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# Pre-service teachers' perceptions of generic proofs in elementary number theory

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*In this paper, we present our concept of the usage of four different types of proofs to engage students in the proving process: the generic proof (with numbers), the generic proof in the context of figurate numbers, the so-called “formal proof”, and the proof in the context of figurate numbers using “geometric variables”. Further, we report from our case study, where 12 pre-service teachers were interviewed after attending our bridging-course, which had a focus on argumentation and proof. We investigated students' perceptions of the generic proof with numbers. Our findings suggest the classification of perceptions in three categories: “Logical acceptance and psychological conviction”, “general acceptance of the concept and psychological uncertainty”, and “inappropriate understanding of the concept”.*

**Keywords:** Generic proof, conviction, transition, figurate numbers.

## INTRODUCTION

Constructing mathematical proofs is said to be a major hurdle for many university freshmen. It still remains a challenge of tertiary education to impart knowledge about the notion of proof. Selden (2012) stresses: “Understanding and constructing such proofs entails a major transition for students but one that is often supported by relatively little explicit instruction” (Ibid., 392). To tackle this problem, the University of Paderborn offers the course “Introduction into the culture of mathematics” as an obligatory course for the first year secondary pre-service teachers (non grammar schools). This course has been developed and taught by the second author in collaboration with the first author. Its content was selected in order to help students to successfully get to know the way of mathematical proving that is prevalent at university. During this course, the students are to investigate mathematical problems (e.g., concerning figurate

numbers) and to construct generic proofs and formal proofs. Refining and evaluating the course are a main focus of the first author's dissertation. In this context, students' perceptions of the different kinds of proofs were investigated.

## THEORETICAL FRAMEWORK AND RELATED RESEARCH

In their article concerning generic examples, Mason and Pimm (1984) describe the basic feature of a generic proof: “The generic proof, although given in terms of a particular number, nowhere relies on any specific properties of that number” (Mason & Pimm, 1984, 284). This concept of a general verification in a concrete context has mainly been substantiated in Leron and Zaslavsky (2009) and Rowland (1998). But still there seems to be no overarching consensus regarding the notion of generic proofs as valid mathematical proofs. However, generic proofs are said to be especially useful in the learning of reasoning and proving in the field of number theory (e.g., Karunakaran et al., 2014; Rowland 2002; Stylianides, 2012). The advantages are mainly seen in the accessibility even for low-performing students because of the absence of algebraic variables. But as shown by Biehler and Kempen (2013), the concept and the construction of a generic proof remain problematic for many pre-service teachers.

It is very common to give examples of generic proofs with the use of arrays or patterns of dots. This is a valid mathematical approach, since these patterns of dots can be considered as a notation system (Dörfler, 2008). In a semiotic view, concrete numbers, or patterns of dots, with their different properties and rules for operations, can build the notation system to perform diagrammatic reasoning and to construct valid mathematical proofs (e.g., Stjernfelt, 2000).

In contrast to the recommended use of generic proofs, it seems surprising, that students' perception of generic proofs has not been investigated in detail yet. The research of Tabach and colleagues (2010) indicates that teachers often reject correct verbal justifications (including generic proofs), because of a claimed lack of generality and an assumed overemphasis of concrete examples. But as was shown by Malek and Movshovitz-Hadar (2011), students can benefit from being presented to generic proofs concerning proof comprehension and proof construction.

### THE CONCEPT OF OUR BRIDGING COURSE

The course is meant to introduce students to the culture of the science mathematics, as it is practiced at university. We want to illustrate the procedural aspects and also give considerations to the ready-made knowledge of mathematics. Therefore, the students are to investigate assertions, build up hypotheses, test conjectures and form lines of arguments to finally achieve a mathematical proof. We also want to deal with the topics, which are known to be a barrier in the transition to tertiary level, so its content covers (1) discovery and proving in arithmetic, (2) figurate numbers, (3) proof by induction, (4) assertions, reasoning, types of proofs, (5) equations, and (6) functions.

The first chapter starts with the question "Someone claims: The sum of three consecutive natural numbers is always divisible by three. Is this correct?". Here, the students are engaged in testing concrete examples and in discussing their informative value. In this context of discovery and justification, we distinguish the significance of examples in a logical and in a psychological way: Logically, it is not important how many concrete examples one has tested, as the assumption is made for all natural numbers. Psychologically, the testing of several concrete examples can be seen positively to understand the assertion, to strengthen one's presumption on the validity of the assertion and maybe to get an idea why the statement is true (in all cases). Following these considerations, we introduce a generic proof of the statement as a possible student's answer (see below).

The concept of the generic proof is then exposed with an emphasis of its general argumentation. As a generic proof, we consider the combination of the following three parts: (1) there are operations on concrete examples that can be generalized, (2) one gives a

(generic) argumentation, why the assumption is true in these specific cases and finally, (3) one has to point out, why this argumentation also fits all possible cases. In doing so, it becomes possible to highlight the difference between purely empirical verifications and general valid arguments. And in using generic proofs, the phase of examples-based exploration becomes an intuitive part of the proving process. Referring to the generic proof, we formulate a correspondent formal proof using algebraic variables (see below). Although this proof production seems to be an almost trivial task for a mathematician, our experience has shown, that our freshmen are not used to this kind of argumentation and are not familiar with this usage of algebraic variables.

We also construct the correspondent proofs in the notation system of figurate numbers (see below). Afterwards, the assertion with three consecutive numbers gets generalized: "Is the sum of four consecutive numbers always divisible by four" and so on. At the end of the chapter, we attain the statement that the sum of  $k$  consecutive natural numbers is divisible by  $k$  if and only if  $k$  is odd.

In this first chapter, arithmetic is the area for doing research and proving conjectures. Here, the notation system of figurate numbers is a tool for constructing alternative types of proofs (see below). But in the second chapter, the figurate numbers themselves become the object of investigation, where arithmetic and algebra can be treated as tools for proving. The field of figurate numbers (e.g., triangular numbers and square numbers) offers excellent possibilities for exploration, forming conjectures and proving. Here, it seems very natural to argue with the arrangement and the number of dots.

### FOUR TYPES OF PROOFS

As we pointed out above, the generic proof is firstly presented in the context of the value of testing several examples. (See the following generic proof to the claim: The sum of three consecutive natural numbers is always divisible by three).

#### Generic proof with numbers:

$$1 + 2 + 3 = (2 - 1) + 2 + (2 + 1) = 3 \times 2$$

$$4 + 5 + 6 = (5 - 1) + 5 + (5 + 1) = 3 \times 5$$

You can always write the sum of three consecutive numbers as: (“number in the middle“-1) + “number in the middle” + (“number in the middle“+1). Since this sum equals three times the „number in the middle“, the sum is always divisible by three.

It is the narrative reasoning that follows the generic examples, which makes a generic proof a valid general argument. So it gets possible to stress the differences between purely empirical examples and valid general arguments. In this way, we work against the misconception that examples on their own can form a valid proof. But the use of generic proofs is also meant to pick up a form of argumentation that is said to be used at school (e.g., Leiß & Blum, 2006). So our students could be somehow familiar with this type of proof and if not, they get equipped with this appropriate way of proving for their future work. In the following transition to the so-called formal proof (see below), it gets possible to introduce and promote the mathematical symbolic language: (1) to express generality, (2) to communicate general incidents, (3) to explore further a supposed relationship, (4) to fulfill arguing and proving and (5) to provide a complete verification of a given statement (as recommended by Malle (1993) and Mason and colleagues (2005)).

**Formal proof:**

For all  $n \in \mathbb{N} \setminus \{1\}$ :  $(n - 1) + n + (n + 1) = n + n + n = 3 \times n$ . This sum is divisible by three, because  $n \in \mathbb{N} \setminus \{1\}$ .

In the whole course we establish figurate numbers (geometrical representations and operations with arrays or patterns of dots or squares) as another notation system. So the students are also asked to construct generic proofs using figurate numbers (see below) and the proof with geometric variables (see below).

**Generic proof in the context of figurate numbers:**

In the example, one can see the sum  $3 + 4 + 5$  and  $5 + 6 + 7$ . In every sum of three consecutive numbers, one obtains the same steps, independent from the starting number. After the transposition of the square at the far right, one always obtains three equal lines of squares. So, the sum is always divisible by three.



Table 1: The sum of three consecutive numbers represented by figurate numbers

**Proof with geometric variables:**

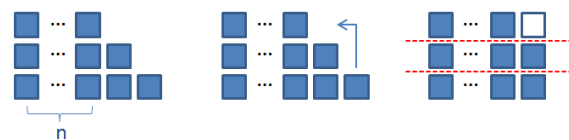


Table 2: A proof with “geometric variables” and figurate numbers

We defined the representation of an arbitrary number by inserting three little dots as a “geometrical variable” to have an analogy to the algebraic variables in the notation system of figurate numbers. This parallel treatment of arithmetic/algebra and geometric representation is said to be useful to ease the transition from arithmetic to algebra. Flores (2002) suggests a similar approach. Moreover, by using these alternative notation systems, it gets possible to stress the immanent quality of an argument, independent from its representation. While in the generic proof, the argument is given in a concrete context, which has to be generalized, the variables in the formal proof imply generality. So, it gets possible to emphasize the notion of variables in algebra and also in the geometrical context.

**RESEARCH QUESTIONS**

We will address the following research questions in this paper:

- 1) With which type of proof do the students start their proving process?
- 2) How can students’ different perceptions of generic proofs with numbers be characterized?
- 3) How can the different perceptions of generic proofs with numbers be distinguished with regard to the logical and psychological aspects mentioned above?

The research presented in this paper is part of a wider research project, which comprises the following further research questions: Are the students able to construct the four different kinds of proofs? Are there common pitfalls in their proof productions and if so, what are these? How can students’ different perceptions of proofs in the context of figural numbers be

characterized? How can these perceptions be distinguished in regard to the logical and the psychological aspects? How do the students judge the different proofs in comparison to each other? By knowing our entire research interest, the following research design becomes coherent.

## RESEARCH-DESIGN

Our study took place in the last week of the semester. We conducted sessions including proof construction and interviews with six different pairs of students, which were randomly selected from a group of volunteers. In the beginning of each session, the students were asked to prove a theorem (see “task analysis and expected solutions”) and afterwards, to construct the remaining three types of proof that they had not used spontaneously. For every proof production, the students were given two sheets of papers, the “draft paper” and the “clean copy”. The first sheet was meant to be used for their individual work, e.g., tests, explorations and proofs. After having finished the task individually, the two students were asked to develop a joint solution on the basis of their draft solutions and to write it correctly on the “clean copy”. After having constructed all proofs, the students and the interviewer looked through the proofs, correcting gaps or inaccuracies. This phase was included in order to ensure the reference to correct proofs in the following interview phase. At the beginning of the interview, the students had to answer a questionnaire and to rate the four proofs with respect to their persuasiveness, their validity, their quality of explanation and their appropriateness for school mathematics on a six-level Likert scale. The last part of the session was an interview. Firstly, the students were asked about the reasons for their spontaneously chosen type of proof at the beginning. Secondly, the interviewer asked questions based on the students' responses in the questionnaire to get to know their perceptions of the different types of proof. All sessions were recorded with two cameras, one in front, in order to capture gestures as well as motions in particular, and one in the back, filming students' writings.

## TASK AND EXPECTED SOLUTIONS

At the beginning of each session, the following task was given to the students:

*Prove or disprove: If one takes a natural number and adds its square, the result will always be divisible by 2.*

Since we only discuss students' different perceptions of generic proofs with numbers in this paper, we will only give one possible solution for the generic proof with numbers:

### Generic proof (with numbers):

$$3 + 3^2 = 3 \times (1 + 3) = 3 \times 4 = 12$$

$$4 + 4^2 = 4 \times (1 + 4) = 4 \times 5 = 20$$

*The sum of a natural number and its square always equals the product of the initial number and its successor. One of two consecutive natural numbers is always even and the other one is odd. Since the product of an odd and an even number is always even, the sum is even, i.e. divisible by two.*

## DATA-ANALYSIS

We transcribed each session and analyzed the transcripts and students' proof productions. For this case study investigating students' perceptions of the generic proof with numbers, we followed the quasi-judicial procedure developed by Bromley (1986). We focused on common and characteristic patterns in students' comments to ultimately categorize them as cases of a certain type. The findings suggest three different types of students' perceptions of generic proofs with numbers.

## RESULTS

### Students' spontaneous choice of types of proof at the beginning of the session

Out of the 12 students participating in the study, nine started immediately to construct a formal proof. In one group, the two students started with testing the statement with two concrete examples. Afterwards, one student explained, referring to their examples ( $3+9=12$  and  $4+16=20$ ), that the sum must be even, because the square of an odd number is always odd. And if you add any two odd numbers, the sum is always an even number. So, she discovered a part of a general argument, she could use for a generic proof with numbers. But afterwards, both students started to construct a formal proof with algebraic variables to verify the statement.



All participants started to use algebraic variables when writing down the assertion. For their choice of the formal proof with algebraic variables, they named different reasons: their socialization in school and/or university, that the formal proof is easier, because one does not have to have an “idea”, or they thought that the construction of the formal proof would be the intended task.

Only one student immediately wrote “generic proof” and started to investigate concrete examples. She gave the following reasons for her choice in the interview:

*Yes, for me it is a support. [...] When I'm writing these things down, I recognise how it works. And afterwards, I insert variables. [...] I'm looking for regularity or something similar.*

### **Students' perceptions of generic proof with numbers**

Analyzing the data, we could identify three different kinds of perceptions of generic proofs with numbers, which are described briefly and illustrated by examples from the transcripts. (The following transcripts were translated and linguistically smoothed.)

#### **(1) Logical acceptance and psychological conviction**

A student with this perception fully understands the concept of the generic proof and accepts the immanent general verification. The generic proof convinces the student without any doubt that the statement holds in any case. Moreover, he gets an insight, why the statement is true.

Michael reports his perception of generic proofs:

Michael: So, I marked “strongly agree” [for validity]. Because ... step by step, one can immediately follow the idea, e.g., you add this one and then you add the next one and so one. It is not the same as in a formal proof: It is just there and you have to prove it and that's it. This [the generic proof] is not [as] illustrative [as a proof] with pictures, but one can clearly see what happens. [...] That's what I like about a generic proof. If you see this, you can easily understand the way it has been done. There are always some examples given which are used for

the following argumentation you can easily understand.

#### **(2) General acceptance of the concept and psychological uncertainty**

In this case a student understands the concept of the generic proof and is willing to accept the immanent general verification. In contrast to this conviction, a subjective, intuitive doubt remains, but this emotional uncertainty is considered as unnecessary. Sarah mentions her perception of generic proof with regard to the persuasiveness:

Sarah: I do understand its [the generic proof's] general validity and overall meaning. But with regard to the persuasiveness: if one submits a generic proof to me, I would say: “Can I also have a formal one?”. For me, it is more convincing.

Christin describes her intuitive need to test the assumption for all natural numbers:

Christin: But this one [the formal proof] – for me – is somehow a more correct and coherent proof. For me, I would have to test it for all “n” - but it is nonsense, because one recognises the scheme, but...

#### **(3) Inappropriate understanding of the concept**

A student with this perception does not (fully) understand the concept of the generic proof. The student focuses on the concrete examples, without noticing the whole wider scheme and its general argument. The generic proof gets misinterpreted as a purely empirical verification.

We cite Paul and Amy as examples of perception (3). Paul quotes, that in a generic proof you are only testing specific examples:

Paul: It [the formal proof] is just – let me say – more correct.

Interviewer: What does it mean “more correct”?

Paul: Yes, correct in the sense that it shows its [the statement's] accuracy, so the validity, yes. And that the proposition is valid. We also show this in the generic proof,

but this only applies to the numbers, we have tested.

Amy mentions her perception of generic proofs, which illustrates her misconception:

Amy: Yes. If it [the generic proof] is sufficient for me? I'm not sure. So I wouldn't say that there is 100%-validity. Since there has been a homework, where we had to refute or to prove a statement. And then we had to test examples. I found two examples that worked, but maybe, there is a third one that doesn't. So for me, it [the generic proof] is not sufficient for I have not proven it. I did only prove it for these two examples and not for all numbers. So there is no validity.

Considering all interviews, we only found one student holding perception (1), identified four participants with perception (2) and five with perception (3). In two cases we could not categorize students' answers, as they did not state clear positions.

## DISCUSSION AND FINAL REMARKS

Nearly all students in our study started to work on the given task with formalisation and the construction of a formal proof using algebraic variables. Their socialization during their time in school and the reputation of advanced mathematics may be considered as reasons for this result. But one has to stress, that we explicitly did not try to convey an overemphasis of the formal proof in our course, but tried to highlight the validity of all four types of proofs. However, in this simple task the formal proof can be considered the easiest one. Another possible explanation could be that students think that "to provide a proof" implies the use of algebraic variables as we did not ask them to provide a "generic proof". Students may have misinterpreted this as a semiotic norm in the sense of Dimmel and Herbst (2014). Although this is incoherent to the semiotic norm the course has intended to establish.

Our findings concerning students' perceptions of generic proofs with numbers indicate that even after having passed our course, the notion of generic proof still remains problematic for a substantial proportion of our students. Only about half of the students in our

study hold a perception, where the generic proof is a general and logical valid argument [perception (1) and (2)]. Five students did not realize the generic aspect of the investigated examples and still hold the view of generic proofs as purely empirical verifications [perception (3)]. One can identify an important pitfall in the usage of generic proofs: When students do not understand the important difference between testing several examples and generic examples combined with valid narrative reasoning, they might be confirmed in their understanding that checking several examples may constitute a proof. Here, the question "why is it true" in combination with the explanatory power of proofs seem to be a promising way to address this misconception. One has to point out, that perceptions (1) and (2) are desired effects of our course. While there are students, who are completely convinced by a generic proof, others accept its logic, but still feel an intuitive doubt. For the latter group, the meaning of variables and the usage of the mathematical language can be pointed out even clearer. While we argue, that our proposed use of generic proofs and formal proof gives a meaningful introduction into the process of proving and the mathematical language, it becomes clear, that generic proofs are not "generic" by themselves.

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