



HAL
open science

Deviation inequalities for Banach space valued martingales differences sequences and random field

Davide Giraud

► **To cite this version:**

Davide Giraud. Deviation inequalities for Banach space valued martingales differences sequences and random field. 2016. hal-01280885v2

HAL Id: hal-01280885

<https://hal.science/hal-01280885v2>

Preprint submitted on 14 Dec 2016 (v2), last revised 11 Jan 2018 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DEVIATION INEQUALITIES FOR BANACH SPACE VALUED MARTINGALES DIFFERENCES SEQUENCES AND RANDOM FIELDS

DAVIDE GIRAUDO

ABSTRACT. We establish deviation inequalities for the maxima of partial sums of a martingale differences sequence, and of a strictly stationary orthomartingale random field. These inequalities can be used to establish complete convergence of normalized maxima of partial sums.

1. INTRODUCTION AND MAIN RESULTS

Probability inequalities play an important role in the study of properties of partial sums of random variables. A particular attention has been given to martingales. In Burkholder's paper [Bur73], distribution function inequalities for maximum of martingales are established, and moment inequalities are derived from them. Sharp results has been obtained for martingales with bounded increments [Hoe63, Azu67]. When the increments of the considered martingale are unbounded but square integrable, it is possible to control the tail function of the martingale by that of the increments and of the sum of conditional variances, like in [Bur73, Hae84, dIP99, FGL12, FGL15]. When the tail of increments have a polynomial decay, it seems that Nagaev's inequality [Nag03] gives the most satisfactory results. However, the involved martingales have to be square integrable. A natural question is thus the extension to martingales with a finite moment of order $p \in (1, 2)$, which is the main motivation of this note.

The paper is organized as follows: in Subsection 1.1, we state a deviation inequality for any Banach space valued martingale differences sequence, then for stationarity sequence. In Subsection 1.2, we review orthomartingales, and state a deviation inequality for stationary orthomartingale differences random fields. Section 2 is devoted to applications to complete convergence for martingale differences arrays and large deviations for strictly stationary martingale differences sequence and orthomartingale differences random fields. All these results are proven in Section 3.

1.1. Martingale differences sequences.

1.1.1. General case.

Definition 1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(B, \|\cdot\|_B)$ be a separable Banach space. For any $p \geq 1$, we denote by \mathbb{L}_B^p the space of B -valued random variables such that $\|X\|_{\mathbb{L}_B^p}^p = \mathbb{E}[\|X\|_B^p]$ is finite. Let $(\mathcal{F}_i)_{i \geq 1}$ be an non-decreasing sequence of sub- σ -algebras of*

Date: December 14, 2016.

Key words and phrases. Martingales, random fields, orthomartingales, deviation inequalities, complete convergence.

\mathcal{F} . We say that a sequence of B -valued random variables $(X_i)_{i \geq 1}$ is a martingale differences sequence with respect to the filtration $(\mathcal{F}_i)_{i \geq 1}$ if

- (1) for any $i \geq 1$, X_i is \mathcal{F}_i -measurable and belongs to \mathbb{L}_B^1 ;
- (2) for any $i \geq 2$, $\mathbb{E}[X_i | \mathcal{F}_{i-1}] = X_{i-1}$ almost surely.

Definition 1.2. Following [Pis75], we say that a Banach space $(B, \|\cdot\|_B)$ is r -smooth ($1 < r \leq 2$) if there exists an equivalent norm $\|\cdot\|$ such that

$$\sup_{t>0} \frac{1}{t^r} \sup \{ \|x + ty\| + \|x - ty\| - 2\|x\| = \|y\| = 1 \} < \infty.$$

From [Ass75], we know that if B is r -smooth and separable, then there exists a constant D such that for any sequence of B -valued martingale differences $(X_i)_{i \geq 1}$,

$$\mathbb{E} \left[\left\| \sum_{i=1}^n X_i \right\|_B^r \right] \leq D \sum_{i=1}^n \mathbb{E} [\|X_i\|_B^r] \quad (1.1.1)$$

Definition 1.3. We say that a separable Banach space $(B, \|\cdot\|_B)$ is (r, D) -smooth ($1 < r \leq 2$) if $(B, \|\cdot\|_B)$ is r -smooth and any sequence of B -valued martingale differences $(X_i)_{i \geq 1}$ satisfies (1.1.1).

Our first main result is an inequality in the spirit of Theorem 1 in [Nag03].

Theorem 1.4. Let $(B, \|\cdot\|_B)$ be a separable (r, D) -smooth Banach space where $1 < r \leq 2$. For each $1 < p \leq r$ and each $q > 0$, there exist two constants C and c depending only on q and D such that for any martingale differences sequence $(X_i, \mathcal{F}_i)_{i \geq 0}$, the inequality holds for each $n \geq 1$ and $x > 0$:

$$\begin{aligned} \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|S_i\|_B > x \right\} &\leq C \int_0^1 \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|X_i\| > cxu \right\} u^{q-1} du \\ &+ C \int_0^1 \mathbb{P} \left\{ \sum_{i=1}^n \mathbb{E} [\|X_i\|^p | \mathcal{F}_{i-1}] > cx^p u^p \right\} u^{q-1} du. \end{aligned} \quad (1.1.2)$$

Remark 1.5. We can choose $c = \min \left\{ 1/2, 2^{-q}/\sqrt{D} \right\}$ and $C = q8^q/(2^q - 1)$.

On one hand, Nagaev's result [Nag03] applies to real valued supermartingales, while our result is restricted to martingales. On the other hand, when applied to the latter class of random variable, our result gives a generalization in two directions. First, we consider Banach space valued random variables. Second, even when restricted to real-valued random variables, our result can be used to treat martingales whose increments do not necessarily have a finite moment of order 2.

1.1.2. Strictly stationary sequences. We now focus on the case of strictly stationary sequences. We assume that $T: \Omega \rightarrow \Omega$ is a bijective bi-measurable and measure preserving map and that \mathcal{F}_0 is a sub- σ -algebra of \mathcal{F} such that $T\mathcal{F}_0 \subset \mathcal{F}_0$. We assume that $X_i = m \circ T^i$ for some function $m: \Omega \rightarrow B$, since any strictly stationary sequence can be represented in this way (see [Doo53], p. 456, second paragraph). Furthermore, we assume that the filtration \mathcal{F}_j has the form $\mathcal{F}_j = T^{-j}\mathcal{F}_0$. In this way, the second term of the right hand side of (1.1.2) involves

a Birkhoff sum, for which the maximal ergodic theorem applies. Consequently, the right hand side of (1.1.2) can be written only with the distribution function of $\|m\|_B$ and $\mathbb{E}[\|m\|_B^p | T\mathcal{F}_0]$.

For a function $f: \Omega \rightarrow B$, we denote Uf the function $f \circ T$, $S_0(f) = 0$ and $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j = \sum_{j=0}^{n-1} U^j f$.

Theorem 1.6. *Let $(B, \|\cdot\|_B)$ be an (r, D) -smooth separable Banach space. For each $1 < p \leq r$ and each $q > p$, there exist two constants C and c depending only on p, q and D such that for any strictly stationary martingale differences sequence $(m \circ T^i, T^{-i}\mathcal{F}_0)_{i \geq 0}$ with values in B , the following inequalities hold for any $n \geq 1$ and $x > 0$:*

$$\begin{aligned} \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|S_i(m)\|_B > xn^{1/p} \right\} &\leq Cn \int_0^1 \mathbb{P} \left\{ \|m\|_B > cxun^{1/p} \right\} u^{q-1} du \\ &\quad + C \int_0^{+\infty} \mathbb{P} \left\{ \mathbb{E}[\|m\|_B^p | T\mathcal{F}_0] > cx^p u^p \right\} \min \{u^{q-1}, u^{p-1}\} du; \end{aligned} \quad (1.1.3)$$

$$\mathbb{P} \left\{ \max_{1 \leq i \leq n} \|S_i\|_B > xn^{1/p} \right\} \leq C \int_0^{+\infty} \mathbb{P} \left\{ \|m\|_B > xu \right\} \min \{u^{q-1}, u^{p-1}\} du. \quad (1.1.4)$$

1.2. Stationary orthomartingale differences random fields. Let $(T_q)_{q=1}^d$ be bijective, bi-measurable and measure preserving transformations on $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $T_q \circ T_{q'} = T_{q'} \circ T_q$ for each $q, q' \in \{1, \dots, d\}$. Let \mathcal{F}_0 be a sub- σ -algebra of \mathcal{F} such that for each $q \in \{1, \dots, d\}$, $\mathcal{F}_0 \subset T_q^{-1}\mathcal{F}_0$. In this way, $\mathcal{F}_i := T^{-i}\mathcal{F}_0$, $\mathbf{i} \in \mathbb{Z}^d$, yields a filtration for the coordinatewise order (that is, $\mathbf{i} \preceq \mathbf{j}$ if $i_q \leq j_q$ for each $q \in \{1, \dots, d\}$).

We also denote for \mathbf{k} and $\mathbf{l} \in \mathbb{Z}^d$ the coordinatewise minimum by $\mathbf{k} \wedge \mathbf{l} = (\min \{k_i, l_i\})_{i=1}^d$.

If for each $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d$ and each integrable and \mathcal{F}_1 -measurable random variable Y ,

$$\mathbb{E}[Y | \mathcal{F}_{\mathbf{k}}] = \mathbb{E}[Y | \mathcal{F}_{\mathbf{k} \wedge \mathbf{l}}] \text{ almost surely,} \quad (1.2.1)$$

the transformations $(T_q)_{q=1}^d$ are said to be *completely commuting*.

The collection of random variables $\{M_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is said to be an orthomartingale random field with respect to the completely commuting filtration $(T^{-i}\mathcal{F}_0)_{\mathbf{i} \in \mathbb{Z}^d}$ if for each $\mathbf{n} \in \mathbb{N}^d$, $M_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$ -measurable, integrable and for each $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^d$ such that $\mathbf{i} \preceq \mathbf{j}$,

$$\mathbb{E}[M_{\mathbf{j}} | \mathcal{F}_{\mathbf{i}}] = M_{\mathbf{i}}. \quad (1.2.2)$$

We denote for $\mathbf{n} \succcurlyeq \mathbf{1}$ the partial sums by $S_{\mathbf{n}}(m) := \sum_{\mathbf{0} \preceq \mathbf{i} \prec \mathbf{n}} m \circ T^{\mathbf{i}}$

Definition 1.7. *Let $m: \Omega \rightarrow \mathbb{R}$ be a measurable function. The random field $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale difference random field with respect to the completely commuting filtration $(T^{-i}\mathcal{F}_0)$ if the random field $(M_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^d}$ defined by $M_{\mathbf{n}} := S_{\mathbf{n}}(m)$ is an orthomartingale random field.*

Orthomartingale random fields have good properties with respect to marginal filtrations $\mathcal{F}_q^{(d)} := \sigma(\mathcal{F}_{\mathbf{k}}, k_q \leq j, \mathbf{k} \in \mathbb{Z}^d)$, $q \in \{1, \dots, d\}$. Furthermore, when a coordinate is fixed, we still have an orthomartingale random field with respect to a completely commuting filtration (see [Kho02], p.37, Theorem 3.5.1).

Lemma 1.8. *Let $m: \Omega \rightarrow \mathbb{R}$ be a measurable function. Let $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be an orthomartingale difference random field with respect to the completely commuting filtration $(T^{-i}\mathcal{F}_0)_{\mathbf{i} \in \mathbb{Z}^d}$. Then the following properties hold.*

- (1) For any $\mathbf{n} = (n_1, \dots, n_{d-1}) \in \mathbb{N}^{d-1}$, the sequence $(S_{(\mathbf{n},j)}(m))_{j \geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_j^{(d)})_{j \geq 0}$.
- (2) For any $\mathbf{n} = (n_1, \dots, n_{d-1}) \in \mathbb{N}^{d-1}$, the sequence $(\max_{i \in [1, \mathbf{n}]} |S_{(i,j)}(m)|)_{j \geq 0}$ is a non-negative submartingale with respect to the filtration $(\mathcal{F}_j^{(d)})_{j \geq 0}$.
- (3) For any $j \in \mathbb{N}$, the random field $(S_{(\mathbf{n},j)}(m))_{\mathbf{n} \geq \mathbf{1}}$ is an orthomartingale with respect to the completely commuting filtration $(T^{-i}\mathcal{F}'_0)_{i \in \mathbb{Z}^{d-1}}$, where \mathcal{F}'_0 is the σ -algebra generated by $\bigcup_{i \in \mathbb{Z}} T^{i\mathbf{e}_d} \mathcal{F}_0$.

We now state the analogue of Theorem 1.6 for strictly stationary orthomartingale differences random fields.

Theorem 1.9. *Let $(B, \|\cdot\|_B)$ be an (r, D) -smooth separable Banach space. For each $1 < p \leq r$, $q > p$ and positive integer d , there exist a constant C depending only on p, q, d and D such that for any strictly stationary orthomartingale differences random field $(m \circ T^i, T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ with values in B , the following inequalities hold for any $n \in \mathbb{N}^d$, $x > 0$ and $j \in \{1, \dots, d\}$:*

$$\begin{aligned} & \mathbb{P} \left\{ \max_{1 \leq i \leq \mathbf{n}} \|S_i(m)\|_B > x |\mathbf{n}|^{1/p} \right\} \\ & \leq C n_j \int_0^1 \mathbb{P} \left\{ \|m\|_B > x u n_j^{1/p} \right\} (1 + |\log u|)^{d-1} u^{q-1} du \\ & + C \int_0^{+\infty} \mathbb{P} \left\{ \mathbb{E} [\|m\|_B^p | T_j \mathcal{F}_0] > x^p u^p \right\} \min \{u^{q-1}, u^{p-1}\} (1 + |\log u|)^{d-1} du; \end{aligned} \quad (1.2.3)$$

$$\begin{aligned} & \mathbb{P} \left\{ \max_{1 \leq i \leq \mathbf{n}} \|S_i(m)\|_B > x |\mathbf{n}|^{1/p} \right\} \\ & \leq C \int_0^{+\infty} \mathbb{P} \left\{ \|m\|_B > x u \right\} \min \{u^{q-1}, u^{p-1}\} (1 + |\log u|)^{d-1} du. \end{aligned} \quad (1.2.4)$$

Remark 1.10. One can integrate the previously obtained inequalities to get moment inequalities. For example, it is possible to recover a multidimensional Burkholder's inequality in the stationary case, like in [Faz05]. Like in the one dimensional case, it is also possible to establish inequalities in weak \mathbb{L}^p spaces like in [JS88], Remark 6.

2. APPLICATIONS

2.1. Complete convergence for martingale differences sequences. A immediate application of Theorem 1.4 is a sufficient condition for the complete convergence of martingale differences arrays.

Theorem 2.1. *Let $(X_{n,k}, \mathcal{F}_{n,k})_{k \geq 1}$ be sequences of square integrable martingale differences with values in a separable $(2, D)$ -smooth Banach space B . Let $(c_n)_{n \geq 1}$ be a sequence of non-negative real numbers. Assume that there exists some $q > 0$ such that for any positive R ,*

$$\sum_{n=1}^{+\infty} c_n \sum_{k=1}^{+\infty} \int_0^R \mathbb{P} \left\{ \|X_{n,k}\|_B > u \right\} u^{q-1} du < +\infty \text{ and} \quad (2.1.1)$$

$$\sum_{n=1}^{+\infty} c_n \int_0^R \mathbb{P} \left\{ \sum_{k=1}^{+\infty} \mathbb{E} \left[\|X_{n,k}\|_B^2 \mid \mathcal{F}_{n,k-1} \right] > u^2 \right\} u^{q-1} du < \infty. \quad (2.1.2)$$

Then for each positive ε ,

$$\sum_{n=1}^{+\infty} c_n \mathbb{P} \left\{ \sup_{k \geq 1} \left\| \sum_{i=1}^k X_{n,i} \right\|_B > \varepsilon \right\} < +\infty. \quad (2.1.3)$$

Remark 2.2. The assumptions of Theorem 2.1 can hold even if the conditional variances $\mathbb{E} \left[\|X_{n,k}\|_B^2 \mid \mathcal{F}_{n,k-1} \right]$ are not bounded, which was assumed in [WH14] and [GC98].

2.2. Large deviations for stationary martingale differences sequence and orthomartingale differences random fields.

2.2.1. *Stationary martingale differences sequences.* Let $(m \circ T^i, T^{-i} \mathcal{F}_0)$ be a strictly stationary martingale differences sequence with values in an (r, D) -smooth separable Banach space. We are interested in upper bounds for the quantity $\mathbb{P} \{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \}$, $n \geq 1$.

For $s > 1$, we denote by $\mathbb{L}^{s, \infty}$ (respectively $\mathbb{L}_0^{s, \infty}$) the set of random variables X such that $\sup_{t > 0} t^s \mathbb{P} \{ \|X\|_B > t \} < +\infty$ (respectively $\lim_{t \rightarrow +\infty} t^s \mathbb{P} \{ \|X\|_B > t \} = 0$).

Theorem 2.3. *Let $(m \circ T^i, T^{-i} \mathcal{F}_0)$ be a strictly stationary martingale differences sequence with values in an (r, D) -smooth separable Banach space B . Assume that $1 < p < r$ and that $(m \circ T^i, T^{-i} \mathcal{F}_0)$ is a B -valued strictly stationary martingale differences sequence. If m belongs to \mathbb{L}^p , then*

$$\lim_{n \rightarrow +\infty} 2^{n(p-1)} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} = 0. \quad (2.2.1)$$

Boundedness of the sequence $(2^{n(p-1)} \mathbb{P} \{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \})_{n \geq 1}$ was established in Theorem 2.1 of [Li03], where the stationarity assumption is replaced by boundedness in \mathbb{L}^p .

Remark 2.4. It seems that we cannot derive the results in [DM07] from Theorem 1.6, even with the help of stationarity.

When a moment of order greater than two is finite, we can formulate precise results in terms of integrability of the increments and of the conditional variance term.

Theorem 2.5. *Let $s > 2$ and let B be a separable $(2, D)$ -smooth Banach space. There exists a constant $C(s, B)$ such that the following holds: for each B -valued strictly stationary martingale differences sequence $(m \circ T^i, T^{-i} \mathcal{F}_0)$,*

(1) *if $m \in \mathbb{L}^{s/2+1, \infty}$ and $\mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] \in \mathbb{L}^{s/2, \infty}$, then for each $x > 0$,*

$$\begin{aligned} & \sup_{n \geq 1} 2^{ns/2} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} \\ & \leq C(s, B) \left(\sup_{t > 0} t^{s/2+1} \mathbb{P} \{ \|m\|_B > t \} x^{-s/2-1} + \sup_{t > 0} t^{s/2} \mathbb{P} \left\{ \mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] > t \right\} x^{-s} \right); \end{aligned} \quad (2.2.2)$$

(2) *if $m \in \mathbb{L}_0^{s/2+1, \infty}$ and $\mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] \in \mathbb{L}^{s/2, \infty}$, then for each $x > 0$,*

$$\lim_{n \rightarrow +\infty} 2^{ns/2} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} = 0; \quad (2.2.3)$$

(3) if $m \in \mathbb{L}^{s/2+1}$ and $\mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] \in \mathbb{L}^{s/2}$, then

$$\sum_{n=1}^{+\infty} 2^{ns/2} a_n(x) \leq C(s, B) x^{-s} \left(\mathbb{E} \left[\|m\|_B^{s/2+1} \right] + \mathbb{E} \left[\left(\mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] \right)^{s/2} \right] \right). \quad (2.2.4)$$

For any non-negative random variable Y , sub- σ -algebra \mathcal{G} and any $s > 2$, $\mathbb{E} [Y^2 \mid \mathcal{G}]$ belongs to $\mathbb{L}^{s/2, \infty}$ (respectively $\mathbb{L}_0^{s/2, \infty}$, $\mathbb{L}^{s/2}$) provided that Y belongs to $\mathbb{L}^{s, \infty}$ (respectively $\mathbb{L}_0^{s, \infty}$, \mathbb{L}^s). As a consequence, we can derive from Theorem 2.5 similar results whose assumptions does not directly carry on the conditional variance.

Corollary 2.6. *Let $s > 2$ and let B be a separable $(2, D)$ -smooth Banach space. There exists a constant $C(s, B)$ such that the following holds: for each B -valued strictly stationary martingale differences sequence $(m \circ T^i, T^{-i}\mathcal{F}_0)$,*

(1) if m belongs to $\mathbb{L}^{s, \infty}$, then

$$\sup_{n \geq 1} \sup_{x > 0} x^s 2^{ns/2} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} \leq C(s, B) \sup_{t > 0} t^s \mathbb{P} \{ \|m\|_B > t \}; \quad (2.2.5)$$

(2) if m belongs to $\mathbb{L}_0^{s, \infty}$ then for each $x > 0$,

$$\lim_{n \rightarrow +\infty} 2^{n(s/2)} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} = 0; \quad (2.2.6)$$

(3) if m belongs to \mathbb{L}^s , then

$$\sum_{n=1}^{+\infty} 2^{ns/2} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} \leq C(s, B) x^{-s} \mathbb{E} [\|m\|_B^s]. \quad (2.2.7)$$

Remark 2.7. In view of Theorem 3.7 in [LV01], the weight $2^{ns/2}$ is optimal for stationary ergodic martingale differences sequences in the following sense: if $(R_n)_{n \geq 1}$ is a sequence of real numbers with goes to infinity and $s > 2$, then there exists a martingale differences sequence $(m \circ T^i)_{i \geq 0}$ such that m belongs to \mathbb{L}^s but the sequence $(2^{ns/2} R_n \mathbb{P} \{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n \})_{n \geq 1}$ does not converge to 0.

2.2.2. *Orthomartingale differences random fields.* Deviation inequalities has been used in [KL11, Lag16] for the question of complete convergence of orthomartingale differences random fields.

Similar results as in Subsubsection 2.2.1 can be proved for strictly stationary orthomartingale differences sequences.

Theorem 2.8. *Let $(m \circ T^i, T^{-i}\mathcal{F}_0, \mathbf{i} \in \mathbb{N}^d)$ be a strictly stationary martingale differences sequence with values in an (r, D) -smooth separable Banach space B . Assume that $1 < p < r$. If $\mathbb{E} \left[|m|^p (\log(1 + |m|))^{d-1} \right]$ is finite, then for any $x > 0$,*

$$\lim_{\max \mathbf{n} \rightarrow +\infty} |\mathbf{2}^{\mathbf{n}}|^{p-1} \mathbb{P} \left\{ \max_{1 \leq i \leq \mathbf{2}^{\mathbf{n}}} \|S_{\mathbf{i}}(m)\|_B > |\mathbf{2}^{\mathbf{n}}| x \right\} = 0. \quad (2.2.8)$$

Theorem 2.9. *Let $s > 2$, let B be a separable $(2, D)$ -smooth Banach space and let d be an integer greater than 1. There exists a constant $C(s, d, B)$ such that the following holds: for each B -valued strictly stationary orthomartingale differences random field $(m \circ T^{\mathbf{i}}, T^{-\mathbf{i}}\mathcal{F}_0, \mathbf{i} \in \mathbb{N}^d)$,*

(1) if m belongs to $\mathbb{L}^{s,\infty}$, then

$$\sup_{n \geq 1} |\mathbf{2}^n|^{s/2} \mathbb{P} \left\{ \max_{1 \leq i \leq \mathbf{2}^n} \|S_i(m)\|_B > |\mathbf{2}^n| x \right\} \leq C(s, d, B) \left(x^{-s} \sup_{t > 0} t^s \mathbb{P} \{ \|m\|_B > t \} \right); \quad (2.2.9)$$

(2) If m belongs to $\mathbb{L}_0^{s,\infty}$ then for each $x > 0$,

$$\lim_{\max n \rightarrow +\infty} |\mathbf{2}^n|^{s/2} \mathbb{P} \left\{ \max_{1 \leq i \leq \mathbf{2}^n} \|S_i(m)\|_B > |\mathbf{2}^n| x \right\} = 0. \quad (2.2.10)$$

(3) If m belongs to \mathbb{L}^s then for each $i \in \{1, \dots, d\}$,

$$\sup_{n \geq 1} \max_{\substack{1 \leq j \leq d \\ j \neq i}} \sum_{n_i=1}^{+\infty} |\mathbf{2}^n|^{s/2} \mathbb{P} \left\{ \max_{1 \leq i \leq \mathbf{2}^n} \|S_i(m)\|_B > |\mathbf{2}^n| x \right\} \leq C(s, d, B) x^{-s} \mathbb{E} [\|m\|_B^s]. \quad (2.2.11)$$

3. PROOFS

3.1. Proofs of Theorems 1.4 and 1.6.

Proof of Theorem 1.4. We first start by a distribution function inequality, which was first established in the real valued case and $p = 2$ in [Bur73] (see also [Pis75], p. 24 for a proof).

Lemma 3.1. *Let $(X_i, \mathcal{F}_i)_{i \geq 1}$ be a martingale differences sequence with values in an (r, D) -smooth separable Banach space and let $S_j := \sum_{i=j}^n X_i$, $\mathcal{F}_0 := \{\emptyset, \Omega\}$. Then for each $p \leq r$, $\beta > 1$, $0 < \delta < \beta - 1$, $x > 0$ and $n \geq 1$, the following inequality holds:*

$$\begin{aligned} \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|S_i\|_B > \beta x \right\} &\leq \frac{\delta^p}{(\beta - 1 - \delta)^p} D \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|S_i\|_B > x \right\} \\ &+ \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|X_i\|_B > \delta x \right\} + \mathbb{P} \left\{ \sum_{i=1}^n \mathbb{E} [\|X_i\|_B^p | \mathcal{F}_{i-1}] > (\delta x)^p \right\}. \end{aligned} \quad (3.1.1)$$

Remark 3.2. A better result can be obtain when the increments are assumed to be conditionally symmetric (see Lemma 4.2 in [Pin94]).

Proof of Lemma 3.1. We follow the proof of equation (21.2) in [Bur73]. Let n be fixed and assume that $X_i \equiv 0$ for $i \geq n + 1$. We define the random variables

$$\sigma := \inf \left\{ l \geq 1 : \|X_l\|_B > \delta x \text{ or } \sum_{i=1}^{l+1} \mathbb{E} [\|X_i\|_B^p | \mathcal{F}_{i-1}] > \delta x \right\}, \quad (3.1.2)$$

$$\mu := \inf \{ l \geq 1 : \|S_l\|_B > x \} \text{ and} \quad (3.1.3)$$

$$\nu := \inf \{ l \geq 1 : \|S_l\|_B > \beta x \}, \quad (3.1.4)$$

and we consider the martingale started at μ and stopped at $\min \{\nu, \sigma\}$:

$$Y_n := \sum_{i=1}^n \mathbf{1}_{\{\mu < i \leq \min \{\nu, \sigma\}\}} X_i. \quad (3.1.5)$$

Then we have the inequalities

$$\begin{aligned} & \mathbb{P} \left(\left\{ \max_{1 \leq i \leq n} \|S_i\|_B > \beta x \right\} \cap \left\{ \max_{1 \leq i \leq n} \|X_i\|_B \leq \delta x \right\} \cap \left\{ \sum_{i=1}^{n+1} \mathbb{E} [\|X_i\|_B^p \mid \mathcal{F}_{i-1}] \leq \delta x \right\} \right) \\ & \leq \mathbb{P} (\{\mu \leq \nu < +\infty\} \cap \{\sigma = +\infty\}) \leq \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|Y_i\|_B > (\beta - \delta - 1)x \right\} \\ & \leq \frac{1}{((\beta - \delta - 1)x)^p} \mathbb{E} \left[\max_{1 \leq i \leq n} \|Y_i\|_B^p \right]. \end{aligned} \quad (3.1.6)$$

Using Doob's inequality followed by (1.1.1), we derive that

$$\begin{aligned} & \mathbb{P} \left(\left\{ \max_{1 \leq i \leq n} \|S_i\|_B > \beta x \right\} \cap \left\{ \max_{1 \leq i \leq n} \|X_i\|_B \leq \delta x \right\} \cap \left\{ \sum_{i=1}^{n+1} \mathbb{E} [\|X_i\|_B^p \mid \mathcal{F}_{i-1}] \leq \delta x \right\} \right) \\ & \leq \frac{1}{((\beta - \delta - 1)x)^p} \frac{p}{p-1} D \sum_{i=1}^n \mathbb{E} [\mathbf{1}_{\{\mu < i \leq \min\{\nu, \sigma\}\}} \|X_i\|_B^p]; \end{aligned} \quad (3.1.7)$$

and since the event $\{\mu < i \leq \min\{\nu, \sigma\}\}$ is \mathcal{F}_{i-1} -measurable, we get

$$\begin{aligned} & \mathbb{P} \left(\left\{ \max_{1 \leq i \leq n} \|S_i\|_B > \beta x \right\} \cap \left\{ \max_{1 \leq i \leq n} \|X_i\|_B \leq \delta x \right\} \cap \left\{ \sum_{i=1}^{n+1} \mathbb{E} [\|X_i\|_B^p \mid \mathcal{F}_{i-1}] \leq \delta x \right\} \right) \\ & \leq \frac{1}{((\beta - \delta - 1)x)^p} \frac{p}{p-1} D \sum_{i=1}^n \mathbb{E} [\mathbf{1}_{\{\mu < i \leq \min\{\nu, \sigma\}\}} \mathbb{E} [\|X_i\|_B^p \mid \mathcal{F}_{i-1}]]. \end{aligned} \quad (3.1.8)$$

Since $\mathbf{1}_{\{\mu < i \leq \min\{\nu, \sigma\}\}} \mathbb{E} [\|X_i\|_B^p \mid \mathcal{F}_{i-1}] \leq \mathbf{1}_{\{\mu < \infty\}} \mathbf{1}_{\{i \leq \sigma\}} \mathbb{E} [\|X_i\|_B^p \mid \mathcal{F}_{i-1}]$, we infer that

$$\begin{aligned} & \mathbb{P} \left(\left\{ \max_{1 \leq i \leq n} \|S_i\|_B > \beta x \right\} \cap \left\{ \max_{1 \leq i \leq n} \|X_i\|_B \leq \delta x \right\} \cap \left\{ \sum_{i=1}^{n+1} \mathbb{E} [\|X_i\|_B^p \mid \mathcal{F}_{i-1}] \leq \delta x \right\} \right) \\ & \leq \frac{1}{((\beta - \delta - 1)x)^p} \frac{p}{p-1} D \mathbb{E} \left[\mathbf{1}_{\{\mu < \infty\}} \sum_{i=1}^{\sigma} \mathbb{E} [\|X_i\|_B^p \mid \mathcal{F}_{i-1}] \right] \\ & \leq \frac{(\delta x)^p}{((\beta - \delta - 1)x)^p} \frac{p}{p-1} D \mathbb{P} \{\mu < \infty\}, \end{aligned} \quad (3.1.9)$$

hence

$$\begin{aligned} & \mathbb{P} \left(\left\{ \max_{1 \leq i \leq n} \|S_i\|_B > \beta x \right\} \cap \left\{ \max_{1 \leq i \leq n} \|X_i\|_B \leq \delta x \right\} \cap \left\{ \sum_{i=1}^{n+1} \mathbb{E} [\|X_i\|_B^p \mid \mathcal{F}_{i-1}] \leq \delta x \right\} \right) \\ & \leq \frac{\delta^p}{(\beta - \delta - 1)^p} \frac{p}{p-1} D \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|S_i\|_B > x \right\}, \end{aligned} \quad (3.1.10)$$

which gives (3.1.1). This ends the proof of Lemma 3.1. \square

Lemma 3.3. *Let f and g be two non-increasing functions defined on the positive real line, taking their values between 0 and 2. Assume that for any $q > 0$, there exists a positive constant b such that for any positive x ,*

$$f(2x) \leq 2^{-q} \cdot f(x) + g(bx). \quad (3.1.11)$$

Then for each $q > 0$, there exists C such that for any $x > 0$,

$$f(x) \leq C \int_0^1 g(bxt) t^{q-1} dt. \quad (3.1.12)$$

As the proof will show, we can choose an explicit valued for C , namely, $C = q8^q/(2^q - 1)$.

Proof of Lemma 3.3. Let $q > 0$ be fixed. We define for a fixed $t_0 > 0$ the sequence of numbers $(t_k)_{k \geq 0}$ by $t_k := 2^k t_0$. By assumption, we derive that for any integer $k \geq 0$,

$$f(t_{k+1}) \leq 2^{-q} f(t_k) + g(bt_k). \quad (3.1.13)$$

Now, define a sequence $(c_k)_{k \geq 0}$ by $c_k := 2^{qk} f(t_k)$. We then have in view of (3.1.13) and $2t_k = t_{k+1}$,

$$c_{k+1} = 2^{q(k+1)} f(t_{k+1}) \leq 2^{q(k+1)} 2^{-q} f(t_k) + 2^{q(k+1)} g(bt_k) = c_k + 2^{q(k+1)} g(bt_k). \quad (3.1.14)$$

It follows that for any $n \geq 0$,

$$c_{n+1} = \sum_{k=0}^n c_{k+1} - c_k + c_0 \leq \sum_{k=0}^n 2^{q(k+1)} g(bt_k) + f(t_0), \quad (3.1.15)$$

hence

$$f(t_{n+1}) \leq \sum_{k=0}^n 2^{q(k-n)} g(bt_k) + 2^{-q(n+1)} f(t_0). \quad (3.1.16)$$

Notice that since g is non-increasing,

$$\sum_{k=1}^n 2^{q(k-n)} g(bt_k) \leq \sum_{k=1}^n 2^{q(k-n)} \int_{t_{k-1}}^{t_k} g(bu) u^{q-1} du \left(\int_{t_{k-1}}^{t_k} u^{q-1} du \right)^{-1} \quad (3.1.17)$$

$$= \sum_{k=1}^n 2^{q(k-n)} \int_{t_{k-1}}^{t_k} g(bu) u^{q-1} du (t_k^q - t_{k-1}^q)^{-1} \quad (3.1.18)$$

$$= q \sum_{k=1}^n 2^{q(k-n)} (2^{k-1} t_0)^{-q} \int_{t_{k-1}}^{t_k} g(bu) u^{q-1} du (2^q - 1)^{-1} \quad (3.1.19)$$

$$= \frac{q}{2^q - 1} t_0^{-q} 2^q 2^{-nq} \int_{t_0}^{t_n} g(bu) u^{q-1} du, \quad (3.1.20)$$

which yields, by (3.1.16),

$$f(t_{n+1}) \leq \frac{q}{2^q - 1} 2^q t_n^{-q} \int_{t_0}^{t_n} g(bu) u^{q-1} du + 2^{-nq} g(bt_0) + 2^{-q(n+1)} f(t_0). \quad (3.1.21)$$

Let t be a real number such that $t_{n+1} \leq t < t_{n+2}$ for some $n \geq -1$. Since f is non-increasing, we have

$$f(t) \leq f(t_{n+1}) \quad (3.1.22)$$

$$\leq \frac{q}{2^q - 1} 2^q 4^q t_{n+2}^{-q} \int_{t_0}^{t_n} g(bu) u^{q-1} du + 2^{-nq} g(bt_0) + 2^{-q(n+1)} f(t_0) \quad (3.1.23)$$

$$\leq \frac{q8^q}{2^q - 1} \frac{1}{t^q} \int_0^t g(bu) u^{q-1} du + 2^{-nq} g(bt_0) + 2^{-q(n+1)} f(t_0). \quad (3.1.24)$$

We thus get

$$\begin{aligned} f(t) \sum_{n=-1}^{+\infty} \mathbf{1}_{[2^{n+1}t_0, 2^{n+2}t_0)}(t) &\leq \sum_{n=-1}^{+\infty} \mathbf{1}_{[2^{n+1}t_0, 2^{n+2}t_0)}(t) \frac{q8^q}{2^q-1} \frac{1}{t^q} \int_0^t g(bu) u^{q-1} du \\ &+ g(bt_0) \sum_{n=-1}^{+\infty} 2^{-nq} \mathbf{1}_{[2^{n+1}t_0, 2^{n+2}t_0)}(t) + f(t_0) \sum_{n=-1}^{+\infty} 2^{-q(n+1)} \mathbf{1}_{[2^{n+1}t_0, 2^{n+2}t_0)}(t), \end{aligned} \quad (3.1.25)$$

and since $\sum_{n=-1}^{+\infty} \mathbf{1}_{[2^{n+1}t_0, 2^{n+2}t_0)}(t) = \mathbf{1}_{[t_0, +\infty)}(t)$, this can be rewritten as

$$\begin{aligned} f(t) \mathbf{1}_{[t_0, +\infty)}(t) &\leq \mathbf{1}_{[t_0, +\infty)}(t) \frac{q8^q}{2^q-1} \frac{1}{t^q} \int_0^t g(bu) u^{q-1} du \\ &+ g(bt_0) \sum_{n=-1}^{+\infty} 2^{-nq} \mathbf{1}_{[2^{n+1}t_0, 2^{n+2}t_0)}(t) + f(t_0) \sum_{n=-1}^{+\infty} 2^{-q(n+1)} \mathbf{1}_{[2^{n+1}t_0, 2^{n+2}t_0)}(t), \end{aligned} \quad (3.1.26)$$

Notice that if t belongs to $[2^{n+1}t_0, 2^{n+2}t_0)$, we have $2^n > 4/(t_0)$ hence $2^{-nq} < (t/(4t_0))^{-q} = 4^q t_0^q / t^q$. Since f and g are bounded above by 2, we derive that

$$g(bt_0) \sum_{n=-1}^{+\infty} 2^{-nq} \mathbf{1}_{[2^{n+1}t_0, 2^{n+2}t_0)}(t) + f(t_0) \sum_{n=-1}^{+\infty} 2^{-q(n+1)} \mathbf{1}_{[2^{n+1}t_0, 2^{n+2}t_0)}(t) \leq 4^{q+1} t_0^q / t^q \quad (3.1.27)$$

hence in view of (3.1.26), we get that

$$f(t) \mathbf{1}_{[t_0, +\infty)}(t) \leq \mathbf{1}_{[t_0, +\infty)}(t) \frac{q8^q}{2^q-1} \frac{1}{t^q} \int_0^t g(bu) u^{q-1} du + \mathbf{1}_{[t_0, +\infty)}(t) 4^{q+1} t_0^q / t^q, \quad (3.1.28)$$

Since t_0 is arbitrary, we get (3.1.12). This ends the proof of Lemma 3.3. \square

In order to prove Theorem 1.4, we define

$$f(x) := \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|S_i\|_B > x \right\} \quad (3.1.29)$$

$$g(x) := \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|X_i\|_B > x \right\} + \mathbb{P} \left\{ \sum_{i=1}^n \mathbb{E} [\|X_i\|_B^p \mid \mathcal{F}_{i-1}] > x^p \right\}. \quad (3.1.30)$$

An application of Lemma 3.1 with δ such that

$$\frac{\delta^2}{(1-\delta)^2} \frac{p}{p-1} D \leq 2^{-q} \quad (3.1.31)$$

shows that the functions f and g satisfy the assumptions of Lemma 3.3. Its use gives (1.1.2), which finishes the proof of Theorem 1.4. \square

Proof of Theorem 1.6. We shall need the following lemma.

Lemma 3.4. *Assume that X and Y are two non-negative random variables such that for each positive x , we have*

$$x \mathbb{P} \{X > x\} \leq \mathbb{E} [Y \mathbf{1} \{X \geq x\}]. \quad (3.1.32)$$

Then for each t , the following inequality holds:

$$\mathbb{P} \{X > 2t\} \leq \int_1^{+\infty} \mathbb{P} \{Y > st\} ds. \quad (3.1.33)$$

Proof of Lemma 3.4. Rewriting the expectation as

$$\mathbb{E} [Y \mathbf{1} \{X \geq 2t\}] = \int_0^{+\infty} \mathbb{P} \{Y \mathbf{1} \{X \geq 2t\} > u\} du \leq t \mathbb{P} \{X \geq 2t\} + \int_t^{+\infty} \mathbb{P} \{Y > u\} du, \quad (3.134)$$

we derive by the assumption the bound

$$2t \mathbb{P} \{X > 2t\} \leq t \mathbb{P} \{X \geq 2t\} + \int_t^{+\infty} \mathbb{P} \{Y > u\} du. \quad (3.135)$$

We conclude using the substitution $ts := u$. \square

We apply Theorem 1.4 with $X_i := m \circ T^i$ for $i \geq 0$ and $\mathcal{F}_i := T^{-i}\mathcal{F}_0$. We get

$$\begin{aligned} & \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|S_i(m)\|_B > xn^{1/p} \right\} \\ & \leq C \int_0^1 \mathbb{P} \left\{ \max_{1 \leq i \leq n} \|m \circ T^i\| > cxun^{1/p} \right\} u^{q-1} du + C \int_0^1 \mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{E} [\|f\|^p \mid T\mathcal{F}_0] \circ T^i > cx^p u^p \right\} u^{q-1} du. \end{aligned} \quad (3.136)$$

We bound the term $\int_0^1 \mathbb{P} \{ \max_{1 \leq i \leq n} \|m\|_B > cxun^{1/p} \} u^{q-1} du$ by $n \int_0^1 \mathbb{P} \{ \|m\|_B > cxun^{1/p} \} u^{q-1} du$.

By the maximal ergodic theorem, inequality (3.134) holds with $X := \sup_{n \geq 1} n^{-1} S_n (\mathbb{E} [\|m\|_B^p \mid T\mathcal{F}_0])$

and $Y = \mathbb{E} [\|m\|_B^p \mid T\mathcal{F}_0]$, hence by Lemma 3.4 the estimate

$$\mathbb{P} \{ n^{-1} S_n (\mathbb{E} [\|m\|_B^p \mid T\mathcal{F}_0]) > 2u^p t^p \} \leq \int_1^{+\infty} \mathbb{P} \{ \mathbb{E} [\|m\|_B^p \mid T\mathcal{F}_0] > u^p t^p s \} ds \quad (3.137)$$

is valid for any n . We can deduce from inequalities (3.136) and (3.137) that (1.1.3) is satisfied (after having switched the integrals).

In order to prove (1.1.4), we go back to (3.136) and we define for an integrable function $g: \Omega \rightarrow \mathbb{R}$, $Vg := U\mathbb{E}[g \mid T\mathcal{F}_0]$. Notice that V is an \mathbb{L}^1 - \mathbb{L}^∞ contraction and that $V^n(g) = U^n \mathbb{E}[g \mid T\mathcal{F}_0]$. Therefore, we have the bound

$$\begin{aligned} & \int_0^1 \mathbb{P} \{ n^{-1} S_n (\mathbb{E} [\|m\|_B^p \mid T\mathcal{F}_0]) > 2u^p t^p \} u^{q-1} du \leq \\ & \int_0^1 \int_1^{+\infty} \mathbb{P} \{ \|m\|_B^p > u^p t^p v \} u^{q-1} dv du \leq \int_0^1 \int_1^{+\infty} \mathbb{P} \{ \|m\|_B > utv^{1/p} \} u^{q-1} dv du, \end{aligned} \quad (3.138)$$

and using the substitution $u' := uv^{1/p}$, we get

$$\begin{aligned} \int_0^1 \int_1^{+\infty} \mathbb{P} \{ \|m\|_B > utv^{1/p} \} u^{q-1} dudv &= \int_0^{+\infty} \mathbb{P} \{ \|m\|_B > tu' \} \int_{\max\{1, u'^p\}}^{+\infty} u'^{q-1} v^{-q/p} dv du' \\ &= \frac{p}{q-p} \int_0^{+\infty} \mathbb{P} \{ |m| > tu \} u^{q-1} (\max\{1, u^p\})^{1-q/p} du \end{aligned} \quad (3.139)$$

$$\int_0^1 \int_1^{+\infty} \mathbb{P} \{ \|m\|_B > utv^{1/p} \} u^{q-1} dudv = \frac{p}{q-p} \int_0^{+\infty} \mathbb{P} \{ |m| > tu \} \min \{ u^{q-1}, u^{p-1} \} du. \quad (3.140)$$

We now bound the first term of (3.1.36) independently of n . Putting $v = n^{1/p}u$, we have

$$n \int_0^1 \mathbb{P} \left\{ \|m\|_B \geq n^{1/p}ut \right\} u^{q-1} du = n^{1-q/p} \int_0^{n^{1/p}} \mathbb{P} \left\{ \|m\|_B \geq vt \right\} v^{q-1} dv. \quad (3.1.41)$$

If $v \leq 1$, we use the bound $n^{1-q/p}v^{q-1} \leq v^{q-1}$ (since $q > p$). If $1 < v \leq n^{1/p}$, then $n^{1-q/p}v^{q-1} \leq v^{p-1}$. We thus have

$$n \int_0^1 \mathbb{P} \left\{ \|m\|_B \geq n^{1/p}ut \right\} u^{q-1} du \int_0^{+\infty} \mathbb{P} \left\{ \|m\|_B \geq vt \right\} v^{q-1} dv. \quad (3.1.42)$$

Combining (3.1.36), (3.1.38), (3.1.40) and (3.1.42), we get (1.1.4). This ends the proof of Theorem 1.6. \square

3.2. Proof of Theorem 1.9. We start by proving (1.2.4) by induction on d .

For $d = 1$, this reduces to (1.1.4). Assume that for some $d \geq 2$, we have the following property: if $(m \circ T^i)_{i \in \mathbb{Z}^{d-1}}$ is a strictly stationary orthomartingale difference random field with respect to the completely commuting filtration $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^{d-1}}$ with values in a (r, D) -smooth separable Banach space and $q > p$, $p \leq r$, then for each $t > 0$ and each $\mathbf{n} \in \mathbb{N}^{d-1}$,

$$\begin{aligned} & \mathbb{P} \left\{ \max_{\mathbf{1} \leq i \leq \mathbf{n}} \|S_i(m)\|_B \geq \prod_{l=1}^{d-1} n_l^{1/p} t \right\} \\ & \leq c(q, d-1) \int_0^{+\infty} \mathbb{P} \left\{ \|m\|_B > tv \right\} (1 + |\log v|)^{(d-1)-1} \min \{v^{p-1}, v^{q-1}\} dv. \end{aligned} \quad (3.2.1)$$

We have to prove that this property is still valid for d , namely, that if $(m \circ T^i)_{i \in \mathbb{Z}^d}$ is a strictly stationary orthomartingale difference random field with respect to the completely commuting filtration $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ and $q > p$, $p \leq r$, then for each t , each $\mathbf{n} \in \mathbb{N}^d$ and each $j \in \{1, \dots, d\}$, (1.2.4) holds. Let us take such a random field and $q > p$. Using item 2 of Lemma 1.8 and Lemma 3.4, we obtain this first estimate

$$\mathbb{P} \left\{ \max_{\mathbf{1} \leq i \leq \mathbf{n}} |S_i(m)| \geq \prod_{l=1}^d n_l^{1/p} 2t \right\} \leq \int_1^{+\infty} \mathbb{P} \left\{ \max_{i \in [\mathbf{1}, \mathbf{n}]} |S_{(i, n_d)}(m)| \geq \prod_{l=1}^d n_l^{1/p} ts \right\} ds. \quad (3.2.2)$$

Now, using item 3 of Lemma 1.8 and the induction assumption, we derive

$$\begin{aligned} & \mathbb{P} \left\{ \max_{\mathbf{1} \leq i \leq \mathbf{n}} |S_i(m)| \geq \prod_{l=1}^d n_l^{1/p} 2t \right\} \leq c(q, d-1) \\ & \int_1^{+\infty} \int_0^{+\infty} \mathbb{P} \left\{ |S_{(\mathbf{0}, n_d)}| > ts v n_d^{1/p} \right\} (1 + |\log v|)^{(d-1)-1} \min \{v^{p-1}, v^{q-1}\} dv ds. \end{aligned} \quad (3.2.3)$$

Switching the integrals and letting $s' := sv$ for a fixed v , we get

$$\begin{aligned} & \mathbb{P} \left\{ \max_{\mathbf{1} \leq i \leq \mathbf{n}} |S_i(m)| \geq \prod_{l=1}^d n_l^{1/p} 2t \right\} \leq c(q, d-1) \\ & \int_0^{+\infty} \mathbb{P} \left\{ |S_{(\mathbf{0}, n_d)}| > ts' n_d^{1/p} \right\} \int_0^{+\infty} \mathbf{1}_{\{s' \geq v\}} (1 + |\log v|)^{(d-1)-1} \min \{v^{p-2}, v^{q-2}\} dv ds'. \end{aligned} \quad (3.2.4)$$

The inner integral is equal to

$$\int_0^1 (1 + |\log(s'w)|)^{d-2} \min \left\{ (s'w)^{p-2}, (s'w)^{q-2} \right\} w dw, \quad (3.2.5)$$

which can be bounded by $K(q, d)(1 + |\log(s')|)^{d-2} \min \{s', s'^{q-1}\}$.

It thus follows that

$$\begin{aligned} \mathbb{P} \left\{ \max_{1 \leq i \leq n} |S_i(m)| \geq \prod_{l=1}^d n_l^{1/p} 2t \right\} &\leq c(q, d-1)K(q, d) \\ &\int_0^{+\infty} \mathbb{P} \left\{ |S_{(\mathbf{0}, n_d)}| > t s n_d^{1/p} \right\} (1 + |\log s|)^{(d-1)-1} \min \{s^{p-1}, s^{q-1}\} ds. \end{aligned} \quad (3.2.6)$$

Now, by (1.1.4), we obtain

$$\begin{aligned} \mathbb{P} \left\{ \max_{1 \leq i \leq n} |S_i(m)| \geq \prod_{l=1}^d n_l^{1/p} 2t \right\} &\leq c(q, d-1)K(q, d)c(q) \\ &\int_0^{+\infty} \int_0^{+\infty} \mathbb{P} \{|m| > tsu\} (1 + |\log s|)^{(d-1)-1} \min \{s^{p-1}, s^{q-1}\} \min \{u^{p-1}, u^{q-1}\} dud s \\ &= c(q, d-1)K(q, d)c(q) \int_0^{+\infty} \int_0^{+\infty} \mathbb{P} \{|m| > tv\} I(v) dv, \end{aligned} \quad (3.2.7)$$

where

$$I(v) = \int_0^{+\infty} (1 + |\log s|)^{(d-1)-1} \min \{s^{p-2}, s^{q-2}\} \min \left\{ \left(\frac{v}{s}\right)^{p-1}, \left(\frac{v}{s}\right)^{q-1} \right\} ds. \quad (3.2.8)$$

Assume that $v < 1$. Then

$$\begin{aligned} I(v) &= \int_0^v (1 + |\log s|)^{d-2} s^{q-2} \left(\frac{v}{s}\right)^{p-1} ds + \int_v^1 (1 + |\log s|)^{d-2} s^{q-2} \left(\frac{v}{s}\right)^{q-1} ds \\ &\quad + \int_1^{+\infty} (1 + |\log s|)^{d-2} s^{p-2} \left(\frac{v}{s}\right)^{q-1} ds \\ &\leq v^{p-1} \int_0^1 (1 + |\log s|)^{d-2} s^{q-p-1} ds + v^{q-1} (1 + |\log v|)^{d-2} \int_v^1 \frac{1}{s} ds \\ &\quad + v^{q-1} \int_1^{+\infty} (1 + |\log s|)^{d-2} \frac{1}{s^{q-p-1}} ds, \end{aligned} \quad (3.2.9)$$

and since the $q > p$, the integrals $\int_0^1 (1 + |\log s|)^{d-2} s^{q-p-1} ds$ and $\int_1^{+\infty} (1 + |\log s|)^{d-2} 1/s^{q-p-1} ds$ are convergent. Consequently, there exists a constant K depending only on d, p and q such that for any $v \in (0, 1)$, $I(v) \leq K v^{q-1} (1 + |\log v|)^{d-1}$.

Assume now that $v \geq 1$. Then

$$\begin{aligned} I(v) &= \int_0^1 (1 + |\log s|)^{d-2} s^{q-2} \left(\frac{v}{s}\right)^{p-1} ds + \int_1^v (1 + |\log s|)^{d-2} s^{p-2} \left(\frac{v}{s}\right)^{p-1} ds \\ &\quad + \int_v^{+\infty} (1 + |\log s|)^{d-2} s^{p-2} \left(\frac{v}{s}\right)^{q-1} ds \\ &\leq v^{p-1} \int_0^1 (1 + |\log s|)^{d-2} s^{q-p-1} ds + v^{p-1} (1 + |\log v|)^{d-2} \int_1^v \frac{1}{s} ds \\ &\quad + v^{p-1} \int_1^{+\infty} (1 + |\log(u)| + |\log(v)|)^{d-2} \frac{1}{u^{q-p+1}} du, \end{aligned}$$

and here again, there exists a constant K depending only on d and q such that for any $v \geq 1$, $I(v) \leq K v^{p-1} (1 + |\log v|)^{d-1}$. This shows the estimate

$$I(v) \leq K (1 + |\log v|)^{d-1} \min \{v^{p-1}, v^{q-1}\}, v > 0. \quad (3.2.10)$$

The combination of (3.2.7) and (3.2.10) gives (1.2.4), which shows that the considered property is valid for d .

We now prove (1.2.3). We start from (3.2.6). Using this time (1.1.3), we obtain

$$\begin{aligned} &\mathbb{P} \left\{ \max_{1 \leq i \leq n} |S_i(m)| \geq 2 \prod_{l=1}^d n_l^{1/p} t \right\} \leq c(q, d-1) K(q, d) c(q) \\ n_d \int_0^{+\infty} \int_0^{+\infty} &\mathbb{P} \left\{ |m| > t s u n_d^{1/p} \right\} (1 + |\log s|)^{(d-1)-1} \min \{s^{p-1}, s^{q-1}\} \min \{u^{p-1}, u^{q-1}\} du ds \\ &\quad + c(q, d-1) K(q, d) c(q) \int_0^{+\infty} \int_0^{+\infty} \mathbb{P} \left\{ (\mathbb{E}[m^2 | T_d \mathcal{F}_0])^{1/2} > t u s \right\} \\ &\quad (1 + |\log s|)^{(d-1)-1} \min \{u^{p-1}, u^{q-1}\} du \min \{s^{p-1}, s^{q-1}\} ds. \quad (3.2.11) \end{aligned}$$

After having done the substitution $v := us$ for a fixed s and rearranged the integrals, we get (1.2.3) when $j = d$. The proof for a general $j \in \{1, \dots, d\}$ can be carried out similarly. This concludes the proof of Theorem 1.9.

3.3. Proof of the results of Section 2.

Proof of Theorem 2.1. We write start from the equality

$$\mathbb{P} \left\{ \sup_{k \geq 1} \left\| \sum_{i=1}^k X_{n,i} \right\|_B > \varepsilon \right\} = \lim_{N \rightarrow +\infty} \mathbb{P} \left\{ \sup_{1 \leq k \leq N} \left\| \sum_{i=1}^k X_{n,i} \right\|_B > \varepsilon \right\} \quad (3.3.1)$$

and we bound $\mathbb{P} \left\{ \sup_{1 \leq k \leq N} \left\| \sum_{i=1}^k X_{n,i} \right\|_B > \varepsilon \right\}$ by Theorem 1.4. \square

Proof of Theorem 2.3. We use inequality (1.1.4) with $q > p$ to get

$$\begin{aligned} &2^{n(p-1)} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} \\ &\leq C \int_0^1 2^{n(p-1)} \mathbb{P} \left\{ \|m\|_B > 2^{n(1-1/p)} x u \right\} u^{q-1} du \\ &\quad + C \int_1^{+\infty} 2^{n(p-1)} \mathbb{P} \left\{ \|m\|_B > 2^{n(1-1/p)} x u \right\} u^{p-1} du. \quad (3.3.2) \end{aligned}$$

One bounds the first term by

$$Cx^{-p} \int_0^1 \sup_{t > 2^{n(1-1/p)}xu} t^p \mathbb{P} \{ \|m\|_B > t \} u^{q-p-1} du,$$

and the second one by $Cx^{-p}/p \mathbb{E} [\|m\|_B^p \mathbf{1} \{ \|m\|_B \geq 2^{n(1-1/p)}x \}]$. Therefore, we can find a constant K such that for any integer n ,

$$2^{n(p-1)} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} \leq K \mathbb{E} \left[\|m\|_B^p \mathbf{1} \left\{ \|m\|_B \geq 2^{n(1-1/p)}x \right\} \right],$$

which ends the proof of Theorem 2.3. \square

Proof of Theorem 2.5. We use inequality (1.1.3) with $p = 2$ and $q = 2s$. We get

$$\begin{aligned} 2^{ns/2} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} &\leq C 2^{ns/2} 2^n \int_0^1 \mathbb{P} \{ \|m\|_B > cxu2^n \} u^{2s-1} du \\ &+ C 2^{ns/2} \int_0^{+\infty} \mathbb{P} \left\{ \mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] > cx^2 u^2 2^n \right\} \min \{ u^{2s-1}, u \} du. \end{aligned} \quad (3.3.3)$$

(1) Assume that m belongs to $\mathbb{L}^{s/2+1, \infty}$ and $\mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] \in \mathbb{L}^{s/2, \infty}$. One bounds the first term of the right hand side of (3.3.3) by

$$\begin{aligned} C 2^{ns/2} 2^n \sup_{t > 0} t^{s/2+1} \mathbb{P} \{ \|m\|_B > t \} &\int_0^1 (cxu2^n)^{-(s/2+1)} u^{2s-1} du \\ &= C \sup_{t > 0} t^{s/2+1} \mathbb{P} \{ \|m\|_B > t \} (cx)^{-(s/2+1)} \int_0^1 u^{3s/2-2} du. \end{aligned} \quad (3.3.4)$$

One bounds the second term of the right hand side of (3.3.3) by

$$\begin{aligned} C 2^{ns/2} \sup_{t > 0} t^{s/2} \mathbb{P} \left\{ \mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] > t \right\} &\int_0^{+\infty} (cx^2 u^2 2^n)^{-s/2} \min \{ u^{2s-1}, u \} du \\ &= C (cx)^{-s} \sup_{t > 0} t^{s/2} \mathbb{P} \left\{ \mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] > t \right\} \int_0^{+\infty} \min \{ u^{s-1}, u^{1-s} \} du, \end{aligned} \quad (3.3.5)$$

and since $s > 2$, the latter integral is finite.

(2) Assume that $m \in \mathbb{L}_0^{s/2+1, \infty}$ and $\mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] \in \mathbb{L}^{s/2, \infty}$. Plugging the bounds

$$\mathbb{P} \{ \|m\|_B > cxu2^n \} \leq (cxu2^n)^{-s/2-1} \sup_{t > cxu2^n} t^{s/2+1} \mathbb{P} \{ \|m\|_B > t \} \quad \text{and} \quad (3.3.6)$$

$$\mathbb{P} \left\{ \mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] > cx^2 u^2 2^n \right\} \leq (cx^2 u^2 2^n)^{-s/2} \sup_{t > cx^2 u^2 2^n} t^{s/2} \mathbb{P} \left\{ \mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] > t \right\}. \quad (3.3.7)$$

into (3.3.3), we get

$$\begin{aligned} 2^{ns/2} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} &\leq C (cx)^{-s/2-1} \int_0^1 \sup_{t > cxu2^n} t^{s/2+1} \mathbb{P} \{ \|m\|_B > t \} u^{3s/2-2} du \\ &+ C (cx^2)^{-s/2} \int_0^{+\infty} \sup_{t > cx^2 u^2 2^n} t^{s/2} \mathbb{P} \left\{ \mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] > t \right\} \max \{ u^{s-1}, u^{1-s} \} du, \end{aligned} \quad (3.3.8)$$

and the right hand side goes to zero by monotone convergence.

(3) Assume that $m \in \mathbb{L}^{s/2+1}$ and $\mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] \in \mathbb{L}^{s/2}$. In view of (3.3.3), we have

$$\begin{aligned} \sum_{n=1}^{+\infty} 2^{ns/2} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > 2^n x \right\} &\leq C \sum_{n=1}^{+\infty} 2^{ns/2} 2^n \int_0^1 \mathbb{P} \{ \|m\|_B > cxu2^n \} u^{2s-1} du \\ &+ C \sum_{n=1}^{+\infty} 2^{ns/2} \int_0^{+\infty} \mathbb{P} \left\{ \mathbb{E} \left[\|m\|_B^2 \mid T\mathcal{F}_0 \right] > cx^2 u^2 2^n \right\} \min \{ u^{2s-1}, u \} du. \end{aligned} \quad (3.3.9)$$

Since for any non-negative random variable Y and any $q > 2$, $\sum_{n=1}^{+\infty} 2^{qn} \mathbb{P} \{ Y > 2^n \} \leq 2\mathbb{E}[Y^q]$, we get the conclusion of item 3 of Theorem 2.5. \square

Proof of Theorem 2.8. We use inequality (1.2.4) with a $q > p$ to obtain

$$\begin{aligned} |\mathbf{2}^n|^{p-1} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > |\mathbf{2}^n| x \right\} \\ \leq C |\mathbf{2}^n|^{p-1} \int_0^1 \mathbb{P} \left\{ \|m\|_B > xu |\mathbf{2}^n|^{1-1/p} \right\} u^{q-1} (1 + |\log u|)^{d-1} du \\ + C |\mathbf{2}^n|^{p-1} \int_1^{+\infty} \mathbb{P} \left\{ \|m\|_B > xu |\mathbf{2}^n|^{1-1/p} \right\} u^{p-1} (1 + |\log u|)^{d-1} du. \end{aligned} \quad (3.3.10)$$

Using Markov's inequality, the first term of the right hand side of (3.3.10) does not exceed

$$Cx^{-p} \int_0^1 \mathbb{E} \left[\|m\|_B^p \mathbf{1} \left\{ \|m\|_B > xu |\mathbf{2}^n|^{1-1/p} \right\} \right] u^{q-p-1} (1 + |\log u|)^{d-1} du$$

and by monotone convergence, this quantity goes to 0 as $\max \mathbf{n} \rightarrow +\infty$.

The second term of the right hand side of (3.3.10) does not exceed

$$C \mathbb{E} \left[\|m\|_B^p \left(\log \left(1 + \frac{\|m\|_B}{x |\mathbf{2}^n|^{1-1/p}} \right) \right)^{d-1} \right]$$

which goes to 0 as $\max \mathbf{n}$ goes to infinity. This ends the proof of Theorem 2.3. \square

Proof of Theorem 2.9. We use inequality (1.2.4) with $p = 2$ in order to obtain

$$\begin{aligned} |\mathbf{2}^n|^{s/2} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^n} \|S_i(m)\|_B > |\mathbf{2}^n| x \right\} \\ \leq C |\mathbf{2}^n|^{s/2} \int_0^{+\infty} \mathbb{P} \left\{ \|m\|_B > xu |\mathbf{2}^n|^{1/2} \right\} (1 + |\log u|)^{d-1} u^{q-1} du. \end{aligned} \quad (3.3.11)$$

(1) Assume that m belongs to $\mathbb{L}^{s,\infty}$. Then we use the bound

$$\mathbb{P} \left\{ \|m\|_B > xu |\mathbf{2}^n|^{1/2} \right\} \leq \sup_{t>0} t^s \mathbb{P} \{ \|m\|_B > t \} \left(xu |\mathbf{2}^n|^{1/2} \right)^{-s} \quad (3.3.12)$$

to get (2.2.9).

(2) Assume that m belongs to $\mathbb{L}_0^{s,\infty}$. Then we use the bound

$$\mathbb{P} \left\{ \|m\|_B > xu |\mathbf{2}^n|^{1/2} \right\} \leq \sup_{t \geq xu |\mathbf{2}^n|^{1/2}} t^s \mathbb{P} \{ \|m\|_B \geq t \} \left(xu |\mathbf{2}^n|^{1/2} \right)^{-s} \quad (3.3.13)$$

to get (2.2.10).

- (3) Assume that m belongs to \mathbb{L}^s . Then for each $i \in \{1, \dots, d\}$, denoting $\mathbf{n}' := \mathbf{n} - n_i \mathbf{e}_i$, the following inequalities hold:

$$\begin{aligned} & \sum_{n_i=1}^{+\infty} |\mathbf{2}^{\mathbf{n}}|^{s/2} \mathbb{P} \left\{ \max_{1 \leq i \leq 2^{\mathbf{n}}} \|S_i(m)\|_B > |\mathbf{2}^{\mathbf{n}}| x \right\} \\ & \leq C \sum_{n_i=1}^{+\infty} |\mathbf{2}^{\mathbf{n}'}|^{s/2} 2^{n_i s/2} \int_0^{+\infty} \mathbb{P} \left\{ \|m\|_B > xu 2^{n_i s/2} |\mathbf{2}^{\mathbf{n}'}|^{1/2} \right\} (1 + |\log u|)^{d-1} \min \{u^{q-1}, u\} du \\ & \leq 2C |\mathbf{2}^{\mathbf{n}'}|^{s/2} x^{-s} \int_0^{+\infty} u^{-s-1} |\mathbf{2}^{\mathbf{n}'}|^{-s/2} (1 + |\log u|)^{d-1} \min \{u^{q-1}, u\} du, \quad (3.3.14) \end{aligned}$$

from which (2.2.11) follows.

This ends the proof of Theorem 2.9. \square

REFERENCES

- [Ass75] P. Assouad, *Espaces p -lisses et q -convexes, inégalités de Burkholder*, Séminaire Maurey-Schwartz 1974–1975: Espaces L^p , applications radonifiantes et géométrie des espaces de Banach, Exp. No. XV, Centre Math., École Polytech., Paris, 1975, p. 8. MR 0407963 2
- [Azu67] Kazuoki Azuma, *Weighted sums of certain dependent random variables*, Tôhoku Math. J. (2) **19** (1967), 357–367. MR 0221571 1
- [Bur73] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probability **1** (1973), 19–42. MR 0365692 (51 #1944) 1, 7
- [dlP99] Victor H. de la Peña, *A general class of exponential inequalities for martingales and ratios*, Ann. Probab. **27** (1999), no. 1, 537–564. MR 1681153 1
- [DM07] J. Dedecker and F. Merlevède, *Convergence rates in the law of large numbers for Banach-valued dependent variables*, Teor. Veroyatn. Primen. **52** (2007), no. 3, 562–587. MR 2743029 (2011k:60096) 5
- [Doo53] J. L. Doob, *Stochastic processes*, John Wiley & Sons, Inc., New York; Chapman & Hall, Limited, London, 1953. MR 0058896 (15,445b) 2
- [Faz05] István Fazekas, *Burkholder’s inequality for multiindex martingales*, Ann. Math. Inform. **32** (2005), 45–51. MR 2264866 (2008a:60105) 4
- [FGL12] Xiequan Fan, Ion Grama, and Quansheng Liu, *Large deviation exponential inequalities for supermartingales*, Electron. Commun. Probab. **17** (2012), no. 59, 8. MR 3005732 1
- [FGL15] ———, *Exponential inequalities for martingales with applications*, Electron. J. Probab. **20** (2015), no. 1, 22. MR 3311214 1
- [GC98] Subhashis Ghosal and Tapas K. Chandra, *Complete convergence of martingale arrays*, J. Theoret. Probab. **11** (1998), no. 3, 621–631. MR 1633378 5
- [Hae84] Erich Haeusler, *An exact rate of convergence in the functional central limit theorem for special martingale difference arrays*, Z. Wahrsch. Verw. Gebiete **65** (1984), no. 4, 523–534. MR 736144 1
- [Hoe63] Wassily Hoeffding, *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. **58** (1963), 13–30. MR 0144363 1
- [JS88] W. B. Johnson and G. Schechtman, *Martingale inequalities in rearrangement invariant function spaces*, Israel J. Math. **64** (1988), no. 3, 267–275 (1989). MR 995572 (90g:60048) 4
- [Kho02] Davar Khoshnevisan, *Multiparameter processes*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2002, An introduction to random fields. MR 1914748 (2004a:60003) 3
- [KL11] Anna Kuczmaszewska and Zbigniew A. Lagodowski, *Convergence rates in the SLLN for some classes of dependent random fields*, J. Math. Anal. Appl. **380** (2011), no. 2, 571–584. MR 2794415 6

- [Lag16] Zbigniew A. Lagodowski, *An approach to complete convergence theorems for dependent random fields via application of Fuk-Nagaev inequality*, J. Math. Anal. Appl. **437** (2016), no. 1, 380–395. MR 3451971 [6](#)
- [Li03] Yulin Li, *A martingale inequality and large deviations*, Statist. Probab. Lett. **62** (2003), no. 3, 317–321. MR 1967664 [5](#)
- [LV01] Emmanuel Lesigne and Dalibor Volný, *Large deviations for martingales*, Stochastic Process. Appl. **96** (2001), no. 1, 143–159. MR 1856684 (2002k:60080) [6](#)
- [Nag03] S. V. Nagaev, *On probability and moment inequalities for supermartingales and martingales*, Proceedings of the Eighth Vilnius Conference on Probability Theory and Mathematical Statistics, Part II (2002), vol. 79, 2003, pp. 35–46. MR 2021875 (2005f:60098) [1](#), [2](#)
- [Pin94] Iosif Pinelis, *Optimum bounds for the distributions of martingales in Banach spaces*, Ann. Probab. **22** (1994), no. 4, 1679–1706. MR 1331198 (96b:60010) [7](#)
- [Pis75] Gilles Pisier, *Martingales with values in uniformly convex spaces*, Israel J. Math. **20** (1975), no. 3-4, 326–350. MR 0394135 [2](#), [7](#)
- [WH14] Xue Jun Wang and Shu He Hu, *Complete convergence and complete moment convergence for martingale difference sequence*, Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 1, 119–132. MR 3151420 [5](#)

NORMANDIE UNIVERSITÉ, UNIVERSITÉ DE ROUEN, LABORATOIRE DE MATHÉMATIQUES RAPHAËL SALEM, CNRS, UMR 6085, AVENUE DE L'UNIVERSITÉ, BP 12, 76801 SAINT-ETIENNE DU ROUVRAY CEDEX, FRANCE.
E-mail address: davide.giraud01@univ-rouen.fr