Textbook explanations: Modes of reasoning in 7 th grade Israeli mathematics textbooks

Boaz Silverman, Ruhama Even

Textbook explanations: Modes of reasoning in 7th grade Israeli mathematics textbooks

Boaz Silverman and Ruhama Even

Weizmann Institute of Science, Rehovot, Israel, boaz.silverman@weizmann.ac.il

The goal of this study is to characterize the justifications and explanations offered in 7th grade Israeli textbooks for mathematical statements. The justifications and explanations offered in eight 7th grade Israeli textbooks for ten selected mathematical statements were analysed, using the modes of reasoning framework (Stacey & Vincent, 2009). The analysis revealed that the textbooks commonly used several modes of reasoning in explanations for each statement. Nearly every justification was deductive or empirical, yet different modes of reasoning were used for geometric and for algebraic statements. It was also found that empirical justifications were more prevalent in textbooks of limited scope, whereas deductive justifications were typically offered in textbooks of regular/ extended scope.

KEYWORDS: Modes of reasoning, textbook analysis, justifications, explanations, empirical and deductive justifications.

INTRODUCTION

Proving, justifying, and explaining are important components of doing and learning mathematics. However, the extensive research on students' conceptions of proof and ways of justifying mathematical claims reveals students' difficulties in understanding the need for justification and in distinguishing between deductive and other types of justification (e.g., Harel & Sowder, 1998). One factor that considerably influences classroom instruction and students' opportunities to learn mathematics is the textbook used in class (Eisenmann & Even, 2011; Haggarty & Pepin, 2002).

Studies of justification and proof in mathematics textbooks have examined various aspects, such as reasoning and proving activities (e.g., Davis, Smith, Roy, & Bilgic, 2014; Fujita & Jones, 2014), the nature of proof (e.g., Miyakawa, 2012), and the opportunities provided in the textbook for students to learn reasoning and proof (e.g., Dolev & Even, 2012; Stacey & Vincent, 2009). To better understand students' opportunities to develop the habit of justifying, and to learn how to justify in mathematics, this study examines the justifications and explanations to key mathematical statements offered in mathematics textbooks, specifically 7th grade Israeli textbooks.

BACKGROUND

Justifications of mathematical statements vary in their nature, from informal and intuitive explanations to rigorous deductive proofs (e.g., Blum & Kirsch, 1991; Harel & Sowder, 2007; Sierpinska, 1994). Research on the issue of justifications in school mathematics attends to a wide range of aspects. Some researchers focus on the formality of justifications (Blum & Kirsch, 1991); others consider the community addressed (Sierpinska, 1994); and yet others focus on the proof scheme of justifications (Harel & Sowder, 2007).

Studies of the opportunities for students to read justifications and explanations in textbooks show that textbooks justify mathematical statements in several ways, and that valid proofs are rare. Building on Harel and Sowder's (2007) framework, Stacey and Vincent (2009) developed the modes of reasoning framework and used it to analyse Australian textbook explanations. Stacey and Vincent identified seven modes of reasoning in textbook explanations:

- *Appeal to authority*: null explanation or reliance on an external source of authority.
- *Qualitative analogy*: reliance on a surface similarity to non-mathematical situations.

- *Experimental demonstration*: identifying a pattern after checking selected examples.
- *Concordance of a rule with a model*: comparing specific results of a rule and a model.
- *Deduction using a model*: a model that serves to illustrate a mathematical structure.
- *Deduction using a specific case*: an inference process conducted using a special case.
- *Deduction using a general case*: an inference process conducted using a general case.

These seven modes can be generally divided into three categories: External sources (*appeal to authority* and *qualitative analogy*); Empirical justifications (*experimental demonstration* and *concordance of a rule with a model*); and Deductive justifications (*deduction using a model*, *a specific case*, or *a general case*). Stacey and Vincent found that justifications offered in the analysed textbooks used several modes of reasoning, yet students were given no indication regarding which can be classified as deductive proofs and which can only serve as supportive empirical evidence at best.

Drawing on Stacey and Vincent's (2009) conceptual framework, Dolev (2011) analysed the modes of reasoning in justifications offered for three mathematical claims in six 7th grade Israeli textbooks (experimental version). She found that all the textbooks offered justifications for the sampled claims, at times using several modes of reasoning. Additionally, Dolev found a difference between the modes of reasoning used in algebra and in geometry – geometric claims were often justified by *deduction using a general case*, whereas algebraic claims were rarely justified that way. A similar pattern was noted in other studies – an abundance of proofs in geometry (e.g., Fujita & Jones, 2014; Hanna & de Bruyn, 1999), and a small number of formal proofs in algebra (e.g., Davis et al., 2014; Hanna & de Bruyn, 1999).

This study builds on these studies, and expands their scope. The research objective is to examine the justifications and explanations for key mathematical statements offered in mathematics textbooks, centring on 7th grade Israeli textbooks (approved version). We examine three aspects: (1) the modes of reasoning offered, (2) the nature of justifications of algebraic vs. geometric statements, and (3) the nature of justifications in textbooks of limited vs. regular/extended scope, designed for students with different achievement levels.

METHODOLOGY

Ten key mathematical statements were selected for analysis from the Israeli 7th grade mathematics national curriculum (Ministry of Education, 2009), across several curricular topics, similar to those in Stacey and Vincent's (2009) study on Australian textbooks. The selection criteria for each analysed statement were: (1) it contains a mathematical idea or concept that requires justification in the national mathematics curriculum, and (2) it is considered to be an important result in the curriculum and in mathematics education literature. The statements are listed in the following:

Algebra:

- The distributive property: $a \cdot (b + c) = ab + ac$ for every three numbers a, b, c.
- The product of two negative numbers is a positive number.
- Division by zero is undefined.
- Performing a basic operation on both sides of an equation maintains their balance.
- Two algebraic expressions are equivalent if for arbitrary values of the symbols in them the equality holds.

Geometry:

- Vertically opposite angles are congruent.
- The area formula for a trapezium with bases a, b and altitude h is $(a + b) \cdot h/2$.
- The area formula for a circle with radius r is πr^2.
- Angle sum of a triangle is 180°.
- The corresponding angles between parallel lines are equal.

Analysis included all eight approved 7th grade textbooks for Hebrew speakers and their accompanying

Reminder: The **perimeter** of a circle with radius r is $2 \cdot \pi \cdot r$.
The **area** of a disc with radius r is $\pi \cdot r^2$.

Figure 1: A justification using an appeal to authority

teacher guides and supplementary materials. Six textbooks are of regular/extended scope, designed for the general student population (labelled A-F); two are of limited scope, designed for students with low achievements (labelled G-H). Data analysis included the textbook chapters introducing the statements – a total of 677 textbook pages; 57–110 pages (9–15%) from each textbook.

We analysed the explanations and justifications in the explanatory texts and those embedded in tasks and problems in the related task pools. Over 70% of the selected sections in the textbooks were analysed and discussed by 2–6 members of our research team until a consensus was achieved, and the remaining sections were analysed by the first author alone. First we identified distinct justifications of the statements in each section of the textbooks (i.e., explanatory texts and task pools). We then classified each justification for its mode of reasoning (following Stacey and Vincent, 2009) and compared frequencies relevant to the examined aspects: (1) the modes of reasoning offered; (2) algebra vs. geometry; and (3) limited vs. regular/extended scope.

FINDINGS

A total of 200 distinct justifications of statements were found in the textbooks. Comparison of the justifications by textbook section revealed that justifications were typically included in the explanatory texts, and seldom in tasks intended for student individual or small-group work, as shown in Table 1. This pattern was found in all textbooks (except textbook A) – regardless of the target student population, and in all statements – both in geometry and in algebra.

The modes of reasoning offered

Six of the seven modes of reasoning included in Stacey and Vincent's framework (2009) were identified in the Israeli textbook justifications, all but *concordance of a rule with a model*. Figures 1–6 exemplify justifications representing each of the identified modes of reasoning (translated from Hebrew).

Figure 1 illustrates a justification using an *appeal to authority* – a null explanation. The textbook merely presented a reminder for the area formula of a disc.

Figure 2 illustrates a justification using a *qualitative analogy*. The textbook included – among justifications using other modes of reasoning for the law of signs – an analogy that relies on a superficial similarity to the proverb "the enemy of my enemy is my friend". This justification does not reflect the mathematical structure of multiplication.

Figure 3 illustrates a justification using an *experimental demonstration*. The students were asked to tear paper triangles and rearrange the three angles, in order to convince themselves that the angle sum in a triangle is a straight angle.

15 Discussion **Wordplay:** match each statement with the corresponding sign rule.

- The friend of my friend is my friend.
- The friend of my enemy is my enemy.
- The enemy of my friend is my enemy.
- The enemy of my enemy is my friend.

Figure 2: A justification using a qualitative analogy

Section	Textbook								Total
	A	B	C	D	E	F	G	H	
Explanatory texts	16	22	26	17	25	28	20	23	177
Task pools	14	3	0	5	0	0	1	0	23
Total	30	25	26	22	25	28	21	23	200

Table 1: Number of distinct justifications by textbook section for each textbook

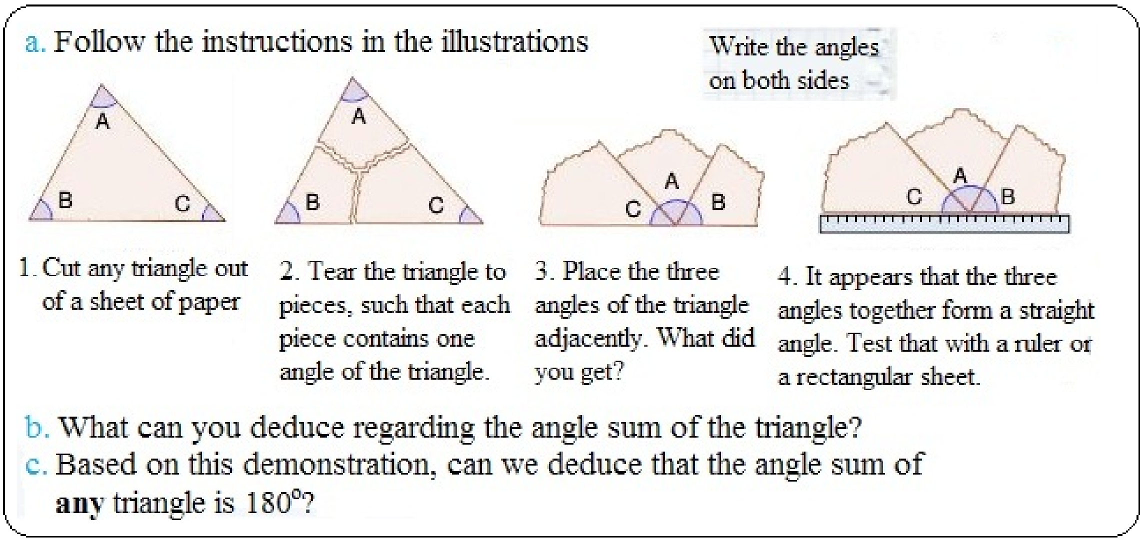


Figure 3: A justification using an experimental demonstration

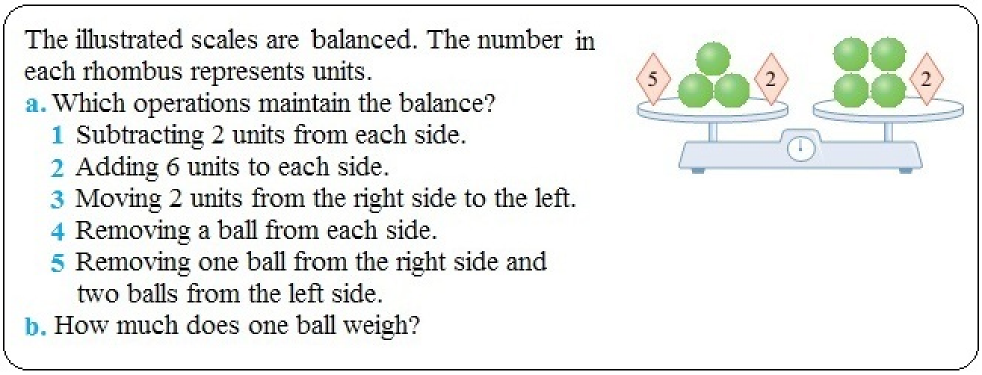


Figure 4: A justification using a deduction using a model

Figure 4 illustrates a justification using a *deduction using a model*. The textbook relied on a structural similarity between the scales (the model) and balancing equations (the mathematics involved) to justify the statement.

Figure 5 illustrates a justification using a *deduction using a special case*. The textbook justified the area formula of a trapezium by splitting the area of a specific trapezium and forming a chain of reasoning, in which each step is logically deduced from previous steps. The given lengths are intended as a generic case (i.e., the specific values can be replaced without loss of generality).

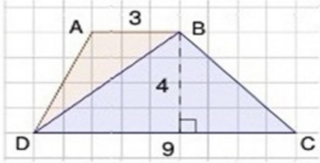
Figure 6 illustrates a justification using a *deduction using a general case*. The textbook justified the area formula of a trapezium by splitting the area of a general trapezium and forming a chain of reasoning, in which each step is logically deduced from previous steps. Pronumerals are used to note the lengths of the bases and of the altitude in the trapezium.

The textbooks provided justifications for all analysed statements (with one exception in Textbook E), commonly using several modes of reasoning in justifications for each statement, as shown in Table 2. Most of the justifications were of empirical (in blue) or deductive (in green, dark green, and olive green) modes, and only three instances of external sources (in red and dark red) were found. As Table 2 depicts, textbooks often justified a statement by using a certain mode of reasoning multiple times.

Algebra vs. geometry

Analysis of the modes of reasoning in the explanations offered in the textbooks revealed that different modes of reasoning were used for statements in algebra and in geometry. As Table 2 shows, algebraic statements (five top rows) were typically justified by deductive modes of reasoning, whereas geometric statements (five bottom rows) were usually justified by both deductive and empirical modes of reasoning. This analysis further shows that *deduction using a general case* appeared more in justifications of geometric

Draw the diagonal BD and split the area of the trapezium to two triangles: ΔABD and ΔDBC. The two triangles share an altitude (4 cm). The area of the trapezium ABCD is the sum of the areas of the two triangles:



$$S_{\Delta ABD} = \frac{3 \cdot 4}{2} \quad S_{\Delta DBC} = \frac{9 \cdot 4}{2}$$

$$S_{ABCD} = S_{\Delta ADB} + S_{\Delta DBC} = \frac{3 \cdot 4 + 9 \cdot 4}{2} = \frac{(3+9) \cdot 4}{2} = 24 \text{ cm}^2$$

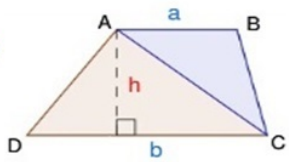
Let's observe the last step and identify the components of the formula. We see that we can find the area of a trapezium by finding the sum of lengths of the bases, times the length of the altitude, and dividing the product by 2.

$$S_{ABCD} = \frac{(\underset{\text{short base}}{3} + \underset{\text{long base}}{9}) \cdot \underset{\text{altitude}}{4}}{2}$$

Figure 5: A justification using a deduction using a specific case

Let's show that the area formula of a trapezium is true in all cases.
Fill in the missing steps in the proof.

- Draw the diagonal AC in the trapezium. It divides the trapezium ABCD into two triangles ΔABC and ΔADC.
- Therefore: $S_{ABCD} = S_{\Delta ABC} + S_{\Delta ADC}$
- The triangles ΔABC and ΔADC share an altitude h (why?)
- Let's write the area formula. Fill in the missing steps:



$$S_{ABCD} = S_{\Delta ABC} + S_{\Delta ADC} = \frac{a \cdot h}{2} + \frac{}{2} = \frac{ah+}{2} = \frac{(a+b) \cdot h}{2}$$

In other words: **The area of a trapezium is the sum of the lengths of the bases, times the length of the altitude, divided by 2.**

Figure 6: A justification using a deduction using a general case

Statement	Textbook							
	A	B	C	D	E	F	G	H
Distributive law	m,m	e,m	m,s	m,m ,s	e,m	m,m,s,g	e,m	m,s
Multiplication of negative integers	q,m,m, m,s,g	s,s	s,g	s,g	s,g	g	a,e	g
Division by zero	m,s,s	s	s,g	s,g	s,g	s,g	s	s,g
Balancing equations	e,m	s	e,m,s	s	e,m,g	m,s	e	m,s
Equivalent expressions	e,m,m,s	e,m,m,s	e,e,m,m	e,s	e,m,m,m,s	e,s	e,m,m,s	e,e,m,m
Opposite angles	e,g	e,g	s,g	g	s,g	e,g	e,s	s,g
Area of a trapezium	s,s,s,g	s,s, g,g,g	e,s,s, s,g	e,e,e, s,s	e,e,e,s	e,e,s,g	e,e,e,g	e,e,e, s
Area of a disc	g	g	g	g	a	e,g	g	g
Angle sum of a triangle	e,g,g	e,e,e, g,g	e,e,g	e,g,g	e,e,g	e,e,e,g, g,g,g	e,e,g	e,e,e
Corresponding angles	e,s,g	e,g	e,e	e,g	e	e	e	e,e

Legend: a= appeal to authority; q= qualitative analogy; e= experimental demonstration; m= deduction using a model; s= deduction using a specific case; g= deduction using a general case.

Table 2: Distribution of modes of reasoning in textbook explanations (n=200)

statements than in justifications of algebraic statements. Figure 7 presents the frequencies of modes of reasoning in textbook explanations in each textbook by content topic – algebra and geometry.

Limited scope vs. regular/extended Scope

Comparison of the explanations offered in textbooks of limited scope and of regular/extended scope was conducted on seven of the ten statements due to the textbooks' structure, a total of 136 distinct justifications. Figure 8 presents the distribution of the modes of reasoning in textbook explanations for these seven statements. A chi-square test of independence was used to test the association between textbook scope and empirical justifications. Textbooks of limited scope offered significantly more empirical justifi-

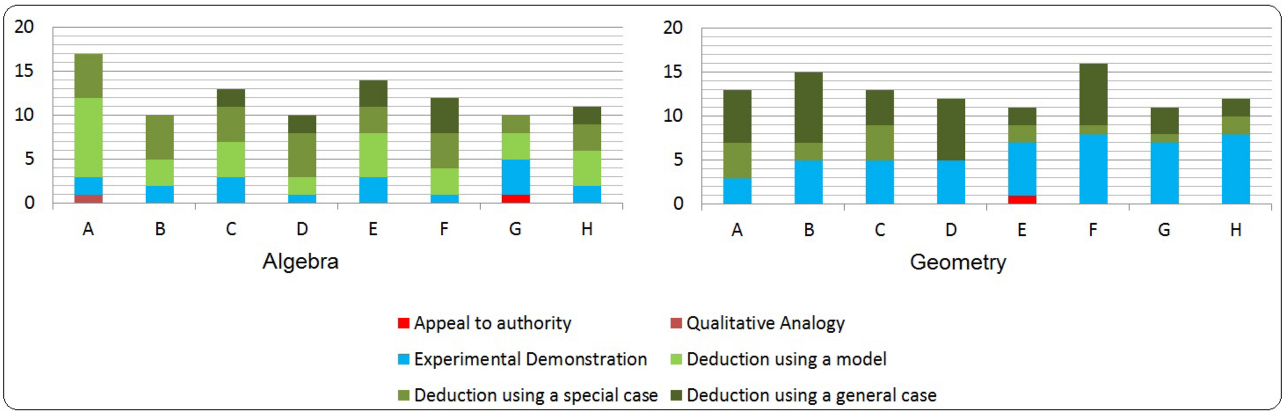


Figure 7: Frequencies of modes of reasoning by textbook and topic: algebra / geometry

cations than textbooks of regular/extended scope – roughly twice as much, $\chi^2_1 = 6.49$ (p = 0.01). An additional chi-square test was used to test the association between textbook scope and deductive justifications. Textbooks of regular/extended scope offered significantly more deductive justifications than textbooks of limited scope – roughly twice as much, $\chi^2_1 = 7.16$ ($p < 0.01$).

DISCUSSION

The findings reveal that Israeli 7th grade mathematics textbooks provide justifications for all the analysed key statements (but one statement in one textbook), commonly using several modes of reasoning in justifications for each statement. The inclusion of multiple modes of reasoning in textbooks might indicate an attentiveness of the textbook authors to the complexity of developing mathematical understanding, as the richness and diversity of textbook justifications might provide opportunities for students with different abilities, strengths, and backgrounds to learn and understand mathematics. Our results are in line with Stacey and Vincent's (2009) results on similar topics in Australian textbooks and with Dolev's (2011) results in Israeli textbooks (experimental version).

Our study shows that while most of the justifications for the analysed mathematical statements in 7th grade Israeli textbooks were included in the explanatory texts, some were embedded in the task pools intended for student individual or small-group work. However, our findings show that inclusion of the latter type of justifications in our analysis did not change the emerging patterns.

We found that 197 out of the 200 justifications analysed in the Israeli textbooks were deductive or empirical, implying that the textbooks typically explain each statement rather than present rules without reason. This finding does not comply with the findings reported in Stacey and Vincent's (2009) study, where 17% of the explanations for similar topics were neither deductive nor empirical.

Despite their emphasis on mathematical reasoning, our analysis shows that Israeli textbooks generally

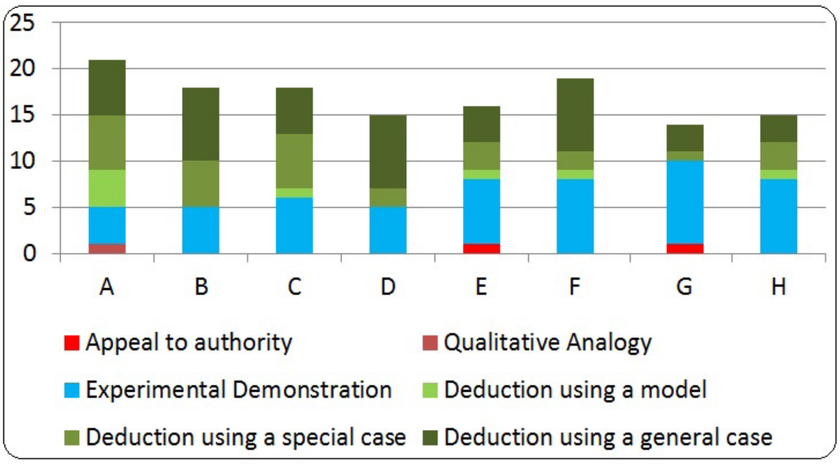


Figure 8: Frequency of modes of reasoning by textbook

give no indication regarding which justification can be considered a mathematically valid deductive proof and which serves only as a didactical tool to assist students' learning.

Comparison of the justifications for geometric statements and for algebraic statements in the Israeli textbooks showed that the textbooks typically use different modes of reasoning: deductive for algebraic statements, and both deductive and empirical for geometric statements. This finding is surprising at first glance due to the historic bias toward geometry as a subject suitable for teaching proof. However, careful analysis shows that the mode of reasoning closest to formal proof – *deduction using a general case* – appeared more in justifications of geometry statements than in justifications of algebra statements. Similar results were reported in Dolev (2011).

This study also shows that empirical justifications were more prevalent in textbooks of limited scope, whereas deductive justifications were predominant in textbooks of regular/extended scope. This pattern implies that student with different achievement levels are exposed to different types of justifications, and specifically that students with low achievements have fewer opportunities to deal with higher-order thinking and reasoning. These differences may have the potential to considerably limit the opportunities of students with low achievements to learn how to justify in mathematics, because teachers often follow teaching sequences suggested by textbooks (Eisenmann & Even, 2011; Haggarty & Pepin, 2002).

Our study focused on 7th grade textbooks. As Thompson (2014) notes, the similarities and differences we identified in this particular grade level – between textbook sections, between target student populations, and between curricular topics – might change over a textbook series. Additional research is needed to characterize the modes of reasoning in textbooks intended for higher grades.

ACKNOWLEDGEMENT

This research was supported by the Israel Science Foundation (grant No. 221/12).

REFERENCES

Blum, W., & Kirsch, A. (1991). Preformal proving: Examples and reflections. *Educational Studies in Mathematics*, *22*, 183–203.

Davis, J. D., Smith, D. O., Roy, A. R., & Bilgic, Y. K. (2014). Reasoning-and-proving in algebra: The case of two reform-oriented US textbooks. *International Journal of Educational Research*, *64*, 92–106.

Dolev, S. (2011). *Justifications and proofs of mathematical statements in 7th grade textbooks*. Rehovot: MSc thesis - Weizmann Institute of Science.

Dolev, S., & Even, R. (2012). Justifications and explanations in Israeli 7th grade math textbooks. *International Journal of Science and Mathematics Education*, 1–19.

Eisenmann, T., & Even, R. (2011). Enacted types of algebraic activity in different classes taught by the same teacher. *International Journal of Science and Mathematics Education*, *9*, 867–891.

Fujita, T., & Jones, K. (2014). Reasoning-and-proving in geometry in school mathematics textbooks in Japan. *International Journal of Educational Research*, *64*, 81–91.

Haggarty, L., & Pepin, B. (2002). An investigation of mathematics textbooks and their use in English, French and German classrooms: who gets an opportunity to learn what? *British Educational Research Journal*, *28*(4), 567–590.

Hanna, G., & de Bruyn, Y. (1999). Opportunity to learn proof in Ontario grade twelve mathematics texts. *Ontario Mathematics Gazette*, *37*, 23–29.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234–283). Washington, DC: Mathematical Association of America.

Harel, G., & Sowder, L. (2007). Toward Comprehensive Perspectives on the Learning and Teaching of Proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 805–842). Charlotte, NC: Information Age Publishing.

Ministry of Education. (2009). *Mathematics curriculum for grades 7–9*. Retrieved from http://meyda.education.gov.il/files/Tochniyot_Limudim/Math/Hatab/Mavo.doc (In Hebrew)

Miyakawa, T. (2012). Proof in geometry: A comparative analysis of French and Japanese textbooks. In Tso, T. Y. (Ed.) *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 225–232). Taipei, Taiwan: PME.

Sierpinska, A. (1994). *Understanding in Mathematics, Vol. 2*. Psychology Press.

Stacey, K., & Vincent, J. (2009). Modes of reasoning in explanations in Australian eighth-grade mathematics textbooks. *Educational Studies in Mathematics*, *72*, 271–288.

Thompson, D. R. (2014). Reasoning-and-proving in the written curriculum: Lessons and implications for teachers, curriculum designers, and researchers. *International Journal of Educational Research*, *64*, 141–148.