Mathematical fit: A first approximation

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We discuss here the notion of mathematical fit, a concept that might relate to mathematical explanation and mathematical beauty. We specify two kinds of fit a proof can have, intrinsic and extrinsic, and provide characteristics that help distinguish different proofs of the same theorem.

Keywords: Proof, fit, explanation, beauty, aesthetics.

INTRODUCTION

Mathematics, as a subject, stands out from other fields in several ways. Even if proofs derive from axioms, there is a sense – which is grounded in the rigorous process of proving – that certain claims can be shown to be true or false. This is part of what makes mathematics satisfying. There are answers that can be shown to be, beyond a shadow of a doubt, true. Similar to the feeling one gets from establishing the truth of a claim, one can often experience a feeling in mathematics that a claim is right, that a certain proof fits a theorem, or that a particular argument is exactly the one needed. This is a stronger and somewhat more mysterious requirement than that a particular claim is true. There might be many arguments that establish the truth of a claim, but what is that makes us feel that some arguments are right? Is this just a subjective feeling or is there some objective grounding for this feeling? The purpose of this paper is to sketch some criteria for what it might mean for the (somewhat more limited and hence manageable) question of what it means for a proof to fit a theorem. While the criteria grew originally from empirical data (polling data and discussions with mathematicians) we present the paper as a theoretical one, with the idea of building, or at least starting to build, a framework that could be tested more broadly.

FIT IN THE LITERATURE

The notion of fit has been discussed in both science and mathematics communities. For instance, Wechsler (1978) compares the experience of "fit" to more aesthetic experiences often considered to be more artistic (and less scientific):

> Scientists talking about their own work and that of other scientists use the terms "beauty," "elegance," and "economy" with the euphoria of praise more characteristically applied to painting, music, and poetry. Or there is the exclamation of recognition — the "Aha" that accompanies the discovery of a connection or an unexpected but utterly right realization in art and science. These are epithets of the sense of "fit" — of finding the most appropriate, evocative and correspondent expression for a reality heretofore unarticulated and unperceived, but strongly sensed and actively probed.

Sinclair (2002) has discussed the role of fit in the process of mathematical discovery. She discusses several different kinds of fit. One kind of fit has to do with recognizing a particular theorem in a larger class of theorems (a case of a square was a specific case of the more general category of polygon.) Another kind of fit has to do with a corporeal sensation of physically putting together a vertex and an edge in a diagram to get a desired result. Sinclair connects these experiences of fit to a more general aesthetic sensibility. She quotes Beardsley who said that the first feature of aesthetic experience was "a feeling that things are working or have worked themselves out fittingly" (Sinclair, 2002, p. 288).

Fit, or the related notion of fitting (more on this distinction in the discussion), are natural, but somewhat vague terms to describe an aesthetic experience in mathematics, of something being appropriate or right, or something sharing some sort of family membership or having some kind of inner coherence. In this paper we try to clarify what some of the characteristics of fit might be (we do not claim that the list of characteristics is exhaustive, but it is at least a start). We illustrate our analysis with two contrasting proofs of the Pythagorean theorem.

THEORETICAL MODEL

Here we discuss two ways that a proof can be fitting in mathematics, which we will refer to as intrinsic and extrinsic fit. Proofs are not the only mathematical objects that can possess fit. Definitions, diagrams, even theories, might be fitting, but in this paper we will limit the discussion to proofs. In this section we list some criteria for determining if a proof has intrinsic or extrinsic fit.

Criteria for intrinsic fit

Intrinsic fit refers to the relationship between a theorem and the underlying ideas in a proof of the theorem. It is what gives one a sense of what is going on in a proof, and how accessible the underlying ideas are. There are (at least) three criteria for intrinsic fit.

> **I$_1$**: Economy. The underlying ideas are represented as concisely as possible. We say a proof is economic (or not economic).
>
> The number of words is not really what determines economy. Sometimes a proof can be too terse to see what is going on. The proof should be as short as possible, but a knowledgeable reader should still be able to follow it, filling in the missing details as appropriate.
>
> **I$_2$**: Transparency. The proof allows the underlying idea to be easily grasped. The structure of the argument is clear. We say a proof is transparent (or not transparent).
>
> This criterion deals with the question of how easy is it to see what is going on in a proof. This has two components: that the general logical structure of the proof is clearly presented, and that the underlying idea that makes the proof work is clearly stated.
>
> **I$_3$**: Coherence. The proof is stated in the same terms as the theorem. We say that a proof is coherent (or not).
>
> A proof that coheres has a clear underlying idea that is explicitly put to use in proving the theorem. The terms with which one would naturally state the proof idea (such as areas or graph cycles or eigenvalues of matrices) are the same terms which are stated in the theorem, allowing one to easily see why that particular idea is essential to the theorem.

Criteria for extrinsic fit

Extrinsic fit refers to the relationship between a particular proof and a family of proofs. The single case and the family might be related via an idea, but what stands out is not as much the idea as the family membership. When you realize that a proof is in the family you think, “Oh it is one of those!” There are (at least) three criteria for extrinsic fit:

> **E$_1$**: Generality. The idea of the proof generalizes to a larger class of theorems. We say that a proof is general (or not), though we often mean that the idea of the proof is general.
>
> This criterion deals with how well an underlying idea generalizes to prove a class of theorems. The proof at hand is seen as a specific instance of a more general claim. Generality is not the same as explanation. For instance in category theory, theorems might be perfectly general but not at all explanatory. See Steiner (1978) for other examples.
>
> **E$_2$**: Specificity. The proof requires a specific tool, or a particular technical approach, to make it tractable. We say that a proof is specific (or not).
>
> This criterion also deals with how well a proof fits into a class of theorems, but not through the idea, as with the criterion of generality above, but through the specific choice of technical tool that makes the proof work. Whereas with the criterion of generality the focus is on the family membership (This is one of those kinds of proofs!), here the focus is on the appropriateness of the specific technical tool for the job (We found a tool that works!). The tool itself is not as much of interest as is the fact that the proof is now within reach.
>
> **E$_3$**: Connectedness. The proof idea connects to proof ideas of other theorems. We say a proof is connected (or not).

We see this particular proof as one of family of proofs. This criterion is at the heart of what is meant by family membership. The idea in one proof is the same as the idea in a family of proofs, and the common idea is what makes the proofs hang together as a family.

AN EXAMPLE

We take as an example a classic, and much discussed theorem — the Pythagorean theorem. We will show two proofs of the theorem and use the model described above to discuss the extent to which each proof fits the theorem. The first proof comes from Euclid (300 B.C./2008, VI. 31) and has been discussed, for instance, in Polya (1954), as a particularly nice proof of the theorem. We suggest this proof is a clear example of a proof that really fits the theorem. It has been suggested that this proof captures exactly what the theorem is about (e.g., Steiner, 1978). The second proof, which uses a clever argument based on trigonometry, does not fit. Our model helps clarify why this is the case.

Theorem: In a right triangle with sides a and b and hypotenuse c, $a^2 + b^2 = c^2$.

Proof 1: We are given a triangle with sides a and b and hypotenuse c. Also line h ^ line c.

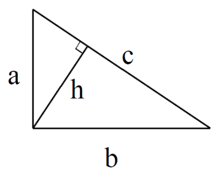

Figure 1

We can see that the sum of the areas of the smaller triangles is the same as area of the large triangle (by construction). We can also see (you can imagine folding out each of the three triangles over its longest side), that the sum of the areas of the triangles on sides a and b is equal the area of the triangle on side c. This relationship will hold for any similar figures on those sides, in particular squares, so $a^2 + b^2 = c^2$. QED

Why does this proof exhibit fit? Let us consider the criteria:

$\mathbf{I}_1$: Economy. The proof is economic. It gives relevant information simply and concisely. It could also be made more concise assuming more knowledge of the reader. (In this paper we have added some details for ease of reading.)

$\mathbf{I}_2$: Transparency. This proof is transparent. The proof consists of two main ideas, clearly presented, namely the dissection of the triangle into similar triangles, and that the equality of the areas carries over to arbitrary shapes.

$\mathbf{I}_3$: Coherence. The proof is coherent because the proof and the theorem are in the same terms, namely area. The theorem is a statement about the relationship of certain areas, and the proof directly relates these areas using properties of the triangles.

We also note that the idea of preserving areas is in line with the more famous proof in Euclid, where the areas of the squares constructed on either side are shown to be equal by area-preserving steps. A "Greek" proof that two areas are equal should ideally show that the one shape can be transformed into the other shape, using only area-preserving steps. The area-preserving step in the present proof is the reflection of either small triangle in the corresponding side of the large triangle. The scaling argument can also be traced to Euclid, and the present proof is also (less famously) given in Euclid.

$\mathbf{E}_1$: Generality. The proof is general. Steiner (1978) gives an account for this generality. This proof happens to be, according to Steiner, the proof that is most explanatory and most general. The generality comes from the fact that the proof works for arbitrary similar shapes constructed on the sides of the triangle.

$\mathbf{E}_2$: Specificity. The proof fulfils the criterion of specificity. The technical tool that works in this case is dividing the original triangle into similar triangles. Put more generally, this could be described as dissection. This division allows us to see the crucial relationship, namely that all three triangles are similar and their areas add up. The move to arbitrary similar figures, including squares, is a relatively small one.

$\mathbf{E}_3$: Connectedness. This criterion is not as easy to apply here as the other criteria, but we are inclined to say that this proof is connected. The class of proofs to which the proof can be seen to belong (other classes may be possible) might be taken to be proofs by area preservation, for instance the one referred to as "Greek" above.

Now consider the second proof, from Zimba (2009), which uses trigonometry.

Proof 2: Suppose we are given the subtraction formulas for sine and cosine:

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$, and
$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

Let α be the angle opposite to side a, and β be the angle opposite to side b, and without loss of generality, assume that $0 < \alpha \leq \beta < 90°$.

We now have:

$$\cos(\beta) = \cos(\alpha - (\alpha - \beta)) = \cos(\alpha)\cos(\alpha - \beta) + \sin(\alpha)\sin(\alpha - \beta) = \cos(\alpha)(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) + \sin(\alpha)(\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)) = (\cos^2(\alpha) + \sin^2(\alpha))\cos(\beta)$$

from which it follows that $\cos^2(\alpha) + \sin^2(\alpha) = 1$, since $\cos(\beta)$ is the ratio between one leg and the hypotenuse of a right triangle, and as such is never zero. The theorem now follows from the definitions of sine and cosine and scaling. QED

To what extent does this proof fulfill the criteria for intrinsic and extrinsic fit?

$\mathbf{I_1}$: Economy. The proof is economic. It gives relevant information simply and concisely, and here some details are left out.

$\mathbf{I_2}$: Transparency. This proof is not transparent. There is no clear sense of direction in the calculations performed. The structure of the proof is clear enough, and each step can easily be verified, but it seems that there is little in the way of a natural sequence of ideas, and the introduction of trigonometric quantities seems extraneous. Also, it is hard to see, for instance, why one would initially want to rewrite $\cos(\beta)$ as $\cos(\alpha - (\alpha - \beta))$.

$\mathbf{I_3}$: Coherence. The proof is not coherent. The trigonometry used in this proof is not in the same terms as the theorem, which is about areas. The work of the proof, that is to say the algebra, takes place in the language of trigonometry. We translate in and out of that language to see that the trigonometric manipulations establish the theorem.

$\mathbf{E_1}$: Generality. The proof as it stands is not general. It is true that once $\cos^2(\alpha) + \sin^2(\alpha) = 1$ is established, one can add the scaling argument to show that the result holds for arbitrary similar shapes, but the scaling argument is not an integral part of the proof.

$\mathbf{E_2}$: Specificity. The proof exhibits specificity, in that the tool used (the subtraction formulas), surprisingly works out to be adequate for the conclusion to be drawn.

$\mathbf{E_3}$: Connectedness. The proof is not connected. It is, as far as we know, the only trigonometric proof of the Pythagorean theorem, so there is no obvious family of proofs it would belong to.

Comparison

Here is how the two proofs compare in terms of fit, according to the six criteria. An X indicates that the proof satisfies the given criterion.

Criterion	I_1	I_2	I_3	E_1	E_2	E_3
Proof 1	X	X	X	X	X	X
Proof 2	X				X	

We conclude from this analysis that Proof 1 fits the Pythagorean theorem better than Proof 2 does. Notice that our notion of fit appears to be gradable. It seems natural to say that one proof fits better than another without all the criteria being fulfilled (or not fulfilled). What is less clear is how many criteria must be fulfilled to say that a proof exhibits fit at all. Further, the criteria are not equally weighted. It seems more central to the notion of fit to be coherent than to be economic.

DISCUSSION

We will now take up a few issues relating to fit that have a more general nature than those discussed above. First, we will discuss the relation between the terms 'fit' and 'fitting'. Next we will discuss the relation between fit and two other concepts, explanation and beauty. Finally we will discuss the applicability of the model given here.

Fit, Fitting, Fitness

There are several words related to fit which differ in meaning and use. With the examples above in mind of what it means for a proof to fit a theorem, or to fit into a class of theorems, we will explore the relation between these words. First, fit and fitting: Fit appears, commonly, to be a relation between two objects. A glove fits a hand. A model fits the data. The objects may be abstract, such as: The experience of going to Rome fits my expectations. The term can also be used metaphorically: Anna is a good fit for Roberto. In all of these cases, the objects that fit work like puzzle pieces. One set of objects has features that complement the features of the other object. When the match is found, we get a sense of satisfaction from having made and accomplished that match. However the fit might be more or less good, as in the case of a glove fitting a hand, or might be a perfect match, as in the case of a key fitting a lock.

Fitting, which has similar meaning to 'fit' has a slightly different connotation. Fitting often means 'appropriate', such as "that behavior was fitting for a man of his stature". Unlike fit, fitting often has a connotation of being socially appropriate. One would not say that a square is a fitting choice for this particular tessellation. 'Fit' refers more broadly to patterns found in nature, mathematics, etc. while 'fitting' is more restricted to the human sphere.

Fitness might not seem as obviously related to fit, but we mention it here to raise a question about whether the notion of 'fit' in mathematics might be at all related to the notion of 'fitness', say, in natural selection. One use of the term fitness has to do with physical aptitude. One trains to stay fit. When one is fit, one has achieved some level of fitness. In Darwinian terms, fitness is related to adaptability. The more adapted a species is for the environment the better it will fit. This reading of 'fit' or 'fitness' is not so different from what we call extrinsic fit above. The features that make a proof fit into a family of other proofs might be the ones that make it 'survive' in some sense, that it is more likely to be remembered, cited, and/or developed in mathematics. (This discussion bears resemblance to Gopnik (2000) who links understanding with sexual reproduction.)

Relation to explanation

Intrinsic fit and extrinsic fit have some parallels with contemporary treatments of mathematical explanation. We make no attempt here to summarize the literature in this area, and we do not try to spell out in detail any of the main models of mathematical explanation, but simply point to a few places in some of the more prominent theories where there are similarities with our account of mathematical fit.

Steiner (1978) provides an account of mathematical explanation in terms of 'characterizing property'. He described this as "a property unique to a given entity or structure within a family or domain of such entities or structures". This description of characterizing property has an obvious parallel with our notion of 'coherence'. Familial membership is central, both in identifying an entity as one that could explain, as well as in finding the grounds for the explanation. As Steiner continues, "an explanatory proof makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property." Similar, but not identical, with our notion of coherence, the relationship between the entity or structure and the result is central for determining if a proof explains (or has fit). A proof that has the same terms as a theorem, which is how we have characterized coherence, seems similar to a proof that evidently gives rise to a particular result.

We will mention very briefly Steiner's notion of 'deformation' because it is essential to his view of explanation (though it has been criticized elsewhere, e.g., Hafner & Mancosu, 2005). To Steiner, a proof that explains can be modified for members of a particular family (e.g. the set of all polygons) while keeping the proof idea the same. An explanatory proof can be deformed to produce a new theorem [3]. While problematic, the idea behind 'deformation', that an explanatory proof contains an idea that is invariant to certain inter-family sorts of transformations, is not completely counter-intuitive. The focus on characterizing properties and deformation seems to build on an intuition similar to that which underlies our distinction between intrinsic and extrinsic fit. In both Steiner's account of explanation and our account of fit there is some aspect that is internal and related to proof idea (and its accessibility, in the case of fit), and there is some aspect that is external and related to situating a proof as a member of a larger family.

Contemporary philosophy of mathematics has reached no consensus on mathematical explanation, but we will consider two more theories, one of which is more aligned with our notion of connectedness, and the other with our notion of coherence. These two aspects of fit correlate most closely related with current accounts of explanation.

Kitcher (1989) offers a view of explanation that is considered to be a counter proposal to Steiner's view, based on the notion of unification. Kitcher says that explanation arises from the use of arguments that have the same form (see Lange (in press) for a summary of this view). These explanations can be found in what Kitcher calls "the explanatory store" and the main task of a theory of explanation is to "specify conditions on the explanatory store" (Kitcher, 1989, p. 80). While the details of what gives rise to an explanation differ greatly in Kitcher's and Steiner's account, one similarity seems to be the emphasis on familial membership, or what we would call connectedness. In Steiner's account the membership comes about via characterizing properties, and in Kitcher's account it comes about via the explanatory store. The fact that there is some kind of unification or some sort of family traits that naturally carry over to similar entities or structures seems central both in these two accounts of mathematical explanation and in our account of mathematical fit.

In contrast to Steiner and Kitcher, whose views of explanation seem to have some component similar to that of connectedness, Lange (in press) suggests a view with three components (unity, salience, and symmetry), the first of which seems to be related to coherence. To Lange, "A proof is unified when it exploits a property that all of the cases covered by the theorem have in common and treats all of those cases in the same way" (personal communication). Unlike Kitcher and Steiner, whose unification and characterizing property ideas involve family membership, Lange's notion of unity is one that is intrinsic to the proof. Lange's concept of salience might also overlap with our criteria for intrinsic fit. Salience is a feature that is "worthy of attention" (Lange, p. 27). Transparency includes a feature that makes a certain idea accessible. While not exactly the same concept, the idea that certain features of a proof make it more readily processed by the mind seem to be a commonality between our account of mathematical fit and Lange's account of mathematical explanation.

Relation to beauty

Less clear than the relation between fit and explanation is the relation between fit and beauty. The opening quote by Wechsler hints at a connection, as does the quote by Beardsley (that the aesthetic experience arises from things working themselves out *fittingly*). Could there be any motivation in mathematics for aesthetic experiences to arise from some sense of fit or fittingness?

We have gathered a small amount of pilot data related to this question. The proofs of the Pythagorean theorem, given above, were shown to a group of six mathematicians and three mathematics educators, along with several other proofs of the theorems. Participants were asked to rank the proofs according to which was most aesthetically pleasing. All of the mathematicians ranked the first proof higher than the second. The two math educators who chose the second proof stated that they did so because they felt they did not fully understand the first proof, but they could follow the steps of the second proof. The words given by the mathematicians to describe the first proof included "simple", "beautiful" and "conceptually correct". The words given for the second proof included "ugly", "clever" and "unnatural". Note that a proof can be clever without it being beautiful.

While far from conclusive, this data seems to suggest that there might be reasons to believe that fit has something to do with beauty. The criteria of economy and coherence seem to be similar what the mathematicians meant by simplicity and conceptual correctness.

Some applications of the model

The model proposed here for mathematical fit is only a sketch, but it may be a small step toward clarifying a few open questions related to the philosophy and aesthetics of mathematics. First, it helps identify some ways that current theories of explanation are at odds with each other. Steiner's account of explanation, which overlaps with our category of coherence, and Kitcher's account, which is more related to our category of connectedness, might be, like the parable of the blind men and the elephant, in which the men describe different parts of the beast, characterising different, but not necessarily contradictory, aspects of fit.

Second, our model takes a modest step towards clarifying what beauty in mathematics might have to do with explanation. If Wechsler is right that beauty involve some sort of fit, we have sketched two different kinds of fit that might play a role in our aesthetic judgments. This may also be a way to identify whether beauty is an objective quality.

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ENDNOTES

1. Algebraic details: Let A, B, and C be the areas on the sides a, b, and c, respectively. Then $A/a^2 = B/b^2 = C/c^2$, which implies $A+B=a^2C/c^2+b^2C/c^2$. Since $A + B = C$, it follows that $(a^2+b^2)/c^2 = 1$ which implies $a^2+b^2= c^2$.

2. Note that for angles between 0 and 180 degrees, the subtraction formulas can be proven without recourse to the notion of distance, and are hence not dependent on the Pythagorean theorem. This proof is therefore not circular.

3. Counterexamples to this claim have been suggested, e.g. in Lange (in press).