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# Approximation of linear distributed parameter systems by delay systems<sup>★</sup>

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## Abstract

The present work addresses continuous-time approximation of distributed parameter systems governed by linear one-dimensional partial differential equations. While approximation is usually realized by lumped systems, that is finite dimensional systems, we propose to approximate the plant by a time-delay system. Within the graph topology, we prove that, if the plant admits a coprime factorization in the algebra of BIBO-stable systems, any linear distributed parameter plant can be approximated by a time-delay system, governed by coupled differential-difference equations. Considerations on stabilization and state-space realization are carried out. A numerical method for constructive approximation is also proposed and illustrated.

*Key words:* Distributed parameter systems, time-delay systems, approximation, stabilization

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## 1 Introduction

In the input-output approach, approximating a plant means to find a more tractable model with respect to given specifications for analysis, simulation and control synthesis. For distributed parameter plants governed by linear 1-D partial differential equations [5,8,38,4], approximation is, in general, realized by lumped models described by linear ordinary differential equations [25]. In feedback and interconnection perspectives, the graph topology of input-output systems, introduced in [33,34] to handle a topology where feedback is a robust property, gives a supporting structure for approximation. See also [37,27,22,2] for generalizations and metrics of the graph topology for distributed parameter plants. Approximation by lumped models in the graph topology was formalized in [35], and studied further in [23,24,26] for various metrics. A preliminary result in lumped stabilization for time-delay systems, viewed as a subclass of distributed parameter plants, was also obtained in [14]. These methods are mainly based on algebraic representations of input-output plants, and in particular on coprime factorizations (see [34,18,29] and [20] for time-delay systems). In [35], existence of a lumped approx-

imation in the graph topology is studied, and gives a clear interpretation of purely atomic parts in coprime factorizations that cannot be closed to lumped distributions, in the BIBO-stability framework. Instead of relaxing graph topology, we propose in this paper to generalize the approximation class of lumped systems by time-delay systems. While this class is still of infinite dimension and can exactly represent some partial differential equations [30], state-space realization and advanced tools for analysis and control reach nowadays maturity, and can therefore be of benefit to the study of distributed parameter systems. The main contribution of this paper is to show that, from a quantitative point of view, such a time-delay approximation always exists if a coprime factorization in the Wiener algebra of bounded input bounded output (BIBO) plants exists. This condition appears to be of practical relevance in constructive feedback perspective [32].

The paper is organized as follows. In Section 2, we briefly recall definitions and introduce some notations. Approximation by time-delay systems is developed in Section 3. In Section 4, we discuss on the state-space realization of the approximation class, and propose a numerical method with explicit construction. Two examples are illustrated in Section 5.

## 2 Preliminaries

In this section, some basic background and notations, that will be used throughout the paper, are introduced.

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## 2.1 Input-Output representation

A causal input-output convolution system is a dynamical system described by an equation of the form

$$y(t) = (Pu)(t) \doteq \int_0^t P(\tau)u(t-\tau) d\tau \quad (1)$$

where  $y(\cdot)$ ,  $u(\cdot)$  and  $P(\cdot)$  are said to be the output, input and kernel of the plant, respectively. For our purpose, we consider the Wiener algebra of BIBO-stable systems [3], and we denote it by  $\mathcal{A}$ . We say that  $P \in \mathcal{A}$  if  $P = P_{\text{na}} + P_{\text{pa}}$  for  $t \geq 0$  and 0 elsewhere, where  $P_{\text{na}}(\cdot) \in L_1(0, \infty)$  (non atomic part), that is  $P_{\text{na}}$  is a real-valued function such that  $\|P_{\text{na}}\|_{L_1} = \int_0^\infty |P_{\text{na}}(t)| dt$  is finite. The real-valued distribution  $P_{\text{pa}}$  stands for the purely atomic part

$$P_{\text{pa}}(t) = \sum_{n=0}^{\infty} P_n \delta(t - r_n) \doteq \sum_{n=0}^{\infty} P_n \delta_{r_n}(t), \quad (2)$$

with elements  $\{P_n\}_{n \in \mathbb{N}}$  in  $\mathbb{R}$ , ordered delays  $0 = r_0 < r_1 < \dots$ ,  $\delta(t - r_n) = \delta_{r_n}(t)$  denoting the Dirac delta distribution centered in  $r_n$ , and  $\sum_{n \geq 0} |P_n| < \infty$ . The set  $\mathcal{A}$  is a commutative convolution Banach algebra for the norm defined by

$$\|P\|_{\mathcal{A}} = \|P_{\text{na}}\|_{L_1} + \sum_{n=0}^{\infty} |P_n|, \quad (3)$$

with unit element the Dirac delta distribution  $\delta$ . Denoting  $\hat{P}$  the Laplace transform of  $P$ ,  $\hat{\mathcal{A}}$  is the set of Laplace transforms of elements in  $\mathcal{A}$ . The set  $\mathcal{A}^{p \times m}$  stands for the set of  $p \times m$  matrices  $P$  with entries in  $\mathcal{A}$ , and the matrix norm  $\|P\|_{\mathcal{A}} = \max_{i=1, \dots, p} \sum_{j=1}^m \|P_{ij}\|_{\mathcal{A}}$ , where  $P_{ij}$  denotes the  $(i, j)$  entry of  $P$ . Two matrices  $N \in \mathcal{A}^{p \times m}$  and  $D \in \mathcal{A}^{m \times m}$  are said to be right-coprime if there exist  $X \in \mathcal{A}^{m \times p}$  and  $Y \in \mathcal{A}^{m \times m}$  such that

$$XN + YD = I_m \delta, \quad (4)$$

where  $I_m$  stands for the  $m \times m$  identity matrix. A pair  $(N, D)$  is a right-coprime factorization (r.c.f.) in  $\mathcal{A}$  of a given plant  $P$  if  $D$  is non singular,  $P = ND^{-1}$ , and  $N, D$  are right-coprime over  $\mathcal{A}$ . Analogously,  $(\tilde{D}, \tilde{N})$  is a left-coprime factorization (l.c.f.) in  $\mathcal{A}$  of  $P$  if  $\tilde{D}$  is non singular,  $P = \tilde{D}^{-1}\tilde{N}$ , and  $\tilde{D} \in \mathcal{A}^{p \times p}$ ,  $\tilde{N} \in \mathcal{A}^{p \times m}$  are left coprime over  $\mathcal{A}$ , that is there exist  $\tilde{X} \in \mathcal{A}^{m \times p}$  and  $\tilde{Y} \in \mathcal{A}^{p \times p}$  such that  $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_p \delta$ .

## 2.2 Properness, BIBO-stability and stabilization

Elements  $P$  in  $\mathcal{A}$  are causal and proper, in the sense that  $\sup_{\text{Re}(s) \geq 0, |s| \geq r} |\hat{P}(s)|$  is bounded for some  $r > 0$  [5, 4]. Elements  $P_{\text{na}}$  in the ideal  $L_1(0, \infty)$  of  $\mathcal{A}$  are strictly

proper, that is  $\lim_{r \rightarrow \infty} \sup_{\text{Re}(s) \geq 0, |s| \geq r} |\hat{P}_{\text{na}}(s)|$  is zero [5, Property A.6.2 pp. 636]. A matrix of transfer functions  $\hat{P}$  is said to be proper (strictly proper) if its entries are proper (strictly proper). Let  $P$  be a  $p \times m$  causal plant, and assume that  $P$  admits a r.c.f.  $(N, D)$ , with  $N \in \mathcal{A}^{p \times m}$  and  $D \in \mathcal{A}^{m \times m}$ . Using the Bézout identity (4), it is readily verified that  $P$  is proper if and only if  $D^{-1}$  is causal and proper. For  $D = D_{\text{pa}} + D_{\text{na}}$ , with  $D_{\text{pa}} = \sum_{i \geq 0} D_i \delta_{r_i}$ , this in turn implies that  $D_0$  is invertible in  $\mathbb{R}^{m \times m}$ .

A causal convolution system  $y = Pu$  is said to be BIBO stable if  $P \in \mathcal{A}^{p \times m}$  [34], in the sense that any bounded input yields a bounded output. In control perspective, let  $C$  be a controller with a l.c.f.  $(\tilde{D}_c, \tilde{N}_c)$  in  $\mathcal{A}^{m \times m} \times \mathcal{A}^{m \times p}$ , interconnected with the plant  $P$  as represented in Figure 1. The feedback system in Figure 1 satisfies  $\mathbf{y} = W_{P,C} \mathbf{u}$ , where

$$W_{P,C} = \begin{bmatrix} I_p \delta - N \Lambda^{-1} \tilde{N}_c & -N \Lambda^{-1} \tilde{D}_c \\ D \Lambda^{-1} \tilde{N}_c & D \Lambda^{-1} \tilde{D}_c \end{bmatrix}, \quad (5)$$

with  $\mathbf{y} = \begin{bmatrix} y_1^T & y_2^T \end{bmatrix}^T$ ,  $\mathbf{u} = \begin{bmatrix} u_1^T & u_2^T \end{bmatrix}^T$  and

$$\Lambda = \tilde{N}_c N + \tilde{D}_c D. \quad (6)$$

The pair  $(P, C)$  is said to be stable, or  $C$  stabilizes  $P$ , if  $W_{P,C} \in \mathcal{A}^{(p+m) \times (p+m)}$ . It is well known that  $(P, C)$  is stable if and only if  $\Lambda$  in (6) is a unit in  $\mathcal{A}^{m \times m}$ , that is  $\Lambda^{-1}$  exists and lies in  $\mathcal{A}^{m \times m}$  [34, 29].

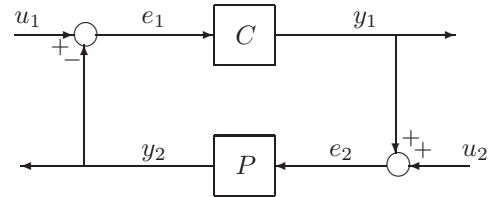


Fig. 1. Interconnection  $(P, C)$  by output feedback.

As an illustration, an acoustic wave in a duct with Dirichlet boundary conditions and a boundary constant impedance writes [4, Section 3.3 pp. 1106]

$$\hat{P}(s) = \frac{1 + \eta e^{-rs}}{1 - \eta e^{-rs}} \quad (7)$$

where  $\eta$  is the reflection coefficient such that  $|\eta| < 1$ . This proper and stable plant admits the coprime factorization in  $\mathcal{A}$  described by

$$\hat{N}(s) = 1 + \eta e^{-rs}, \text{ and } \hat{D}(s) = 1 - \eta e^{-rs}, \quad (8)$$

where  $D = \delta - \eta \delta_r$  is a unit in  $\mathcal{A}$ . This coprime factorization satisfies (4) with

$$\hat{X}(s) = -\hat{Q}(s)\hat{D}(s), \quad \hat{Y}(s) = \hat{D}(s)^{-1} + \hat{Q}(s)\hat{N}(s),$$

for any  $Q \in \mathcal{A}$ . Another example issued from [1] and the theory of transmission lines is

$$\hat{P}(s) = \frac{e^{-\sqrt{s}}}{s-1} \quad (9)$$

with a simple pole at  $s = 1$ . A coprime factorization in  $\mathcal{A}$  is  $\hat{N}(s) = \frac{e^{-\sqrt{s}}}{s+1}$  and  $\hat{D}(s) = \frac{s-1}{s+1}$ , which satisfies (4) for

$$\hat{X}(s) = -\frac{4e}{s+1} \text{ and } \hat{Y}(s) = \frac{(s+1)^2 - 4ee^{-\sqrt{s}}}{(s+1)(s-1)},$$

where it is noted that  $\hat{Y}(s) \in \mathcal{A}$  has a removable singularity at  $s = 1$ .

### 2.3 Graph topology and approximation

Graph topology gives a suitable framework for approximation in feedback perspective and robustness issues. We refer to [33,34,37] for details on this topology. For a causal plant  $P$  with a r.c.f.  $(N, D)$ ,  $N \in \mathcal{A}^{p \times m}$ ,  $D \in \mathcal{A}^{m \times m}$ , a neighbourhood of  $P$  in the graph topology can be characterized by convergence of coprime factorizations [34, ch. 7 p. 238]. For this, we consider the following definition.

**Definition 1** *For a subalgebra  $\mathcal{S}$  in  $\mathcal{A}$ , we say that the sequence  $\{P_\nu\}_{\nu \in \mathbb{N}}$ , with  $P_\nu = N_\nu D_\nu^{-1}$ ,  $N_\nu \in \mathcal{S}^{p \times m}$  and  $D_\nu \in \mathcal{S}^{m \times m}$ , is an approximation of  $P = ND^{-1}$  (in the graph topology) if for any  $\epsilon > 0$ , there exists  $\nu_\epsilon$  such that  $P_{\nu_\epsilon} \in \mathcal{B}(P, \epsilon)$ , where*

$$\mathcal{B}(P, \epsilon) = \{P_\nu = N_\nu D_\nu^{-1}, \left\| \begin{bmatrix} N - N_\nu \\ D - D_\nu \end{bmatrix} \right\|_{\mathcal{A}} < \epsilon\}. \quad (10)$$

It is noted that, for the r.c.f.  $(N, D)$  in  $\mathcal{A}$  of  $P$ , its approximation  $(N_\nu, D_\nu)$  is a r.c.f. in  $\mathcal{S}$  of  $P_\nu$ , for any sufficiently small  $\epsilon$  [34, p. 235]. Approximation by lumped systems in graph topology was studied in [35], and comes down to define the approximation subset  $\mathcal{S}$  in Definition 1 as the set of BIBO-stable lumped systems in the form

$$P(t) = P_0 \delta(t) + \sum_{i,j} P_{ij} t^j e^{-\lambda_i t}, \quad t \geq 0, \quad (11)$$

where  $P_0$  and  $P_{ij}$  are complex numbers,  $\text{Re}(\lambda_i) > 0$  for all  $i$ , and the sums are finite. A necessary and sufficient condition for the existence of a lumped approximation was established in the following result.

**Theorem 2** [35] *Let  $P$  be a plant with a r.c.f.  $(N, D)$ ,  $N \in \mathcal{A}^{p \times m}$ ,  $D \in \mathcal{A}^{m \times m}$ . Then  $P$  can be approximated by a lumped plant in the graph topology if and only if there exists a constant matrix  $M \in \mathbb{R}^{p \times (m+p)}$  of rank  $p$  such that*

$$M \begin{bmatrix} N_{\text{pa}} \\ D_{\text{pa}} \end{bmatrix} = 0.$$

For illustration, consider the plant  $P$  in (7) with the coprime factorization (8), where  $N = N_{\text{pa}} = \delta + \eta \delta_r$  and  $D = D_{\text{pa}} = \delta - \eta \delta_r$ . It is trivial to see that Theorem 2 is not fulfilled, or equivalently that  $P$  cannot be approximated by a lumped plant in the graph topology.

A direct consequence of Theorem 2 is that any element in  $L_1(0, \infty)$ , which is strictly proper, can be approximated in the graph topology by a lumped plant. See also [15].

## 3 Approximation by delay systems

The limits that appear in lumped approximation motivate to consider a larger class of approximation, with the introduction of a time-delay operator.

### 3.1 Class of Approximation

Take the class of approximation as systems governed by continuous-time equations with time-delay in the form

$$\dot{\chi}(t) = \sum_{i=1}^s A_i \chi(t - r_i) + Ew(t) + Bu(t) \quad (12)$$

$$\dot{w}(t) = Hw(t) + G\chi(t) \quad (13)$$

$$y_\nu(t) = \sum_{i=0}^s C_i \chi(t - r_i) + Lw(t) \quad (14)$$

for  $t \geq 0$  and ordered delays  $0 = r_0 < r_1 < \dots < r_s$ . The initial conditions are  $\chi(t) = \varphi_\chi(t)$  for  $t \in [-r_s, 0)$  (continuous and bounded) and  $w(0) = w_0$ . The equations (12)-(13) are (a particular case of) coupled differential-difference equations in standard form [11,12]. As a preliminary result which will be used in the general approximation procedure, let us start with the particular case of stable plants with only purely atomic part, which can be approximated in the graph topology by an input-delay system.

**Lemma 3** *Let  $P \in \mathcal{A}^{p \times m}$  and assume that  $P = P_{\text{pa}}$ . Then  $P$  can be approximated in the graph topology by a coupled differential-difference system in the form (12)-(14).*

**PROOF.** From assumption  $P = P_{\text{pa}}$ , the input-output plant  $y = Pu$  is described by  $P = \sum_{i \geq 0} P_i \delta_{r_i}$ . Since

$\{P_i\}_{i \in \mathbb{N}}$  is convergent (see (2)-(3)), for any  $\epsilon > 0$ , there exists  $\varsigma$  in  $\mathbb{N}$  such that the truncation  $\sum_{i=0}^{\varsigma} P_i \delta_{r_i}$  lies in  $\mathcal{B}(P, \epsilon)$ . Such an approximation admits a state-space realization in the form (12)–(14), with  $\chi = u$ ,  $w = 0$  and  $C_i = P_i$ , for  $i = 0, \dots, \varsigma$ .  $\square$

### 3.2 Approximation by coupled differential-difference systems

From this preliminary result, let us consider the general case.

**Theorem 4** *Let  $P$  be a causal and proper plant with a r.c.f.  $(N, D)$ ,  $N \in \mathcal{A}^{p \times m}$ ,  $D \in \mathcal{A}^{m \times m}$ . Then  $P$  can be approximated in the graph topology by a model governed by coupled differential-difference equations.*

**PROOF.** Let  $y = Pu$ , and introduce the partial state  $\xi(\cdot)$  in  $\mathbb{R}^m$  such that  $u = D\xi$  and  $y = N\xi$ . This proof aims to construct a coprime factorization  $(N_\nu, D_\nu)$  for the approximation such that  $P_\nu = N_\nu D_\nu^{-1}$  lies in  $\mathcal{B}(P, \epsilon)$ , for some fixed arbitrary  $\epsilon > 0$ . For this, denote  $D = \sum_{i \geq 0} D_i \delta_{r_i} + D_{\text{na}}$ . From Section 2.2, since  $P$  is causal and proper,  $D_0$  is nonsingular, so that  $u = D\xi$  can be put in the form

$$\begin{aligned} \hat{\xi}(s) = & - \sum_{i \geq 1} D_0^{-1} D_i e^{-sr_i} \hat{\xi}(s) - D_0^{-1} \hat{D}_{\text{na}}(s) \hat{\xi}(s) \\ & + D_0^{-1} \hat{u}(s). \end{aligned} \quad (15)$$

The term  $D_0^{-1} \hat{D}_{\text{na}}$  is strictly proper (and BIBO-stable), and can therefore be approximated by a lumped system in the graph topology [35]. It follows that there exist  $\kappa_D \in \mathbb{N}$  and real matrices  $(E_{\kappa_D}, H_{\kappa_D}, G_{\kappa_D})$ , respectively in  $\mathbb{R}^{m \times \kappa_D}$ ,  $\mathbb{R}^{\kappa_D \times \kappa_D}$  and  $\mathbb{R}^{\kappa_D \times m}$ , such that  $E_{\kappa_D}(sI_{\kappa_D} - H_{\kappa_D})^{-1}G_{\kappa_D}$  realizes a lumped approximation of  $D_0^{-1} \hat{D}_{\text{na}}$  in the graph topology. For the approximation of the purely atomic part  $\sum_{i \geq 1} D_0^{-1} D_i \delta_{r_i}$  in (15), we use Lemma 3 with the truncation of the first  $\varsigma_D$  terms, for some  $\varsigma_D \in \mathbb{N}$ . Hence, an approximation in the graph topology of  $u = D\xi$  writes  $u = D_\nu \chi$ , for  $\chi(\cdot) \in \mathbb{R}^m$ , and is realized by the state-space equations

$$\begin{aligned} \chi(t) = & - \sum_{i=1}^{\varsigma_D} D_0^{-1} D_i \chi(t - r_i) - E_{\kappa_D} w_D(t) + D_0^{-1} u(t) \\ \dot{w}_D(t) = & H_{\kappa_D} w_D(t) + G_{\kappa_D} \chi(t). \end{aligned}$$

For the relationship  $y = N\xi$ , we have  $N = N_{\text{pa}} + N_{\text{na}}$ . Similarly,  $N_{\text{na}}$  is approximated by a lumped system realized by  $(L_{\kappa_N}, H_{\kappa_N}, G_{\kappa_N})$ , in respectively  $\mathbb{R}^{p \times \kappa_N}$ ,  $\mathbb{R}^{\kappa_N \times \kappa_N}$  and  $\mathbb{R}^{\kappa_N \times m}$ , for some  $\kappa_N \in \mathbb{N}$ .  $N_{\text{pa}}$  is truncated to its first  $\varsigma_N$  terms, with  $\varsigma_N \in \mathbb{N}$ . The approxi-

mation of  $y = N\xi$  is therefore  $y_\nu = N_\nu \chi$ , governed by

$$\begin{aligned} \dot{w}_N(t) = & H_{\kappa_N} w_N(t) + G_{\kappa_N} \chi(t) \\ y_\nu(t) = & \sum_{i=0}^{\varsigma_N} N_i \chi(t - r_i) + L_{\kappa_N} w_N(t). \end{aligned}$$

Let  $w^T = [w_D^T \ w_N^T]$  and  $\varsigma = \max(\varsigma_D, \varsigma_N)$ . Combining the previous identities leads to the state-space equations in the form (12)–(14), where  $A_i = -D_0^{-1} D_i$ ,  $C_i = N_i$ ,  $B = D_0^{-1}$ ,  $E = [-E_{\kappa_D} \ 0]$ ,  $L = [0 \ L_{\kappa_N}]$  and

$$H = \begin{bmatrix} H_{\kappa_D} & 0 \\ 0 & H_{\kappa_N} \end{bmatrix}, \quad G = \begin{bmatrix} G_{\kappa_D} \\ G_{\kappa_N} \end{bmatrix}. \quad (16)$$

By convention, the terms  $C_i$  and  $A_i$  are zero if they are not defined.  $\square$

In the proposed approximation, the partial state  $\chi(\cdot)$  lies in  $\mathbb{R}^m$ , while  $w(\cdot) \in \mathbb{R}^{\kappa_D + \kappa_N}$  has a dependent dimension with respect to the desired accuracy of the approximation for the non atomic parts. Purely atomic parts lead to  $\varsigma$  pointwise (*a priori* independent) delays in (12)–(14). The accuracy of the approximation  $P_\nu$  is completely characterized by the parameters  $(\kappa_N, \varsigma_N, \kappa_D, \varsigma_D)$ , which define the order of the approximation.

## 4 Discussion and numerical method

### 4.1 Stabilization and input-output performances

Let  $P_\nu$  be an approximation in the graph topology of  $P$ , realized, as explained in Section 3.2, by a time-delay system governed by the coupled differential-difference equations (12)–(14). Following Section 2.2, assume that  $C$ , which is causal and proper with a l.c.f.  $(\tilde{D}_c, \tilde{N}_c)$ , stabilizes  $P_\nu$ , with a r.c.f.  $(N_\nu, D_\nu)$ . Such synthesis may be obtained by advanced tools in delay systems, see [20, 28, 16, 9, 6] and references therein. Denote

$$\Lambda_\nu = \tilde{N}_c N_\nu + \tilde{D}_c D_\nu \quad (17)$$

the unit matrix in  $\mathcal{A}^{m \times m}$ . For  $\Lambda$  defined in (6), we have

$$\Lambda = \Lambda_\nu \left( I_m \delta + \Lambda_\nu^{-1} \begin{bmatrix} \tilde{N}_c & \tilde{D}_c \end{bmatrix} \begin{bmatrix} N - N_\nu \\ D - D_\nu \end{bmatrix} \right). \quad (18)$$

It follows that  $(P, C)$  is stable if and only if  $\Lambda$  is a unit in  $\mathcal{A}^{m \times m}$ . Hence, a sufficient condition for stabilization is that

$$\left\| \begin{bmatrix} N - N_\nu \\ D - D_\nu \end{bmatrix} \right\|_{\mathcal{A}} < \frac{1}{\left\| \Lambda_\nu^{-1} \begin{bmatrix} \tilde{N}_c & \tilde{D}_c \end{bmatrix} \right\|_{\mathcal{A}}}, \quad (19)$$



which always admits a solution for  $P_\nu \in \mathcal{B}(P, \epsilon)$  and a sufficiently small  $\epsilon > 0$ . In other words, if a coprime factorization  $(N, D)$  exists in  $\mathcal{A}$ , stabilization may be achieved using an approximation  $(N_\nu, D_\nu)$  as in Theorem 4 and a controller synthesized by time-delay tools. Remark also that, for input-output performance, the realized approximation leads to the input-output norm of the closed-loop plants  $\|W_{P,C} - W_{P_\nu,C}\|_{\mathcal{A}}$  that can be arbitrarily reduced as far as the approximation is closed, where  $W_{P,C}$  is defined in (5). The norm in  $\mathcal{A}$  being an upper bound for  $L_2$ -induced norm, this shows in particular that gain and phase approximations in frequency domain hold.

#### 4.2 State-space realization

Few comments on the state-space realization (12)–(13) are in order. For notations convenience, we only consider the single delay case ( $\varsigma = 1$  and  $r = r_1$ ). While (12)–(13) appear in the standard form of coupled differential-difference equations (see [11] and references therein) for analysis, it may be preferred the canonical form where all terms in the right-hand side of (12) are delayed. For this, the realization of coupled differential-difference equations (12)–(13) is equivalent to the realization

$$\begin{aligned}\phi(t) &= A_1\phi(t-r) + A_1Ew(t-r) + A_1Bu(t-r) \\ \dot{w}(t) &= (H + GE)w(t) + G\phi(t) + GBu(t)\end{aligned}\quad (20)$$

using the combination of variables

$$\phi(t) = \chi(t) - Ew(t) - Bu(t),$$

for any  $t \geq 0$ . This form is the so-called Roesser canonical form for 2-D systems [7, 19]. It is well known that such a realization using a finite bank of operators is equivalent to neutral time-delay systems [19]. Note that the initial conditions for (20) are  $\phi(t) = \varphi_\phi(t)$  for  $t \in [-r, 0)$ ,  $w(t) = \varphi_w(t)$  for  $t \in [-r, 0]$ , where  $\varphi_\phi = \varphi_\chi - E\varphi_w$  (by convention  $u(t) = 0$  for  $t \in [-r, 0)$ ) and

$$\varphi_w(t) = e^{Ht}w_0 + \int_0^t e^{H(t-\tau)}G\varphi_\chi(\tau) d\tau. \quad (21)$$

The differential equation (13) can also be transformed into a difference equation with distributed delay. For this, the constant variation formula applied to (13) writes as

$$w(t) = e^{Hr}w(t-r) + \int_{t-r}^t e^{H(t-\tau)}G\chi(\tau) d\tau \quad (22)$$

for  $t \geq 0$  and the initial condition  $\varphi_w(\cdot)$  in (21). Combining (12) and (22) leads to the difference equation with distributed delay

$$x(t) = A_d x(t-r) + B_d u(t) + \int_{t-r}^t G_d(t-\tau)x(\tau) d\tau \quad (23)$$

for  $t \geq 0$ , where  $x^T = [\chi^T \ w^T]$ ,  $B_d^T = [B^T \ 0]$ ,

$$A_d = \begin{bmatrix} A_1 & E e^{Hr} \\ 0 & e^{Hr} \end{bmatrix}, G_d(t) = \begin{bmatrix} E e^{Ht} G & 0 \\ e^{Ht} G & 0 \end{bmatrix}.$$

In reformulating the coupled differential-difference equations (12)–(13) in difference equations with distributed delay (23), additional dynamics appear. Indeed, the characteristic polynomial associated to the initial formulation (12)–(13) is (in the single delay case)

$$\pi_c(\lambda) = \det(\lambda I_m - H - G(I_m - A_1 e^{-\lambda r})^{-1} E), \quad (24)$$

while for the difference formulation (23), the characteristic equation writes

$$\pi_d(\lambda) = \pi_c(\lambda)\pi_{dd}(\lambda), \quad (25)$$

with

$$\pi_{dd}(\lambda) = \det((I_m - e^{-(\lambda I_m - H)r})(\lambda I_m - H)^{-1}). \quad (26)$$

$\pi_{dd}(\lambda)$  is actually a strictly proper pseudo-polynomial in  $\lambda$  [20], and more precisely it is an entire function and has no pole. Additional zeros of  $\pi_{dd}(\lambda)$  with respect to (24) are obtained in (26), and lie in the set

$$\lambda \in \{\sigma(H) + i \frac{2\pi k}{r}, \ k = \pm 1, \pm 2, \dots\}, \quad (27)$$

where  $\lambda \notin \sigma(H)$ ,  $\sigma(H)$  denoting the eigenvalues of  $H$ . Additional dynamics are, in the complex plane, along vertical chains with real parts located at  $\text{Re}(\sigma(H))$ . It follows that, since  $H$  is Hurwitz (see (16) and (31)), stability of the realizations (12)–(13) and (23) are equivalent.

#### 4.3 Numerical approximation

The numerical computation of the time-delay system that approximates  $P$  in the graph topology requires, from a coprime factorization  $(N, D)$ , to truncate the atomic parts  $(N_{pa}, D_{pa})$ , and to realize a  $L_1(0, \infty)$  approximation of the non atomic parts  $(N_{na}, D_{na})$ . This last part may be realized by any convergent method in  $L_1(0, \infty)$ , as for instance in [10, 13]. For direct computations in the Laplace domain, Laguerre polynomials are applied. See [31] and [21] for Laguerre approximation techniques of distributed parameter systems. Let us denote the Laguerre polynomials

$$p_\kappa(t) = \frac{e^t}{\kappa!} \frac{d^\kappa}{dt^\kappa} (e^{-t} t^\kappa) = \sum_{k=0}^{\kappa} \binom{\kappa}{k} \frac{(-t)^k}{k!}, \quad \kappa \in \mathbb{N}.$$

The elements  $g_\kappa(t) = p_\kappa(t)e^{-t/2}$ , for  $\kappa \geq 0$ , form an orthonormal basis of  $L_2(0, \infty)$ , and any function  $P \in$

$L_2(0, \infty)$  can be expanded as  $P(t) = \sum_{\kappa=0}^{\infty} c_{\kappa}(P)g_{\kappa}(t)$ , where  $c_{\kappa}(P) = \langle P, g_{\kappa} \rangle = \int_0^{\infty} P(t)g_{\kappa}(t)dt$  are the Laguerre coefficients of  $P$  relative to  $g_{\kappa}$  [31]. Assume that there exists  $\alpha_0 > 0$  such that  $e^{\alpha_0 t}P(t) \in L_2(0, \infty)$ . For any  $\alpha \in (0, \alpha_0)$ , it has been shown in [17] that the function

$$P_{\nu, \alpha}(t) = \sum_{\kappa=0}^{\nu} c_{\kappa, \alpha}(P)g_{\kappa}(t)e^{-\alpha t}, \quad (28)$$

with  $c_{\kappa, \alpha}(P) = \langle e^{\alpha t}P(t), g_{\kappa}(t) \rangle$ , lies in  $L_1(0, \infty) \cap L_2(0, \infty)$ , and converges to  $P$  in  $L_1(0, \infty)$ , that is

$$\lim_{\nu \rightarrow \infty} \|P - P_{\nu, \alpha}\|_{L_1} = 0. \quad (29)$$

The computation of the coefficients  $c_{\kappa, \alpha}(P)$  is made directly in the Laplace domain, by the formula

$$c_{\kappa, \alpha}(P) = \sum_{k=0}^{\kappa} \binom{\kappa}{k} \frac{1}{k!} \hat{P}^{(k)}\left(\frac{1}{2} - \alpha\right), \quad (30)$$

where  $\hat{P}^{(k)}$  stands for the  $k$ th derivative of  $\hat{P}$ , and the approximation (28) leads to (see [31, p. 379])

$$\hat{P}_{\nu, \alpha}(s) = \sum_{\kappa=0}^{\nu} c_{\kappa, \alpha}(P) \frac{(s + \alpha - \frac{1}{2})^{\kappa}}{(s + \alpha + \frac{1}{2})^{\kappa+1}}. \quad (31)$$

Particularizing this method for  $N_{\text{na}}$  and  $D_0^{-1}D_{\text{na}}$  and using a state-space realization for (31) leads to a constructive approximation of the quadruple  $(H, E, L, G)$  in (16) and (12)-(14) for the non atomic parts, while  $N_{\text{pa}}$  and  $D_0^{-1}D_{\text{pa}}$  are truncated, as proposed in Lemma 3.

## 5 Illustrations

### 5.1 Coupled heat-transport equation

Consider the coupled partial differential equations with boundary conditions

$$\xi_t(z, t) = \vartheta^2 \xi_{zz}(z, t), \quad z \in (0, \ell), \quad t \geq 0, \quad (32)$$

$$\xi(0, t) = 0, \quad \xi(\ell, t) = u(t) - y(t), \quad (33)$$

for  $\vartheta > 0$ , zero initial conditions and the transport equation

$$w_t(z, t) + w_z(z, t) = 0, \quad z \in (0, \ell), \quad t \geq 0, \quad (34)$$

with the boundary condition  $w(0, t) = \frac{1}{2}y(t)$ . The output signal is defined by  $y(t) = \xi(z_0, t) + w(z_0, t)$ , for a fixed  $z_0 \in (0, \ell)$ . The input-output transfer function for (32)-(34) is  $\hat{y}(s) = \hat{P}(s)\hat{u}(s)$ , where

$$\hat{P}(s) = \frac{\frac{\sinh(\frac{z_0\sqrt{s}}{\vartheta})}{\sinh(\frac{\ell\sqrt{s}}{\vartheta})}}{1 - \frac{1}{2}e^{-z_0s} + \frac{\sinh(\frac{z_0\sqrt{s}}{\vartheta})}{\sinh(\frac{\ell\sqrt{s}}{\vartheta})}}. \quad (35)$$

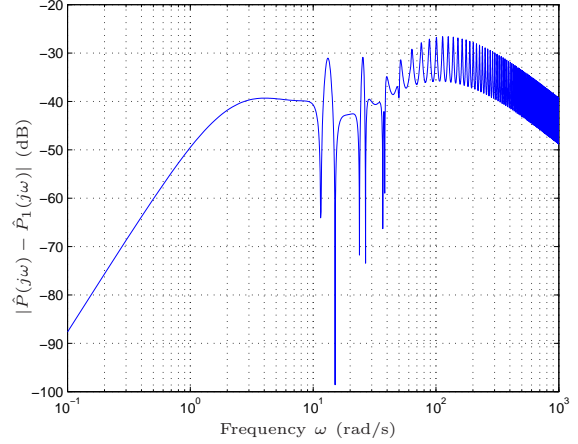


Fig. 2. Approximation error  $|\hat{P}(j\omega) - \hat{P}_1(j\omega)|$  frequency diagram in (35)-(37) for the coupled heat-transport system.

Define  $\hat{N}(s) = \frac{\sinh(\frac{z_0\sqrt{s}}{\vartheta})}{\sinh(\frac{\ell\sqrt{s}}{\vartheta})}$ ,  $\hat{D}(s) = 1 - \frac{1}{2}e^{-z_0s} + \hat{N}(s)$ .

The poles of  $\hat{N}(s)$  are  $-(\frac{k\pi\vartheta}{\ell})^2$ , for  $k = 1, 2, \dots$ . This implies that  $N(\cdot) \in \mathcal{A}$  and is strictly proper [4], and also that  $D(\cdot) \in \mathcal{A}$ . Furthermore,  $(N, D)$  is a coprime factorization of  $\hat{P}(s)$ , since  $\hat{X}(s)\hat{N}(s) + \hat{Y}(s)\hat{D}(s) = 1$ , with  $\hat{X}(s) = -\hat{Y}(s) = -\frac{1}{1 - \frac{1}{2}e^{-z_0s}}$  elements in  $\mathcal{A}$ . The Laguerre approximation in  $L_1(0, \infty)$  is realized for  $N = N_{\text{na}}$  and  $D_{\text{na}} = N$ , since assumptions before (28) hold and  $\alpha_0$  can be taken in  $(0, \frac{\pi^2\vartheta^2}{\ell^2})$ . Simulations are obtained in Figures 2-3, for  $\ell = 1$ ,  $z_0 = \ell/2$ ,  $\vartheta = 1$  and  $\alpha = \frac{\pi^2\vartheta^2}{\ell^2} - \frac{1}{2}$ . The approximation of  $\hat{N}(s)$  with  $\nu = 1$  in (28), and then of  $\hat{D}_{\text{na}}(s)$ , is realized by  $\hat{N}_1(s) = E(sI_2 - H)^{-1}G$ , where

$$H = \begin{bmatrix} -19.74 & -97.41 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 5.97 & 59.1 \end{bmatrix}.$$

The state-space realization of  $\hat{P}_1(s)$  is

$$\chi(t) = \frac{1}{2}\chi(t - z_0) - Ew(t) + Bu(t) \quad (36)$$

$$\dot{w}(t) = Hw(t) + G\chi(t) \quad (37)$$

with  $B = 0.912$  and the output of the approximation plant satisfies  $y_1(t) = Ew(t)$ . The frequency diagram of the error  $|\hat{P}(j\omega) - \hat{P}_1(j\omega)|$  between the exact plant (35) and  $\hat{P}_1(s)$  is reported in Figure 2. This gives an illustration of the approximated magnitude in the frequency domain (see below (19) in Section 4.1), related to the convergence of coprime factors in the  $\mathcal{A}$ -norm, where  $P_1 \in \mathcal{B}(P, 0.05)$ . While the approximation is of first order  $\nu = 1$  (corresponding to the Laguerre polynomial degree), its realization is obtained by a coupled scalar one-delay difference equation with a second order  $(\nu + 1)$  differential equation only. Using [35, Th. 4.1],  $\hat{P}(s)$  can

actually be approximated in the graph topology by a lumped system. However, lumped approximation may involve large-scale realization for a similar level of performance. This can be seen in Figure 3 where the unit step response for (36)-(37) reproduces with fidelity the transport-heat phenomenon considered in (32)-(34).

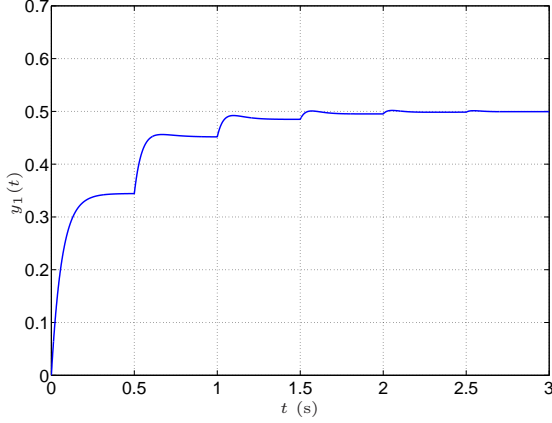


Fig. 3. Unit step response  $y_1(t)$  of  $\hat{P}_1(s)$  in (36)-(37) for the coupled heat-transport system.

## 5.2 Damped wave equation

As a second illustration, consider the damped wave equation

$$\xi_{zz}(z, t) = \frac{1}{c^2} \xi_{tt}(z, t) + \gamma \xi_t(z, t) + \mu \xi(z, t), \quad (38)$$

for  $z \in (0, \ell)$ ,  $t \geq 0$ , with positive reals  $c$ ,  $\gamma$  and  $\mu$ , the boundary conditions

$$\xi(0, t) = 0, \quad \xi(\ell, t) = u(t), \quad (39)$$

and zero initial condition. Let

$$\varpi(s) = s^2 + c^2 \gamma s + c^2 \mu = (s + \frac{c^2 \gamma}{2})^2 + \eta, \quad (40)$$

with  $\eta = c^2 \mu - \frac{c^4 \gamma^2}{4}$ . From (38)-(39), the transfer function  $\hat{\xi}(z, s) = \hat{P}(s) \hat{u}(s)$  is, for any  $z \in (0, \ell)$ ,

$$\hat{P}(s) = \frac{\sinh(\frac{z}{c} \sqrt{\varpi(s)})}{\sinh(\frac{\ell}{c} \sqrt{\varpi(s)})}. \quad (41)$$

The poles of  $\hat{P}(s)$  are, for  $k = 1, 2, \dots$ ,

$$s_k = \begin{cases} -\frac{c^2 \gamma}{2} \pm i \sqrt{\eta + (\frac{k \pi c}{\ell})^2}, & -\frac{\eta \ell^2}{\pi^2 c^2} \leq k^2 \\ -\frac{c^2 \gamma}{2} \pm \sqrt{-\eta - (\frac{k \pi c}{\ell})^2}, & k^2 \leq -\frac{\eta \ell^2}{\pi^2 c^2} \end{cases}, \quad (42)$$

and satisfy  $\text{Re}(s_k) \leq \max(-\frac{\mu}{\gamma}, -\frac{c^2 \gamma}{2})$ . With the Bessel function

$$J_1(t) = \frac{1}{\pi} \int_0^\pi \cos(\tau - t \sin \tau) d\tau, \quad (43)$$

we remark that the following decomposition holds [36, p. 164]

$$e^{-\frac{\beta(z)}{c} \sqrt{\varpi(s)}} = e^{-\frac{c \gamma \beta(z)}{2}} e^{-\frac{\beta(z)}{c} s} - \hat{\psi}_{\beta(z)}(s), \quad (44)$$

where  $\beta(z) = \ell + z$ ,  $\varrho(t, \beta(z)) = \sqrt{t^2 - \left(\frac{\beta(z)}{c}\right)^2}$ ,

$$\psi_{\beta(z)}(t) = \frac{\frac{\beta(z) \sqrt{\eta}}{c} e^{-\frac{c^2 \gamma}{2} t}}{\varrho(t, \beta(z))} J_1(\sqrt{\eta} \varrho(t, \beta(z))) h(t - \frac{\beta(z)}{c})$$

and  $h(\cdot)$  stands for the Heaviside function. The function  $\psi_{\beta(z)}(\cdot)$  lies in  $L_1(0, \infty)$ , so that, using (42), a coprime factorization in  $\hat{\mathcal{A}}$  for  $\hat{P}(s)$  in (41) is

$$\hat{N}(s) = e^{-\frac{\ell-z}{c} \sqrt{\varpi(s)}} - e^{-\frac{\ell+z}{c} \sqrt{\varpi(s)}}, \quad (45)$$

$$\hat{D}(s) = 1 - e^{-\frac{2\ell}{c} \sqrt{\varpi(s)}}, \quad (46)$$

or equivalently

$$N(t) = e^{-\frac{\gamma c \beta(-z)}{2}} \delta(t - \frac{\beta(-z)}{c}) - e^{-\frac{\gamma c \beta(z)}{2}} \delta(t - \frac{\beta(z)}{c}) + \psi_{\beta(z)}(t) - \psi_{\beta(-z)}(t),$$

$$D(t) = \delta(t) - e^{-\frac{\gamma c \beta(\ell)}{2}} \delta(t - \frac{\beta(\ell)}{c}) + \psi_{\beta(\ell)}(t).$$

The non atomic parts of (45)-(46)

$$N_{\text{na}}(t) = \psi_{\beta(z)}(t) - \psi_{\beta(-z)}(t) \quad (47)$$

$$D_{\text{na}}(t) = \psi_{\beta(\ell)}(t) \quad (48)$$

are approximated in  $L_1(0, \infty)$  by lumped plants, as proposed in Section 4.3, while purely atomic parts

$$N_{\text{pa}}(t) = e^{-\frac{\gamma c \beta(-z)}{2}} \delta(t - \frac{\beta(-z)}{c}) - e^{-\frac{\gamma c \beta(z)}{2}} \delta(t - \frac{\beta(z)}{c})$$

$$D_{\text{pa}}(t) = \delta(t) - e^{-\frac{\gamma c \beta(\ell)}{2}} \delta(t - \frac{\beta(\ell)}{c})$$

are conserved in the approximation. Simulations are carried out for  $c = 1$ ,  $\gamma = 1$ ,  $\mu = 0.5$ ,  $\ell = 1$  and  $z = 0.3$ , and the approximation technique proposed in Section 4.3 applies with  $\alpha = \frac{c^2 \gamma}{2}$ . Two approximations  $P_\nu(t)$  of order  $\nu = 1$  and  $\nu = 8$  are reported in Figure 4, where the frequency diagram of the approximation error  $|\hat{P}(j\omega) - \hat{P}_\nu(j\omega)|$  is reported. Approximations satisfy  $P_1 \in \mathcal{B}(P, 0.14)$  and  $P_8 \in \mathcal{B}(P, 0.07)$ . In the first order



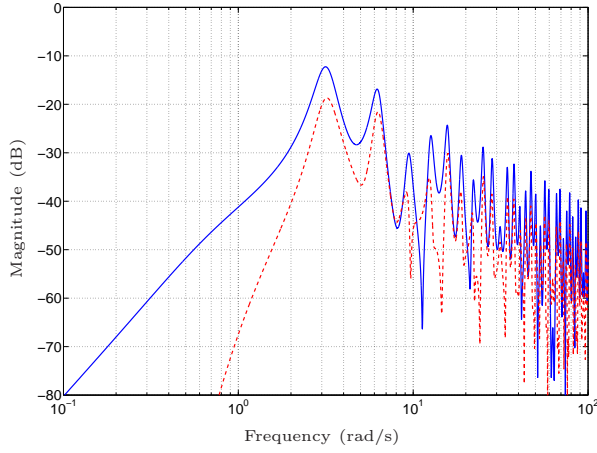


Fig. 4. Approximation error  $|\hat{P}(j\omega) - \hat{P}_\nu(j\omega)|$  diagram in (41) and (47)-(48), with  $\nu = 1$  (cont. line, blue) and  $\nu = 8$  (dot line, red), for the wave equation (38)-(39).

case  $\nu = 1$ , the approximation  $P_1$  is realized by

$$\begin{aligned}\chi(t) &= e^{-\frac{\gamma c \beta(\ell)}{2}} \chi(t - \frac{\beta(\ell)}{c}) - Ew(t) + u(t) \\ \dot{w}(t) &= Hw(t) + G\chi(t) \\ y_1(t) &= e^{-\frac{\gamma c \beta(-z)}{2}} \chi(t - \frac{\beta(-z)}{c}) - e^{-\frac{\gamma c \beta(z)}{2}} \chi(t - \frac{\beta(z)}{c}) \\ &\quad + Lw(t),\end{aligned}$$

where  $L = \begin{bmatrix} -0.06 & 0.03 \end{bmatrix}$ ,  $E = \begin{bmatrix} -0.14 & 0.12 \end{bmatrix}$  and

$$H = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Again, it is remarkable to verify that the first order approximation (corresponding to a second order differential equation coupled with a difference equation with one delay) can approximate the exact plant for all frequencies, which guarantees its pertinence in control perspectives.

## 6 Conclusion

In this paper, we define an approximation in the graph topology using norm convergence of a coprime factorization in the Wiener algebra. Such an approximation is governed by coupled continuous-time differential-difference equations. A constructive state-space realization is established, and a numerical method based on Laguerre polynomials is proposed and illustrated. Properties of this approximation appear to be of practical interest for control and simulation. While the content of this paper is not related to advances in quantitative aspects in topology, its interest is mainly in the existence result of the time-delay approximation and its perspective seen as a trade-off for analysis and synthesis

between distributed parameter systems and time-delay systems.

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