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Lyapunov stability analysis of switching controllers in presence of sliding modes and parametric uncertainties with application to pneumatic systems

Omar Ameer, Paolo Massioni, Gérard Scorletti, Xavier Brun and Mohamed Smaoui

Abstract—This paper concerns the control and the stability analysis of pneumatic actuators, which are nowadays of widespread use in the industry. A problem related to the use of such actuators is the so-called “stick-slip”, due to the presence of dry friction in the system. In this paper we provide an empirical switching control law which avoids this phenomenon, as well as a general approach to the stability analysis of nonlinear systems which will let us prove the stability of the closed-loop system. The approach is based on casting the closed-loop system into a piecewise-affine form and finding a Lyapunov function for it. Such an approach will be able to cope with the special features of the controlled pneumatic system model, namely the presence of sliding modes, a whole equilibrium set and uncertainties on the values of a few parameters.

At the end of the paper we will show how such a method can be successfully applied to our experimental setup under several different hypotheses.

Index Terms—Piecewise-affine systems, piecewise-polynomial Lyapunov functions, switched control, fluid power.

I. INTRODUCTION

Electropneumatic systems are widely used in industry, especially in the form of pneumatic actuators or cylinders. Their efficient use depends on good control laws, which are often non trivial to synthesize due to nonlinear dynamics. In particular, friction is a phenomenon which plays a key role in the dynamic behavior of pneumatic cylinders, as the presence of friction may lead to what is called “stick-slip” [20], [7]. Such a phenomenon typically occurs in the presence of dry friction combined with an integration effect (either in the control law or in the system itself), and it consists, in pneumatic cylinders, in a displacement of the rod a while after it has come to a rest; this is due to the fact that the force acting on the rod initially becomes smaller than the threshold which is necessary for a motion in presence of dry friction, but later on this threshold is overcome due to a slowly growing integrator. For example, Figure 1 features the record of an experiment on a pneumatic cylinder showing an occurrence of stick-slip. We can see that the rod, following the given reference, comes to a rest in the first half second; then it starts moving again, with no changes for the setpoint.

In this case, stick-slip is caused by the presence of dry friction and by the pressure dynamics in the chambers, which continue to evolve (integrating the net incoming mass flow from the servovalves) even after the rod has stopped.

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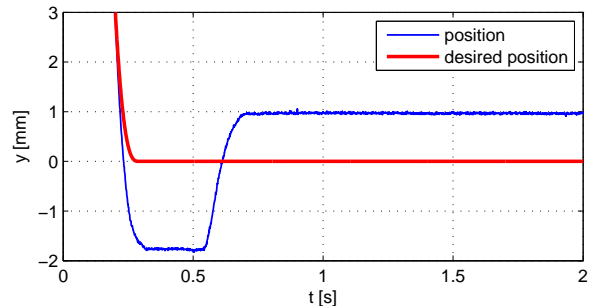


Fig. 1. Stick-slip phenomenon on a pneumatic cylinder.

To avoid this highly undesirable phenomenon, a control law has been proposed recently in [28]. This solution consists in a classic feedback linearization, which compensates for the nonlinearities in the pneumatic model, together with an appropriate switching law. The controller switches from a trajectory tracking control to a pressure control after the rod has come to a rest, offsetting the pressure difference between the two chambers and avoiding an uncontrolled evolution that eventually could make the rod restart. A major problem with this solution, which has been otherwise verified as very effective in the practice, was to find a formal proof of its stability; in fact, such a proof has not been given in the original reference [28].

The difficulty in finding a formal proof of stability lies in the pressure dynamics of the pneumatic chambers, and in the presence of dry friction, which make the electropneumatic actuator a strongly non-linear system. Typically, feedback linearization can be used to cancel the nonlinearities with a feedback control, with the “caveat” that an uncertainty in the model can lead to catastrophic effects. In this study we will analyze a control law, based on the one in [28], which cancels all the nonlinear effects through feedback linearization with the exception of the most difficult to model quantitatively, namely the friction. We will then approach the problem of proving the stability of the closed-loop system by casting it into a piecewise-affine (PWA) form [12], [23], [25], [22]. In order to do that, we will consider two different models for the dry friction; we will consider both a simplified model, where the friction force is a continuous function of the velocity of the rod (in a so-called “saturation” shape), as well as a more realistic, discontinuous model of friction. In fact, as indicated by many works [3], [11], [29], [9], more accurate models of friction feature discontinuities, which can lead to the appearance of sliding modes [14], [13] in the system dynamics.

In the literature, we can find several approaches for dealing with sliding modes in piecewise-affine systems [18], [12], [5], [16], [26] but unfortunately, none of these approaches are enough to deal with the stability of our pneumatic actuator systems. First of all, we have verified that searching for a piecewise-quadratic Lyapunov functions is too conservative and does not yield any valid solution. Secondly, these former methods are not able to cope with the presence of a whole equilibrium set (not only a point) in a sliding mode. In this work, we propose new sufficient conditions for stability based on piecewise-polynomial Lyapunov functions, which can be given in terms of linear matrix inequalities (LMIs) and linear matrix equalities (LMEs). These conditions are also able to cope with the uncertainties in the friction models, and we have used them successfully for proving the stability of the closed-loop system for all the possible models of friction.

Some of the results of this paper have been partially presented in [1], [2], which contain only two special cases of the main, more general result introduced here. This paper analyses all the possible cases and offers a unified point of view.

The remainder of this paper is organized as follows. Section II contains the description of the pneumatic actuator model with the proposed switching control law avoiding “stick-slip”. Section III introduces the uncertain PWA class of systems and the tools that can be used for the robust stability analysis in the presence of sliding modes and parameters variations. Section IV contains the main theoretical result, i.e. a method for proving robust stability, whereas Section V shows its different applications to our test bench model. The conclusions are given in Section VI.

Notation

We denote by \mathbb{R} the field of real numbers and by $\mathbb{R}^{n \times m}$ the set of real $n \times m$ matrices. Let A^T indicate the transpose of a matrix A , and let I be the identity matrix. We use also the notation $A \succ 0$ ($A \prec 0$) to indicate that all the eigenvalues of the symmetric matrix A are strictly positive (negative). The symbol $\binom{n}{k}$ indicates the binomial coefficient, for which we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

II. PNEUMATIC ACTUATOR SYSTEMS

We consider a pneumatic cylinder test bench like the one shown in Figure 2. The setup is intended for applications in rectilinear motion [8], [6], [27]. It comprises an actuator in the form of a pneumatic cylinder (double acting) with a rod connected to a carriage on rails. The actuator is powered by compressed air, with two servovalves for controlling the flow supplied to both the chambers of the cylinder. Two sensors measure the pressures in the chambers.

Based on [28], the physical model of the system in open loop is given by the following equations (see Table I for the notation):

y	position of the piston rod (m)
v	velocity (m s^{-1})
p_P, p_N	pressures in the cylinder chambers (Pa)
a	acceleration (m s^{-2})
F_s	dry friction force (N)
k	gas polytropic constant
l	length of stroke (m)
M	load (carriage and rod mass) (kg)
q_{mP}, q_{mN}	mass flow rate provided by the servovalves to the cylinder chambers (kg s^{-1})
r	perfect gas constant ($\text{J kg}^{-1} \text{K}^{-1}$)
S	area of cylinder bore (m^2)
T	temperature (K)
V	volume (m^3)
V_D	dead volume of cylinder chamber (m^3)

TABLE I
NOTATION.

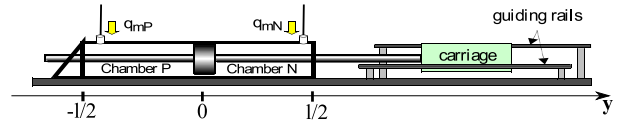


Fig. 2. Electro-pneumatic actuator.

$$\begin{cases} \dot{y} = v \\ \dot{v} = \frac{1}{M}(S(p_P - p_N) - F_f(v)) \\ \dot{p}_N = \frac{krT}{V_N(y)} \left(\frac{S}{rT} p_N v + q_{mN} \right) \\ \dot{p}_P = \frac{krT}{V_P(y)} \left(\frac{-S}{rT} p_P v + q_{mP} \right) \end{cases} \quad (1)$$

where the inputs are the two mass flow rates q_{mP} and q_{mN} , and $V_P(y) = V_0 + Sy$, $V_N(y) = V_0 - Sy$ with $V_0 = V_D + S \frac{l}{2}$. $F_f(v)$ is the dry friction force, whose nonlinear model is given in the literature by several relations [3], [11], [29], [9]. In our case, we consider two possible models of the dry friction forces, one in a continuous saturation form (M_2) and another in a discontinuous, “relay” form (M_1), both shown in Figure 3, such as:

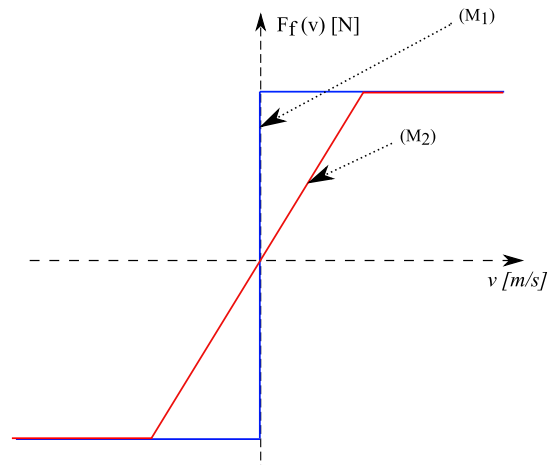


Fig. 3. Dry friction models.

$$M_1: F_f(v) \begin{cases} = +F_s & \text{for } v > 0 \\ \in [-F_s, +F_s] & \text{for } v = 0 \\ = -F_s & \text{for } v < 0. \end{cases} \quad (2)$$

and

$$M_2: F_f(v) = \begin{cases} F_s & \text{for } v > \varepsilon \\ \frac{F_s}{\varepsilon}v & \text{for } -\varepsilon \leq v \leq \varepsilon \\ -F_s & \text{for } v < -\varepsilon \end{cases} \quad (3)$$

In order to overcome the stick-slip phenomenon, we propose the following control law adapted from [28]. This law is based on a feedback linearization of the model in (1), and it switches between the two following modes.

- **Trajectory tracking law (#1)**

$$\begin{cases} q_{mP} = \frac{V_P(y)}{krT} \left[\frac{kS}{V_P(y)} v_{PP} + \dot{p}_{Pd} - k_P e_P \right] \\ q_{mN} = \frac{MV_N(y)}{SkrT} \left[\frac{SkrT}{MV_P(y)} q_{mP} + \frac{S^2kv}{M} \left(-\frac{p_P}{V_P(y)} - \frac{p_N}{V_N(y)} \right) \right. \\ \quad \left. - j_d + k_a e_a + k_v e_v + k_y e_y \right] \end{cases}$$

which allows the tracking of a given time-varying position reference; under the hypothesis of no state or input saturation, it yields the following closed-loop dynamics:

$$\begin{cases} \dot{e}_y = e_v \\ \dot{e}_v = e_{af} - \frac{1}{M} F_f(v) \\ \dot{e}_{af} = -k_a e_{af} - k_v e_v - k_y e_y + \frac{k_a}{M} F_f(v) \\ \dot{e}_P = -k_P e_P \end{cases} \quad (4)$$

- **Pressure control law (#2)**

$$\begin{cases} q_{mP} = \frac{V_P(y)}{krT} \left[\frac{kS}{V_P(y)} v_{PP} + \dot{p}_{Pd} - k_P e_P \right] \\ q_{mN} = \frac{V_N(y)}{krT} \left[-\frac{kS}{V_N(y)} v_{PN} + \dot{p}_{Nd} - k_N e_N \right] \end{cases}$$

which regulates the pressures in the two chambers in order to avoid stick-slip; this law is active when the carriage has arrived at the desired position, resulting in the following closed-loop dynamics:

$$\begin{cases} \dot{e}_y = e_v \\ \dot{e}_v = e_{af} - \frac{1}{M} F_f(v) \\ \dot{e}_{af} = \frac{S}{M} (k_N - k_P) e_P - k_N e_{af} \\ \dot{e}_P = -k_P e_P \end{cases} \quad (5)$$

In all of the equations above, $e_{af} = e_a + \frac{1}{M} F_f(v)$. e_y , e_v , e_a , e_P , e_N are the errors between states and their desired values; the constants k_y , k_v , k_a , k_P and k_N are the state feedback gains chosen by a pole-placement on the feedback-linearized model. The switching criterion is

$$\begin{aligned} \#1 \rightarrow \#2: & |e_y| \leq \varepsilon_1 \wedge |e_v| \leq \varepsilon_2 \wedge |v_d| \leq \varepsilon_3 \\ \#2 \rightarrow \#1: & |e_y| > \varepsilon_1 \vee |e_v| > \varepsilon_2 \vee |v_d| > \varepsilon_3 \end{aligned}$$

where ε_1 , ε_2 and ε_3 are small arbitrary constants. The above switching controller has shown no instability either in simulation or on the test bench and has never caused the occurrence of stick-slip. Figure 4 and Figure 5 show trajectories obtained using this control law, respectively in simulation and on the

test bench; the system can switch to pressure control (#2) only when the trajectory becomes constant.

Remark 1: The chosen control law is based on a feedback linearization that cancels out all the nonlinearities of the system but the ones caused by friction. We avoid canceling them because there is a high degree of uncertainty on the friction forces, so exact cancellation is impossible.

The topic of the rest of this paper is the formal proof of stability. One of the issues that we will need to face is that the stability of the system has to be proven for a special kind of convergence of the state, not to the setpoint, but to a whole set in its neighborhood. This is due to the well-known fact that moving systems subject to dry friction cannot be easily brought to a stop at a desired point, but they will rather stop in its proximity (this can be seen for example in Figure 4, when the rod stops just after $t = 2$ s, as well as in Figure 5, where the error never goes exactly to 0).

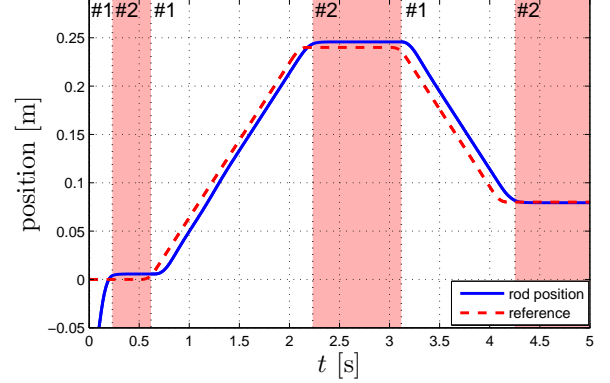


Fig. 4. Simulation of position evolution with respect to the desired position. The background colors are related to the active control mode (#1 or #2).

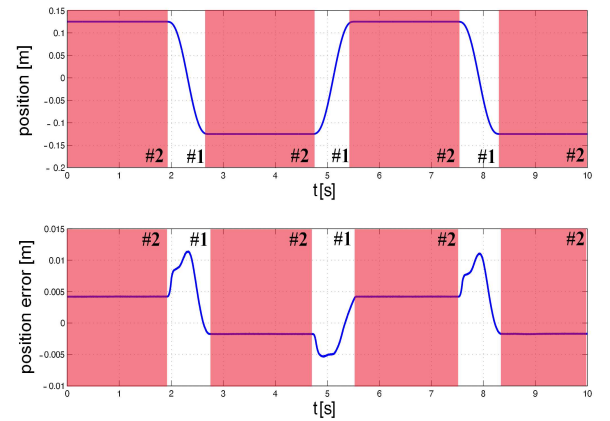


Fig. 5. Experimental results: evolution of the rod position and the position error with respect to the desired reference. The background colors are related to the active control mode (#1 or #2). No stick-slip occurs.

III. PROPOSED APPROACH

Let us introduce a partition of \mathbb{R}^n into N polyhedral cells X_i with disjoint interior, with $i \in \mathcal{I}$, a set of N valid indices.

We partition $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$ (with $\mathcal{I}_0 \cap \mathcal{I}_1 = \emptyset$) such as $0 \in X_i$ if $i \in \mathcal{I}_0$, otherwise $0 \notin X_i$ if $i \in \mathcal{I}_1$ (the cells with index in \mathcal{I}_0 contain the origin).

Definition 2 (Piece-wise affine with polytopic uncertainties): A dynamical system is called ‘‘piecewise affine’’ (PWA) with polytopic uncertainties if it has the following dynamics [1], [19], [15]:

$$\dot{x}(t) = \sum_{k=1}^q \lambda_k(t)(A_{i,k}x(t) + a_{i,k}), \text{ for } x(t) \in X_i, i \in \mathcal{I} \quad (6)$$

where $x(t) \in \mathbb{R}^n$ denotes the state-space vector and $\lambda_k(t)$ are continuously time-varying parameters which can take values according to

$$\begin{cases} \lambda_k(t) \geq 0 \forall k \in \{1, \dots, q\} \\ \sum_{k=1}^q \lambda_k(t) = 1 \end{cases} \quad (7)$$

Equivalently, for $\bar{x}(t)^T = [x(t)^T \ 1]$

$$\dot{\bar{x}}(t) = \sum_{k=1}^q \lambda_k(t) \underbrace{\begin{bmatrix} A_{i,k} & a_{i,k} \\ 0 & 0 \end{bmatrix}}_{\bar{A}_{i,k}} \bar{x}(t), \text{ for } x(t) \in X_i, i \in \mathcal{I} \quad (8)$$

The domain of each cell X_i is described by matrices $\bar{E}_i \in \mathbb{R}^{l_i \times (n+1)}$ such as:

$$x \in X_i \Rightarrow \bar{E}_i \bar{x} \geq 0, \quad (9)$$

and the boundaries between two cells X_i and X_j by matrices $\bar{F}_{ij} \in \mathbb{R}^{r_{ij} \times (n+1)}$ such as

$$\begin{cases} \Gamma = \{(i, j) \mid X_i \cap X_j \neq \emptyset\} \\ \forall (i, j) \in \Gamma, X_i \cap X_j \subseteq \{x \mid \bar{F}_{ij} \bar{x} = 0\} \end{cases} \quad (10)$$

We define $\lambda^T = [\lambda_1 \dots \lambda_q]^T$. We then formulate a set of assumptions which will hold throughout the paper.

Assumption 1: A sliding mode may occur on any boundary in Γ between two neighboring cells X_i and X_j (see Figure 6), i.e. for $x \in X_i \cap X_j$, $(i, j) \in \Gamma$, we have the following dynamics (according to Filippov [13]):

$$\begin{aligned} \dot{\bar{x}}(t) &= \left(\theta(\lambda) \sum_{k=1}^q \lambda_k(t) \bar{A}_{i,k} + (1 - \theta(\lambda)) \sum_{k=1}^q \lambda_k(t) \bar{A}_{j,k} \right) \bar{x}(t), \\ 0 &\leq \theta(\lambda) \leq 1. \end{aligned} \quad (11)$$

which means that the matrix describing the system dynamics at a boundary is a convex combination of the matrices of the neighboring cells.

Assumption 2: The set $E_{eq} \supseteq \{0\}$ of the equilibrium points for (8) is a subset of $\bigcup_{i \in \mathcal{I}_0} X_i$ (notice that the equilibrium set might also be on a sliding mode).

Assumption 3: Two cases can be distinguished:

- if the origin is in the interior of a cell X_i (for which we have then that $\mathcal{I}_0 = \{i\}$): $\sum_{k=1}^q \lambda_k(t) \bar{A}_{i,k}$ does not have any constant terms ($a_{i,k} = 0$, for $k = 1, \dots, q$); we define $\bar{A}_0(\lambda) = \sum_{k=1}^q \lambda_k(t) \bar{A}_{i,k}$;
- if the origin is in the boundary between cells X_i and X_j : there exists a $\bar{\theta}(\lambda)$, $0 \leq \bar{\theta}(\lambda) \leq 1$, for which

$$\begin{aligned} \bar{\theta}(\lambda) \sum_{k=1}^q \lambda_k(t) a_{i,k} + (1 - \bar{\theta}(\lambda)) \sum_{k=1}^q \lambda_k(t) a_{j,k} &= 0; \\ \text{we define } \bar{A}_0(\lambda) &= \bar{\theta}(\lambda) \sum_{k=1}^q \lambda_k(t) \bar{A}_{i,k} + (1 - \bar{\theta}(\lambda)) \sum_{k=1}^q \lambda_k(t) \bar{A}_{j,k}. \end{aligned}$$

Moreover, $\bar{A}_0(\lambda)$ has \tilde{h} eigenvectors not depending from λ (shared with $\sum_{k=1}^q \lambda_k(t) \bar{A}_{i,k}$, and with $\sum_{k=1}^q \lambda_k(t) \bar{A}_{j,k}$ in the second case) that have 0 as associated eigenvalue. We call $\bar{Z} \in \mathbb{R}^{(n+1) \times \tilde{h}}$ the full-rank matrix of these eigenvectors ($\bar{A}_0(\lambda) \bar{Z} = \sum_{k=1}^q \lambda_k(t) \bar{A}_{i,k} \bar{Z} = \sum_{k=1}^q \lambda_k(t) \bar{A}_{j,k} \bar{Z} = 0$; this implies that the equilibrium set can be larger than the origin alone, i.e. $E_{eq} \supseteq \{0\}$). We then define $\bar{\Pi} \in \mathbb{R}^{(n+1) \times ((n+1) - \tilde{h})}$ the orthogonal complement of \bar{Z} , with $\bar{\Pi}^T \bar{\Pi} = I$.

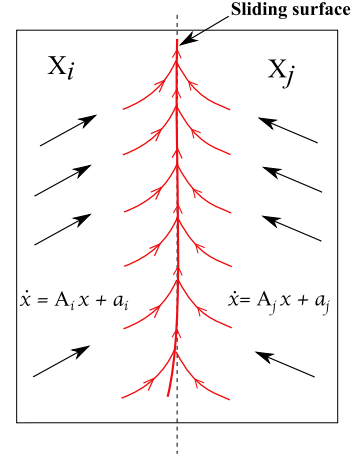


Fig. 6. Sliding modes.

The objective is to find a set of conditions ensuring the exponential convergence of the system state $x(t)$ to the set of equilibrium points E_{eq} , notwithstanding the presence of sliding modes on the boundaries and parametric variations in the model. Basically we have to upgrade the methods in [19], [15] in order to cope with sliding modes, equilibrium sets and parametric variations; we will use a Lyapunov function of degree higher than 2 (as done in [2]), i.e. a piecewise-polynomial Lyapunov function, which is less conservative than the classical piecewise-quadratic Lyapunov functions. Such functions are given by the expression

$$V(x) = V_i(x) \text{ for } x \in X_i, i \in \mathcal{I} \quad (12)$$

such that $V_i(x)$ is a polynomial in n variables with degree $2m$ given by:

$$V_i(x) = \chi(x)^T P_i \chi(x) \quad (13)$$

where $P_i = P_i^T \in \mathbb{R}^{\rho \times \rho}$, and $\chi(x) \in \mathbb{R}^{\rho \times 1}$ is the vector of all the monomials of degree less than or equal to m that can be made from the elements of x ; $\rho = \binom{m+n}{m}$. For example,

$$\begin{aligned} \chi(x) &= [x_1, x_2, \dots, x_n, x_1 x_2, x_1 x_3, \dots, x_1 x_n, x_2 x_3, \dots \\ &\dots, x_2 x_n, \dots, x_{n-1} x_n, \dots, x_1^m, x_2^m, \dots, x_n^m, 1]^T. \end{aligned} \quad (14)$$

As seen in [24], the choice of P_i is not unique for a given polynomial $V_i(x)$; there exists a number ι of linearly inde-

pendent matrices $Q_\nu = Q_\nu^T \in \mathbb{R}^{\rho \times \rho}$ (defined up to a scalar factor), for which

$$\chi(x)^T Q_\nu \chi(x) = 0 \text{ for } \nu = 1, \dots, \iota \quad (15)$$

with

$$\iota = \frac{1}{2} \left(\binom{m+n}{m}^2 + \binom{m+n}{m} \right) - \binom{n+2m}{2m}. \quad (16)$$

This implies that $V_i(x) = \chi(x)^T P_i \chi(x) = \chi(x)^T (P_i + \sum_{\nu=1}^{\iota} Q_\nu \tau(\nu)) \chi(x)$ for any real scalars $\tau(\nu)$. This is due to the fact that some monomials in V_i can be obtained as products of different elements in χ , e.g. $x_1^2 = x_1 \cdot x_1 = 1 \cdot x_1^2 = x_1^2 \cdot 1$.

Following the procedure of the ‘‘power transformation’’ [32], [30], we can obtain the dynamics of $\chi(x(t)) = \chi(t)$, with which we will be able to compute the time derivative of the Lyapunov function. Namely, the uncertain dynamics will still be PWA, i.e. we will have

$$\dot{\chi}(t) = \sum_{k=1}^{q_i} \lambda_k(t) \tilde{A}_{i,k} \chi(t), \quad \forall k \in \{1, \dots, q\}, \text{ for } \chi \in \tilde{X}_i, i \in \mathcal{I} \quad (17)$$

and we can construct a new description of the cells \tilde{X}_i , with matrices $\tilde{E}_i \in \mathbb{R}^{\tilde{h} \times \rho}$ such as

$$\begin{cases} x \in X_i \Rightarrow \chi \in \tilde{X}_i \\ \chi \in \tilde{X}_i \Rightarrow \tilde{E}_i \chi \geq 0, \end{cases} \quad (18)$$

and defining the boundaries, with $\tilde{F}_{ij} \in \mathbb{R}^{\tilde{h} \times \rho}$, as

$$\forall (i, j) \in \Gamma, \quad \tilde{X}_i \cap \tilde{X}_j \subseteq \left\{ \chi \mid \tilde{F}_{ij} \chi = 0 \right\} \quad (19)$$

Notice that, counterintuitively, the power transformation leaves λ as linear terms in the dynamic equations for χ . We define $\tilde{A}_i(\lambda) = \sum_{k=1}^q \lambda_k \tilde{A}_{i,k}$. Assumptions 1, Assumption 2 and Assumption 3 are also still holding for the new system in (17):

1) Sliding modes may occur, with the following dynamics:

$$\dot{\chi}(t) = \left(\theta(\lambda) \sum_{k=1}^q \lambda_k(t) \tilde{A}_{i,k} + (1 - \theta(\lambda)) \sum_{k=1}^q \lambda_k(t) \tilde{A}_{j,k} \right) \chi(t), \quad 0 \leq \theta(\lambda) \leq 1 \quad (20)$$

2) There exists an equilibrium set \tilde{E}_{eq} such that $\chi \in \tilde{E}_{eq} \Leftrightarrow \dot{\chi} = 0$; $\chi \in \tilde{E}_{eq} \Leftrightarrow x \in E_{eq}$; $\tilde{E}_{eq} \subset \bigcup_{i \in \mathcal{I}_0} \tilde{X}_i$

3) Again we distinguish two cases:

- if the origin is in the interior of a cell \tilde{X}_i ; $\tilde{A}_{i,k}(\lambda)$ does not have any constant terms for all λ ; $\tilde{A}_0(\lambda) = \sum_{k=1}^q \lambda_k(t) \tilde{A}_{i,k} = \tilde{A}_{i,k}(\lambda)$;
- if the origin is in the boundary between cells \tilde{X}_i and \tilde{X}_j : there exists a $\tilde{\theta}(\lambda)$, $0 \leq \tilde{\theta}(\lambda) \leq 1$, for which the constant terms disappear; we define $\tilde{A}_0(\lambda) = \tilde{\theta}(\lambda) \sum_{k=1}^q \lambda_k(t) \tilde{A}_{i,k} + (1 - \tilde{\theta}(\lambda)) \sum_{k=1}^q \lambda_k(t) \tilde{A}_{j,k} = \tilde{\theta}(\lambda) \tilde{A}_i(\lambda) + (1 - \tilde{\theta}(\lambda)) \tilde{A}_j(\lambda)$.

Identically, $\tilde{A}_0(\lambda)$ has \tilde{h} eigenvectors not depending from λ (shared with $\sum_{k=1}^q \lambda_k(t) \tilde{A}_{i,k}$ and with $\sum_{k=1}^q \lambda_k(t) \tilde{A}_{j,k}$ in the second case) that have 0 as associated eigenvalue. We can also call $\tilde{Z} \in \mathbb{R}^{\rho \times \tilde{h}}$

the full-rank matrix of these eigenvectors ($\tilde{A}_0(\lambda) \tilde{Z} = \sum_{k=1}^q \lambda_k(t) \tilde{A}_{i,k} \tilde{Z} = \sum_{k=1}^q \lambda_k(t) \tilde{A}_{j,k} \tilde{Z} = 0$); we define $\tilde{\Pi} \in \mathbb{R}^{\rho \times (\rho - \tilde{h})}$ as the orthogonal complement of \tilde{Z} , with $\tilde{\Pi}^T \tilde{\Pi} = I$.

$V(\chi)$ in (12) and (13) is a candidate Lyapunov function for the system in (8). In order to ensure its continuity on the boundary between two cells \tilde{X}_i and \tilde{X}_j , $(i, j) \in \Gamma$, the following condition has to be satisfied:

$$V_i(\chi) = V_j(\chi) \quad \forall \chi(x) \in \tilde{X}_i \cap \tilde{X}_j. \quad (21)$$

As a direct consequence of LaSalle’s theorem ([21], [31]), the conditions that the candidate time-invariant Lyapunov function must satisfy in order to prove the convergence to the equilibrium set are in the following lemma.

Lemma 3: For the system in (8), the convergence of x to E_{eq} (which coincides with the convergence of χ in (17) to \tilde{E}_{eq}) for $t > 0$, $t \rightarrow +\infty$ is assured under the following conditions.

1) The Lyapunov function is continuous, i.e. it satisfies (21);

2) The Lyapunov function is positive outside \tilde{E}_{eq} ,

$$V_i(\chi(x)) > 0 \text{ for } \chi(x) \in \tilde{X}_i / \tilde{E}_{eq}, i \in \mathcal{I}; \quad (22)$$

3) The derivative of the Lyapunov function is negative outside \tilde{E}_{eq} , i.e.

$$\dot{V}_i(\chi(x)) < 0 \text{ for } \chi(x) \in \tilde{X}_i / \tilde{E}_{eq}, i \in \mathcal{I}; \quad (23)$$

even in the case of sliding-mode dynamics;

4) The derivative of the Lyapunov function is null in \tilde{E}_{eq} , i.e.

$$\dot{V}_i(\chi(x)) = 0 \text{ for } \chi(x) \in \tilde{E}_{eq} \cap \tilde{X}_i, i \in \mathcal{I}_0. \quad (24)$$

We will also try and evaluate the speed of the exponential convergence of the system trajectories to the equilibrium set E_{eq} . For this purpose, we can define the distance d from a state x to any point in E_{eq} as

$$d(x, E_{eq}) = \inf_{x_{eq} \in E_{eq}} \|x - x_{eq}\|.$$

We can then introduce the following definition.

Definition 4 (Decay rate): We call ‘‘decay rate’’ the largest positive number α such that for any initial condition x_0 :

$$\lim_{t \rightarrow \infty} e^{\alpha t} d(x(t), E_{eq}) = 0.$$

Corollary 5: The decay rate of system in (8) is larger than α if the conditions in (23) are replaced by

$$\begin{cases} \dot{V}_i(\tilde{\Pi} \tilde{\Pi}^T \chi(x)) < -2\alpha V_i(\tilde{\Pi} \tilde{\Pi}^T \chi(x)) \text{ for } \chi(x) \in \tilde{X}_i / \tilde{E}_{eq}, i \in \mathcal{I}_0 \\ \dot{V}_i(\chi(x)) < -2\alpha V_i(\chi(x)) \text{ for } \chi(x) \in \tilde{X}_i, i \in \mathcal{I}_1. \end{cases} \quad (25)$$

IV. MAIN RESULT

Given the previous considerations and using the S-procedure [4], we can arrive at our main theoretical result, which reveals that a Lyapunov function can be computed through an LMI optimization.

Theorem 6: Let us assume that the system in (8) satisfies Assumption 1, Assumption 2 and Assumption 3. Let $\tilde{A}_{i,k}$,

$\forall k \in \{1, \dots, q\}$ be the uncertain matrices describing the dynamics of the vector χ of degree m deriving from x , as in (17), with $\tilde{X}_i, \tilde{E}_i, \tilde{F}_{ij}, \tilde{Z}_i$, etc. as previously defined.

If there exist

- $P_i = P_i^T \in \mathbb{R}^{\rho \times \rho}$ for $i \in \mathcal{I}$;
- $U_{i,k} = U_{i,k}^T, W_i = W_i^T \in \mathbb{R}^{\tilde{l}_i \times \tilde{l}_i}$ and $\mathcal{T}_{i,k}, \mathcal{T}'_i \in \mathbb{R}^{\tilde{l}_i}$; all with non-negative entries for $i \in \mathcal{I}, k = 1, \dots, q$;
- $U_{ij,k}, U'_{ij,k} \in \mathbb{R}^{\tilde{l}_i \times \tilde{l}_i}, \mathcal{T}_{ij,k}, \mathcal{T}'_{ij,k} \in \mathbb{R}^{\tilde{l}_i}$, all with non-negative entries for $i, j \mid (i, j) \in \Gamma, k = 1, \dots, q$;
- $L_{ij} \in \mathbb{R}^{\rho \times i_{ij}}$ for $i, j \mid (i, j) \in \Gamma$;
- $\tau_i(\nu), \tau'_i(\nu), \tau''_{i,k}(\nu), \tau_{ij,k}(\nu), \tau'_{ij,k}(\nu) \in \mathbb{R}$, for $i, j \in \mathcal{I}, \nu = 1, \dots, \iota, k = 1, \dots, q$

such that

$$P_j = P_i + \tilde{F}_{ij}^T L_{ij}^T + L_{ij} \tilde{F}_{ij} + H(\tau_{ij}) \text{ for } (i, j) \in \Gamma \quad (26)$$

$$\begin{cases} \tilde{\Pi}^T \tilde{A}_{i,k}^T P_i \tilde{Z} = 0 \\ \tilde{E}_i \tilde{Z} = 0 \end{cases} \text{ for } i \in \mathcal{I}_0, k = 1, \dots, q \quad (27)$$

and it holds that

$$\begin{cases} \tilde{\Pi}^T (P_i + H(\tau'_i) - N_i(\mathcal{T}'_i) - \tilde{E}_i^T W_i \tilde{E}_i) \tilde{\Pi} \succ 0 \\ \tilde{\Pi}^T (\tilde{A}_{i,k}^T P_i + P_i \tilde{A}_{i,k} + H(\tau''_{i,k}) + N_i(\mathcal{T}_{i,k}) + \tilde{E}_i^T U_{i,k} \tilde{E}_i) \tilde{\Pi} \\ + 2\alpha \tilde{\Pi}^T P_i \tilde{\Pi} \prec 0 \end{cases} \text{ for } i \in \mathcal{I}_0, k = 1, \dots, q \quad (28)$$

$$\begin{cases} P_i + H(\tau'_i) - N_i(\mathcal{T}'_i) - \tilde{E}_i^T W_i \tilde{E}_i \succ 0 \\ \tilde{A}_{i,k}^T P_i + P_i \tilde{A}_{i,k} + H(\tau''_{i,k}) + N_i(\mathcal{T}_{i,k}) + \tilde{E}_i^T U_{i,k} \tilde{E}_i \\ + 2\alpha P_i \prec 0 \end{cases} \text{ for } i \in \mathcal{I}_1, k = 1, \dots, q \quad (29)$$

$$\begin{cases} \tilde{\Pi}^T (\tilde{A}_{i,k}^T P_j + P_j \tilde{A}_{i,k} + H(\tau_{ij,k}) + N_i(\mathcal{T}_{ij,k}) + N_j(\mathcal{T}'_{ij,k}) \\ + \tilde{E}_i^T U_{ij,k} \tilde{E}_i + \tilde{E}_j^T U'_{ij,k} \tilde{E}_j) \tilde{\Pi} + 2\alpha \tilde{\Pi}^T P_j \tilde{\Pi} \prec 0 \\ \tilde{\Pi}^T (\tilde{A}_{j,k}^T P_i + P_i \tilde{A}_{j,k} + H(\tau_{ji,k}) + N_i(\mathcal{T}_{ji,k}) + N_j(\mathcal{T}'_{ji,k}) \\ + \tilde{E}_j^T U_{ji,k} \tilde{E}_j + \tilde{E}_i^T U'_{ji,k} \tilde{E}_i) \tilde{\Pi} + 2\alpha \tilde{\Pi}^T P_i \tilde{\Pi} \prec 0 \\ \tilde{\Pi}^T \tilde{A}_{i,k}^T P_j \tilde{Z} = 0, \tilde{\Pi}^T \tilde{A}_{j,k}^T P_i \tilde{Z} = 0 \end{cases} \text{ for } (i, j) \in \Gamma, i, j \in \mathcal{I}_0, k = 1, \dots, q \quad (30)$$

$$\begin{cases} \tilde{A}_{i,k}^T P_j + P_j \tilde{A}_{i,k} + H(\tau_{ij,k}) + N_i(\mathcal{T}_{ij,k}) + N_j(\mathcal{T}'_{ij,k}) \\ + \tilde{E}_i^T U_{ij,k} \tilde{E}_i + \tilde{E}_j^T U'_{ij,k} \tilde{E}_j + 2\alpha P_j \prec 0 \\ \tilde{A}_{j,k}^T P_i + P_i \tilde{A}_{j,k} + H(\tau_{ji,k}) + N_i(\mathcal{T}_{ji,k}) + N_j(\mathcal{T}'_{ji,k}) \\ + \tilde{E}_j^T U_{ji,k} \tilde{E}_j + \tilde{E}_i^T U'_{ji,k} \tilde{E}_i + 2\alpha P_i \prec 0 \end{cases} \text{ for } (i, j) \in \Gamma, i \notin \mathcal{I}_0 \vee j \notin \mathcal{I}_0, k = 1, \dots, q \quad (31)$$

with

$$N_i(\mathcal{T}) = \begin{bmatrix} 0 \\ \mathcal{T}^T \tilde{E}_i \end{bmatrix} + [0 \quad \tilde{E}_i^T \mathcal{T}]$$

and

$$H(\tau) = \sum_{\nu=1}^{\iota} \tau(\nu) Q_{\nu}$$

then the trajectories $x(t)$ of the system (8) converge exponentially to the equilibrium set E_{eq} , with a decay rate larger than α , with the Lyapunov function (12).

Proof: Consider the Lyapunov function candidate $V(\chi(x))$ defined by (12).

To prove (26), we use the same approach as in [15]. So, for $\chi(x) \in \tilde{X}_i \cap \tilde{X}_j$, we have $\tilde{F}_{ij} \chi(x) = 0$; replacing this into (26), multiplying on the right by $\chi(x)$ and on the left by $\chi(x)^T$, and remembering that $\chi(x)^T H(\tau) \chi(x) = 0$ for any τ , we get $\chi(x)^T P_j \chi(x) = \chi(x)^T P_i \chi(x)$, that is (21), i.e. (26) implies continuity of the Lyapunov function on the boundaries. Let us now consider the inequalities; all of the inequalities involving $\tilde{A}_{i,k}$ are present for all the values of k , assuring that such equalities will also hold for $\tilde{A}_i(\lambda)$ for any valid λ . For example, if $\tilde{A}_{i,k}^T P_i + P_i \tilde{A}_{i,k} + H(\tau''_{i,k}) + N_i(\mathcal{T}_{i,k}) + \tilde{E}_i^T U_{i,k} \tilde{E}_i + 2\alpha P_i \prec 0$ holds for all k , then $\tilde{A}_i(\lambda)^T P_i + P_i \tilde{A}_i(\lambda) + H(\tau''_i) + N_i(\mathcal{T}_i) + \tilde{E}_i^T U_i \tilde{E}_i + 2\alpha P_i \prec 0$ holds for all λ , with $\tau''_i, \mathcal{T}_i, U_i$ appropriately chosen. We will consider this as granted for the follow-up of the proof, for all of the inequalities.

Let us now consider (27) and (28). Based on Assumption 3, we can always write a decomposition for $\chi(x)$, of the kind $\chi(x) = \tilde{Z}\xi + \tilde{\Pi}\zeta$, with $\chi(x) \in \tilde{X}_i, i \in \mathcal{I}_0$. As $\tilde{A}_0(\lambda)\tilde{Z} = 0$ for $\chi(x) \in \tilde{E}_{eq}$ (i.e. $\chi(x) = \tilde{Z}\xi$), then the first in (27) implies (24). For $\chi(x) \notin \tilde{E}_{eq}$ instead, when the active dynamics is $\tilde{A}_i(\lambda)$, with $i \in \mathcal{I}_0$, we have $\dot{V}(\chi(x)) = (\tilde{Z}\xi + \tilde{\Pi}\zeta)^T (\tilde{A}_i(\lambda)^T P_i + P_i \tilde{A}_i(\lambda)) (\tilde{Z}\xi + \tilde{\Pi}\zeta) = \zeta^T \tilde{\Pi}^T (\tilde{A}_i(\lambda)^T P_i + P_i \tilde{A}_i(\lambda)^T) \tilde{\Pi} \zeta$ thanks to (27) (notice that: $\tilde{\Pi}^T \tilde{A}_i(\lambda)^T P_i \tilde{Z} = 0$). Notice also that the second in (27) implies that $\tilde{E}_i \chi \geq 0 \Rightarrow \tilde{E}_i \tilde{\Pi} \zeta \geq 0$. Then, thanks to the S-procedure, the second inequality in (28) implies $\dot{V}_i(\tilde{\Pi} \tilde{\Pi}^T \chi(x)) < -2\alpha V_i(\tilde{\Pi} \tilde{\Pi}^T \chi(x))$ for $\chi(x) \notin \tilde{E}_{eq}$, which is the first in (25), as we have $\zeta^T \tilde{\Pi}^T \tilde{E}_i^T U_{i,k} \tilde{E}_i \tilde{\Pi} \zeta \geq 0$ (true thanks to (18)) and (27)), $\zeta^T \tilde{\Pi}^T N_i(\mathcal{T}'_{i,k}) \tilde{\Pi} \zeta \geq 0$ (true thanks to (18) and (27)) and $\chi(x)^T H(\tau'_{i,k}) \chi(x) = 0$ (true thanks to (15)). In a similar way, the first inequality in (28) naturally implies the inequality (22) for the cells with $i \in \mathcal{I}_0$.

Subsequently, again thanks to the S-procedure, the first inequality in (29) implies that $\chi(x)^T P_i \chi(x) > 0$ when $\chi(x)^T \tilde{E}_i^T W_i \tilde{E}_i \chi(x) \geq 0$ (true thanks to (18) if $\chi(x) \in \tilde{X}_i$), $\chi(x)^T N_i(\mathcal{T}'_i) \chi(x) \geq 0$ (true thanks to (18)) $\chi(x)^T H(\tau'_i) \chi(x) = 0$ (true thanks to (15)), which in turn implies that $V_i(\chi(x)) > 0$ for $\chi(x) \in \tilde{X}_i$ i.e. the inequality (22) for the cells with $i \in \mathcal{I}_1$. Similarly, the second inequality in (29) ensures the second inequality in (25).

The conditions (30) and (31) concern possible sliding modes on the boundary between two cells. Indeed, multiplying the second inequality in (31) by θ (omitting the dependency on λ) and the second in (28) by $1 - \theta$ and summing, we get (thanks to the S-procedure, as seen above)

$$\chi(x)^T ((\theta \tilde{A}_j(\lambda)^T + (1 - \theta) \tilde{A}_i(\lambda)^T) P_i + P_i (\theta \tilde{A}_j(\lambda) + (1 - \theta) \tilde{A}_i(\lambda)) + 2\alpha P_i) \chi(x) < 0$$

for any $\chi \in \tilde{X}_i \cap \tilde{X}_j$, which ensures that the derivative of V_i is sufficiently negative for the given decay rate for any possible sliding mode dynamics. Similarly, one can get the complementary inequality

$$\chi(x)^T ((\theta \tilde{A}_j(\lambda)^T + (1 - \theta) \tilde{A}_i(\lambda)^T) P_j + P_j (\theta \tilde{A}_j(\lambda) + (1 - \theta) \tilde{A}_i(\lambda)) + 2\alpha P_j) \chi(x) < 0$$

from the first in (31). Condition (30) concerns a sliding mode containing the equilibrium set, the inequalities can be proven with the same reasoning as for (31) and (28).

So we have shown that the conditions required by the theorem imply (21), (22), (24) and (25), satisfying the hypotheses of Lemma 3, which proves the theorem statement. ■

The theorem involves a feasibility problem under linear matrix inequalities (LMIs) in (28), (29), (30) and (31), as well as linear matrix equalities (LMEs), (26) and (27), and the last in (30). Such LMEs can be resolved by an appropriate parameterization of the unknowns. In the case of $\alpha = 0$, the theorem proves the simple asymptotic stability of the system in (8) with respect to E_{eq} . An interesting problem is then to find the largest α such that Theorem 6 is satisfied. This problem can be solved through a dichotomic search with respect to α .

Remark 7: The first conditions in (28) and (29) can be interpreted as a “sum of squares” property (SOS) [17], [24], [10]. In fact we look for a positive definite polynomial $V_i(\chi)$ as a square product of a positive definite matrix times a vector of monomials χ . The terms in $H(\tau)$ reduce the conservatism of the inequality.

Remark 8: Theorem 6 includes several previous results as special cases.

- In the case of no sliding modes, no uncertain parameters, for $m = 1$, $E_{eq} = \{0\}$, $\mathcal{T}_\bullet = 0$ and $\alpha = 0$, we have Theorem 1 in [19].
- In the case of no sliding modes, no uncertain parameters, for $m = 1$, $E_{eq} = \{0\}$, $U_\bullet = 0$ and $\alpha = 0$, we have the theorem in Section 5 of [15].
- In the case of no sliding modes, no uncertain parameters, for $m = 1$, we have Theorem 9 in [1].
- With sliding modes allowed, no uncertain parameters, for $m \geq 1$ and $\alpha = 0$, we have Theorem 3 in [2].

In this sense, Theorem 6 is an extension of these previous results; it is less conservative as it features a more general condition, and at the same times it allows the analysis of the convergence to equilibrium sets other than the origin alone even in the presence of sliding modes and parametric uncertainties.

V. APPLICATION TO THE PNEUMATIC ACTUATOR

As shown in [1], and according to (4) and (5), the dynamical behavior of the pneumatic system described in Section II can then be cast into the form of a PWA system as in (8). Concerning the parameters of the test bench, we have $l = 0.5$ m, $M = 17$ kg, $F_s = 38$ N, $S = 7.27 \cdot 10^{-4}$ m², $\varepsilon = 0.1$ m/s, $\varepsilon_1 = 0.005$ m (which implies that the final positioning error is smaller in modulus than this value), $\varepsilon_2 = 0.01$ m/s and $\varepsilon_3 = 10^{-4}$ m/s. Through a pole placement, according to the system specifications, we have set $k_y = 50$ s⁻³, $k_v = 71$ s⁻², $k_a = 51.4$ s⁻¹, $k_P = 10$ s⁻¹ and $k_N = 10$ s⁻¹.

Theorem 6 can be applied in several different cases, according to the friction model chosen (saturation form or relay form), the type of problem under consideration (stability or performance study), and the value of m . We will present the results of the possible cases comparing the outcome of using piecewise-quadratic Lyapunov functions ($m = 1$) and piecewise-quartic ones ($m = 2$). We study the asymptotic

and the exponential stability of the equilibrium set $E_{eq} = [x_1^e \ 0 \ 0 \ 0]^T$, where x_1^e represents a steady position error between $-\varepsilon_1$ and ε_1 . Finding a valid Lyapunov function is equivalent to proving the closed-loop stability of the pneumatic actuator (for any possible initial condition, including any possible switching sequence).

A. Friction in saturation form

We begin our study choosing friction in saturation form as in (3), which presents no sliding modes on the boundaries. For this model, the results of the theorem applications are presented as follows.

• Stability study

As we study the convergence of the state to E_{eq} , we consider a static setpoint, i.e. y_d constant and $v_d = 0$, $a_d = 0$ (which implies $e_v = v$, $e_a = a$).

The switching criteria divide the state-space into seven cells (see Figure 7): a central cell X_0 , for which $0 \in X_0$, and six external cells X_1, X_2, X_3, X_4, X_5 and X_6 . Certain couples of cells are symmetric to each other with respect to the origin: X_1 and X_2, X_3 and X_4 , and X_5 and X_6 . We can use this property by imposing conditions only for a single cell of each couple, and by symmetry these conditions will necessarily be verified for the other ones. So, we need to impose the conditions of Theorem 6 only for the central cell ($i = 0$) and for $i = 1, 3, 5$. First we consider that there is no variation on friction; so $k = 1$, $\lambda_1 = 1$.

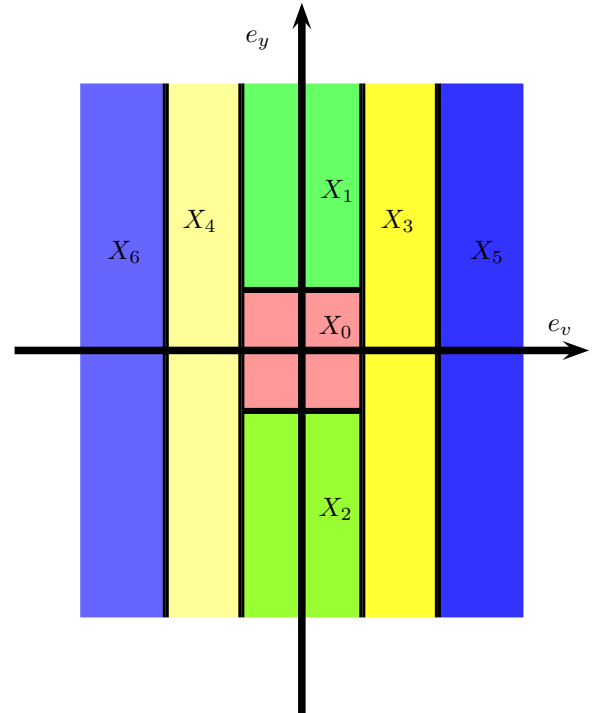


Fig. 7. The cells X_i in the case of friction in saturation form, intersection with (e_v, e_y) plane.

1) Piecewise quadratic Lyapunov function ($m = 1$)

By applying Theorem 6 for $m = 1$, the lower bound value on the decay rate found is: $\alpha = 0.0897$.

Figure 8 presents the 2-dimensional level curves of the obtained Lyapunov function in the (e_v, e_y) plane. We can clearly see that the level curves do not have a simple ellipsoidal shape; this suggests that a simple common quadratic Lyapunov function (i.e., the same matrix P for all the cells) could be not sufficient to obtain such shapes. In fact, we have run our test also in order to look for a common quadratic Lyapunov function, and the test failed to find any. This justifies the effort in finding less conservative conditions as the ones of Theorem 6.

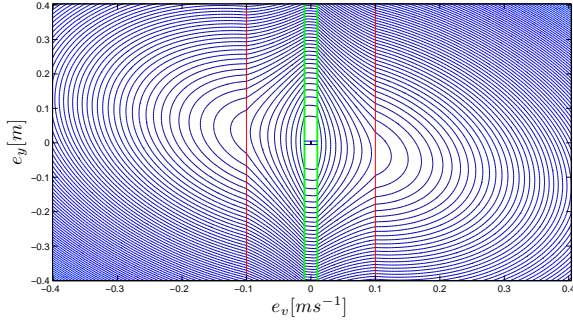


Fig. 8. Lyapunov function level curves, stability study, friction in saturation form, $m = 1$.

2) Piecewise polynomial Lyapunov function ($m = 2$)

By applying Theorem 6 for $m = 2$, the lower bound value on the decay rate found is: $\alpha = 0.4751$; which is larger than the one obtained in the case of $m = 1$. This result shows the effectiveness of using a piecewise-polynomial Lyapunov function ($m = 2$) compared to the piecewise-quadratic one ($m = 1$). Figure 9 presents the 2-dimensional level curves of the obtained Lyapunov function in the (e_v, e_y) plane

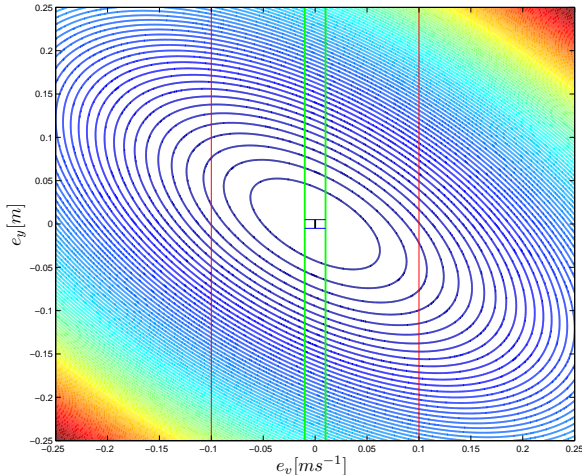


Fig. 9. Lyapunov function level curves, stability study, friction in saturation form, $m = 2$.

On the other hand, we have also verified that for $m = 2$ it is possible to find a common polynomial Lyapunov function, thanks to the lower conservatism with respect to a common quadratic one.

• Robust stability

For the robust stability study, we have modified the chosen model of friction by inserting a range of variation of the parameter F_s (see for example Figure 10).

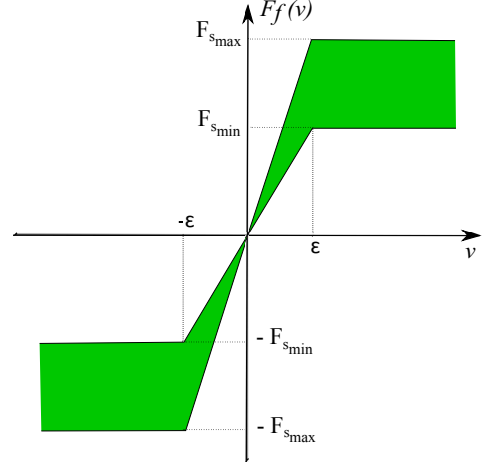


Fig. 10. Friction variation for the saturation case.

In our case study, we took a variation between $F_{s,max} = 40 \text{ N}$ and $F_{s,min} = 20 \text{ N}$. In this case, the friction model is polytopic with $k = 1, 2$, and can be presented according to (7) as:

$$\begin{cases} F_s(\lambda) = \lambda_1 F_{s,max} + \lambda_2 F_{s,min} \\ \lambda_1 + \lambda_2 = 1 \end{cases} \quad (32)$$

In this way, the model of the system is PWA with polytopic uncertainties.

1) Piecewise quadratic Lyapunov function ($m = 1$)

By applying Theorem 6 for $m = 1$, the lower bound value on the decay rate found is: $\alpha = 0.0671$. The level curves of the obtained Lyapunov function are presented in the Figure 11.

2) Piecewise polynomial Lyapunov function ($m = 2$)

By applying Theorem 6 for $m = 2$, the lower bound value on the decay rate found is: $\alpha = 0.4750$. For this case, the level curves of the Lyapunov function are pretty much the same compared to the case without variations.

For both cases, the stability was proven despite variations in friction, with a slight reduction of the decay rate.

• Performance study

We study the trajectory tracking by taking specified desired trajectories (see Figure 12), i.e in this case y_d variable and $v_d \neq 0$, $a_d \neq 0$.

The cells are divided with respect to desired velocity switching criteria resulting into 9 cells (not shown for brevity).

1) Piecewise quadratic Lyapunov function ($m = 1$)

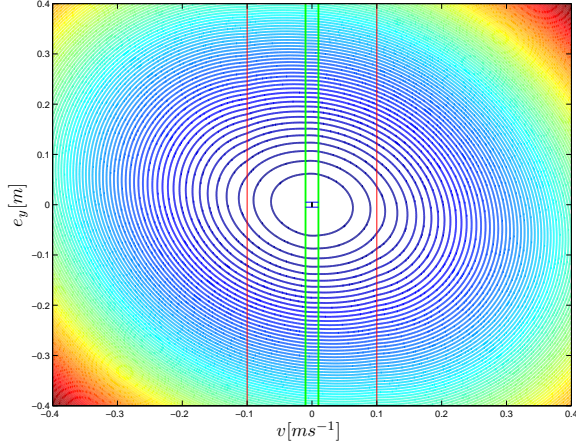


Fig. 11. Lyapunov function level curves, robust stability study, friction in saturation form, $m = 1$.

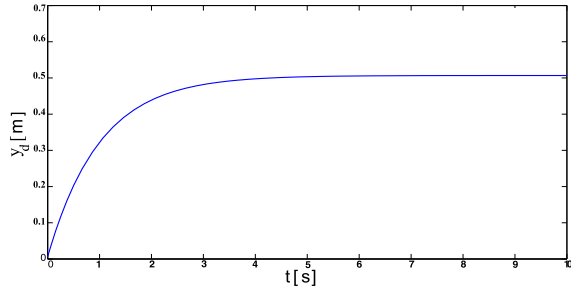


Fig. 12. Desired position y_d

By applying Theorem 6 for $m = 1$ and $\alpha = 0$, the asymptotic stability has been verified with a level curve of the obtained Lyapunov function presented in Figure 13 in the (e_y, v, v_d) space. Again, we can point out that the shape is quite different from a simple ellipsoid, which makes it impossible to obtain by means of a common quadratic Lyapunov function.

- 2) **Piecewise polynomial Lyapunov function ($m = 2$)**
The asymptotic stability has been also verified for $m = 2$ and a level curve of the obtained Lyapunov function is presented in Figure 14 in the (e_y, v, v_d) space.

B. Friction in relay form

The second friction model chosen for our study is in the relay form, as in (2), for which we study all the possible cases as done for the other friction model.

The switching of the system (due to the friction model and due to the control law) divides the state-space into a set of several cells; 8 cells considering only two switching criteria (see Figure 15), or 12 cells with respect to desired velocity switching criteria (not shown for brevity; more details on the PWA formulation of the model can be found in [2]).

The discontinuity of the friction model on $v = 0$ can generate a sliding mode on the corresponding boundary, which

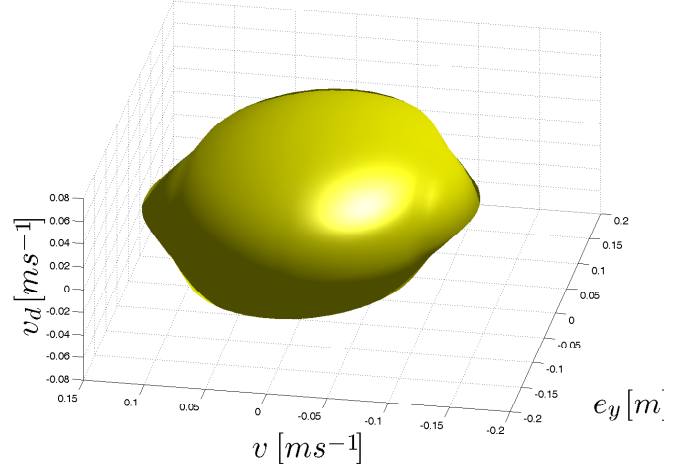


Fig. 13. A Lyapunov function level curve, performance study, friction in saturation form, $m = 1$.

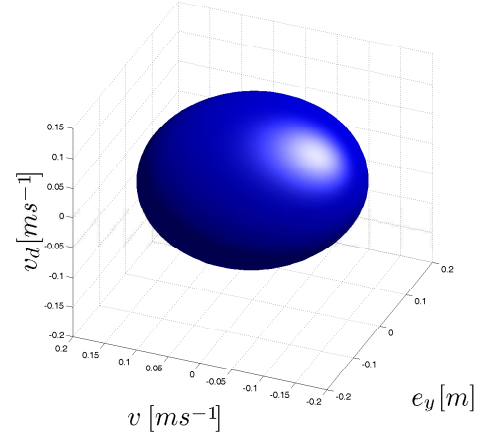


Fig. 14. A Lyapunov function level curve, performance study, friction in saturation form, $m = 2$.

we need to take into account while proving stability. All the results of the theorem application concerning this friction model are presented in Table II.

To suggest an idea of the computations involved, we provide a brief discussion only for the performance study (trajectory tracking). In this case, the piecewise-polynomial function is of degree $2m = 4$, which yields a vector $\chi(x)$ of the extended state, according to (14), in the following form (remember that $x = [x_1, x_2, x_3, x_4, x_5, x_6]^T$, where x_5, x_6 are states associated to the reference trajectory): $\chi(x) = [x_1, x_2, x_3, x_4, x_5, x_6, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_3x_4, x_3x_5, x_3x_6, x_4x_5, x_4x_6, x_5x_6, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, 1]^T$. The level curves of the piecewise-polynomial Lyapunov function found have been plotted in Figure 16 in the (v, e_y) plane and in Figure 17 in the (e_y, v, v_d) space. We can see that the level curves do not have a simple ellipsoidal shape; this implies again that a simple common quadratic Lyapunov function is not sufficient to obtain such shapes. Moreover, we have verified that the test fails for the case of $m = 1$, which

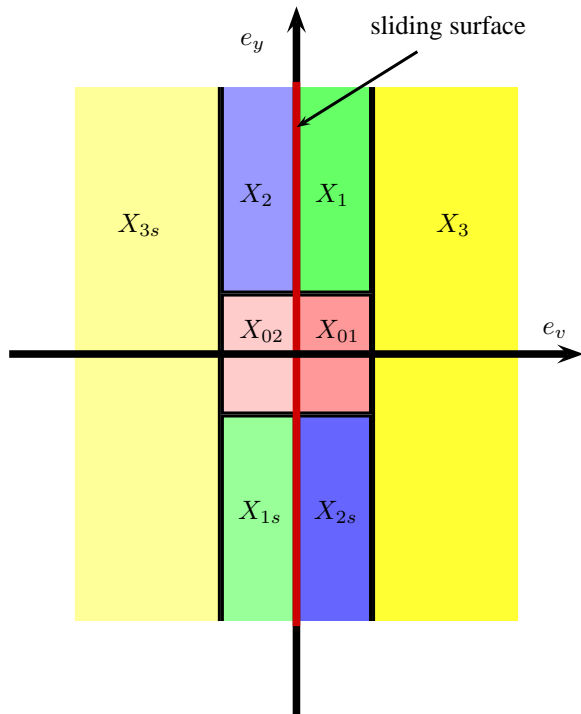


Fig. 15. The cells X_i in the case of friction in relay form, intersection with (e_v, e_y) plane.

justifies the need of a piecewise-polynomial function instead of a piecewise-quadratic one.

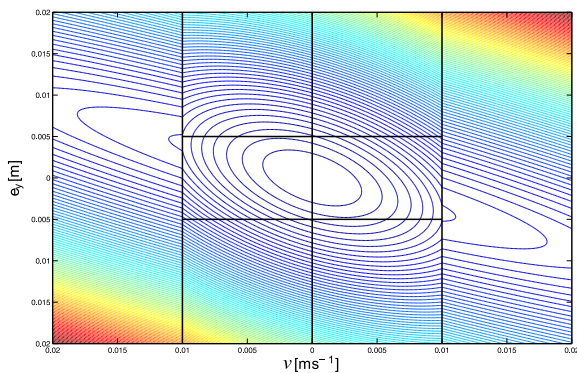


Fig. 16. Lyapunov function level curves, friction in relay form, performance study, $m = 2$.

C. Summary

Table II contains the results for all cases of the theorem application to our electropneumatic system; namely stability, robust stability, performance and robust performance under the two models of friction and two forms of Lyapunov function are considered.

We can notice the effectiveness of the piecewise-polynomial Lyapunov function approach for proving stability even in the presence of sliding modes and parametric uncertainties, finding a more reliable upper bound on the decay rate compared to piecewise-quadratic Lyapunov functions.

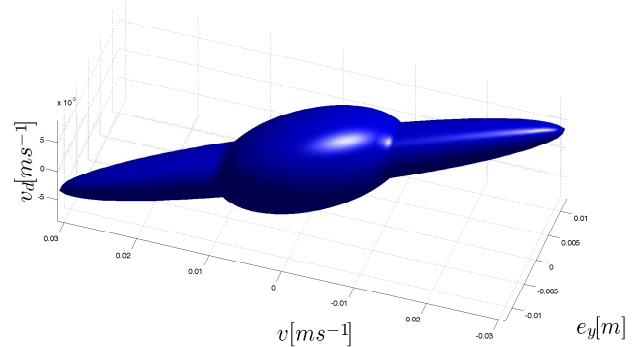


Fig. 17. A Lyapunov function level curve on the (e_y, v, v_d) space, friction in relay form, performance study, $m = 2$.

	Lyapunov function degree	Stability	Tracking
	$m = 1$	feasible with $\alpha = 0.0897$	feasible
	$m = 2$	feasible with $\alpha = 0.4751$	feasible
	$m = 1$	feasible with $\alpha = 0.0671$	feasible
	$m = 2$	feasible with $\alpha = 0.4750$	feasible
	$m = 1$	infeasible	infeasible
	$m = 2$	feasible with $\alpha \approx 0$	feasible
	$m = 1$	infeasible	infeasible
	$m = 2$	feasible with $\alpha \approx 0$	feasible

TABLE II
ALL STUDY CASES ON ELECTROPNEUMATIC ACTUATOR

VI. CONCLUSIONS

In this paper, we have investigated the problem of proving the stability of an electropneumatic system in closed-loop with a switching control law modeled as a PWA system with sliding modes due to presence of friction. Some sufficient conditions have been presented for finding piecewise-polynomial Lyapunov functions for PWA systems with sliding modes, uncertainties on parameters and equilibrium sets greater than the mere origin, and potentially on a sliding mode. The new extended conditions successfully found a 4-th degree piecewise-polynomial Lyapunov function through a convex optimization problem in terms of linear matrix inequalities. We have shown that this method is able to prove stability for a model of a real pneumatic test bench.

ACKNOWLEDGMENT

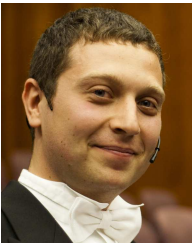
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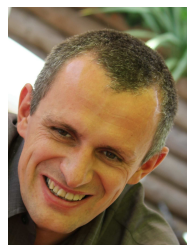
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