of length N , and the corresponding encoding functions $f_1^{(1)}, \dots, f_1^{(N)}$ and $f_2^{(1)}, \dots, f_2^{(N)}$ such that the average bit error probability can be made arbitrarily small by letting the block length N grow to infinity.

Denote by $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ the capacity region of the two-user LD-IC-NOF with parameters $\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}$, and \overleftarrow{n}_{22} . Lemma 1 fully characterizes the set $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$.

Lemma 1 (Lemma 6 in [10]) The capacity region $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ of the two-user LD-IC-NOF is the set of non-negative rate pairs (R_1, R_2) that satisfy $\forall i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$:

$$R_i \leq \min(\max(\vec{n}_{ii}, n_{ji}), \max(\vec{n}_{ii}, n_{ij})), \quad (7a)$$

$$R_i \leq \min(\max(\vec{n}_{ii}, n_{ji}), \max(\vec{n}_{ii}, \overleftarrow{n}_{jj} - (\vec{n}_{jj} - n_{ji})^+)), \quad (7b)$$

$$R_1 + R_2 \leq \min(\max(\vec{n}_{22}, n_{12}) + (\vec{n}_{11} - n_{12})^+, \max(\vec{n}_{11}, n_{21}) + (\vec{n}_{22} - n_{21})^+), \quad (7c)$$

$$R_1 + R_2 \leq \max\left((\vec{n}_{11} - n_{12})^+, n_{21}, \vec{n}_{11} - (\max(\vec{n}_{11}, n_{12}) - \overleftarrow{n}_{11})^+\right) \\ + \max\left((\vec{n}_{22} - n_{21})^+, n_{12}, \vec{n}_{22} - (\max(\vec{n}_{22}, n_{21}) - \overleftarrow{n}_{22})^+\right), \quad (7d)$$

$$2R_i + R_j \leq \max(\vec{n}_{ii}, n_{ji}) + (\vec{n}_{ii} - n_{ij})^+ \\ + \max\left((\vec{n}_{jj} - n_{ji})^+, n_{ij}, \vec{n}_{jj} - (\max(\vec{n}_{jj}, n_{ji}) - \overleftarrow{n}_{jj})^+\right). \quad (7e)$$

3 Preliminaries

3.1 Definitions

Let $\alpha_i \in \mathbb{Q}$ be the interference regime of transmitter-receiver pair i , with $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$,

$$\alpha_i = \frac{n_{ij}}{\vec{n}_{ii}}. \quad (8)$$

For each transmitter-receiver pair i , there exist five possible interference regimes (IRs): very weak IR (VWIR), i.e., $\alpha_i \leq \frac{1}{2}$, weak IR (WIR), i.e., $\frac{1}{2} < \alpha_i \leq \frac{2}{3}$, moderate IR (MIR), i.e., $\frac{2}{3} < \alpha_i \leq 1$, strong IR (SIR), i.e., $1 < \alpha_i \leq 2$ and very strong IR (VSIR), i.e., $\alpha_i > 2$ [11]. The scenarios in which the desired signal is stronger than or equal to the interference ($\alpha_i \leq 1$), namely VWIR, WIR and MIR, are referred to as low-interference regimes (LIRs). Conversely, the scenarios in which the desired signal is weaker than the interference ($\alpha_i > 1$), namely SIR and VSIR, are referred to as high-interference regimes (HIRs). In the two-user LD-IC, it is possible to observe up to twenty-five possible interference regimes, given α_1 and α_2 . However, only twelve cases are of real interest. This is because the transmitter-receiver pairs can be indifferently labeled and thus, for instance, studying the case in which $\alpha_1 \leq \frac{1}{2}$ and $\alpha_2 > 2$ is the same as studying the case in which $\alpha_1 > 2$ and $\alpha_2 \leq \frac{1}{2}$.

The main results of this technical report are presented using a list of events (Boolean variables) that are fully determined by the parameters $\vec{n}_{11}, \vec{n}_{22}, n_{12}$, and n_{21} . For instance, given the parameters $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21})$, the events (9)-(25) describe some combinations of interference

regimes that are particularly interesting. Let $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$ and define the following events:

$$E_1 : \quad \alpha_1 \leq 1 \wedge \alpha_2 \leq 1, \quad (i \text{ and } j \text{ in LIR}) \quad (9)$$

$$E_{2,i} : \quad \alpha_i \leq \frac{1}{2} \wedge 1 < \alpha_j \leq 2, \quad (i \text{ in VWIR and } j \text{ in SIR}) \quad (10)$$

$$E_{3,i} : \quad \alpha_i \leq \frac{1}{2} \wedge \alpha_j > 2, \quad (i \text{ in VWIR and } j \text{ in VSIR}) \quad (11)$$

$$E_{4,i} : \quad \frac{1}{2} < \alpha_i \leq \frac{2}{3} \wedge \alpha_j > 1, \quad (i \text{ in WIR and } j \text{ in HIR}) \quad (12)$$

$$E_{5,i} : \quad \frac{2}{3} < \alpha_i \leq 1 \wedge 1 < \alpha_j \leq 2, \quad (i \text{ in MIR and } j \text{ in SIR}) \quad (13)$$

$$E_{6,i} : \quad \frac{2}{3} < \alpha_i \leq 1 \wedge \alpha_j > 2, \quad (i \text{ in MIR and } j \text{ in VSIR}) \quad (14)$$

$$E_{7,i} : \quad \frac{1}{2} < \alpha_i \leq 1 \wedge \alpha_j > 1, \quad (i \text{ in WIR or MIR and } j \text{ in HIR}) \quad (15)$$

$$E_{8,i} : \quad \alpha_i > 1 \wedge \alpha_j \leq 1, \quad (i \text{ in HIR and } j \text{ in LIR}) \quad (16)$$

$$E_9 : \quad \alpha_1 \leq \frac{2}{3} \wedge \alpha_2 \leq \frac{2}{3}, \quad (i \text{ and } j \text{ in VWIR or WIR}) \quad (17)$$

$$E_{10,i} : \quad \alpha_i \leq \frac{2}{3} \wedge \frac{2}{3} < \alpha_j \leq 1, \quad (i \text{ in VWIR or WIR and } j \text{ in MIR}) \quad (18)$$

$$E_{11,i} : \quad \frac{2}{3} < \alpha_i \leq 1 \wedge \alpha_j \leq \frac{2}{3}, \quad (i \text{ in MIR and } j \text{ in VWIR or WIR}) \quad (19)$$

$$E_{12} : \quad \alpha_1 > 2 \wedge \alpha_2 > 2, \quad (i \text{ and } j \text{ in VSIR}). \quad (20)$$

Some other auxiliary events are considered. The event in which the signal from transmitter i is stronger (resp. weaker) in its intended receiver than in its non-intended receiver is denoted by $E_{13,i}$ (resp. $\widetilde{E}_{13,i}$), i.e.,

$$E_{13,i} : \vec{n}_{ii} > n_{ji}, \quad (21)$$

$$\widetilde{E}_{13,i} : \vec{n}_{ii} < n_{ji}. \quad (22)$$

The event in which the sum of the number of bit-pipes in the direct links is bigger (resp. smaller) than the sum of the number of bit-pipes in the cross-interference links is denoted by $E_{14,i}$ (resp. $\widetilde{E}_{14,i}$), i.e.,

$$E_{14} : \vec{n}_{11} + \vec{n}_{22} > n_{12} + n_{21}, \quad (23)$$

$$\widetilde{E}_{14} : \vec{n}_{11} + \vec{n}_{22} < n_{12} + n_{21}. \quad (24)$$

The event in which the number of bit-pipes in the direct link j is bigger than the sum of bit-pipes in both cross-interference links is denoted by

$$E_{15,i} : \vec{n}_{jj} > n_{ij} + n_{ji}. \quad (25)$$

The event in which the sum of the number of bit-pipes in the direct links is bigger than the sum of the number of bit-pipes in one cross-interference link and twice the number of the bit-pipes in the other cross-interference link is denoted by

$$E_{16,i} : \vec{n}_{ii} + \vec{n}_{jj} > n_{ij} + 2n_{ji}. \quad (26)$$

Finally, the event in which the sum of the number of bit-pipes in the direct links is bigger than the number of bit-pipes in one cross-interference link is denoted by

$$E_{17,i} : \vec{n}_{ii} + \vec{n}_{jj} < n_{ij}. \quad (27)$$

Combining the events (9)-(27), five main events are identified:

$$S_{1,i}: (E_1 \wedge E_{13,i}) \vee (E_{2,i} \wedge E_{13,i}) \vee (E_{3,i} \wedge E_{13,i} \wedge E_{14}) \vee (E_{4,i} \wedge E_{13,i} \wedge E_{14}) \vee (E_{5,i} \wedge E_{13,i} \wedge E_{14}) \\ \vee (E_{6,i} \wedge \widetilde{E}_{13,j} \wedge E_{14}), \quad (28)$$

$$S_{2,i}: (E_{3,i} \wedge \widetilde{E}_{13,j} \wedge \widetilde{E}_{14}) \vee (E_{7,i} \wedge \widetilde{E}_{13,j} \wedge \widetilde{E}_{14}) \vee (\overline{E}_1 \wedge \widetilde{E}_{13,j}), \quad (29)$$

$$S_{3,i}: (E_1 \wedge \overline{E}_{13,i}) \vee (E_{2,i} \wedge \overline{E}_{13,i}) \vee (E_{3,i} \wedge E_{13,j} \wedge \overline{E}_{13,i}) \vee (E_{4,i} \wedge E_{13,j} \wedge \overline{E}_{13,i}) \\ \vee (E_{5,i} \wedge E_{13,j} \wedge \overline{E}_{13,i}) \vee (E_{6,i} \wedge E_{13,j}) \vee (\overline{E}_1 \wedge E_{13,j}) \vee (E_{8,i}), \quad (30)$$

$$S_{4,i}: (E_9 \wedge E_{13,i} \wedge E_{13,j}) \vee (E_{10,i} \wedge E_{13,i} \wedge E_{13,j} \wedge E_{16,i} \wedge E_{16,j}) \\ \vee (E_{11,i} \wedge E_{13,i} \wedge E_{13,j} \wedge E_{16,i} \wedge E_{16,j}) \quad (31)$$

$$S_{5,i}: E_{12} \wedge E_{17,i} \wedge E_{17,j}. \quad (32)$$

For all $i \in \{1, 2\}$ the events $S_{1,i}$, $S_{2,i}$ and $S_{3,i}$ exhibit the property stated by the following lemma.

Lemma 2 For all $i \in \{1, 2\}$ and for all $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$, only one of the events $S_{1,i}$, $S_{2,i}$ and $S_{3,i}$ is true.

Proof: The proof follows from verifying that for all $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, the events (28)-(30) are mutually exclusive. For instance, consider that the event $(E_1 \wedge E_{13,i})$ in (28) is true. Then, $S_{1,i}$ is true and $E_{2,i}$, $E_{3,i}$, $E_{4,i}$, $E_{5,i}$, $E_{6,i}$, $E_{7,i}$ and $E_{8,i}$ hold false, which implies that $S_{2,i}$ and $S_{3,i}$ hold false as well, since all events in (29) and (30) hold false. The same verification can be made for all the remaining events in (28). This proves that if $S_{1,i}$ is true then $S_{2,i}$ and $S_{3,i}$ hold simultaneously false. The same verification can be done for showing that when $S_{2,i}$ holds true (resp. $S_{3,i}$), both events $S_{1,i}$ and $S_{3,i}$ (resp. $S_{1,i}$ and $S_{2,i}$) hold simultaneously false. Finally following the same reasoning it can be verified that if any pair of the events $\{S_{1,i}, S_{2,i}, S_{3,i}\}$ is false, the remaining event is necessarily true. This completes the proof. ■

For all $i \in \{1, 2\}$ the events $S_{4,i}$ and $S_{5,i}$ exhibit the property stated by the following lemma.

Lemma 3 For all $i \in \{1, 2\}$ if one of the events $S_{4,i}$ or $S_{5,i}$ holds true, then the other necessarily holds false.

Proof: The proof of Lemma 3 follows along the same lines of the proof of Lemma 2. ■

3.2 Rate Improvement Metrics

The rate improvements are given in terms of the following metrics [9, 10]: (a) maximum individual rate improvements Δ_1 and Δ_2 ; and (b) maximum sum-rate improvement Σ , with $\Delta_i \in \mathbb{R}_+$ and $\Sigma \in \mathbb{R}_+$ for $i \in \{1, 2\}$.

Let $\mathcal{C}_1 = \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ and $\mathcal{C}_2 = \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$ be the capacity region with noisy channel-output feedback and without feedback, respectively. In order to formally define Δ_1 , Δ_2 and Σ , consider a two-user LD-IC-NOF with parameters \vec{n}_{11} , \vec{n}_{22} , n_{12} , n_{21} , \overleftarrow{n}_{11} , and \overleftarrow{n}_{22} . The maximum improvement of the individual rate R_i , $\Delta_i(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$, due to the effect of channel-output feedback with respect to the case without feedback is

$$\Delta_i(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) = \max_{R_j > 0} \sup_{\substack{(R_i, R_j) \in \mathcal{C}_1 \\ (R_i^\dagger, R_j) \in \mathcal{C}_2}} R_i - R_i^\dagger, \quad (33)$$

and the maximum improvement of the sum rate $\Sigma(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ with respect to the case without feedback is

$$\Sigma(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}) = \sup_{\substack{(R_1, R_2) \in \mathcal{C}_1 \\ (R_1^\dagger, R_2^\dagger) \in \mathcal{C}_2}} R_1 + R_2 - (R_1^\dagger + R_2^\dagger). \quad (34)$$

In the following, when feedback is exclusively used by transmitter-receiver pair i , i.e., $\overleftarrow{n}_{ii} > 0$ and $\overleftarrow{n}_{jj} = 0$, then the maximum improvement of the individual rate of transmitter-receiver k , with $k \in \{1, 2\}$, and the maximum improvement of the sum rate are denoted by $\Delta_k(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$ and $\Sigma(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$, respectively. Hence, this notation $\Delta_k(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$ replaces either $\Delta_k(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ or $\Delta_k(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$, when $i = 1$ or $i = 2$, respectively. The same holds for the notation $\Sigma(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$ that replaces $\Sigma(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ or $\Sigma(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$, when $i = 1$ or $i = 2$, respectively.

4 Main Results

4.1 Enlargement of the Capacity Region

In this subsection, the capacity region of a two-user LD-IC-NOF with parameters $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21})$, when feedback is available only at transmitter-receiver pair i , i.e., $\overleftarrow{n}_{ii} > 0$ and $\overleftarrow{n}_{jj} = 0$, is denoted by $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$ instead of $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ or $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$, when $i = 1$ or $i = 2$, respectively. Following this notation, Theorem 1 identifies the exact values of \overleftarrow{n}_{ii} for which the strict inclusion $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$ holds, with $i \in \{1, 2\}$.

Theorem 1 Let $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$ and $\overleftarrow{n}_{ii}^* \in \mathbb{N}$ be

$$\overleftarrow{n}_{ii}^* = \begin{cases} \max(n_{ji}, (\vec{n}_{ii} - n_{ij})^+) & \text{if } S_{1,i} = \text{True} \\ \vec{n}_{jj} + (\vec{n}_{ii} - n_{ij})^+ & \text{if } S_{2,i} = \text{True}. \end{cases} \quad (35)$$

Assume that $S_{3,i} = \text{True}$. Then, for all $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) = \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$. Assume that either $S_{1,i} = \text{True}$ or $S_{2,i} = \text{True}$. Then, for all $\overleftarrow{n}_{ii} \leq \overleftarrow{n}_{ii}^*$, $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) = \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$ and for all $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^*$, $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii})$.

Proof: The proof of Theorem 1 is presented in Appendix A. ■

Theorem 1 shows that under event $S_{3,i}$ in (30), implementing feedback in transmitter-receiver pair i does not bring any capacity region enlargement. Alternatively, under events $S_{1,i}$ in (28) and $S_{2,i}$ in (29), the capacity region can be enlarged whenever $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^*$. That is, there exists a threshold on the SNR of the feedback link beyond which it is possible to observe a capacity region enlargement.

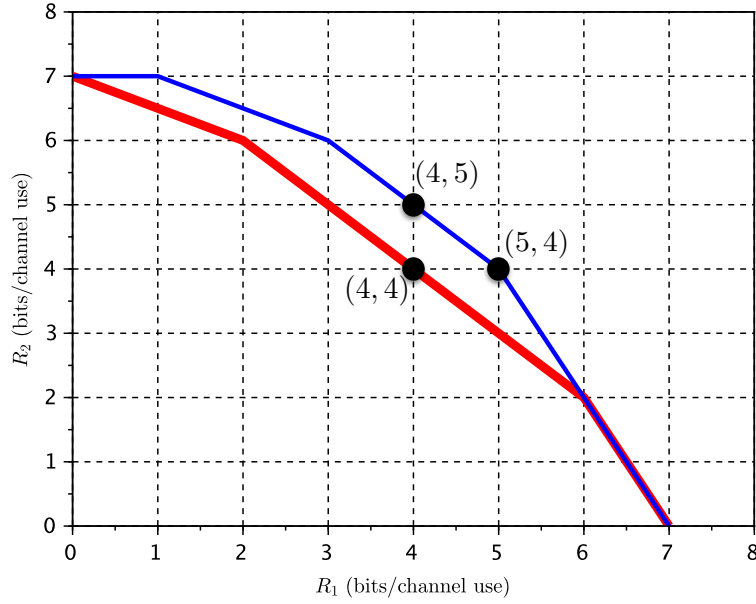


Figure 2: Capacity regions of $\mathcal{C}(7, 7, 3, 5, 0, 0)$ (thick red line) and $\mathcal{C}(7, 7, 3, 5, 6, 0)$ (thin blue line).

Consider an example in which transmitter-receiver pair i is in SIR ($1 < \alpha_i \leq 2$), transmitter-receiver pair j is in VWIR ($\alpha_j \leq \frac{1}{2}$) and $\vec{n}_{jj} > n_{ij}$. In this case, the event $E_{8,i}$ holds true (and so does $S_{3,i}$). Thus, implementing feedback in the transmitter-receiver pair i does not enlarge the capacity region. Note also that the events $E_{2,j}$ and $E_{13,j}$ hold true (and so does $S_{1,j}$). In view of this, implementing feedback in transmitter-receiver pair j enlarges the capacity region whenever the feedback link j satisfies that $\overleftarrow{n}_{jj} > \max(n_{ij}, (\vec{n}_{jj} - n_{ji})^+)$.

It is important to highlight that in the cases in which feedback might potentially enlarge the capacity region of the two-user LD-IC-NOF, that is, in events $S_{1,1}$, $S_{2,1}$, $S_{1,2}$ or $S_{2,2}$, the following is always true for all $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$:

$$\overleftarrow{n}_{ii}^* > (\vec{n}_{ii} - n_{ij})^+. \quad (36)$$

The inequality in (36) unveils the fact that feedback is useful for enlarging the capacity region of the two-user LD-IC-NOF when at least one receiver is capable of feeding back to its intended transmitter at least part of the information transmitted by the other transmitter. Essentially, when $\overleftarrow{n}_{ii} \leq (\vec{n}_{ii} - n_{ij})^+$, the noise over the feedback link i impairs all the bit-pipes that contain information sent by transmitter j . This implies that transmitter i cannot obtain any information sent by transmitter j , see for instance Fig. 1.

Another interesting observation is that the threshold \overleftarrow{n}_{ii}^* beyond which feedback is useful is different under event $S_{1,i}$ in (28) and event $S_{2,i}$ in (29). In general when $S_{1,i}$ holds true, the enlargement of the capacity region is due to the fact that feedback allows using *interference as side information* [12]. More specifically, when feedback is used at transmitter-receiver pair i and $\overleftarrow{n}_{ii} > \max(n_{ji}, (\vec{n}_{ii} - n_{ij})^+)$, transmitter i obtains part of the information sent by transmitter j . This information can be re-transmitted by transmitter i to cancel the interference it produced at receiver i when it was first transmitted by transmitter j . Interestingly, the interference perceived at receiver j due to this re-transmission can be cancelled given that this

information was reliably decoded when it was first sent by transmitter j . This allows transmitter-receiver pair i or j to improve its individual rate.

Consider an example with parameters $\vec{n}_{11} = 7$, $\vec{n}_{22} = 7$, $n_{12} = 3$, and $n_{21} = 5$. This implies that both $S_{1,1}$ and $S_{1,2}$ hold true. Hence, from Theorem 1, it follows that the capacity region can be enlarged by either using feedback in transmitter-receiver pair 1 or in transmitter-receiver pair 2, with $\overleftarrow{n}_{11} > \max(n_{21}, (\vec{n}_{11} - n_{12})^+) = 5$ or $\overleftarrow{n}_{22} > \max(n_{12}, (\vec{n}_{22} - n_{21})^+) = 3$, respectively. Figure 2 shows the capacity region of $\mathcal{C}(7, 7, 3, 5, 0, 0)$ (thick red line) and $\mathcal{C}(7, 7, 3, 5, 6, 0)$ (thin blue line). Note that the capacity region is enlarged when $\overleftarrow{n}_{11} = 6$, with respect to the case in which $\overleftarrow{n}_{11} = 0$. Figure 3 shows the achievability of (4, 4) without feedback, whereas Figures 4 and 5 show the achievability of (5, 4) and (4, 5) with feedback in transmitter-receiver pair 1, respectively.

In the following, a bit-pipe is said to be private if it carries bits uniquely to the intended receiver. A bit-pipe that is not private is referred to as a common bit-pipe. In the achievability of (4, 4) without feedback in Fig. 3, both transmitters use all the private bit-pipes that are effectively seen without interference at the intended receivers. That is, bit-pipes 6 and 7 in transmitter-receiver pair 1 and bit-pipes 5, 6 and 7 in transmitter-receiver pair 2. Note that bit-pipe 4 in transmitter-receiver pair 2 is private but it is impaired by the interference of bit-pipe 2 from transmitter 1. In the achievability of (4, 5), transmitter 2 uses the bit-pipe 2 to send bits b_2, b_7, b_{12}, \dots that produce interference to the private bit-pipe 6 of transmitter-receiver pair 1. Note that bits b_2, b_7, b_{12}, \dots are received interference-free at receiver 2 via bit-pipe 2. Thanks to the existence of the feedback link, transmitter 1 can obtain the interfering bits b_2, b_7, b_{12}, \dots and re-transmit them over common-bit pipes that are interference-free at receiver 1. This re-transmission, allows to cancel the interference at receiver 1. The interference produced by this re-transmission over bit-pipe 5 at receiver 2 can be eliminated since bits b_2, b_7, b_{12}, \dots have been previously received interference-free. Hence, transmitter-receiver pair 1 maintains the rate $R_1 = 4$, whereas transmitter-receiver pair 2 improves its rate from $R_2 = 4$ to $R_2 = 5$.

A similar analysis holds for the achievability of (5, 4) in Fig. 4, in which the individual rate improvement is also thanks to the use of interference as side information.

Alternatively, when $S_{2,i}$ in (29) holds true, the enlargement of the capacity region occurs thanks to the fact that some of the bits that cannot be transmitted directly from transmitter j to receiver j , that is, those transmitted via the bit-pipes $\vec{n}_{jj} + 1, \dots, \max(\vec{n}_{jj}, n_{ij})$, can arrive to receiver j via an alternative path: transmitter j - receiver i - transmitter i - receiver j . For this to be possible at least the $(\vec{n}_{jj} + (\vec{n}_{ii} - n_{ij})^+ + 1)$ -th (feedback) bit-pipe from receiver i to transmitter i must be noise-free, i.e., $\overleftarrow{n}_{ii} > \vec{n}_{jj} + (\vec{n}_{ii} - n_{ij})^+$.

Consider for instance an example with parameters $\vec{n}_{11} = 5$, $\vec{n}_{22} = 1$, $\vec{n}_{12} = 3$, and $\vec{n}_{21} = 4$. In this case, $S_{2,1}$ in (29) and $S_{3,2}$ in (30) hold true. Hence from Theorem 1, it follows that the capacity region cannot be enlarged by using feedback in transmitter-receiver pair 2 ($S_{3,2}$ holds true). At the same time, it follows that the capacity region can be enlarged using feedback in transmitter-receiver pair 1, whenever $\overleftarrow{n}_{11} > \vec{n}_{22} + (\vec{n}_{11} - n_{12})^+ = 3$ ($S_{2,1}$ holds true). Figure 6 shows the capacity region of $\mathcal{C}(5, 1, 3, 4, 0, 0)$ (thick red line) and $\mathcal{C}(5, 1, 3, 4, 4, 0)$ (thin blue line). Note that the capacity region is indeed enlarged when $\overleftarrow{n}_{11} = 4$, with respect to $\overleftarrow{n}_{11} = 0$. Figure 7 and Figure 8 show the achievability of the rate pairs (3, 1) and (3, 2) without and with feedback, respectively. Note that the main difference between both schemes is that the existence of a feedback link between transmitter 1 and receiver 1, with $\overleftarrow{n}_{11} = 4$, allows transmitter 2 to send an additional bit per channel use, i.e., bits b_2, b_4, b_6, \dots , via the alternative path (transmitter 2 - receiver 1 - transmitter 1 - receiver 2). This accounts for transmitter-receiver pair 2 increasing its rate from one to two bits per channel use.

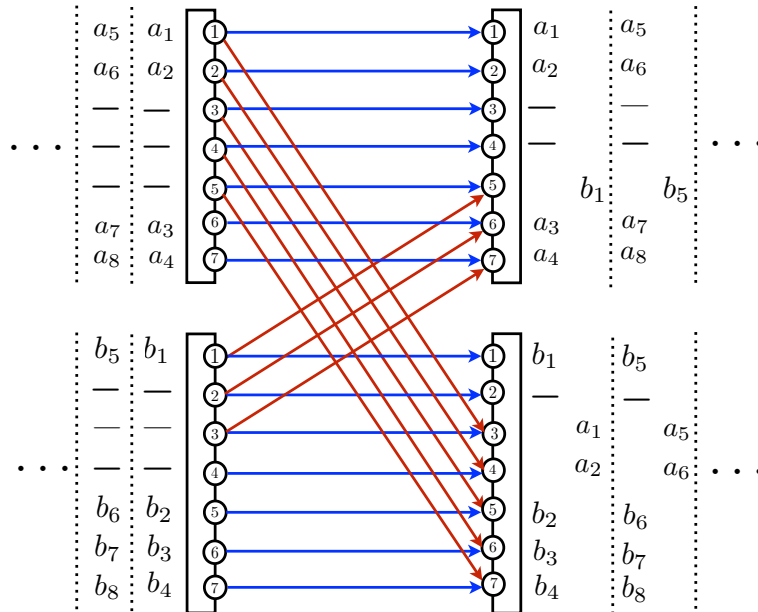


Figure 3: Achievability of the rate pair $(4, 4)$ in an LD-IC with parameters $\vec{n}_{11} = 7$, $\vec{n}_{22} = 7$, $n_{12} = 3$, $n_{21} = 5$, $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$.

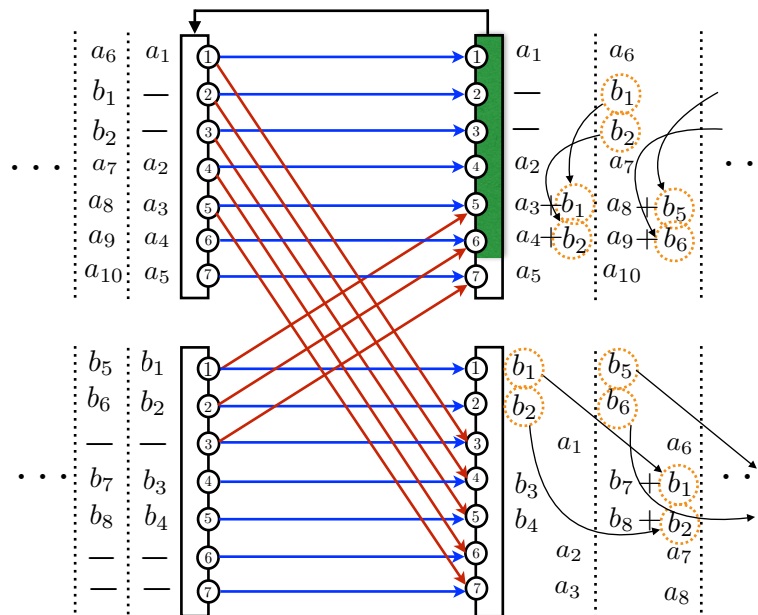


Figure 4: Achievability of the rate pair $(5, 4)$ in an LD-IC with parameters $\vec{n}_{11} = 7$, $\vec{n}_{22} = 7$, $n_{12} = 3$, $n_{21} = 5$, $\overleftarrow{n}_{11} = 6$ and $\overleftarrow{n}_{22} = 0$.

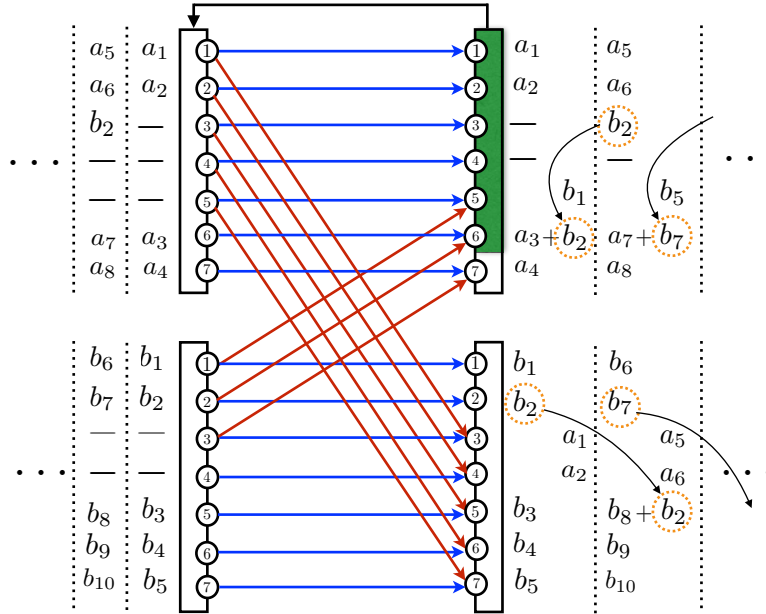


Figure 5: Achievability of the rate pair (4, 5) in an LD-IC with parameters $\vec{n}_{11} = 7$, $\vec{n}_{22} = 7$, $n_{12} = 3$, $n_{21} = 5$, $\overleftarrow{n}_{11} = 6$ and $\overleftarrow{n}_{22} = 0$.

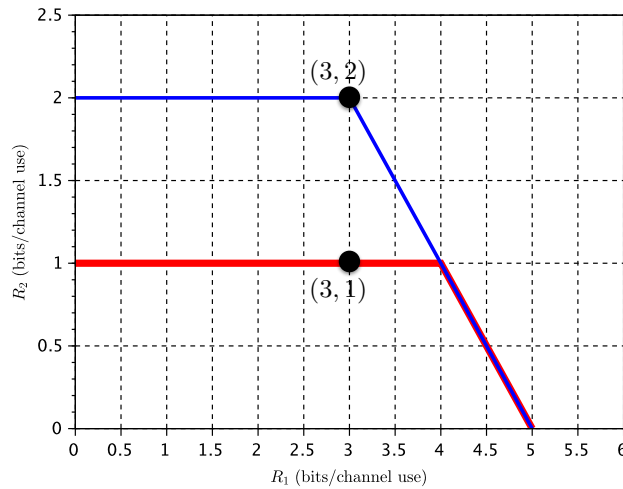


Figure 6: Capacity regions of $\mathcal{C}(5, 1, 3, 4, 0, 0)$ (thick red line) and $\mathcal{C}(5, 1, 3, 4, 4, 0)$ (thin blue line).

4.2 Improvement of the Individual Rate R_i by Using Feedback in Link i

Implementing channel output feedback in transmitter-receiver pair i might allow increasing the individual rate of either transmitter-receiver pair i or j . Theorem 2 identifies the exact values of \overleftarrow{n}_{ii} for which the individual rate R_i can be improved, given the parameters $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21})$

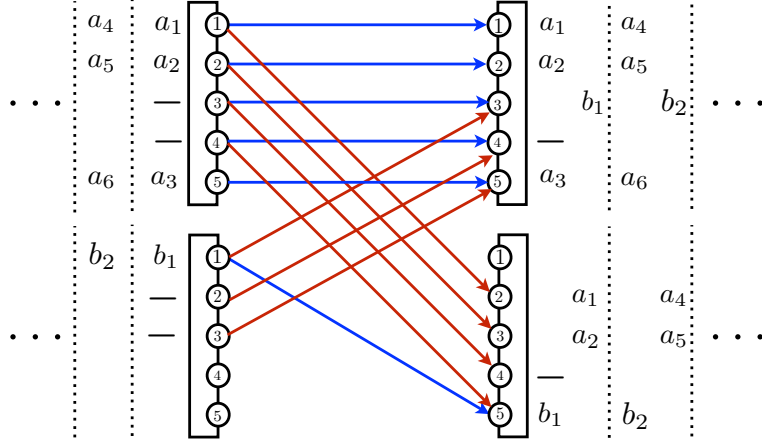


Figure 7: Achievability of the rate pair $(3, 1)$ in an LD-IC with parameters $\vec{n}_{11} = 5$, $\vec{n}_{22} = 1$, $n_{12} = 3$, $n_{21} = 4$, $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$ (no feedback links).

and $\overleftarrow{n}_{jj} = 0$ in the two-user LD-IC-NOF.

Theorem 2 Let $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$ and $\overleftarrow{n}_{ii}^\dagger \in \mathbb{N}$ be

$$\overleftarrow{n}_{ii}^\dagger = \max \left(n_{ji}, (\vec{n}_{ii} - n_{ij})^+ \right). \quad (37)$$

Assume that either $S_{2,i} = \text{True}$ or $S_{3,i} = \text{True}$. Then, for all $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\Delta_i(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$. Assume that $S_{1,i} = \text{True}$. Then, when $\overleftarrow{n}_{ii} \leq \overleftarrow{n}_{ii}^\dagger$, it holds that $\Delta_i(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$; and when $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^\dagger$, it holds that $\Delta_i(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$.

Proof: The proof of Theorem 2 is presented in Appendix B. \blacksquare

Theorem 2 highlights that under events $S_{2,i}$ in (29) and $S_{3,i}$ in (30), the individual rate R_i cannot be improved by using feedback in transmitter-receiver pair i , i.e., $\Delta_i(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$. Alternatively, under event $S_{1,i}$ in (28), the individual rate R_i can be improved, i.e., $\Delta_i(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$, whenever $\overleftarrow{n}_{ii} > \max \left(n_{ji}, (\vec{n}_{ii} - n_{ij})^+ \right)$.

An example in which for all $(R'_1, R'_2) \in \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$ there always exists a rate pair $(R_1, R_2) \in \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ such that $R'_1 < R_1$ and $R'_2 = R_2$, is presented in Fig. 2.

It is worth noting that under event $S_{3,i}$ in (30), the capacity region cannot be improved via feedback (Theorem 1) and thus, none of the individual rates can be improved as suggested by Theorem 2. Alternatively, under event $S_{2,i}$ in (29), the capacity region can be enlarged (Theorem 1) but the individual rate R_i cannot be improved (Theorem 2). This implies that the capacity improvement occurs due to the fact that R_j can be improved. More specifically, in this case: $\Delta_i(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$ and $\Delta_j(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$. This implies that using feedback in transmitter-receiver pair i is exclusively beneficial for transmitter-receiver pair j , as shown in the following section.

4.3 Improvement of the Individual Rate R_j by Using Feedback in Link i

Implementing channel output feedback in transmitter-receiver pair i might allow increasing the individual rate of transmitter-receiver pair i or j ; or both individual rates. This reveals the

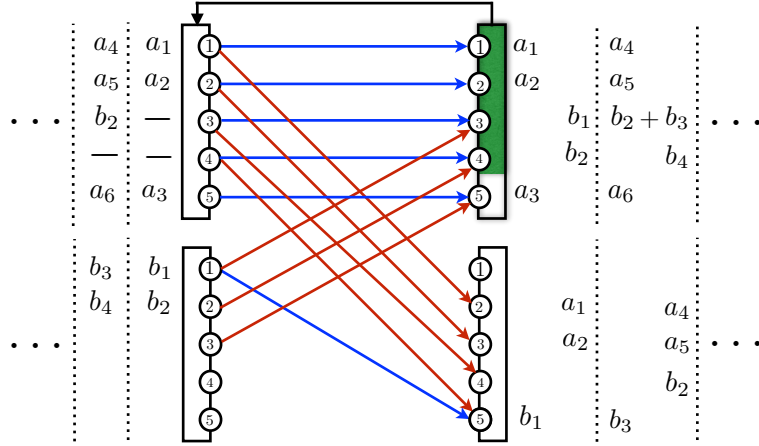


Figure 8: Achievability of the rate pair (3, 2) in an LD-IC with parameters $\vec{n}_{11} = 5$, $\vec{n}_{22} = 1$, $n_{12} = 3$, $n_{21} = 4$, $\overleftarrow{n}_{11} = 4$ and $\overleftarrow{n}_{22} = 0$.

altruistic nature of implementing feedback as suggested in [2]. Theorem 3 identifies the exact values of \overleftarrow{n}_{ii} for which the individual rate R_j can be improved by using feedback in transmitter-receiver pair i , given the parameters $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21})$ and $\overleftarrow{n}_{jj} = 0$ in the two-user LD-IC-NOF.

Theorem 3 Let $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$ and $\overleftarrow{n}_{ii}^\dagger \in \mathbb{N}$ be

$$\overleftarrow{n}_{ii}^\dagger = \begin{cases} \max(n_{ji}, (\vec{n}_{ii} - n_{ij})^+) & \text{if } S_{1,i} = \text{True} \\ \vec{n}_{jj} + (\vec{n}_{ii} - n_{ij})^+ & \text{if } S_{2,i} = \text{True}. \end{cases} \quad (38)$$

Assume that $S_{3,i} = \text{True}$. Then, for all $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\Delta_j(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$. Assume that either $S_{1,i} = \text{True}$ or $S_{2,i} = \text{True}$. Then, when $\overleftarrow{n}_{ii} \leq \overleftarrow{n}_{ii}^\dagger$, it holds that $\Delta_j(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$; and when $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^\dagger$, it holds that $\Delta_j(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$.

Proof: The proof of Theorem 3 is presented in Appendix C. \blacksquare

Theorem 3 shows that under event $S_{3,i}$ in (30), implementing feedback in transmitter-receiver pair i does not bring any improvement on the rate R_j . This is in line with the results of Theorem 1 that states that under event $S_{3,i}$ in (30), implementing feedback in transmitter-receiver pair i does not enlarge the capacity region.

In contrast, under events $S_{1,i}$ in (28) and $S_{2,i}$ in (29), the individual rate R_j can be improved ($\Delta_j(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$) whenever $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^\dagger$. It is important to highlight that under event $S_{1,i}$, the threshold on \overleftarrow{n}_{ii} for increasing the individual rate R_i i.e., $(\overleftarrow{n}_{ii}^\dagger)$, and R_j i.e., $(\overleftarrow{n}_{ii}^\dagger)$, are identical, see Theorem 2 and Theorem 3. This shows that in this case, the use of feedback in transmitter-receiver pair i , with $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^\dagger = \overleftarrow{n}_{ii}^\dagger$, simultaneously improves both individual rates. Under event $S_{2,i}$, using feedback in transmitter-receiver pair i , with $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^\dagger$, exclusively benefits transmitter-receiver pair j , which can improve its own individual rate. An example in which for all $(R'_1, R'_2) \in \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$ there always exists a rate pair $(R_1, R_2) \in \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ such that $R'_2 < R_2$, is presented in Fig. 6.

4.4 Improvement of the Sum-Capacity

Implementing channel output feedback in transmitter-receiver pair i might allow increasing the sum-capacity. Theorem 4 identifies the exact values of \overleftarrow{n}_{ii} for which the sum-capacity can be improved, for parameters $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21})$ and $\overleftarrow{n}_{jj} = 0$ in the two-user LD-IC-NOF.

Theorem 4 Let $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$ and $\overleftarrow{n}_{ii}^+ \in \mathbb{N}$ be

$$\overleftarrow{n}_{ii}^+ = \begin{cases} \max(n_{ji}, (\vec{n}_{ii} - n_{ij})^+) & \text{if } S_{4,i} = \text{True} \\ \vec{n}_{jj} + (\vec{n}_{ii} - n_{ij})^+ & \text{if } S_{5,i} = \text{True}. \end{cases} \quad (39)$$

Assume that $S_{4,i} = \text{False}$ and $S_{5,i} = \text{False}$. Then, for all $\overleftarrow{n}_{ii} \in \mathbb{N}$, $\Sigma(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$. Assume that $S_{4,i} = \text{True}$ or $S_{5,i} = \text{True}$. Then, when $\overleftarrow{n}_{ii} \leq \overleftarrow{n}_{ii}^+$, it holds that $\Sigma(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) = 0$; and when $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^+$, it holds that $\Sigma(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$.

Proof: The proof of Theorem 4 is presented in Appendix D. ■

Theorem 4 identifies the conditions under which implementing feedback in transmitter-receiver pair i improves the sum-capacity whenever $\overleftarrow{n}_{ii} > \overleftarrow{n}_{ii}^+$, that is, $\Sigma(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{ii}) > 0$. Theorem 4 highlights that one of the necessary but not sufficient conditions for improving the sum-capacity by implementing feedback in transmitter-receiver pair i is that either (a) at least one transmitter-receiver pair must be in VWIR or WIR; or (b) both transmitter-receiver pairs must be in VSIR. This follows immediately from observing that for $S_{4,i}$ or $S_{5,i}$ to hold true, at least one of the events E_9 , $E_{10,i}$, $E_{11,i}$ or E_{12} must hold true. An example in which $C_{\text{sum}}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) < C_{\text{sum}}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$, with $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^+$, is presented in Fig. 2.

Interestingly, Theorem 4 shows that if at least one transmitter-receiver pair is in SIR, then the sum-capacity cannot be improved. Finally, note that the thresholds \overleftarrow{n}_{ii}^+ in the events $S_{4,i}$ and $S_{5,i}$ coincide with those observed in Theorem 1.

5 Generalized Degrees of Freedom

This section focuses on the analysis of the number of GDoF of the LD-IC-NOF for studying the case in which feedback is simultaneously implemented in both transmitter-receiver pairs. Moreover, the analysis is only performed for the symmetric case described by:

$$\vec{n} = \vec{n}_{11} = \vec{n}_{22}, \quad (40)$$

$$m = \vec{n}_{12} = \vec{n}_{21}, \quad (41)$$

$$\overleftarrow{n} = \overleftarrow{n}_{11} = \overleftarrow{n}_{22}, \quad (42)$$

with $(\vec{n}, m, \overleftarrow{n}) \in \mathbb{N}^3$. The results in Lemma 1 allow a more general analysis of the GDoF, e.g., non-symmetric case. However, the symmetric case captures some of the most important insights about how the capacity region is enlarged when feedback is used in both transmitter-receiver pairs.

Essentially, given the parameters \vec{n} , m and \overleftarrow{n} , with

$$\alpha = \frac{m}{\vec{n}} \text{ and} \quad (43)$$

$$\beta = \frac{\overleftarrow{n}}{\vec{n}}, \quad (44)$$

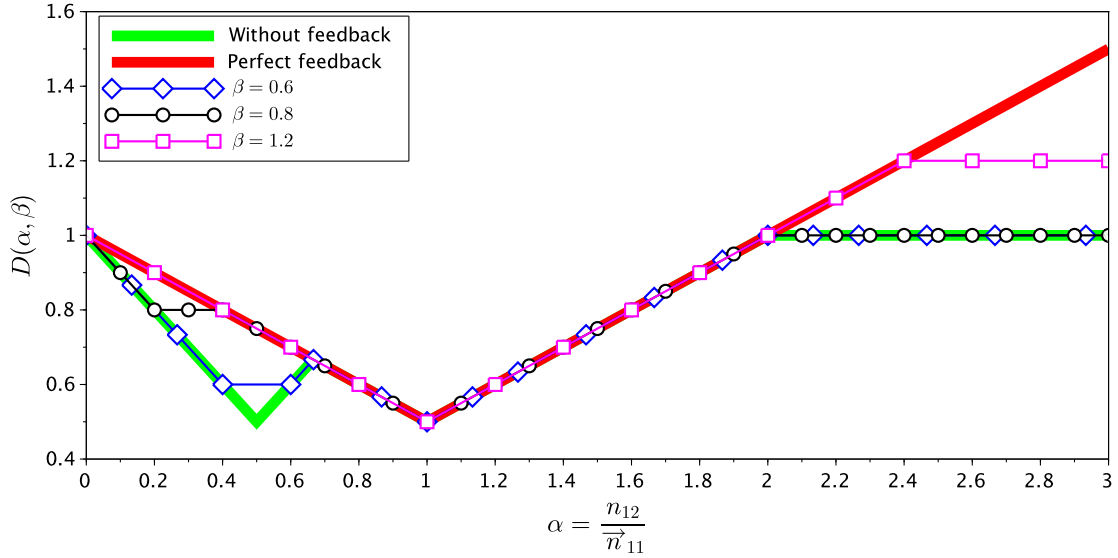


Figure 9: Generalized Degrees of Freedom (GDoF) as a function of parameters α and β , with $0 \leq \alpha \leq 3$ and $\beta \in \{\frac{3}{5}, \frac{4}{5}, \frac{6}{5}\}$, of the symmetric LD-IC-NOF. The plot without feedback is obtained from [11] and the plot with perfect-output feedback is obtained from [1].

the number of GDoF, denoted by $D(\alpha, \beta)$, is the ratio between the symmetric sum-capacity, i.e.,

$$C_{\text{sym}}(\vec{n}, m, \overleftarrow{n}) = \sup\{R : (R, R) \in \mathcal{C}(\vec{n}, \vec{n}, m, m, \overleftarrow{n}, \overleftarrow{n})\}, \quad (45)$$

and the individual interference-free point-to-point capacity, i.e., \vec{n} , when $(\vec{n}, m, \overleftarrow{n}) \rightarrow (\infty, \infty, \infty)$ at constant ratios $\alpha = \frac{m}{\vec{n}}$ and $\beta = \frac{\overleftarrow{n}}{\vec{n}}$. More specifically, the number of GDoF is

$$D(\alpha, \beta) = \lim_{\vec{n}, m, \overleftarrow{n} \rightarrow \infty} \frac{C_{\text{sym}}(\vec{n}, m, \overleftarrow{n})}{\vec{n}}. \quad (46)$$

Theorem 5 determines the number of GDoF for the two-user LD-IC-NOF.

Theorem 5 The number of GDoF for the two user symmetric LD-IC-NOF with parameters α and β is given by

$$D(\alpha, \beta) = \min \left(\max(1, \alpha), \max(1, \beta - (1 - \alpha)^+), \frac{1}{2} (\max(1, \alpha) + (1 - \alpha)^+), \right. \\ \left. \max((1 - \alpha)^+, \alpha, 1 - (\max(1, \alpha) - \beta)^+), \frac{1}{3} (\max(1, \alpha) + (1 - \alpha)^+ + \max((1 - \alpha)^+, \right. \\ \left. \alpha, 1 - (\max(1, \alpha) - \beta)^+)) \right). \quad (47)$$

Proof: The proof of Theorem 5 is presented in Appendix E. ■

The following properties are a direct consequence of Theorem 4:

$$\forall \alpha \in \left[0, \frac{2}{3}\right], \quad \max\left(\frac{1}{2}, \beta\right) \leq D(\alpha, \beta) \leq 1, \quad (48a)$$

$$\forall \alpha \in \left(\frac{2}{3}, 2\right], \quad D(\alpha, 0) = D(\alpha, \beta) = D(\alpha, \max(1, \alpha)), \quad (48b)$$

$$\forall \alpha \in (2, \infty), \quad 1 \leq D(\alpha, \beta) \leq \min\left(\frac{\alpha}{2}, \beta\right). \quad (48c)$$

Property (48a) highlights the fact that the existence of feedback links in the symmetric LD-IC in the VWIR and WIR does not have any impact in the GDoF when $\beta \leq \frac{1}{2}$. Property (48b) underlines that in the symmetric LD-IC in MIR and SIR, the number of GDoF is identical in both extreme cases: without feedback ($\beta = 0$) and with perfect-output feedback ($\beta = \max(1, \alpha)$). Finally, from (48c), it follows that for observing an improvement in the GDoF of the LD-IC-NOF in VSIR, the following condition must be met: $\beta > 1$. That is, the number of bit-pipes in the feedback links must be strictly bigger than the number of bit-pipes in the direct links.

Figure 9 shows the number of GDoF for the two user symmetric LD-IC-NOF for the case in which $0 \leq \alpha \leq 3$ and $\beta \in \{\frac{3}{5}, \frac{4}{5}, \frac{6}{5}\}$.

6 Conclusions

This technical report presented the exact conditions on the feedback parameters \overleftarrow{n}_{11} and \overleftarrow{n}_{22} , beyond which the capacity region of the two-user LD-IC-NOF can be enlarged for any 4-tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$. More specifically, the exact values of \overleftarrow{n}_{11} (resp. \overleftarrow{n}_{22}) for which $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ (resp. $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, \overleftarrow{n}_{22})$), with strict inclusion. The exact conditions on \overleftarrow{n}_{11} (resp. \overleftarrow{n}_{22}) to observe an improvement on a single rate or the sum-rate capacity, for any 4-tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$ were also presented. Interestingly, there exist conditions in the two-user LD-IC-NOF in which the use of feedback does not enlarge the capacity region. The gain in the capacity of the two user LD-IC-NOF under symmetric conditions and for the VSIR can be bounded by the noise in the feedback links. Finally, closed-form expressions for the number of GDoF are presented in the case of symmetric LD-IC-NOF. As observed before in [1] and [11], the number of GDoF is invariant with respect to the use of output feedback links in both moderate and strong interference regimes. In all the other interference regimes, the exact improvement on the GDoF is perfectly identified.

Appendices

A Proof of Theorem 1: Enlargement of the Capacity Region by Using Feedback in one Transmitter-Receiver Pair

For fix parameters \vec{n}_{11} , \vec{n}_{22} , n_{12} , n_{21} and $\overleftarrow{n}_{22} = 0$, this appendix presents the procedure to obtain the threshold, denoted by \overleftarrow{n}_{11}^* , on the feedback parameter \overleftarrow{n}_{11} beyond which the capacity region can be enlarged, when both transmitter-receiver pairs are in VWIR. The proof of Theorem 1 is completed by performing the above mentioned procedure for all cases of possible combinations of interference regimes. This procedure is tedious and repetitive, and thus, in this appendix only one combination of interference regimes is fully studied, e.g., VWIR - VWIR.

Proof:

Denote by $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$ the capacity region of the LD-IC without feedback and parameters \vec{n}_{11} , \vec{n}_{22} , n_{12} and n_{21} . This region is fully described by the following corollary.

Corollary 1 (Lemma 4 in [13]) The capacity region $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$ of the two-user LD-IC without feedback is included in the set of non-negative rate pairs (R_1, R_2) satisfying

$$R_1 \leq \vec{n}_{11}, \quad (49a)$$

$$R_2 \leq \vec{n}_{22}, \quad (49b)$$

$$R_1 + R_2 \leq (\vec{n}_{11} - n_{12})^+ + \max(\vec{n}_{22}, n_{12}), \quad (49c)$$

$$R_1 + R_2 \leq (\vec{n}_{22} - n_{21})^+ + \max(\vec{n}_{11}, n_{21}), \quad (49d)$$

$$R_1 + R_2 \leq \max(n_{21}, (\vec{n}_{11} - n_{12})^+) + \max(n_{12}, (\vec{n}_{22} - n_{21})^+), \quad (49e)$$

$$2R_1 + R_2 \leq \max(\vec{n}_{11}, n_{21}) + (\vec{n}_{11} - n_{12})^+ + \max(n_{12}, (\vec{n}_{22} - n_{21})^+), \quad (49f)$$

$$R_1 + 2R_2 \leq \max(\vec{n}_{22}, n_{12}) + (\vec{n}_{22} - n_{21})^+ + \max(n_{21}, (\vec{n}_{11} - n_{12})^+). \quad (49g)$$

Corollary 1 follows immediately from Lemma 1 by setting $\overleftarrow{n}_{11} = 0$ and $\overleftarrow{n}_{22} = 0$. Consider that both transmitter-receiver pairs are in VWIR, that is,

$$\alpha_1 = \frac{n_{12}}{\vec{n}_{11}} \leq \frac{1}{2}, \quad \text{and} \quad (50a)$$

$$\alpha_2 = \frac{n_{21}}{\vec{n}_{22}} \leq \frac{1}{2}. \quad (50b)$$

From (50), it follows that:

$$\vec{n}_{11} \geq 2n_{12} \geq n_{12}, \quad \text{and} \quad (51a)$$

$$\vec{n}_{22} \geq 2n_{21} \geq n_{21}. \quad (51b)$$

Note that for all $(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{22}) \in \mathbb{N}^5$ and $\overleftarrow{n}_{11} > \max(\vec{n}_{11}, n_{12})$, it follows that $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \max(\vec{n}_{11}, n_{12}), \overleftarrow{n}_{22}) = \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$. Using this obser-

vation and plugging (51) and $\overleftarrow{n}_{22} = 0$ into (52) yields

$$R_1 \leq \overrightarrow{n}_{11}, \quad (52a)$$

$$R_2 \leq \overrightarrow{n}_{22} \quad (52b)$$

$$R_1 + R_2 \leq \min(\max(\overrightarrow{n}_{22}, n_{12}) + \overrightarrow{n}_{11} - n_{12}, \max(\overrightarrow{n}_{11}, n_{21}) + \overrightarrow{n}_{22} - n_{21}), \quad (52c)$$

$$R_1 + R_2 \leq \max(\overrightarrow{n}_{11} - n_{12}, n_{21}, \overleftarrow{n}_{11}) + \max(\overrightarrow{n}_{22} - n_{21}, n_{12}), \quad (52d)$$

$$2R_1 + R_2 \leq \max(\overrightarrow{n}_{11}, n_{21}) + \overrightarrow{n}_{11} - n_{12} + \max(\overrightarrow{n}_{22} - n_{21}, n_{12}), \quad (52e)$$

$$R_1 + 2R_2 \leq \max(\overrightarrow{n}_{22}, n_{12}) + \overrightarrow{n}_{22} - n_{21} + \max(\overrightarrow{n}_{11} - n_{12}, n_{21}, \overleftarrow{n}_{11}). \quad (52f)$$

Plugging (51) into (49) yields

$$R_1 \leq \overrightarrow{n}_{11}, \quad (53a)$$

$$R_2 \leq \overrightarrow{n}_{22}, \quad (53b)$$

$$R_1 + R_2 \leq \overrightarrow{n}_{11} - n_{12} + \max(\overrightarrow{n}_{22}, n_{12}), \quad (53c)$$

$$R_1 + R_2 \leq \overrightarrow{n}_{22} - n_{21} + \max(\overrightarrow{n}_{11}, n_{21}), \quad (53d)$$

$$R_1 + R_2 \leq \max(n_{21}, \overrightarrow{n}_{11} - n_{12}) + \max(n_{12}, \overrightarrow{n}_{22} - n_{21}), \quad (53e)$$

$$2R_1 + R_2 \leq \max(\overrightarrow{n}_{11}, n_{21}) + \overrightarrow{n}_{11} - n_{12} + \max(n_{12}, \overrightarrow{n}_{22} - n_{21}), \quad (53f)$$

$$R_1 + 2R_2 \leq \max(\overrightarrow{n}_{22}, n_{12}) + \overrightarrow{n}_{22} - n_{21} + \max(n_{21}, \overrightarrow{n}_{11} - n_{12}). \quad (53g)$$

To simplify the inequalities containing the operator $\max(\cdot, \cdot)$ in (52) and (53), the following 4 cases are identified:

$$\text{Case 1: } \overrightarrow{n}_{11} - n_{12} < n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} < n_{12}; \quad (54)$$

$$\text{Case 2: } \overrightarrow{n}_{11} - n_{12} < n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} > n_{12}; \quad (55)$$

$$\text{Case 3: } \overrightarrow{n}_{11} - n_{12} > n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} < n_{12}; \text{ and} \quad (56)$$

$$\text{Case 4: } \overrightarrow{n}_{11} - n_{12} > n_{21} \text{ and } \overrightarrow{n}_{22} - n_{21} > n_{12}. \quad (57)$$

Case 1

Under conditions (51) and (54), it is proved that this case is not possible. From (54) and (51), it follows that:

$$\begin{aligned} n_{21} &> \overrightarrow{n}_{11} - n_{12} \\ &> 2n_{12} - n_{12} \\ &= n_{12}, \end{aligned} \quad (58)$$

and

$$\begin{aligned} n_{12} &> \overrightarrow{n}_{22} - n_{21} \\ &> 2n_{21} - n_{21} \\ &= n_{21}. \end{aligned} \quad (59)$$

The inequalities (58) and (59) are contradictory, and thus, Case 1 is vacuous.

Case 2

By plugging (51) into (55), it follows that:

$$\begin{aligned} n_{21} &> \vec{n}_{11} - n_{12} \\ &> 2n_{12} - n_{12} \\ &= n_{12}, \end{aligned} \tag{60}$$

and

$$\vec{n}_{22} > n_{12} + n_{21}, \tag{61}$$

which implies that Case 2 is always possible. Note that from (61), it follows that:

$$\vec{n}_{22} > n_{12}. \tag{62}$$

Plugging (55) and (62) into (52) yields:

$$R_1 \leq \vec{n}_{11}, \tag{63a}$$

$$R_2 \leq \vec{n}_{22} \tag{63b}$$

$$R_1 + R_2 \leq \min(\vec{n}_{22} + \vec{n}_{11} - n_{12}, \max(\vec{n}_{11}, n_{21}) + \vec{n}_{22} - n_{21}), \tag{63c}$$

$$R_1 + R_2 \leq \max(n_{21}, \overleftarrow{n}_{11}) + \vec{n}_{22} - n_{21}, \tag{63d}$$

$$2R_1 + R_2 \leq \max(\vec{n}_{11}, n_{21}) + \vec{n}_{11} - n_{12} + \vec{n}_{22} - n_{21}, \tag{63e}$$

$$R_1 + 2R_2 \leq 2\vec{n}_{22} - n_{21} + \max(n_{21}, \overleftarrow{n}_{11}). \tag{63f}$$

Plugging (55) and (62) into (53) yields:

$$R_1 \leq \vec{n}_{11}, \tag{64a}$$

$$R_2 \leq \vec{n}_{22}, \tag{64b}$$

$$R_1 + R_2 \leq \vec{n}_{22}, \tag{64c}$$

$$2R_1 + R_2 \leq \max(\vec{n}_{11}, n_{21}) + \vec{n}_{11} - n_{12} + \vec{n}_{22} - n_{21}, \tag{64d}$$

$$R_1 + 2R_2 \leq 2\vec{n}_{22}. \tag{64e}$$

To simplify the inequalities containing the operator $\max(\cdot, \cdot)$ in (63) and (64), the following 2 cases are identified:

$$\text{Case 2a : } \vec{n}_{11} > n_{21}; \text{ and} \tag{65}$$

$$\text{Case 2b : } \vec{n}_{11} \leq n_{21}. \tag{66}$$

Case 2a

Plugging (65) into (63) yields:

$$R_1 \leq \vec{n}_{11}, \tag{67a}$$

$$R_2 \leq \vec{n}_{22} \tag{67b}$$

$$R_1 + R_2 \leq \vec{n}_{11} + \vec{n}_{22} - n_{21}, \tag{67c}$$

$$R_1 + R_2 \leq \max(n_{21}, \overleftarrow{n}_{11}) + \vec{n}_{22} - n_{21}, \tag{67d}$$

$$2R_1 + R_2 \leq 2\vec{n}_{11} - n_{12} + \vec{n}_{22} - n_{21}, \tag{67e}$$

$$R_1 + 2R_2 \leq 2\vec{n}_{22} - n_{21} + \max(n_{21}, \overleftarrow{n}_{11}). \tag{67f}$$

Plugging (65) into (64) yields:

$$R_1 \leq \vec{n}_{11}, \quad (68a)$$

$$R_2 \leq \vec{n}_{22}, \quad (68b)$$

$$R_1 + R_2 \leq \vec{n}_{22}, \quad (68c)$$

$$2R_1 + R_2 \leq 2\vec{n}_{11} - n_{12} + \vec{n}_{22} - n_{21}, \quad (68d)$$

$$R_1 + 2R_2 \leq 2\vec{n}_{22}. \quad (68e)$$

Comparing inequalities (67c) and (67d) with inequality (68c) under conditions (50), (55) and (65), it can be verified that:

$$\min(\vec{n}_{11} + \vec{n}_{22} - n_{21}, \max(n_{21}, \overleftarrow{n}_{11}) + \vec{n}_{22} - n_{21}) > \vec{n}_{22}, \quad (69)$$

when $\overleftarrow{n}_{11} > n_{21}$.

Comparing inequalities (67f) and (68e) under conditions (50), (55) and (65), it can be verified that:

$$2\vec{n}_{22} - n_{21} + \max(n_{21}, \overleftarrow{n}_{11}) > 2\vec{n}_{22}, \quad (70)$$

when $\overleftarrow{n}_{11} > n_{21}$.

All the other inequalities in (67) and (68) are identical. Therefore, $\overleftarrow{n}_{11}^* = n_{21}$ under conditions (50), (55) and (65).

Case 2b

Plugging (66) into (63) yields:

$$R_1 \leq \vec{n}_{11}, \quad (71a)$$

$$R_2 \leq \vec{n}_{22} \quad (71b)$$

$$R_1 + R_2 \leq \vec{n}_{22}, \quad (71c)$$

$$R_1 + R_2 \leq \max(n_{21}, \overleftarrow{n}_{11}) + \vec{n}_{22} - n_{21}, \quad (71d)$$

$$2R_1 + R_2 \leq \vec{n}_{11} - n_{12} + \vec{n}_{22}, \quad (71e)$$

$$R_1 + 2R_2 \leq 2\vec{n}_{22} - n_{21} + \max(n_{21}, \overleftarrow{n}_{11}). \quad (71f)$$

Plugging (65) into (64) yields:

$$R_1 \leq \vec{n}_{11}, \quad (72a)$$

$$R_2 \leq \vec{n}_{22}, \quad (72b)$$

$$R_1 + R_2 \leq \vec{n}_{22}, \quad (72c)$$

$$2R_1 + R_2 \leq \vec{n}_{11} - n_{12} + \vec{n}_{22}, \quad (72d)$$

$$R_1 + 2R_2 \leq 2\vec{n}_{22}. \quad (72e)$$

Comparing inequalities (71c) and (71d) with inequality (72c) under conditions (50), (55) and (66), it can be verified that:

$$\min(\vec{n}_{22}, \max(n_{21}, \overleftarrow{n}_{11}) + \vec{n}_{22} - n_{21}) = \vec{n}_{22}, \quad (73)$$

for all $\overleftarrow{n}_{11} \in \mathbb{N}$.

Comparing inequalities (71f) and (72e) under conditions (50), (55) and (66), it can be verified that:

$$2\vec{n}_{22} - n_{21} + \max(n_{21}, \overleftarrow{n}_{11}) > 2\vec{n}_{22}, \quad (74)$$

when $\overleftarrow{n}_{11} > n_{21}$. However, the improvement of the bound on $R_1 + 2R_2$ due to the use of feedback in transmitter-receiver 1, does not represent an effective improvement of the capacity region $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ with respect to $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$. This is because when $\overleftarrow{n}_{11} > n_{21}$, it follows from (66) that $\overleftarrow{n}_{11} > \max(\overrightarrow{n}_{11}, n_{12})$. Nonetheless, for all $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}) \in \mathbb{N}^4$, $\overleftarrow{n}_{22} = 0$ and $\overleftarrow{n}_{11} > \max(\overrightarrow{n}_{11}, n_{12})$, it follows that $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \max(\overrightarrow{n}_{11}, n_{12}), 0) = \mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$.

All the other inequalities in (71) and (72) are identical. Therefore, for all $\overleftarrow{n}_{11} \in \mathbb{N}$, the capacity region cannot be enlarged under conditions (50), (55) and (66).

Case 3

By plugging (51) into (56), it follows that:

$$\begin{aligned} n_{12} &> \overrightarrow{n}_{22} - n_{21} \\ &> 2n_{21} - n_{21} \\ &= n_{21}, \end{aligned} \tag{75}$$

and

$$\overrightarrow{n}_{11} > n_{12} + n_{21}, \tag{76}$$

which implies that Case 3 is always possible. Note that from (76), it follows that:

$$\overrightarrow{n}_{11} > n_{21}. \tag{77}$$

Plugging (56) and (77) into (52) yields:

$$R_1 \leq \overrightarrow{n}_{11}, \tag{78a}$$

$$R_2 \leq \overrightarrow{n}_{22} \tag{78b}$$

$$R_1 + R_2 \leq \min(\max(\overrightarrow{n}_{22}, n_{12}) + \overrightarrow{n}_{11} - n_{12}, \overrightarrow{n}_{11} + \overrightarrow{n}_{22} - n_{21}), \tag{78c}$$

$$R_1 + R_2 \leq \max(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}) + n_{12}, \tag{78d}$$

$$2R_1 + R_2 \leq 2\overrightarrow{n}_{11}, \tag{78e}$$

$$R_1 + 2R_2 \leq \max(\overrightarrow{n}_{22}, n_{12}) + \overrightarrow{n}_{22} - n_{21} + \max(\overrightarrow{n}_{11} - n_{12}, \overleftarrow{n}_{11}). \tag{78f}$$

Plugging (56) and (77) into (53) yields:

$$R_1 \leq \overrightarrow{n}_{11}, \tag{79a}$$

$$R_2 \leq \overrightarrow{n}_{22}, \tag{79b}$$

$$R_1 + R_2 \leq \overrightarrow{n}_{11}, \tag{79c}$$

$$2R_1 + R_2 \leq 2\overrightarrow{n}_{11}, \tag{79d}$$

$$R_1 + 2R_2 \leq \max(\overrightarrow{n}_{22}, n_{12}) + \overrightarrow{n}_{22} - n_{21} + \overrightarrow{n}_{11} - n_{12}. \tag{79e}$$

To simplify the inequalities containing the operator $\max(\cdot, \cdot)$ in (78) and (79), the following 2 cases are identified:

$$\text{Case 3a : } \overrightarrow{n}_{22} > n_{12}; \text{ and} \tag{80}$$

$$\text{Case 3b : } \overrightarrow{n}_{22} \leq n_{12}. \tag{81}$$

Case 3a

Plugging (80) into (78) yields:

$$R_1 \leq \vec{n}_{11}, \quad (82a)$$

$$R_2 \leq \vec{n}_{22} \quad (82b)$$

$$R_1 + R_2 \leq \vec{n}_{22} + \vec{n}_{11} - n_{12}, \quad (82c)$$

$$R_1 + R_2 \leq \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}) + n_{12}, \quad (82d)$$

$$2R_1 + R_2 \leq 2\vec{n}_{11}, \quad (82e)$$

$$R_1 + 2R_2 \leq 2\vec{n}_{22} - n_{21} + \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}). \quad (82f)$$

Plugging (80) into (79) yields:

$$R_1 \leq \vec{n}_{11}, \quad (83a)$$

$$R_2 \leq \vec{n}_{22}, \quad (83b)$$

$$R_1 + R_2 \leq \vec{n}_{11}, \quad (83c)$$

$$2R_1 + R_2 \leq 2\vec{n}_{11}, \quad (83d)$$

$$R_1 + 2R_2 \leq 2\vec{n}_{22} - n_{21} + \vec{n}_{11} - n_{12}. \quad (83e)$$

Comparing inequalities (82c) and (82d) with inequality (83c) under conditions (50), (56) and (80), it can be verified that:

$$\min(\vec{n}_{22} + \vec{n}_{11} - n_{12}, \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}) + n_{12}) > \vec{n}_{11}, \quad (84)$$

when $\overleftarrow{n}_{11} > \vec{n}_{11} - n_{12}$.

Comparing inequalities (82f) and (83e) under conditions (50), (56) and (80), it can be verified that:

$$2\vec{n}_{22} - n_{21} + \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}) > 2\vec{n}_{22} - n_{21} + \vec{n}_{11} - n_{12}, \quad (85)$$

when $\overleftarrow{n}_{11} > \vec{n}_{11} - n_{12}$.

All the other inequalities in (82) and (83) are identical. Therefore, $\overleftarrow{n}_{11}^* = \vec{n}_{11} - n_{12}$ under conditions (50), (56) and (80).

Case 3b

Plugging (81) into (78) yields:

$$R_1 \leq \vec{n}_{11}, \quad (86a)$$

$$R_2 \leq \vec{n}_{22} \quad (86b)$$

$$R_1 + R_2 \leq \vec{n}_{11}, \quad (86c)$$

$$R_1 + R_2 \leq \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}) + n_{12}, \quad (86d)$$

$$2R_1 + R_2 \leq 2\vec{n}_{11}, \quad (86e)$$

$$R_1 + 2R_2 \leq n_{12} + \vec{n}_{22} - n_{21} + \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}). \quad (86f)$$

Plugging (80) into (79) yields:

$$R_1 \leq \vec{n}_{11}, \quad (87a)$$

$$R_2 \leq \vec{n}_{22}, \quad (87b)$$

$$R_1 + R_2 \leq \vec{n}_{11}, \quad (87c)$$

$$2R_1 + R_2 \leq 2\vec{n}_{11}, \quad (87d)$$

$$R_1 + 2R_2 \leq \vec{n}_{22} - n_{21} + \vec{n}_{11}. \quad (87e)$$

Comparing inequalities (86c) and (86d) with inequality (87c) under conditions (50), (56) and (81), it can be verified that:

$$\min(\vec{n}_{11}, \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}) + n_{12}) = \vec{n}_{11}, \quad (88)$$

for all $\overleftarrow{n}_{11} \in \mathbb{N}$.

Comparing inequalities (86f) and (87e) under conditions (50), (56) and (81), it can be verified that:

$$n_{12} + \vec{n}_{22} - n_{21} + \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}) > \vec{n}_{22} - n_{21} + \vec{n}_{11}, \quad (89)$$

when $\overleftarrow{n}_{11} > \vec{n}_{11} - n_{12}$.

All the other inequalities in (86) and (87) are identical. Therefore, $\overleftarrow{n}_{11}^* = \vec{n}_{11} - n_{12}$ under conditions (50), (56) and (81).

Case 4

By plugging (51) into (57), it follows that:

$$\vec{n}_{22} > n_{12} + n_{21}, \quad (90)$$

and

$$\vec{n}_{11} > n_{12} + n_{21}, \quad (91)$$

which implies that Case 4 is always possible. Note that from (90) and (91), it follows that:

$$\vec{n}_{22} > n_{12} \quad (92)$$

and

$$\vec{n}_{11} > n_{21}, \quad (93)$$

respectively. Plugging (57), (92) and (93) into (52) yields:

$$R_1 \leq \vec{n}_{11}, \quad (94a)$$

$$R_2 \leq \vec{n}_{22} \quad (94b)$$

$$R_1 + R_2 \leq \min(\vec{n}_{22} + \vec{n}_{11} - n_{12}, \vec{n}_{11} + \vec{n}_{22} - n_{21}), \quad (94c)$$

$$R_1 + R_2 \leq \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}) + \vec{n}_{22} - n_{21}, \quad (94d)$$

$$2R_1 + R_2 \leq 2\vec{n}_{11} - n_{12} + \vec{n}_{22} - n_{21}, \quad (94e)$$

$$R_1 + 2R_2 \leq 2\vec{n}_{22} - n_{21} + \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}). \quad (94f)$$

Plugging (57), (92) and (93) into (53) yields:

$$R_1 \leq \vec{n}_{11}, \quad (95a)$$

$$R_2 \leq \vec{n}_{22}, \quad (95b)$$

$$R_1 + R_2 \leq \vec{n}_{11} - n_{12} + \vec{n}_{22} - n_{21}, \quad (95c)$$

$$2R_1 + R_2 \leq 2\vec{n}_{11} - n_{12} + \vec{n}_{22} - n_{21}, \quad (95d)$$

$$R_1 + 2R_2 \leq 2\vec{n}_{22} - n_{21} + \vec{n}_{11} - n_{12}. \quad (95e)$$

Comparing inequalities (94c) and (94d) with inequality (95c) under conditions (50) and (57), it can be verified that:

$$\min(\min(\vec{n}_{22} + \vec{n}_{11} - n_{12}, \vec{n}_{11} + \vec{n}_{22} - n_{21}), \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}) + \vec{n}_{22} - n_{21}) > \overleftarrow{n}_{11} - n_{12} + \vec{n}_{22} - n_{21}, \quad (96)$$

when $\overleftarrow{n}_{11} > \vec{n}_{11} - n_{12}$.

Comparing inequalities (94f) and (95e) under conditions (50) and (57), it can be verified that:

$$2\vec{n}_{22} - n_{21} + \max(\vec{n}_{11} - n_{12}, \overleftarrow{n}_{11}) > 2\vec{n}_{22} - n_{21} + \vec{n}_{11} - n_{12}, \quad (97)$$

when $\overleftarrow{n}_{11} > \vec{n}_{11} - n_{12}$. All the other inequalities in (94) and (95) are identical. Therefore, $\overleftarrow{n}_{11}^* = \vec{n}_{11} - n_{12}$ under conditions (50) and (57).

From all the observations above, when both transmitter-receiver pairs are in VWIR (event E_1 is True), it follows that for all $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^*$ and $\vec{n}_{11} > n_{21}$ (event $E_{13,1}$ is True), $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) \subset \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$; otherwise $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0) = \mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ with

$$\overleftarrow{n}_{11}^* = \max(\vec{n}_{11} - n_{12}, n_{21}). \quad (98)$$

Note that when events E_1 and $E_{13,1}$ hold simultaneously true, then the event $S_{1,1}$ is true, which verifies the statement of Theorem 1. The same procedure can be applied for all the other combinations of interference regimes. This completes the proof. ■

B Proof of Theorem 2: Improvement of the Individual Rate R_1 by Using Feedback in Link 1

For fix parameters \vec{n}_{11} , \vec{n}_{22} , n_{12} , n_{21} and $\overleftarrow{n}_{22} = 0$, this appendix presents the procedure to obtain the threshold, denoted by $\overleftarrow{n}_{11}^\dagger$, on the feedback parameter \overleftarrow{n}_{11} beyond which the individual rate R_1 is improved, i.e., $\Delta_1(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0) > 0$, when both transmitter-receiver pairs are in VWIR. The proof of Theorem 2 is completed by performing the above mentioned procedure for all cases of possible combinations of interference regimes. This procedure is tedious and repetitive, and thus, in this appendix only one combination of interference regimes is fully studied, e.g., VWIR - VWIR.

Proof:

Consider that both transmitter-receiver pairs are in VWIR, i.e., conditions (50) hold. Under these conditions, the capacity regions $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ and $\mathcal{C}(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, 0, 0)$ are given by (52) and (53), respectively. Note that the bounds on R_1 are identical in both capacity regions. However, an improvement on the sum-rate bound $R_1 + R_2$ or the weighted sum-rate bound $R_1 + 2R_2$ implies an improvement of the individual rate R_1 , i.e., $\Delta_1(\vec{n}_{11}, \vec{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0) > 0$. Note also that the weighted sum-rate bound $2R_1 + R_2$ is identical in both capacity regions, thus, it does not have any effect on the improvement on the individual rate R_1 .

From the analysis presented in Appendix A, it follows that:

- (a) The bounds on $R_1 + R_2$ and $R_1 + 2R_2$ can be improved by using feedback in transmitter-receiver 1 if $\overleftarrow{n}_{11} > n_{21}$ under conditions (50), (55) and (65).
- (b) The bounds on $R_1 + R_2$ and $R_1 + 2R_2$ cannot be improved for all $\overleftarrow{n}_{11} \in \mathbb{N}$ under conditions (50), (55) and (66).
- (c) The bounds on $R_1 + R_2$ and $R_1 + 2R_2$ can be improved by using feedback in transmitter-receiver 1 if $\overleftarrow{n}_{11} > \vec{n}_{11} - n_{12}$ under conditions (50), (56) and (80).

(d) The bound on $R_1 + 2R_2$ can be improved by using feedback in transmitter-receiver 1 if $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$ under conditions (50), (56) and (81).

(e) The bounds on $R_1 + R_2$ and $R_1 + 2R_2$ can be improved by using feedback in transmitter-receiver 1 if $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$ under conditions (50) and (57).

From all the observations above, when both transmitter-receiver pairs are in VWIR (event E_1 is True), it follows that for all $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^\dagger$ and $\overrightarrow{n}_{11} > n_{21}$ (event $E_{13,1}$ is True), $\Delta_1(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0) > 0$; otherwise $\Delta_1(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0) = 0$ with

$$\overleftarrow{n}_{11}^\dagger = \max(\overrightarrow{n}_{11} - n_{12}, n_{21}). \quad (99)$$

Note that when events E_1 and $E_{13,1}$ hold simultaneously true, then the event $S_{1,1}$ is true, which verifies the statement of Theorem 2. The same procedure can be applied for all the other combinations of interference regimes. This completes the proof. \blacksquare

C Proof of Theorem 3: Improvement of the Individual Rate R_2 by Using Feedback in Link 1

The proof of this theorem follows along the same lines of the proof on Theorem 2 in Appendix B. The difference is that the analysis is focused on the individual rate R_2 . In particular, when both transmitter-receiver pairs are in VWIR (event E_1 is True), it follows that for all $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^\ddagger$ and $\overrightarrow{n}_{11} > n_{21}$ (event $E_{13,1}$ is True), $\Delta_2(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0) > 0$; otherwise $\Delta_2(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0) = 0$ with

$$\overleftarrow{n}_{11}^\ddagger = \max(\overrightarrow{n}_{11} - n_{12}, n_{21}). \quad (100)$$

Note that when events E_1 and $E_{13,1}$ hold simultaneously true, then the event $S_{1,1}$ is true, which verifies the statement of Theorem 3.

D Proof of Theorem 4: Improvement of the Sum-Rate Capacity by Using Feedback in one Transmitter-Receiver Pair

For fix parameters \overrightarrow{n}_{11} , \overrightarrow{n}_{22} , n_{12} , n_{21} and $\overleftarrow{n}_{22} = 0$, this appendix presents the procedure to obtain the threshold, denoted by $\overleftarrow{n}_{11}^\ddagger$, on the feedback parameter \overleftarrow{n}_{11} beyond which the sum-rate capacity $R_1 + R_2$ is improved, i.e., $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0) > 0$, when both transmitter-receiver pairs are in VWIR. The proof of Theorem 4 is completed by performing the above mentioned procedure for all cases of possible combinations of interference regimes. This procedure is tedious and repetitive, and thus, in this appendix only one combination of interference regimes is fully studied, e.g., VWIR - VWIR.

Proof:

Consider that both transmitter-receiver pairs are in VWIR, i.e., conditions (50) hold. Under these conditions, the capacity regions $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0)$ and $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, 0, 0)$ are given by (52) and (53), respectively.

From the analysis presented in Appendix A, it follows that:

(a) The bound on $R_1 + R_2$ can be improved by using feedback in transmitter-receiver 1 if $\overleftarrow{n}_{11} > n_{21}$ under conditions (50), (55) and (65).

(b) The bound on $R_1 + R_2$ cannot be improved for all $\overleftarrow{n}_{11} \in \mathbb{N}$ under conditions (50), (55) and (66).

- (c) The bound on $R_1 + R_2$ can be improved by using feedback in transmitter-receiver 1 if $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$ under conditions (50), (56) and (80).
- (d) The bound on $R_1 + R_2$ cannot be improved for all $\overleftarrow{n}_{11} \in \mathbb{N}$ under conditions (50), (56) and (81).
- (e) The bound on $R_1 + R_2$ can be improved by using feedback in transmitter-receiver 1 if $\overleftarrow{n}_{11} > \overrightarrow{n}_{11} - n_{12}$ under conditions (50) and (57).

From all the observations above, when both transmitter-receiver pairs are in VWIR (event E_9 is True), it follows that for all $\overleftarrow{n}_{11} > \overleftarrow{n}_{11}^+$, $\overrightarrow{n}_{11} > n_{21}$ (event $E_{13,1}$ is True) and $\overrightarrow{n}_{22} > n_{12}$ (event $E_{13,2}$ is True), $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0) > 0$; otherwise $\Sigma(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, 0) = 0$ with

$$\overleftarrow{n}_{11}^+ = \max(\overrightarrow{n}_{11} - n_{12}, n_{21}). \quad (101)$$

Note that when events E_9 , $E_{13,1}$ and $E_{13,2}$ hold simultaneously true, then the event $S_{4,1}$ is true, which verifies the statement of Theorem 4. The same procedure can be applied for all the other combinations of interference regimes. This completes the proof. \blacksquare

E Proof of Theorem 5: Generalized Degrees of Freedom

This appendix provides a proof to Theorem 5 for the two user LD-IC-NOF.

Proof:

The following are equivalent expressions for the bounds on the single rate R_1 , on the sum rate $R_1 + R_2$, and the weighted sum-rate $2R_1 + R_2$ from Lemma 1 under symmetric conditions i.e., $\overrightarrow{n} = \overrightarrow{n}_{11} = \overrightarrow{n}_{22}$, $m = n_{12} = n_{21}$ and $\overleftarrow{n} = \overleftarrow{n}_{11} = \overleftarrow{n}_{22}$. From (7a) and (7b) with $i = 1$ and $j = 2$, it follows that:

$$R_1 \leq \min\left(\max(\overrightarrow{n}, m), \max\left(\overrightarrow{n}, \overleftarrow{n} - (\overrightarrow{n} - m)^+\right)\right) \triangleq a_1. \quad (102)$$

From (7c) and (7d), it follows that:

$$R_1 + R_2 \leq \min\left(\max(\overrightarrow{n}, m) + (\overrightarrow{n} - m)^+, 2 \max\left((\overrightarrow{n} - m)^+, m, \overrightarrow{n} - (\max(\overrightarrow{n}, m) - \overleftarrow{n})^+\right)\right) \triangleq a_2. \quad (103)$$

From (7e) with $i = 1$ and $j = 2$, it follows that:

$$2R_1 + R_2 \leq \max(\overrightarrow{n}, m) + (\overrightarrow{n} - m)^+ + \max\left((\overrightarrow{n} - m)^+, m, \overrightarrow{n} - (\max(\overrightarrow{n}, m) - \overleftarrow{n})^+\right) \triangleq a_3. \quad (104)$$

The symmetric sum-capacity, $C_{\text{sym}}(\overrightarrow{n}, m, \overleftarrow{n}) = \sup\{R : (R, R) \in \mathcal{C}(\overrightarrow{n}, \overrightarrow{n}, m, m, \overleftarrow{n}, \overleftarrow{n})\}$, can

be obtained from (102), (103) and (104) as follows

$$\begin{aligned}
C_{\text{sym}} &= \min \left(a_1, \frac{a_2}{2}, \frac{a_3}{2}, \frac{a_4}{3} \right) \\
&= \min \left(\min \left(\max(\bar{n}, m), \max(\bar{n}, \bar{h} - (\bar{n} - m)^+) \right), \frac{1}{2} \min \left(\max(\bar{n}, m) + (\bar{n} - m)^+, \right. \right. \\
&\quad \left. \left. 2 \max \left((\bar{n} - m)^+, m, \bar{n} - (\max(\bar{n}, m) - \bar{h})^+ \right) \right), \frac{1}{3} \left(\max(\bar{n}, m) + (\bar{n} - m)^+ \right. \right. \\
&\quad \left. \left. + \max \left((\bar{n} - m)^+, m, \bar{n} - (\max(\bar{n}, m) - \bar{h})^+ \right) \right) \right), \\
&= \min \left(\max(\bar{n}, m), \max(\bar{n}, \bar{h} - (\bar{n} - m)^+), \frac{1}{2} \left(\max(\bar{n}, m) + (\bar{n} - m)^+ \right), \right. \\
&\quad \max \left((\bar{n} - m)^+, m, \bar{n} - (\max(\bar{n}, m) - \bar{h})^+ \right), \frac{1}{3} \left(\max(\bar{n}, m) + (\bar{n} - m)^+ \right. \\
&\quad \left. \left. \max \left((\bar{n} - m)^+, m, \bar{n} - (\max(\bar{n}, m) - \bar{h})^+ \right) \right) \right). \tag{105}
\end{aligned}$$

Plugging (105) into (46) yields

$$\begin{aligned}
D_{\text{sym}}(\alpha, \beta) &= \min \left(\max(1, \alpha), \max(1, \beta - (1 - \alpha)^+), \frac{1}{2} \left(\max(1, \alpha) + (1 - \alpha)^+ \right), \right. \\
&\quad \max \left((1 - \alpha)^+, \alpha, 1 - (\max(1, \alpha) - \beta)^+ \right), \frac{1}{3} \left(\max(1, \alpha) + (1 - \alpha)^+ \right. \\
&\quad \left. \left. + \max \left((1 - \alpha)^+, \alpha, 1 - (\max(1, \alpha) - \beta)^+ \right) \right) \right). \tag{106}
\end{aligned}$$

where $\alpha = \frac{n}{\bar{n}}$ and $\beta = \frac{\bar{h}}{\bar{n}}$ and this completes the proof. ■

References

- [1] C. Suh and D. N. C. Tse, "Feedback capacity of the Gaussian interference channel to within 2 bits," *IEEE Transactions on Information Theory*, vol. 57, no. 5, pp. 2667–2685, May. 2011.
- [2] S. M. Perlaza, R. Tandon, H. V. Poor, and Z. Han, "Perfect output feedback in the two-user decentralized interference channel," *IEEE Transactions on Information Theory*, vol. 61, no. 10, pp. 5441–5462, Oct. 2015.
- [3] S. A. Jafar, "Interference alignment: A new look at signal dimensions in a communication network," *Foundations and Trends in Communications and Information Theory*, vol. 7, no. 1, pp. 1–134, 2010.
- [4] R. Tandon, S. Mohajer, and H. V. Poor, "On the symmetric feedback capacity of the K-user cyclic Z-interference channel," *IEEE Transactions on Information Theory*, vol. 59, no. 5, pp. 2713–2734, May. 2013.
- [5] S. Mohajer, R. Tandon, and H. V. Poor, "On the feedback capacity of the fully connected-user interference channel," *IEEE Transactions on Information Theory*, vol. 59, no. 5, pp. 2863–2881, May. 2013.
- [6] S.-Q. Le, R. Tandon, M. Motani, and H. V. Poor, "Approximate capacity region for the symmetric Gaussian interference channel with noisy feedback," *IEEE Transactions on Information Theory*, vol. 61, no. 7, pp. 3737–3762, Jul. 2015.
- [7] V. Quintero, S. M. Perlaza, I. Esnaola, and J.-M. Gorce, "Approximated capacity of the two-user Gaussian interference channel with noisy channel-output feedback," INRIA Grenoble - Rhône-Alpes, Lyon, France, Tech. Rep. 8861, Mar. 2016.
- [8] S. Avestimehr, S. Diggavi, and D. N. C. Tse, "Wireless network information flow: A deterministic approach," *IEEE Transactions on Information Theory*, vol. 57, no. 4, pp. 1872–1905, Apr. 2011.
- [9] V. Quintero, S. M. Perlaza, and J.-M. Gorce, "Noisy channel-output feedback capacity of the linear deterministic interference channel," in *IEEE Information Theory Workshop*, Jeju Island, Korea, Oct. 2015.
- [10] —, "Noisy channel-output feedback capacity of the linear deterministic interference channel," INRIA Grenoble - Rhône-Alpes, Tech. Rep. 456, Jan. 2015.
- [11] R. H. Etkin, D. N. C. Tse, and W. Hua, "Gaussian interference channel capacity to within one bit," *IEEE Transactions on Information Theory*, vol. 54, no. 12, pp. 5534–5562, Dec. 2008.
- [12] C. E. Shannon, "Channels with side information at the transmitter," *IBM Journal of Research and Development*, vol. 2, no. 4, pp. 289–293, Oct. 1958.
- [13] G. Bresler and D. N. C. Tse, "The two user Gaussian interference channel: A deterministic view," *European Transactions on Telecommunications*, vol. 19, no. 4, pp. 333–354, Apr. 2008.



**RESEARCH CENTRE
GRENOBLE – RHÔNE-ALPES**

Inovallée
655 avenue de l'Europe Montbonnot
38334 Saint Ismier Cedex

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
inria.fr

ISSN 0249-0803