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# A marvelous contribution from Michel Hénon to globular cluster’s study : the isochrone cluster

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## Sommaire

I The isochrone model .....	7
II A modern lecture of the isochrone model .....	10

### **Bibliographie .....** 12

Globular clusters are cornerstones in the study of the dynamics of gravitationally interacting systems of particles.

As a matter of fact, they are made of thousands of stars – at minimum – and, in the contrary of galaxies, they do not contain gas or other dissipative components. Hence, excepting the fact that stars are not punctual, they corresponds exactly to their mathematical modelisation.

The observation of globular clusters reveals that they are characterized by a spherical distribution of their mass.

By consequence, their volumic mass density is a radial function  $\rho = \rho(r)$ . This density corresponds to the marginal distribution in position space of the total distribution function in the whole phase space : this is a mean field description. In this model the gravitational potential  $\psi$  of the cluster is solution of the Poisson equation  $\Delta\psi = 4\pi G\rho$ , it is then radial too :  $\psi = \psi(r)$ . The  $r$  variable represents the distance between the center of mass of the system and a test star evolving in the mean field potential of the cluster. This test star of mass  $m$  experiences a force given by  $F = -m\nabla\psi$ , which is already radial. In this context, it is very well known that its trajectory is contained in a plane. In this plane, the orbit is determined by the total energy  $E$  and the squared angular momentum  $L^2$  of the star. These two quantities are conserved during the motion.

The description of the whole dynamics of a globular cluster is then possible when its gravitational potential  $\psi$  is given. Three possibilities exists to do this :

- Extract  $\rho(r)$  and  $\psi(r)$  from observational data. It is a rough and empirical manner, but it is necessary to fix ideas.
- Compute  $\rho(r)$  and  $\psi(r)$  from numerical simulations. But the parameter’s space of the result is wide for this kind of experiments.
- Propose fundamental physical arguments in order to select a model through all theoretical possibilities.

The two first ways are very used and a plenty of references give refined to rough descriptions of globular clusters since their formation until the end of their actual or numerical evolution (see [1]). The last way is by far the least used. Michel Hénon’s paper [2] about isochrone cluster, published in french in "Annales d’Astrophysiques" is very representative of this way to do which count no more than a ten of reference in more than two centuries of globular cluster modelisation ! In this paper, we will follow Michel Hénon’s paper in a first part to analyze in a second part its influence under gravitational dynamicists community, giving finally a modern reading of the result.

### I. The isochrone model

As we mention before, the motion of a given star in a spherical globular cluster is contained in a plane. In this plane the two parameters of the orbit are its energy  $E$  and its squared angular momentum  $L^2$ . Both these two parameters contribute to the definition of the gravitational potential of the cluster  $\psi(r)$  and to the computation of the distance  $r$  between the star and the center of mass of the cluster. This contribution is resumed in the definition of the energy of the star

$$(1) \quad E = \frac{m}{2} \left( \frac{dr}{dt} \right)^2 + \frac{L^2}{2mr^2} + m\psi(r) = \text{cste}$$

Imposing that the total mass of the system<sup>1</sup> is finite, the effective potential

$$\psi_e(r) = \frac{L^2}{2mr^2} + m\psi(r)$$

is such that  $\lim_{r \rightarrow 0} \psi_e(r) = +\infty$ .

At the edge of the system, up to an additive constant, the potential is on the form  $\psi(r)_{r \rightarrow +\infty} \sim -GM/r$ , where  $M$  the total mass of the cluster. Hence,  $\lim_{r \rightarrow +\infty} \psi_e(r) = 0^-$ .

This effective potential is also the place where  $\frac{dr}{dt} = 0$ , its behavior is represented on figure 1.

When the considered star belongs to the cluster its total energy  $E$  is negative. The extreme value  $E = E_c < 0$  corresponds to an orbit for which  $r = r_c = \text{cste}$ , it is then circular. For each value of  $E \in ]E_c, 0[$ , the distance is such that  $r \in [r_p, r_a]$ . The extreme values of the radius are the periastron for  $r_p$  and the apoastron for  $r_a$ . The time for the transfer from  $r_p$  to  $r_a$  is given by

$$(2) \quad \frac{\tau}{2} = \int_{t_a}^{t_p} dt = \int_{r_a}^{r_p} \frac{dt}{dr} dr = \int_{r_a}^{r_p} \frac{dr}{\sqrt{2 \left[ \frac{E}{m} - \psi(r) \right] - \frac{L^2}{m^2 r^2}}}$$

where we have used the energy to get  $\frac{dr}{dt}$ . The problem being symmetrical, if this integral is finite, we have  $r_a = r(t_a) = r(t_a + \tau)$ . The expression (1) of the energy is in fact an ordinary differential equation, fulfilled by both  $r(t)$  and  $r(t + \tau)$ . These two functions are then equal when  $t = t_a$ , the Cauchy criterion – which is supposed valid for this problem – then tell us that they are equal at each time : the distance between the center of mass of the cluster and the star is then a  $\tau$ -periodic function. The inspection of the relation (2) shows that, if it exists, the radial period is such that  $\tau = \tau(E, L^2)$ . The mass of the star is another parameter of the period which is not under interest in this study. The radial period is computable at least in two fundamental cases :

1. The total mass  $M$  of a spherical self-gravitating system is given by the integral of the density over the whole space. In spherical coordinates, using the Poisson equation, this gives

$$M = \frac{1}{G} \left[ \lim_{r \rightarrow \infty} \left( r^2 \frac{d\psi}{dr} \right) - \lim_{r \rightarrow 0} \left( r^2 \frac{d\psi}{dr} \right) \right]$$

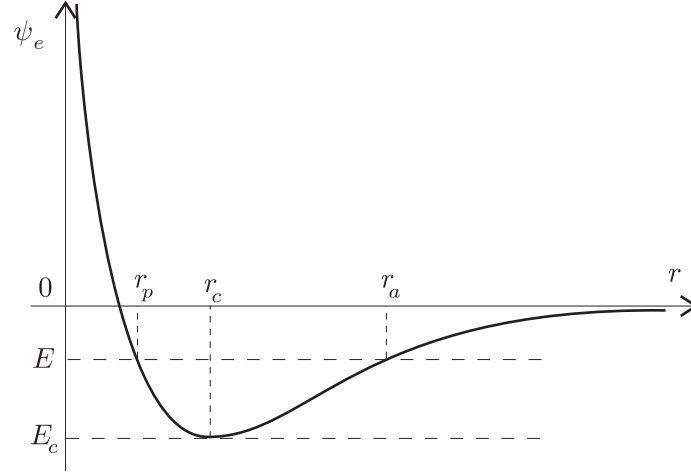


FIGURE 1. Effective potential of a spherical cluster

- The Kepler potential  $\psi(r) = -GM/r$ , where  $M$  is the total mass of the cluster. It is the well-known two body problem for which Kepler’s third law gives  $\tau = \frac{\pi GM}{\sqrt{2}} (-E)^{-3/2}$ . It is also the limit potential viewed by a star of the cluster always evolving in far from center regions.
- The harmonic potential  $\psi(r) = \frac{2}{3}\pi G\rho r^2$ , where  $\rho$  is the constant density of a homogeneous cluster, for which  $\tau = \sqrt{\frac{3\pi}{4G\rho}}$ . Using the virial theorem, one can show that  $\rho$  depends only of the energy of the considered star and the total mass of this homogeneous system. The central regions of a globular clusters are usually considered as homogenous ones, the gravitational potential is then harmonic at the center of such systems. A centrally confined star could then have a radial period depending only of its energy  $E$ .

Both centrally or far from center confined orbits are then characterized by radial periods which do not depends on the squared angular momentum. In his 1958 paper, Michel Hénon conjectures that this property could propagates to all orbits of all stars of a globular system. This so called "isochrone" property, could be a fundamental physical property allowing to determine the other properties of globular clusters.

Thus, Michel Hénon proposes to find the most general potential  $\psi_i(r)$  such that

$$\tau = 2 \int_{r_a}^{r_p} \frac{dr}{\sqrt{2 \left[ \frac{E}{m} - \psi_i(r) \right] - \frac{L^2}{m^2 r^2}}} = \tau(E)$$

To do this he doesn’t use the effective potential but introduces new variables

$$x = 2r^2 \text{ and } y(x) = x\psi_i(x).$$

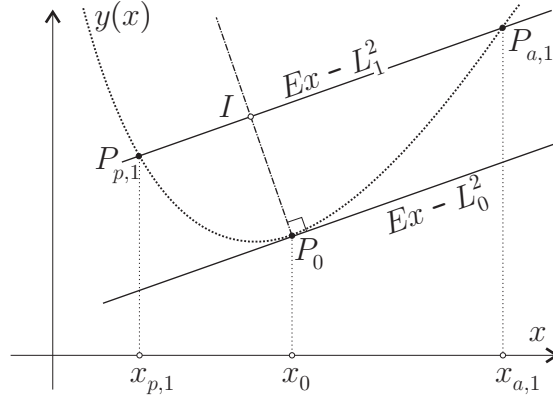


FIGURE 2. Illustration from Michel Hénon paper about the characteristic points and  $y(x)$  function.

Fixing  $m = 1$ , the radial period then writes

$$\tau = \int_{x_a}^{x_p} \frac{dx}{\sqrt{Ex - L^2 - y(x)}}$$

$x_p$  and  $x_a$  being the values of  $x$  to the periastron and apoastron, roots of the equation  $y(x) = Ex - L^2$ . Due to the finite mass of the system,  $y(x)$  must have infinite branches. In order to identify  $x_p$  and  $x_a$ , Michel Hénon then propose a graphical study of the problem. This representation is an alternative of the effective potential theory, it is full of sense and forms a new way for the study of such systems. This graphical study is illustrated in the figure 2.

By an explicit calculus of the radial period in terms of  $x$  and  $y$ , Michel Hénon shows that  $\tau$  depends only on the energy  $E$  if and only if

$$P_0 I \propto (P_{p,1} P_{a,1})^2$$

This condition is fulfilled if and only if  $y(x)$  is a parabola.

The physical parameters of the problem are

$$\psi_0 = \lim_{r \rightarrow 0} \psi(r), \quad \psi_\infty = \lim_{r \rightarrow +\infty} \psi(r) \text{ and the total mass } M.$$

Three cases can be considered :

- If  $\psi_0 \rightarrow -\infty$ , it is always possible to choose a reference for the potential such that  $\psi_\infty = 0$ . The equation for  $y(x)$  is then

$$y(x) = -GM\sqrt{2x} \implies \psi_i(r) = -\frac{GM}{r}$$

It is the keplerian potential associated to a central mass.

- If  $\psi_0 \rightarrow -\infty$ , it is always possible to choose a reference for the potential such that  $\psi_0 = 0$ . The equation for  $y(x)$  is now

$$y(x) = \frac{1}{2}Kx^2 \implies \psi_i(r) = Kr^2$$

It is the harmonic potential.

- If both  $\psi_0$  and  $\psi_\infty$  are finite, choosing a reference for the potential such that  $\psi_\infty = 0$ , the equation for  $y(x)$  becomes

$$y^2 + \frac{G^2 M^2}{2\psi_0} y - 2G^2 M^2 x = 0 \implies \psi_i(r) = \frac{2\psi_0}{1 + \sqrt{1 + \frac{r^2}{b^2}}} \text{ with } b = -\frac{GM}{2\psi_0} > 0$$

this is the general form introduced by Michel Hénon for his potential, the isochrone potential.

The paper ends by a comparison between this new potential and the globular clusters data observation of this epoch. Michel Hénon finds a good agreement and conjectures a physical explanation : a resonance between orbits during the formation process of globular clusters could vanishes the  $L^2$  dependence of the radial period. The history will forget progressively the isochrone model mainly because more and more observations will reveal that the potential density pair is trickier than a simple stationary model. In 1968 the *empirical* King model, with 3 free parameter, became the reference. Does the isochrone model, with only the parameter  $b$ , could be forgotten ? It should be a big error...

## II. A modern lecture of the isochrone model

Isochrone cluster are characterized by a central structure of constant density, namely a core, surrounded by a halo. The size of the core is given by the parameter  $b$  of the model. The radial period of the isochrone model can be explicitly computed, it is given by the relation

$$\tau = \frac{2\pi GM}{(-2E)^{3/2}}$$

Using Poisson equation one can give an explicit formula for the mass density

$$\rho(r) = \frac{M}{4\pi} \frac{3ab(a+b) - br^2}{a^3(a+b)^3} \text{ with } a^2 = r^2 - b^2$$

When  $r \gg b$ , i.e. in the halo, the mass density is given by a power law  $\rho(r) \propto r^{-4}$ . This property correspond to young globular clusters, i.e. characterized by a large two body encounters relaxation time ( $T_c$ ) in comparison of the Hubble time.

In a numerical way, the equilibrium state with a core and a  $-4$  halo slope structure corresponds to the post collapse state of an initial homogeneous sphere. These kind of systems are generally called "Hénon Spheres" since the pioneering numerical experiments of Michel Hénon in the sixties [3]. They could correspond to one of the globular clusters formation process. After this formation process which takes a few dynamical times ( $T_d$ ), such a system evolves under collisional effects under very more longer times, order of them is given by the two body encounters relaxation time ( $T_c \simeq \frac{N}{\ln N} T_d$ , where  $N$  is the number of stars in the system). During this slow evolution the extension of the core shrinks progressively and the slope of the halo passes from  $-4$  to  $-2$ . At the end of this evolution, the system become unstable and the core collapses under the pressure of the halo.

On the century of globular clusters orbiting in our galaxy, 80% have a core-halo structure with slope in the interval  $[-4, -2]$  and 20% are said core-collapsed. The fine mechanisms of this evolution begins to be well understood, but, the question of the initial equilibrium state at the beginning of this evolution is always an open problem. In this context, the Michel Hénon’s isochrone idea seems to be one of the possible physical explanations.

Very few research are done in order to verify Michel Hénon’s conjecture about the resonant mechanism as the origin of the isochrone model. A confirmation of the existence of such a mechanism could be an important and elegant fact in the comprehension of globular clusters, in the direct line of Michel Hénon works.



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