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HÖLDERIAN INVARIANCE PRINCIPLE FOR MARTINGALE DIFFERENCE RANDOM FIELDS

DAVIDE GIRAUDDO

ABSTRACT. We investigate the convergence in Hölder spaces of the summation process based on the collection of products of d intervals and associated to a strictly stationary orthomartingale random field. We give a sufficient condition in terms of the law of the common distribution and the conditional variances, and we discuss its sharpness. The main tools of the proof are a tightness criterion in Hölder spaces for the multidimensional summation processes associated to a strictly stationary random field and a probability inequality for strictly stationary orthomartingale random fields. Finally, we obtain by approximation a Hannan type condition.

1. INTRODUCTION AND NOTATIONS

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Let T be a \mathbb{Z}^d -measure preserving action, where d is an integer greater than 2. We denote the Koopman operator U by $U^{\mathbf{i}}f(\omega) = f(T^{\mathbf{i}}\omega)$, $\omega \in \Omega$, $\mathbf{i} \in \mathbb{Z}^d$. We will consider the partial order \preceq on \mathbb{Z}^d given by $\mathbf{i} \preceq \mathbf{j}$ (respectively $\mathbf{i} \succcurlyeq \mathbf{j}$) if $i_q \leq j_q$ (resp. $i_q \geq j_q$) for each $q \in \{1, \dots, d\} =: \langle d \rangle$. For $\mathbf{i} \succcurlyeq \mathbf{1}$, we denote the unit cube with upper corner at $\mathbf{i} = (i_1, \dots, i_d)$ that is,

$$R_{\mathbf{i}} := \prod_{q=1}^d (i_q - 1, i_q]. \quad (1.0.1)$$

For a function $f: \Omega \rightarrow \mathbb{R}$, we consider the partial sum process defined by

$$S_{\mathbf{n}}(f, \mathbf{t}) := \sum_{\mathbf{i} \in [\mathbf{1}, \mathbf{n}]} \lambda([\mathbf{0}, \mathbf{n} \cdot \mathbf{t}] \cap R_{\mathbf{i}}) U^{\mathbf{i}}f, \quad \mathbf{t} \in [0, 1]^d, \mathbf{n} \in (\mathbb{N}^*)^d, \quad (1.0.2)$$

where λ denotes the Lebesgue on \mathbb{R}^d , $[\mathbf{0}, \mathbf{n} \cdot \mathbf{t}] = \prod_{q=1}^d [0, n_q t_q]$ and

$$[\mathbf{1}, \mathbf{n}] = \{\mathbf{i} \in \mathbb{Z}^d, 1 \leq i_q \leq n_q \text{ for each } q \in \{1, \dots, d\}\}. \quad (1.0.3)$$

We are interested in the functional central limit theorem in Hölder spaces for the net $(S_{\mathbf{n}}(f, \cdot))_{\mathbf{n} \in (\mathbb{N}^*)^d}$ in order to understand the asymptotic behavior of the partial sums of $(f \circ T^{\mathbf{i}})$ over rectangles. By "functional central limit theorem in a function space E ", we mean that for each continuous bounded functional $F: E \rightarrow \mathbb{R}$, the convergence $F(S_{\mathbf{n}}(f, \cdot)/a_{\mathbf{n}}) \rightarrow F(W)$ holds as $\min_{1 \leq i \leq d} n_i$ goes to infinity, where W is a Gaussian process. Usually, the normalizing term $a_{\mathbf{n}}$ will be chosen as $|a_{\mathbf{n}}| := \prod_{q=1}^d n_q$.

The question of the functional central limit theorem in the space of continuous functions for strictly stationary random fields has been studied. Wichura [Wic69] established such a

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result for an i.i.d. centered random field with finite variance, which generalized Donsker's one dimensional result [Don51]. Wichura's result was extended to a class of stationary ergodic martingale differences random fields [BD79, PR98], and Dedecker found a projective condition [Ded01]. Wang and Woodroffe [WW13] attempted to extend the Maxwell and Woodroffe condition [MW00] but found a weaker condition, which was improved by Volný and Wang [VW14]. The later is a multidimensional extension of Hannan's condition [Han73]. In the context of the mentioned works, the limiting process is a standard Brownian sheet when the considered random field is ergodic, that is, a Gaussian process $(W_{\mathbf{t}})_{\mathbf{t} \in [0,1]^d}$ such that $\text{Cov}(W_{\mathbf{t}}; W_{\mathbf{s}}) = \prod_{i=1}^d \min\{t_i, s_i\}$. Since the paths of such a process are almost surely Hölder-continuous with exponent α for each $\alpha \in (0, 1/2)$, it is natural to try to investigate the convergence of $S_{\mathbf{n}}(f, \cdot)/\sqrt{|\mathbf{n}|}$ in Hölder spaces. The i.i.d. case has been considered first by Erickson [Eri81]: he proved that if $(f \circ T^i)_{i \in \mathbb{Z}^d}$ is an i.i.d. centered random field such that $f \in \mathbb{L}^q$ for some $q > d/(1/2 - \alpha)$, then $S_{\mathbf{n}}(f, \cdot)/\sqrt{|\mathbf{n}|}$ converges in distribution to a standard Brownian sheet in the space of α -Hölder continuous functions. This result was improved in the case $d = 2$ [RZ05] and then to any $d \geq 2$ [RSZ07] in the sense that a necessary and sufficient condition for the invariance principle in the space of α -Hölder continuous functions for an i.i.d. centered random field $(f \circ T^i)$ is the boundedness of $t^{(1/2-\alpha)^{-1}} \mu\{|f| > t\}$. This contrasts with the one dimensional case, for which the necessary and sufficient condition reads $t^{(1/2-\alpha)^{-1}} \mu\{|f| > t\} \rightarrow 0$ as t goes to $+\infty$ (see [RS03]).

Once the i.i.d. case is solved, it is natural to seek for sufficient conditions ensuring the Hölderian invariance principle for strictly stationary random fields. In the one dimensional case, like for the space of continuous functions, it is possible to approximate by strictly stationary martingale differences sequences. Indeed, a sufficient condition on the moments of the increments is known (Theorem 2.2 of [Gir16]), namely, a finite strong moment of order $p(\alpha)$. Then some projective conditions guarantee the existence of an approximating martingale: a Hannan type condition (Theorem 2.6 of [Gir16]) and a Maxwell and Woodroffe type condition (Theorem 1.1 of [Gir15b])

The functional central limit theorem in the space of continuous functions holds for strictly stationary orthomartingale differences random fields if one of the transformations is ergodic [Vol15] (see Subsection 1.2 for the definition). Approximating by such random fields, a multidimensional Hannan condition has been obtained [VW14]. It is thus natural to try such an approach for the Hölderian invariance principle. Even in the i.i.d. case, this limit theorem requires a stronger condition on the common law than a finite second moment. In the one dimensional case, the found condition for i.i.d. sequences does not extend to martingales with stationary increments. Therefore, the first task is to find a sharp condition for orthomartingales.

Usually, the main difficulty in the proof of Hölderian invariance principle is tightness (the convergence of finite-dimensional distributions holds because we impose conditions which guarantee the invariance principle in the space of continuous functions). To do so, we can use a tightness criterion, given by (1.4) in [Gir15a]. Although moment inequalities helped in the i.i.d. case, counter-example given in Theorem 2.6 in [Gir15a] shows that Burkholder's type inequalities are not sufficient in general. This is why it is preferable to work with deviation inequalities instead of moment inequalities.

Here we can also formulate a tightness criterion in the multidimensional setting. In order to check it, we have to establish probability inequalities for orthomartingale differences random fields, which are of independent interest.

The paper is organized as follows: in the first section, we give some properties of the Hölder spaces and the definition of orthomartingale random fields as well as that of related projectors. We state the main results in the second section, with applications for a class of random fields. In the third section, we establish a tightness criterion for the normalized partial sum process defined by (1.0.2), which only involves the maxima of partial sums over rectangles. We also prove probability inequalities for strictly stationary orthomartingale differences random fields. The last section contains the proofs of main results.

1.1. Hölder spaces. In this subsection, we recall some properties of Hölder spaces. Define for $x: [0, 1]^d \rightarrow \mathbb{R}$ and $\delta \in (0, 1)$ the modulus of regularity

$$\omega_\alpha(x, \delta) := \sup_{0 < |\mathbf{t} - \mathbf{s}| \leq \delta} \frac{|x(\mathbf{t}) - x(\mathbf{s})|}{\|\mathbf{t} - \mathbf{s}\|^\alpha}. \quad (1.1.1)$$

For $0 < \alpha < 1$, we denote by $\mathcal{H}_\alpha^o([0, 1]^d)$ (respectively $\mathcal{H}_\alpha^o([0, 1]^d)$) the vector space of function $x: [0, 1]^d \rightarrow \mathbb{R}$ such that

$$\|x\|_\alpha := |x(\mathbf{0})| + \omega_\alpha(x, 1) < \infty, \quad (1.1.2)$$

(respectively $\omega_\alpha(x, \delta) \rightarrow 0$ as $\delta \rightarrow 0$).

Set for $j \geq 0$,

$$W_j := \{k2^{-j}, 0 \leq k \leq 2^j\}^d \quad (1.1.3)$$

and

$$V_0 := W_0, \quad V_j := W_j \setminus W_{j-1}, j \geq 1. \quad (1.1.4)$$

We define for $\mathbf{v} \in V_j$ the pyramidal function $\Lambda_{j, \mathbf{v}}$ by

$$\Lambda_{j, \mathbf{v}}(\mathbf{t}) := \Lambda(2^j(\mathbf{t} - \mathbf{v})), \quad \mathbf{t} \in [0, 1]^d, \quad (1.1.5)$$

where

$$\Lambda(\mathbf{t}) := \max \left\{ 0, 1 - \max_{t_i < 0} |t_i| - \max_{t_i > 0} |t_i| \right\}, \quad \mathbf{t} = (t_i)_{i=1}^d \in [-1, 1]^d. \quad (1.1.6)$$

For $x \in \mathcal{H}_\alpha^o([0, 1]^d)$, we define the coefficients $\lambda_{j, \mathbf{v}}(x)$ by $\lambda_{0, \mathbf{v}}(x) = x(\mathbf{v})$, $\mathbf{v} \in V_0$ and for $j \geq 1$ and $\mathbf{v} \in V_j$,

$$\lambda_{j, \mathbf{v}}(x) := x(\mathbf{v}) - \frac{1}{2} (x(\mathbf{v}^-) + x(\mathbf{v}^+)), \quad (1.1.7)$$

where \mathbf{v}^+ and \mathbf{v}^- are define in the following way. Each $\mathbf{v} \in V_j$ is represented in a unique way by $\mathbf{v} = (k_i 2^{-j})_{i=1}^d$. Then $\mathbf{v}^+ := (v_i^+)_{i=1}^d$ and $\mathbf{v}^- := (v_i^-)_{i=1}^d$ are defined by

$$\mathbf{v}_i^- := \begin{cases} v_i - 2^{-j}, & \text{if } k_i \text{ is odd;} \\ v_i, & \text{if } k_i \text{ is even} \end{cases} \quad \mathbf{v}_i^+ := \begin{cases} v_i + 2^{-j}, & \text{if } k_i \text{ is odd;} \\ v_i, & \text{if } k_i \text{ is even.} \end{cases} \quad (1.1.8)$$

The sequential norm is defined by

$$\|x\|_\alpha^{\text{seq}} := \sup_{j \geq 0} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(x)|, \quad x \in \mathcal{H}_\alpha^o([0, 1]^d). \quad (1.1.9)$$

By [RS04], the norm $\|\cdot\|_\alpha^{\text{seq}}$ is equivalent to $\|\cdot\|_\alpha$ on $\mathcal{H}_\alpha^o([0, 1]^d)$.

1.2. Orthomartingales and projection operators.

1.2.1. *Definition of orthomartingales.* Let $(T_q)_{q=1}^d$ be bijective, bi-measurable and measure preserving transformations on $(\Omega, \mathcal{F}, \mu)$. Assume that $T_q \circ T_{q'} = T_{q'} \circ T_q$ for each $q, q' \in \{1, \dots, d\}$. Let \mathcal{M} be a sub- σ -algebra of \mathcal{F} such that for each $q \in \{1, \dots, d\}$, $\mathcal{M} \subset T_q^{-1}\mathcal{M}$. In this way, $\mathcal{F}_i := T^{-i}\mathcal{M}$, $\mathbf{i} \in \mathbb{Z}^d$, yields a filtration. If for each $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d$ and each integrable and \mathcal{F}_i -measurable random variable Y ,

$$\mathbb{E}[Y \mid \mathcal{F}_{\mathbf{k}}] = \mathbb{E}[Y \mid \mathcal{F}_{\mathbf{k} \wedge \mathbf{l}}] \text{ almost surely,} \quad (1.2.1)$$

the transformations $(T_q)_{q=1}^d$ are said to be *completely commuting*.

The collection of random variables $\{M_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is said to be an orthomartingale random field with respect to the completely commuting filtration $(T^{-i}\mathcal{M})$ if for each $\mathbf{n} \in \mathbb{N}^d$, $M_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$ -measurable, integrable and for each $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^d$ such that $\mathbf{i} \preceq \mathbf{j}$,

$$\mathbb{E}[M_{\mathbf{j}} \mid \mathcal{F}_{\mathbf{i}}] = M_{\mathbf{i}}. \quad (1.2.2)$$

Here $\mathbf{i} \preceq \mathbf{j}$ means that $i_q \leq j_q$ for each $q \in \{1, \dots, d\}$.

Definition 1.1. *The random field $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale difference random field with respect to the completely commuting filtration $(T^{-i}\mathcal{M})$ if the random field $(M_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^d}$ defined by $M_{\mathbf{n}} := \sum_{\mathbf{i} \in [0, \mathbf{n}-1]} m \circ T^{\mathbf{i}}$ is an orthomartingale random field.*

1.2.2. *Some properties of orthomartingales.* Orthomartingale random fields have good properties with respect to marginal filtrations $\mathcal{F}_j^{(d)} := \sigma(\mathcal{F}_{\mathbf{k}}, k_q \leq j, \mathbf{k} \in \mathbb{Z}^d)$. Furthermore, when a coordinate is fixed, we still have an orthomartingale random field with respect to a completely commuting filtration.

Lemma 1.2. *Let $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be an orthomartingale difference random field with respect to the completely commuting filtration $(T^{-i}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$. Then the following properties hold.*

- (1) *For any $\mathbf{n} = (n_1, \dots, n_{d-1}) \in \mathbb{N}^{d-1}$, the sequence $(S_{(\mathbf{n}, j)}(m))_{j \geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_j^{(d)})_{j \geq 0}$.*
- (2) *For any $\mathbf{n} = (n_1, \dots, n_{d-1}) \in \mathbb{N}^{d-1}$, the sequence $(\max_{\mathbf{i} \in [1, \mathbf{n}]} |S_{(\mathbf{i}, j)}(m)|)_{j \geq 0}$ is a non-negative submartingale with respect to the filtration $(\mathcal{F}_j^{(d)})_{j \geq 0}$.*
- (3) *For any $j \in \mathbb{N}$, the random field $(S_{(\mathbf{n}, j)}(m))_{\mathbf{n} \succeq \mathbf{1}}$ is an orthomartingale with respect to the completely commuting filtration $(T^{-i}\mathcal{M}')_{\mathbf{i} \in \mathbb{Z}^{d-1}}$, where \mathcal{M}' is the σ -algebra generated by $\bigcup_{j \in \mathbb{Z}} T^{j\mathbf{e}_d}\mathcal{M}$.*

1.2.3. *Projection operators.* The projection operators with respect to a commuting filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ are defined by

$$P_{\mathbf{j}} := \prod_{q=1}^d P_{j_q}^{(q)}, \quad \mathbf{j} \in \mathbb{Z}^d, \quad (1.2.3)$$

where for $l \in \mathbb{Z}$, $P_l^{(q)} : \mathbb{L}^1(\mathcal{F}) \rightarrow \mathbb{L}^1(\mathcal{F})$ is defined by

$$P_l^{(q)}(f) = \mathbb{E}_l^{(q)}[f] - \mathbb{E}_{l-1}^{(q)}[f] \quad (1.2.4)$$

and

$$\mathbb{E}_l^{(q)}[\cdot] = \mathbb{E} \left[\cdot \mid \bigvee_{\substack{\mathbf{i} \in \mathbb{Z}^d \\ i_q \leq l}} \mathcal{F}_{\mathbf{i}} \right], q \in \{1, \dots, d\}, l \in \mathbb{Z}. \quad (1.2.5)$$

2. MAIN RESULTS

2.1. The orthomartingale case. When we try to extend a limit theorem for i.i.d. random to strictly stationary orthomartingale difference random fields, it is natural to determine whether the sufficient condition on the common law in the i.i.d. still is sufficient. In the case of the Hölderian weak invariance principle, like in the one dimensional case, the answer is negative.

Theorem 2.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, T a \mathbb{Z}^d -measure preserving action. We assume that the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ is ergodic and of positive entropy. For each $\alpha \in (0, 1/2)$, there exists a sub- σ -algebra \mathcal{F} such that $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$ is a commuting filtration, a function m and an increasing sequence of positive integers $(N_l)_{l \geq 1}$ such that*

- (1) $\lim_{t \rightarrow \infty} t^{p(\alpha)} \mu \{|m| > t\} = 0$ where $p(\alpha) = (1/2 - \alpha)^{-1}$;
- (2) $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale difference random field with respect to the filtration $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$;
- (3) the sequence $\left((N_l l^{d-1})^{-1/2} S_{N_l, l, \dots, l}(m, \cdot) \right)$ is not tight in $\mathcal{H}_\alpha([0, 1]^d)$.

In the one dimensional case, a sufficient condition for the Hölderian weak invariance principle for stationary martingales differences sequences is expressed in terms of the common distribution of the sequence and the conditional variances. A similar condition can be formulated in the multidimensional setting.

Theorem 2.2. *Assume that $T: \Omega \rightarrow \Omega$ is a \mathbb{Z}^d -measure preserving action with T_q ergodic for some $q \in \langle d \rangle$ and \mathcal{M} is a sub- σ -algebra such that $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$ is a commuting filtration. If m is an orthomartingale martingale difference random field with respect to $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$ such that for some $p > 2$,*

- (1) $\lim_{t \rightarrow +\infty} t^p \mu \{|m| > t\} = 0$;
- (2) for each $q \in \{1, \dots, d\}$, $\mathbb{E}[m^2 \mid T_q \mathcal{M}] \in \mathbb{L}^{p/2}$,

then the convergence

$$\frac{1}{|\mathbf{n}|^{1/2}} S_{\mathbf{n}}(m, \cdot) \xrightarrow{\min \mathbf{n} \rightarrow \infty} \|m\|_2 W \text{ in distribution in } \mathcal{H}_{1/2-1/p}([0, 1]^d) \quad (2.1.1)$$

takes place. In particular, if $m \in \mathbb{L}^p$, then the convergence (2.1.1) holds.

Remark 2.3. Ergodicity of one of the maps T_q is not needed to guarantee asymptotic tightness of $\left(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(m, \cdot) \right)_{\mathbf{n} \in (\mathbb{N} \setminus \{0\})^d}$ in $\mathcal{H}_{1/2-1/p}([0, 1]^d)$. This is only used for the convergence of the finite dimensional distribution.

Remark 2.4. If $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d} =: (\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an i.i.d. centered random field, then we do not exactly recover Theorem 2 of [RSZ07]. Indeed, in this case, the conditions of Theorem 2.2 are equivalent to $\mu \{|m| > t\} = o(t^{-p})$, while that of Theorem 2 of [RSZ07] is $\mu \{|m| > t\} = O(t^{-p})$.

2.2. Hannan type condition. Once a sharp moment condition on the orthomartingale random field is known, it is natural to try to extend it to stationary random fields which are approximable in a satisfying way by orthomartingales. This is the case for Hannan's condition, for which an invariance principle has been established [VW14].

We extend Theorem 2.6 of [Gir16] to random fields.

Theorem 2.5. *Let $p > 2$ and let f be a $\mathcal{F}_{\infty, \dots, \infty}$ -measurable function such $\mathbb{E}[f \mid \mathcal{F}_i] \rightarrow 0$ in \mathbb{L}^p as $\min_{q \in \langle d \rangle} i_q \rightarrow -\infty$. Assume that one of the transformations T_q , $q \in \{1, \dots, d\}$ is ergodic and that*

$$\sum_{\mathbf{i} \in \mathbb{Z}^d} \|P_{\mathbf{i}}(f)\|_p < \infty. \quad (2.2.1)$$

Then

$$\frac{1}{|\mathbf{n}|^{1/2}} S_{\mathbf{n}}(f, \cdot) \xrightarrow{\min \mathbf{n} \rightarrow \infty} \sigma^2 W \text{ in distribution in } \mathcal{H}_{1/2-1/p}([0, 1]^d), \quad (2.2.2)$$

where

$$\sigma^2 = \mathbb{E} \left[\left(\sum_{\mathbf{i} \in \mathbb{Z}^d} P_{\mathbf{0}}(U^{\mathbf{i}} f) \right)^2 \right]. \quad (2.2.3)$$

Remark 2.6. Theorem 2.5 improves Theorem 2.3 in [EMG]. The later can be used only if $p > 2d$, and the convergence is considered along $n_1 = \dots = n_d$, which is less general than $\min_{i \in \langle d \rangle} n_i \rightarrow +\infty$. Applying Theorem 2.5 with $p = 3d$, we obtain the invariance principle in $\mathcal{H}_{1/2-1/(3d)}([0, 1]^d)$ while Theorem 2.3 of [EMG] only gives the invariance principle in $\mathcal{H}_{\gamma}([0, 1]^d)$ for any $\gamma < 1/2 - 1/3$. The proof of Theorem 2.3 of [EMG] rests on a general tightness criterion given in [Kli07]. It turns out that the tightness criterion in Proposition 3.2 is more efficient for partial sum processes associated to a strictly stationary random field.

2.3. Applications. Theorem 2.5 can be applied in the context of functional of i.i.d. random fields. Let $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be an i.i.d. random field and let $g: \mathbb{R}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ be a measurable function. The random field $(X_{\mathbf{j}})_{\mathbf{j} \in \mathbb{Z}^d}$ defined by

$$X_{\mathbf{j}} = g(\varepsilon_{\mathbf{j}-\mathbf{i}}, \mathbf{i} \in \mathbb{Z}^d), \mathbf{j} \in \mathbb{Z}^d \quad (2.3.1)$$

is strictly stationary (such a process is called Bernoulli shift in [Dou03]). Wu [Wu05] introduced the following measure of dependence: let $\varepsilon'_{\mathbf{0}}$ be a copy of $\varepsilon_{\mathbf{0}}$ independent of $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ and define the random field $(\varepsilon_{\mathbf{j}}^*)_{\mathbf{j} \in \mathbb{Z}^d}$ by $\varepsilon_{\mathbf{0}}^* = \varepsilon'_{\mathbf{0}}$ and $\varepsilon_{\mathbf{i}}^* = \varepsilon_{\mathbf{i}}$ if $\mathbf{i} \neq \mathbf{0}$. For each $p \geq 1$, the physical dependence measure $\delta_{\mathbf{j}, p}$ is given by

$$\delta_{\mathbf{j}, p} = \|X_{\mathbf{j}} - X_{\mathbf{j}}^*\|_p, \quad (2.3.2)$$

where $X_{\mathbf{j}}^*$ is the random variable defined by (2.3.1) with $(\varepsilon_{\mathbf{j}}^*)_{\mathbf{j} \in \mathbb{Z}^d}$ instead of $(\varepsilon_{\mathbf{j}})_{\mathbf{j} \in \mathbb{Z}^d}$. The random field $(X_{\mathbf{j}})_{\mathbf{j} \in \mathbb{Z}^d}$ defined by (2.3.1) is called p -stable if $\sum_{\mathbf{j} \in \mathbb{Z}^d} \delta_{\mathbf{j}, p}$ is finite.

In [Wu05], Theorem 1, it is shown that in dimension one, $\|P_{\mathbf{j}}\|_2$ can be bounded by $2\delta_{\mathbf{j}, 2}$, where $P_{\mathbf{j}}$ is defined by (1.2.3). By similar argument, for a general d , defining $\mathcal{M} := \sigma(\varepsilon_{\mathbf{i}}, \mathbf{i} \preccurlyeq \mathbf{0})$ and $P_{\mathbf{j}}$ by (1.2.3), $\|P_{\mathbf{j}}(f)\|_p$ can be controlled by a constant independent of f times $\delta_{\mathbf{j}, p}$, hence the conditions of Theorem 2.5 are satisfied by a $p(\alpha)$ -stable centered random field. We now give some examples of such random fields (the proof of their $p(\alpha)$ -stability is contained in [BD14], Section 2), which were also given in [EVW13].

(1) Linear random fields. Let $X_{\mathbf{j}}$ be defined by

$$X_{\mathbf{j}} = \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} \varepsilon_{\mathbf{j}-\mathbf{i}}, \quad (2.3.3)$$

where $(a_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is a family of real numbers such that $\sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}}^2 < +\infty$ and $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is a i.i.d. centered random field such that $\varepsilon_{\mathbf{0}} \in \mathbb{L}^{p(\alpha)}$. Since $p(\alpha) > 2$, Rosenthal's inequality (see Theorem 3 of [Ros70]) shows the random field $X = (X_{\mathbf{j}})_{\mathbf{j} \in \mathbb{Z}^d}$ is well defined in $\mathbb{L}^{p(\alpha)}$. Moreover, the random field X is $p(\alpha)$ -stable if and only if $\sum_{\mathbf{i} \in \mathbb{Z}^d} |a_{\mathbf{i}}| < +\infty$.

(2) Functionals of linear random fields. Let $X_{\mathbf{j}}$ be a linear random field as above (such that $\sum_{\mathbf{i} \in \mathbb{Z}^d} |a_{\mathbf{i}}| < +\infty$) and $Y = (Y_{\mathbf{j}})_{\mathbf{j} \in \mathbb{Z}^d}$ be the random field defined by $Y_{\mathbf{j}} = f(X_{\mathbf{j}})$, $\mathbf{j} \in \mathbb{Z}^d$, where $f: \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz function. Then Y is $p(\alpha)$ -stable. More generally, if f is H -Hölder continuous with $H \in (0, 1]$, then Y is $p(\alpha)$ -stable if $Y_{\mathbf{0}} \in \mathbb{L}^{p(\alpha)}$, $\varepsilon_{\mathbf{0}} \in \mathbb{L}^{p(\alpha)H}$ and $\sum_{\mathbf{j} \in \mathbb{Z}^d} |a_{\mathbf{j}}|^H < +\infty$.

(3) Volterra fields. They are defined by the equality

$$X_{\mathbf{j}} = \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d} a_{\mathbf{k}, \mathbf{l}} \varepsilon_{\mathbf{j}-\mathbf{k}} \varepsilon_{\mathbf{j}-\mathbf{l}}, \quad (2.3.4)$$

where $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is a i.i.d. centered random field such that $\varepsilon_{\mathbf{0}} \in \mathbb{L}^{p(\alpha)}$ and $(a_{\mathbf{k}, \mathbf{l}})_{\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d}$ is a sequence such that $a_{\mathbf{k}, \mathbf{k}} = 0$, $\mathbf{k} \in \mathbb{Z}^d$ and $\sum_{\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d} a_{\mathbf{k}, \mathbf{l}}^2 < \infty$ (in this way, $X_{\mathbf{j}}$ is well defined in $\mathbb{L}^{p(\alpha)}$). If furthermore

$$\sum_{\mathbf{j} \in \mathbb{Z}^d} \left(\sum_{\mathbf{l} \in \mathbb{Z}^d} |a_{\mathbf{j}, \mathbf{l}} + a_{\mathbf{l}, \mathbf{j}}|^2 \right)^{1/2} < +\infty, \quad (2.3.5)$$

then $(X_{\mathbf{j}})_{\mathbf{j} \in \mathbb{Z}^d}$ is $p(\alpha)$ -stable.

3. TOOLS FOR THE PROOFS

3.1. Tightness criterion in Hölder spaces. In this subsection, we give a sufficient condition for tightness of the partial sum process associated to a strictly stationary random field. No other assumption is done but of course, some dependence will be required for this condition to be satisfied. A general tightness criterion is available.

Theorem 3.1 (Theorem 6, [RSZ07]). *Let $\{\zeta_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ and ζ be random elements with values in the space $\mathcal{H}_{\alpha}([0, 1]^d)$. Assume that the following conditions are satisfied.*

- (1) *For each dyadic $\mathbf{t} \in [0, 1]^d$, the net $\{\zeta_{\mathbf{n}}(\mathbf{t}), \mathbf{n} \in \mathbb{N}^d\}$ is asymptotically tight on \mathbb{R} .*
- (2) *For each positive ε ,*

$$\lim_{J \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(\zeta_{\mathbf{n}})| > \varepsilon \right\} = 0. \quad (3.1.1)$$

Then the net $\{\zeta_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is asymptotically tight in the space $\mathcal{H}_{\alpha}([0, 1]^d)$.

3.1.1. *Statement.* In the sequel, we shall denote by \log the binary logarithm. Let us state our tightness criterion.

Proposition 3.2. *Let $f: \Omega \rightarrow \mathbb{R}$ be a measurable function. Then for each $\mathbf{n} \in \mathbb{N}^d$, $J \geq 1$ and $x > 0$, the following inequality holds*

$$\begin{aligned} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f, \cdot))| > x \right\} \leq \\ \leq 3 \sum_{i=1}^d \sum_{j=J}^{\log n_i} 2^j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} f \right| > Cx \prod_{q=1}^d \sqrt{n_q} \right\}, \end{aligned} \quad (3.1.2)$$

where C depends only on d .

Therefore, if f is such that for each $i \in \{1, \dots, d\}$, and each positive ε

$$\lim_{J \rightarrow \infty} \limsup_{\min \mathbf{n} \rightarrow \infty} \sum_{j=J}^{\log n_i} 2^j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{s} \preccurlyeq \mathbf{1}} U^{\mathbf{s}} f \right| > \varepsilon \prod_{q=1}^d \sqrt{n_q} \right\} = 0, \quad (3.1.3)$$

then the net $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot))$ is asymptotically tight in $\mathcal{H}_{1/2-1/p}([0, 1]^d)$.

3.1.2. *Proof of the tightness criterion.* Consider for $\mathbf{s} := (s_2, \dots, s_d) \in [0, 1]^{d-1}$ and $t, t' \in [0, 1]$ the quantity

$$\Delta_n(t, t', \mathbf{s}) := |S_{\mathbf{n}}(f, (t', \mathbf{s})) - S_{\mathbf{n}}(f, (t, \mathbf{s}))|. \quad (3.1.4)$$

We recall the following lemma:

Lemma 3.3 (Lemma 11, [RSZ07]). *For any $t', t \in [0, 1]$, $t' > t$, we have*

$$\begin{aligned} \sup_{\mathbf{s} \in [0, 1]^{d-1}} \Delta_n(t, t', \mathbf{s}) \leq 3^d \mathbf{1} \left\{ t' - t \geq \frac{1}{n_1} \right\} \max_{\substack{1 \leq k_l \leq n_l \\ 2 \leq l \leq d}} \left| \sum_{i_1=[n_1 t]+1}^{[n_1 t']} \sum_{\substack{1 \leq i_l \leq k_l \\ 2 \leq l \leq d}} U^{\mathbf{i}} f \right| + \\ + 3^d \min \{1, n_1(t' - t)\} \max_{1 \leq i_1 \leq n_1} \max_{\substack{1 \leq k_l \leq n_l \\ 2 \leq l \leq d}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ 2 \leq l \leq d}} U^{\mathbf{i}} f \right|. \end{aligned} \quad (3.1.5)$$

Now, we define for $q \in \langle d \rangle$ and $\mathbf{s} = (s_l)_{l \in \langle d \rangle \setminus \{q\}} \in [0, 1]^{d-1}$,

$$\Delta_n^{(q)}(t, t', \mathbf{s}) := |S_{\mathbf{n}}(f, (s_1, \dots, s_{q-1}, t', s_{q+1}, \dots, s_d)) - S_{\mathbf{n}}(f, (s_1, \dots, s_{q-1}, t, s_{q+1}, \dots, s_d))|.$$

By definition of $\lambda_{j, \mathbf{v}}$ and V_j , the inequality

$$\begin{aligned} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f, \cdot))| > 2^{d+1} 3^d x \right\} \leq \\ \sum_{q=1}^d \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\substack{0 \leq k < 2^j \\ \mathbf{0} \preccurlyeq \mathbf{u} \preccurlyeq 2^j \mathbf{1}}} \Delta_n^{(q)}(t_{k+1}, t_k; \mathbf{s}_{\mathbf{u}}) > 2 \cdot 3^d x \right\} \end{aligned} \quad (3.1.6)$$

takes place, where $t_k = k2^{-j}$ and $\mathbf{s}_{\mathbf{u}} = (u_i 2^{-j})_{i \in \langle d \rangle \setminus \{q\}}$.

We have in view of Lemma 3.3 that for each $q \in \langle d \rangle$,

$$\begin{aligned}
 & \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\substack{0 \leq k < 2^j \\ \mathbf{0} \leq \mathbf{u} \leq 2^j \mathbf{1}}} \Delta_n^{(q)}(t_{k+1}, t_k; \mathbf{s}_{\mathbf{u}}) > 2x |\mathbf{n}|^{1/2} \right\} \\
 & \leq \mu \left\{ \sup_{j \geq J} \max_{0 \leq a < 2^j} 2^{\alpha j} \mathbf{1} \left\{ 2^{-j} \geq \frac{1}{n_q} \right\} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| + \right. \\
 & \quad \left. + \min \{1, n_q 2^{-j}\} \max_{1 \leq i_q \leq n_q} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > 2x |\mathbf{n}|^{1/2} \right\} \\
 & \leq \mu \left\{ \sup_{j \geq J} \max_{0 \leq a < 2^j} 2^{\alpha j} \mathbf{1} \left\{ 2^{-j} \geq \frac{1}{n_q} \right\} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\} + \\
 & \quad + \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \min \{1, n_q 2^{-j}\} \max_{1 \leq i_q \leq n_q} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\}. \quad (3.1.7)
 \end{aligned}$$

Since the indicator in the first term of the right hand side of (3.1.7) vanishes if $j > \log n_q$, we have

$$\begin{aligned}
 & \mu \left\{ \sup_{j \geq J} \max_{0 \leq a < 2^j} 2^{\alpha j} \mathbf{1} \left\{ 2^{-j} \geq \frac{1}{n_q} \right\} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\} \\
 & \leq \mu \left\{ \sup_{J \leq j \leq \log n_q} \max_{0 \leq a < 2^j} 2^{\alpha j} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\} \\
 & \leq \sum_{j=J}^{\log n_q} 2^j \max_{0 \leq a < 2^j} \mu \left\{ 2^{\alpha j} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j}] + 1}^{[n_q(a+1)2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\}, \quad (3.1.8)
 \end{aligned}$$

and by stationarity, it follows that

$$\begin{aligned} & \mu \left\{ \sup_{j \geq J} \max_{0 \leq a < 2^j} 2^{\alpha j} \mathbf{1} \left\{ 2^{-j} \geq \frac{1}{n_q} \right\} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{i_q = [n_q a 2^{-j} + 1]}^{[n_q (a+1) 2^{-j}]} \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\} \\ & \leq \sum_{j=J}^{\log n_q} 2^j \mu \left\{ 2^{\alpha j} \max_{0 \leq i_q \leq 2n_q 2^{-j}} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\}. \end{aligned} \quad (3.1.9)$$

For the second term of the right hand side of (3.1.7), notice that

$$\sup_{j \geq J} 2^{\alpha j} \min \{1, n_q 2^{-j}\} \leq n_q^\alpha. \quad (3.1.10)$$

Indeed, if $j \leq \log n_q$, then $2^j \leq n_q$ hence $2^{\alpha j} \min \{1, n_q 2^{-j}\} \leq n_q^\alpha$, and if $j > \log n_q$, then $2^j > n_q$, hence $\min \{1, n_q 2^{-j}\} = n_q 2^{-j}$ and for such j 's, we have $2^{\alpha j} n_q 2^{-j} = n_q 2^{-(1-\alpha)j} \leq n_q^\alpha$, since $\alpha < 1$. As a consequence, after having bounded the probability of the max over i_q by the sum of probabilities and used stationarity, we obtain

$$\begin{aligned} & \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \min \{1, n_q 2^{-j}\} \max_{1 \leq i_q \leq n_q} \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} \right\} \\ & \leq n_q \mu \left\{ \max_{\substack{1 \leq k_l \leq n_l \\ l \neq q}} \left| \sum_{\substack{1 \leq i_l \leq k_l \\ l \neq q}} U^i f \right| > x |\mathbf{n}|^{1/2} n_q^{-\alpha} \right\}. \end{aligned} \quad (3.1.11)$$

Combining inequalities (3.1.7) with (3.1.9) and (3.1.11), we obtain (3.1.2).

The second part of Proposition 3.2 follows by Theorem 3.1.

3.2. Probability inequalities for orthomartingale random fields.

3.2.1. *The one dimensional case.* A lot of inequalities for martingales involve the maxima of increments and the quadratic variance. Let us mention the following result by Nagaev, which links the tail function of the maxima of a martingale with that of the increments and the quadratic variance.

Theorem 3.4 (Theorem 1, [Nag03]). *Let $q > 0$ and $C(q) := qe^{3qe^{q+1}}/qe^{q+1}$ and let (S_n, \mathcal{F}_n) be a martingale. Then*

$$\mu \left\{ \max_{1 \leq k \leq n} S_k \geq t \right\} \leq C(q) t^{-q} \int_0^t Q(u) u^{q-1} du, \quad (3.2.1)$$

where

$$Q(u) := \mu \left\{ \max_{1 \leq k \leq n} |X_k| > u \right\} + \mu \left\{ \left(\sum_{k=1}^n \mathbb{E} [X_k^2 | \mathcal{F}_{k-1}] \right)^{1/2} > u \right\}. \quad (3.2.2)$$

Let us focus on the stationary case. If q is a real number greater than 2, we take $\delta = \delta(q) > 0$ such that $\delta(1 + \delta + \sqrt{\delta})^q = 1$.

Proposition 3.5. *Let $T: \Omega \rightarrow \Omega$ be a measure-preserving map. Assume that \mathcal{F} is a sub- σ -algebra such that $T\mathcal{F} \subset \mathcal{F}$ and that $(M \circ T^i)_{i \geq 1}$ is a martingale difference sequence with respect to the filtration $(T^{-i}\mathcal{F})_{i \geq 0}$. Then for each real number t , each $q > 2$ and each positive integer n , the following inequalities hold:*

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq 2\sqrt{nt}/\delta \right\} &\leq c(q)n \int_0^1 \mu \{ |M| \geq \sqrt{nut} \} u^{q-1} du + \\ &+ c(q) \int_0^\infty \mu \left\{ (\mathbb{E}[M^2 | T\mathcal{F}])^{1/2} > \sqrt{2vt} \right\} \min \{ v, v^{q-1} \} dv; \end{aligned} \quad (3.2.3)$$

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq \sqrt{nt} \right\} \leq c(q) \int_0^{+\infty} \mu \{ |M| \geq tv \} \min \{ v, v^{q-1} \} dv. \quad (3.2.4)$$

While this can be deduced from a combination of Theorem 3.4 and the maximal ergodic theorem, we shall give a complete proof for the following two reasons. First, Nagaev's result also applies for supermartingales and in the martingale case, his proof can be simplified. Second, the constant $C(q)$ obtained in the general case can be improved when restricted to martingales.

The following lemma will be useful in the sequel.

Lemma 3.6. *Assume that X and Y are two random non-negative variables such that for each positive λ , we have*

$$\lambda \mu \{ X > \lambda \} \leq \mathbb{E}[Y \mathbf{1} \{ X \geq \lambda \}]. \quad (3.2.5)$$

Then for each t , the following inequality holds:

$$\mu \{ X > 2t \} \leq \int_1^{+\infty} \mu \{ Y > st \} ds. \quad (3.2.6)$$

Proof of Lemma 3.6. Rewriting the expectation as

$$\mathbb{E}[Y \mathbf{1} \{ X \geq 2t \}] = \int_0^{+\infty} \mu \{ Y \mathbf{1} \{ X \geq 2t \} > u \} du \leq t \mu \{ X \geq 2t \} + \int_t^{+\infty} \mu \{ Y > u \} du, \quad (3.2.7)$$

we derive by the assumption the bound

$$2t \mu \{ X > 2t \} \leq t \mu \{ X \geq 2t \} + \int_t^{+\infty} \mu \{ Y > u \} du. \quad (3.2.8)$$

We conclude using the substitution $ts := u$. \square

For example, if $(S_i, \mathcal{F}_i)_{i \geq 1}$ is a non-negative submartingale, then (3.2.5) holds with $X = \max_{1 \leq i \leq n} S_i$ and $Y = S_n$ for each $n \geq 1$.

Proof of Proposition 3.5. If $(S_i = \sum_{j \leq i} X_j, \mathcal{F}_i)$ is a martingale, then for each $\beta > 1$, $\delta \in (0, \beta - 1)$ and $\lambda > 0$, and each integer $N \geq 1$, the inequality

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq N} |S_i| > \beta \lambda \right\} &\leq \frac{\delta^2}{(\beta - \delta - 1)^2} \mu \left\{ \max_{1 \leq i \leq N} |S_i| > \lambda \right\} + \\ &+ \mu \left\{ \sum_{i=1}^N \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}] > \delta^2 \lambda^2 \right\} + \mu \left\{ \max_{1 \leq i \leq N} |X_i| > \delta \lambda \right\} \end{aligned} \quad (3.2.9)$$

takes place (see [HH80], p. 28). Let us consider a positive δ such that $\delta(1 + \delta + \sqrt{\delta})^q = 1$.

For fixed x , and a non-negative integer m , we define

$$y_m := (1 + \delta + \sqrt{\delta})^m x. \quad (3.2.10)$$

By (3.2.9) applied with $\beta = 1 + \delta + \sqrt{\delta}$, it follows that for each m ,

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \leq \delta \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_{m-1} \right\} + Q(\delta y_{m-1}), \quad (3.2.11)$$

where Q is defined by (3.2.2) (with $X_k = M \circ T^k$ and $\mathcal{F}_i = T^{-i}\mathcal{F}$). Denoting by a_m the quantity $a_m := \delta^{-m} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\}$, we have $a_m \leq a_{m-1} + Q(\delta y_{m-1})$, hence

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \leq \sum_{i=0}^{m-1} Q(\delta y_i) \delta^{m-i-1} + \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq x \right\} \delta^{m-1}. \quad (3.2.12)$$

Since $Q(\delta y_i) \leq 2$ and $\delta \leq 1$, we derive the bound

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \leq \sum_{i=1}^{m-1} Q(\delta y_i) \delta^{m-i-1} + 3\delta^{m-1}. \quad (3.2.13)$$

Since the function $u \mapsto Q(u)$ is non-increasing, we have

$$Q(\delta y_i) \int_{y_{i-1}}^{y_i} u^{q-1} du \leq \int_{y_{i-1}}^{y_i} Q(\delta u) u^{q-1} du,$$

hence

$$Q(\delta y_i) \leq \int_{y_{i-1}}^{y_i} Q(\delta u) u^{q-1} du \cdot \left((y_i - y_{i-1}) y_i^{q-1} \right)^{-1}. \quad (3.2.14)$$

Plugging this estimate and (3.2.10) into (3.2.13), we obtain

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \\ & \leq \frac{1}{y_m^q} \sum_{i=1}^{m-1} \int_{y_{i-1}}^{y_i} Q(\delta u) u^{q-1} du \cdot y_m^q \left((y_i - y_{i-1}) y_i^{q-1} \right)^{-1} \delta^{m-i-1} + 3\delta^{m-1}. \end{aligned} \quad (3.2.15)$$

Now, since $\delta(1 + \delta + \sqrt{\delta})^q = 1$, it follows that

$$\begin{aligned} y_m^q \left((y_i - y_{i-1}) y_i^{q-1} \right)^{-1} \delta^{m-i-1} &= \frac{y_m^q}{y_i^q} \frac{y_i}{y_i - y_{i-1}} \delta^{m-i-1} \\ &= (1 + \delta + \sqrt{\delta})^{(m-i)q} \frac{(1 + \delta + \sqrt{\delta}) y_{i-1}}{(\delta + \sqrt{\delta}) y_{i-1}} \delta^{m-i-1} \\ &= \frac{(1 + \delta + \sqrt{\delta})}{\delta(\delta + \sqrt{\delta})} \\ &\leq \frac{3}{\delta^2}, \end{aligned}$$

hence by (3.2.15), we derive

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \leq \frac{3}{y_m^q \delta^2} \int_0^{y_{m-1}} Q(\delta u) u^{q-1} du + 3\delta^{m-1}. \quad (3.2.16)$$

Let y be such that $y_m \leq y < y_{m+1}$; then

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y \right\} &\leq \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y_m \right\} \\ &\leq \frac{3}{y_m^q \delta^2} \int_0^{y_m} Q(\delta u) u^{q-1} du + 3\delta^{m-1} \\ &\leq \frac{3}{y^q \delta^2} \left(\frac{y_{m+1}}{y_m} \right)^q \int_0^y Q(\delta u) u^{q-1} du + 3\delta^{m-1} \\ &= \frac{3}{y^q \delta^2} (1 + \delta + \sqrt{\delta})^q \int_0^y Q(\delta u) u^{q-1} du + 3\delta^{m-1}, \end{aligned}$$

and using again the equality $\delta(1 + \delta + \sqrt{\delta})^q = 1$, we derive

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y \right\} \mathbf{1}_{[x, \infty)}(y) &\leq \frac{3}{y^q \delta^3} \int_0^y Q(\delta u) u^{q-1} du \cdot \mathbf{1}_{[x, \infty)}(y) + \\ &+ 3 \sum_{m=0}^{\infty} \delta^{m-1} \cdot \mathbf{1}_{[x(1+\delta+\sqrt{\delta})^m, x(1+\delta+\sqrt{\delta})^{m+1})}(y). \quad (3.2.17) \end{aligned}$$

Since x is arbitrary and $\delta \in (0, 1)$, the second term converges to 0 as x goes to 0, hence

$$\mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq y \right\} \leq \frac{3}{y^q \delta^3} \int_0^y Q(\delta u) u^{q-1} du. \quad (3.2.18)$$

By definition of Q and the fact that T is measure preserving, we obtain

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(M)| \geq \sqrt{nt} \right\} &\leq c(q)n \int_0^1 \mu \left\{ |M| \geq \sqrt{nut} \right\} u^{q-1} du + \\ &+ c(q) \int_0^1 \mu \left\{ n^{-1} \sum_{j=0}^{n-1} U^j (\mathbb{E}[M^2 | T\mathcal{F}]) > 4u^2 t^2 \right\} u^{q-1} du. \quad (3.2.19) \end{aligned}$$

By the maximal ergodic theorem, inequality (3.2.7) holds with $X = \sup_{n \geq 1} n^{-1} S_n (\mathbb{E}[M^2 | T\mathcal{F}])$ and $Y = \mathbb{E}[M^2 | T\mathcal{F}]$, hence by Lemma 3.6 the estimate

$$\mu \left\{ n^{-1} \sum_{j=1}^n U^j (\mathbb{E}[M^2 | T\mathcal{F}]) > 4u^2 t^2 \right\} \leq \int_1^{+\infty} \mu \left\{ \mathbb{E}[M^2 | T\mathcal{F}] > 2u^2 t^2 s \right\} ds \quad (3.2.20)$$

is valid for any n . We can deduce from inequalities (3.2.19) and (3.2.20) that (3.2.3) is satisfied (after having switched the integrals).

In order to prove (3.2.4), we go back to (3.2.19) and we define for an integrable function $g: \Omega \rightarrow \mathbb{R}$, $Vg := U\mathbb{E}[g | T\mathcal{M}]$. Notice that V is an \mathbb{L}^1 - \mathbb{L}^∞ contraction and that $V^n(g) =$

$U^n \mathbb{E}[g \mid T\mathcal{M}]$. Therefore, we have the bound

$$\begin{aligned} \int_0^1 \mu \left\{ n^{-1} \sum_{j=0}^{n-1} U^j (\mathbb{E}[M^2 \mid T\mathcal{F}]) > 4u^2 t^2 \right\} u^{q-1} du &\leq \\ &\int_0^1 \int_1^{+\infty} \mu \{M^2 > 2u^2 t^2 v\} u^{q-1} dudv \leq \\ &\int_0^1 \int_1^{+\infty} \mu \{|M| > ut\sqrt{v}\} u^{q-1} dudv, \end{aligned} \quad (3.2.21)$$

and using the substitution $u' := u\sqrt{v}$, we get

$$\begin{aligned} \int_0^1 \int_1^{+\infty} \mu \{|M| > ut\sqrt{v}\} u^{q-1} dudv &= \int_0^{+\infty} \mu \{|M| > tu\} \int_{\max\{1, u^2\}}^{+\infty} u^{q-1} v^{-q/2} dv du \\ &= \frac{1}{1-q/2} \int_0^{+\infty} \mu \{|M| > tu\} u^{q-1} (\max\{1, u^2\})^{1-q/2} du \\ &= \frac{2}{2-q} \int_0^{+\infty} \mu \{|M| > tu\} \min\{u, u^{q-1}\} du. \end{aligned}$$

We now bound the first term of (3.2.19) independently of n . We have

$$n \int_0^1 \mu \{|M| \geq \sqrt{n}ut\} u^{q-1} du = n^{1-q/2} \int_0^{\sqrt{n}} \mu \{|M| \geq vt\} v^{q-1} dv. \quad (3.2.22)$$

If $v \leq 1$, we use the bound $n^{1-q/2}v^{q-1} \leq v^{q-1}$, and if $1 < v \leq \sqrt{n}$, then $n^{1-q/2}v^{q-1} \leq v$.

This concludes the proof of Proposition 3.5. \square

3.2.2. *The multidimensional case.* We extend Proposition 3.5 to the multidimensional setting.

Proposition 3.7. *Let $(m \circ T^i)_{i \in \mathbb{Z}^d}$ be a strictly stationary orthomartingale difference random field with respect to the completely commuting filtration $(T^{-i}\mathcal{M})_{i \in \mathbb{Z}^d}$ and $q > 2$. Then for each t , each $\mathbf{n} \in \mathbb{N}^d$ and each $j \in \{1, \dots, d\}$, the following inequalities take place:*

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq \mathbf{n}} |S_i(m)| \geq \prod_{l=1}^d \sqrt{n_l} t \right\} \\ \leq c(q, d) n_j \int_0^{+\infty} \mu \{|m| > tv\sqrt{n_j}\} (1 + |\log v|)^{d-1} \min\{v, v^{q-1}\} dv \\ + c(q, d) \int_0^{+\infty} \mu \{(\mathbb{E}[m^2 \mid T_j\mathcal{M}])^{1/2} > tv\} (1 + |\log v|)^{d-1} \min\{v, v^{q-1}\} dv, \end{aligned} \quad (3.2.23)$$

and

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq \mathbf{n}} |S_i(m)| \geq \prod_{l=1}^d \sqrt{n_l} t \right\} \\ \leq c(q, d) \int_0^{+\infty} \mu \{|m| > tv\} (1 + |\log v|)^{d-1} \min\{v, v^{q-1}\} dv. \end{aligned} \quad (3.2.24)$$

Although it will not be needed in the sequel, let us mention an immediate application of the previous result.

Corollary 3.8. *Let $p > 2$ and m a function as in Proposition 3.7 and such that $\mathbb{E}|m|^p$ is finite. Then the family*

$$\mathcal{F} := \left\{ |\mathbf{n}|^{-p/2} \max_{1 \leq j \leq \mathbf{n}} |S_j(m)|^p, \mathbf{n} \succcurlyeq \mathbf{1} \right\} \quad (3.2.25)$$

is uniformly integrable.

Remark 3.9. For a fixed $p > 2$, one can deduce multidimensional Burkholder's inequality (see [Faz05]) by applying Proposition 3.7 with $q > p$. Indeed, we multiply on both sides of inequality (3.2.23) by pt^{p-1} and integrate on $[0, +\infty)$ using the convergence of $\int_0^1 v^{q-p-1} dv$ and $\int_1^\infty v \log v \cdot v^{-p} dv$ (because $2 < p < q$).

However, in this way, the obtained constant is certainly not optimal.

Remark 3.10. It seems that Proposition 3.7 is not efficient for second moments. In particular, the uniform integrability result in [VW14] does not appear as a consequence of (3.2.24). This is due to the fact that the integral $\int_1^\infty 1/v dv$ is divergent.

Proof of Proposition 3.7. We start by proving (3.2.24) by induction on d .

For $d = 1$, this reduces to (3.2.4). Assume that for some $d \geq 2$, we have the following property: if $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^{d-1}}$ is a strictly stationary orthomartingale difference random field with respect to the completely commuting filtration $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^{d-1}}$ and $q > 2$. Then for each $t > 0$ and each $\mathbf{n} \in \mathbb{N}^{d-1}$,

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i \leq \mathbf{n}} |S_i(m)| \geq \prod_{l=1}^d \sqrt{n_l t} \right\} \\ & \leq c(q, d-1) \int_0^{+\infty} \mu \{ |m| > tv \} (1 + |\log v|)^{(d-1)-1} \min \{ v, v^{q-1} \} dv. \end{aligned} \quad (3.2.26)$$

We have to prove that this property is still valid for d , namely, that if $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is a strictly stationary orthomartingale difference random field with respect to the completely commuting filtration $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$ and $q > 2$. Then for each t , each $\mathbf{n} \in \mathbb{N}^d$ and each $j \in \{1, \dots, d\}$, (3.2.24) holds. Let us take such a random field and $q > 2$. Using item 2 of Lemma 1.2 and Lemma 3.6, we obtain this first estimate

$$\mu \left\{ \max_{1 \leq i \leq \mathbf{n}} |S_i(m)| \geq \prod_{l=1}^d \sqrt{n_l 2t} \right\} \leq \int_1^{+\infty} \mu \left\{ \max_{\mathbf{i} \in [1, \mathbf{n}]} |S_{(\mathbf{i}, n_d)}(m)| \geq \prod_{l=1}^d \sqrt{n_l t s} \right\} ds. \quad (3.2.27)$$

Now, using item 3 of Lemma 1.2 and the induction assumption, we derive

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i \leq \mathbf{n}} |S_i(m)| \geq \prod_{l=1}^d \sqrt{n_l 2t} \right\} \leq c(q, d-1) \\ & \int_1^{+\infty} \int_0^{+\infty} \mu \{ |S_{(\mathbf{0}, n_d)}| > tsv\sqrt{n_d} \} (1 + |\log v|)^{(d-1)-1} \min \{ v, v^{q-1} \} dv ds. \end{aligned} \quad (3.2.28)$$

Switching the integrals and letting $s' := sv$ for a fixed v , we get

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(m)| \geq \prod_{l=1}^d \sqrt{n_l} 2t \right\} &\leq c(q, d-1) \\ \int_0^{+\infty} \mu \{ |S_{(\mathbf{0}, n_d)}| > ts' \sqrt{n_d} \} &\int_0^{+\infty} \mathbf{1}_{\{s' \geq v\}} (1 + |\log v|)^{(d-1)-1} \min \{1, v^{q-2}\} dv ds'. \end{aligned} \quad (3.2.29)$$

Treating the cases $s' > 1$ and $s' \leq 1$, we notice that for some constant $K(q, d)$, we have

$$\int_0^{s'} (1 + |\log v|)^{(d-1)-1} \min \{1, v^{q-2}\} dv \leq K(q, d) (1 + |\log(s')|)^{d-2} \min \{s', s'^{q-1}\}, \quad (3.2.30)$$

from which it follows that

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(m)| \geq \prod_{l=1}^d \sqrt{n_l} 2t \right\} &\leq c(q, d-1) K(q, d) \\ \int_0^{+\infty} \mu \{ |S_{(\mathbf{0}, n_d)}| > ts \sqrt{n_d} \} &(1 + |\log s|)^{(d-1)-1} \min \{s, s^{q-1}\} ds. \end{aligned} \quad (3.2.31)$$

Now, by (3.2.4), we obtain

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(m)| \geq \prod_{l=1}^d \sqrt{n_l} 2t \right\} &\leq c(q, d-1) K(q, d) c(q) \\ \int_0^{+\infty} \int_0^{+\infty} \mu \{ |m| > tsu \} &(1 + |\log s|)^{(d-1)-1} \min \{s, s^{q-1}\} \min \{u, u^{q-1}\} dud s \\ &= c(q, d-1) K(q, d) c(q) \int_0^{+\infty} \int_0^{+\infty} \mu \{ |m| > tv \} I(v) dv, \end{aligned} \quad (3.2.32)$$

where

$$I(v) = \int_0^{+\infty} (1 + |\log s|)^{(d-1)-1} \min \{1, s^{q-2}\} \min \left\{ \frac{v}{s}, \left(\frac{v}{s} \right)^{q-1} \right\} ds. \quad (3.2.33)$$

Assume that $v < 1$. Then

$$\begin{aligned} I(v) &= \int_0^v (1 + |\log s|)^{d-2} s^{q-2} \frac{v}{s} ds + \int_v^1 (1 + |\log s|)^{d-2} s^{q-2} \left(\frac{v}{s} \right)^{q-1} ds \\ &\quad + \int_1^{+\infty} (1 + |\log s|)^{d-2} \left(\frac{v}{s} \right)^{q-1} ds \\ &\leq v \int_0^1 (1 + |\log s|)^{d-2} s^{q-3} ds + v^{q-1} (1 + |\log v|)^{d-2} \int_v^1 \frac{1}{s} ds \\ &\quad + v^{q-1} \int_1^{+\infty} (1 + |\log s|)^{d-2} \frac{1}{s^{q-1}} ds, \end{aligned} \quad (3.2.34)$$

and since the $q > 2$, the integrals $\int_0^1 (1 + |\log s|)^{d-2} s^{q-3} ds$ and $\int_1^{+\infty} (1 + |\log s|)^{d-2} \frac{1}{s^{q-1}} ds$ are convergent. Consequently, there exists a constant K depending only on d and q such that for any $v \in (0, 1)$, $I(v) \leq K v^{q-1} (1 + |\log v|)^{d-1}$.

Assume now that $v \geq 1$. Then

$$\begin{aligned} I(v) &= \int_0^1 (1 + |\log s|)^{d-2} s^{q-2} \frac{v}{s} ds + \int_1^v (1 + |\log s|)^{d-2} \frac{v}{s} ds + \int_v^{+\infty} (1 + |\log s|)^{d-2} \left(\frac{v}{s}\right)^{q-1} ds \\ &\leq v \int_0^1 (1 + |\log s|)^{d-2} s^{q-3} ds + v(1 + |\log v|)^{d-2} \int_1^v \frac{1}{s} ds \\ &\quad + v \int_1^{+\infty} (1 + |\log(u)| + |\log(v)|)^{d-2} \frac{1}{u^{q-1}} du, \end{aligned} \quad (3.2.35)$$

and here again, there exists a constant K depending only on d and q such that for any $v \geq 1$, $I(v) \leq Kv(1 + |\log v|)^{d-1}$. This shows the estimate

$$I(v) \leq Kv(1 + |\log v|)^{d-1} \min\{v, v^{q-1}\}, v > 0. \quad (3.2.36)$$

The combination of (3.2.32) and (3.2.36) gives (3.2.24), which shows that the considered property is valid for d .

We now prove (3.2.23). We start from (3.2.31). Using this time (3.2.3), we obtain

$$\begin{aligned} \mu \left\{ \max_{1 \leq i \leq n} |S_i(m)| \geq 2 \prod_{l=1}^d \sqrt{n_l t} \right\} &\leq c(q, d-1)K(q, d)c(q) \\ n_d \int_0^{+\infty} \int_0^{+\infty} \mu \{ |m| > tsu\sqrt{n_d} \} &(1 + |\log s|)^{(d-1)-1} \min\{s, s^{q-1}\} \min\{u, u^{q-1}\} ds du \\ &+ c(q, d-1)K(q, d)c(q) \int_0^{+\infty} \int_0^{+\infty} \mu \{ (\mathbb{E}[m^2 | T_d \mathcal{M}])^{1/2} > tus \} \\ &(1 + |\log s|)^{(d-1)-1} \min\{u, u^{q-1}\} du \min\{s, s^{q-1}\} ds. \end{aligned} \quad (3.2.37)$$

After having done the substitution $v := us$ for a fixed s and rearranged the integrals, we get (3.2.23) when $j = d$. The proof for a general $j \in \{1, \dots, d\}$ can be carried out similarly. This concludes the proof of Proposition 3.7. \square

4. PROOFS

4.1. The orthomartingale case.

4.1.1. *Proof of Theorem 2.1.* In this subsection, we prove Theorem 2.1. The construction will require a result on dynamical systems of positive entropy.

Lemma 4.1 (Lemma 1, [EV03]). *There exist two T -invariant sub- σ -algebras \mathcal{B} and \mathcal{C} of \mathcal{F} and a function g defined on Ω such that*

- *the σ -algebras \mathcal{B} and \mathcal{C} are independent;*
- *the function g is \mathcal{B} -measurable, zero-mean, with values in $\{-1, 0, 1\}$ and the random field $(g \circ T^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ is independent (identically distributed);*
- *the dynamical system $(\Omega, \mathcal{C}, \mu, T)$ is aperiodic: for each $\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}$,*

$$\mu \{ \omega \in \Omega \mid T^{\mathbf{k}} \omega = \omega \} = 0. \quad (4.1.1)$$

Moreover, there exists $0 < a \leq 1$ such that $\mu \{g = 1\} = \mu \{g = -1\} = a/2$ and $\mu \{g = 0\} = 1 - a$.

We consider four increasing sequences of positive integers $(I_l)_{l \geq 1}$, $(J_l)_{l \geq 1}$, $(K_l)_{l \geq 1}$ and $(L_l)_{l \geq 1}$ such that

$$\sum_{l \geq 1} \left(\frac{l^{d+1}}{L_l} \right)^{p(\alpha)} \text{ is convergent ;} \quad (4.1.2)$$

$$\lim_{l \rightarrow \infty} J_l \mu \left\{ |N| > \frac{L_l}{\|g\|_2} \right\} = 1; \quad (4.1.3)$$

$$\lim_{l \rightarrow \infty} I_l = +\infty; \quad (4.1.4)$$

$$\lim_{l \rightarrow \infty} l^{d-1} 2^{I_l+J_l+K_l} \sum_{i>l} 2^{-K_i} = 0 \text{ and} \quad (4.1.5)$$

$$\text{for each } l \geq 1, \sum_{i=1}^{l-1} L_i^{-1} 2^{(J_i+K_i)(1/2-\alpha)} l^{d-1} \leq \frac{1}{2} (2^{I_l+J_l+K_l} l^{d-1})^{1/2-\alpha}. \quad (4.1.6)$$

To construct such sequences, first define $(I_l)_{l \geq 1}$ and $(L_l)_{l \geq 1}$ satisfying (4.1.2) and (4.1.4) respectively. Then consider $(J_l)_{l \geq 1}$ satisfying (4.1.3). Once these three sequences are chosen, we can construct $(K_l)_{l \geq 1}$ inductively such that (4.1.6) holds, $K_{l+1} \geq lK_l$ and $l^{d-1} 2^{I_l+J_l-K_l}$ in order to guarantee (4.1.5).

By the multidimensional version of Rokhlin's lemma (see Theorem 3.1 in [Con73]), we can find for each l a measurable set C_l such that the family

$$(T^i C_l)_{\substack{0 \leq i_1 \leq N_l-1 \\ 0 \leq i_q \leq l, q \geq 2}} \text{ is pairwise disjoint and} \quad (4.1.7)$$

$$\mu \left(\bigcup_{\substack{0 \leq i_1 \leq N_l-1 \\ 0 \leq i_q \leq l, q \geq 2}} T^i C_l \right) > \frac{1}{2}, \quad (4.1.8)$$

where $N_l := 2^{I_l+J_l+K_l}$. We define

$$f_l := \frac{1}{L_l} \left(\frac{N_l}{2^{I_l}} \right)^{1/p(\alpha)} l^{d-1} \sum_{j=1}^{J_l} 2^{-j/p(\alpha)} \sum_{i_1=2^{I_l+j+1}}^{2^{I_l+j+1}} \sum_{0 \leq i_q \leq l, q \geq 2} \mathbf{1} \left(T_1^{N_l-i_1} \prod_{q=2}^d T_q^{l-i_q} C_l \right) \quad (4.1.9)$$

and the function m by the equalities

$$f := \sum_{l \geq 1} f_l, \quad m := gf, \quad (4.1.10)$$

where g is the function obtained in Lemma 4.1.

We define the σ -algebra \mathcal{M} by the equality

$$\mathcal{M} := \sigma(g \circ T^{\mathbf{1}}, \mathbf{1} \preceq \mathbf{0}) \vee \mathcal{C}. \quad (4.1.11)$$

By Lemma 4.1, the σ -algebra \mathcal{C} is T -invariant, hence for each $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ such that $\mathbf{i} \preceq \mathbf{j}$, the inclusion $T^{-\mathbf{i}}\mathcal{M} \subset T^{-\mathbf{j}}\mathcal{M}$ takes place, hence $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$ is a stationary filtration.

Proposition 4.2. *The filtration $(T^{-\mathbf{i}}\mathcal{M})_{\mathbf{i} \in \mathbb{Z}^d}$ is commuting and the random field $(U^{\mathbf{i}}m)_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale difference random field with respect to this filtration.*

Proof. Denoting for $\mathbf{j} \in \mathbb{Z}^d$, $\mathcal{F}_{\mathbf{j}} := T^{-\mathbf{j}}\mathcal{M}$, we have to check that for each $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ and each bounded random variable X ,

$$\mathbb{E}[\mathbb{E}[X | \mathcal{F}_{\mathbf{i}}] | \mathcal{F}_{\mathbf{j}}] = \mathbb{E}[X | \mathcal{F}_{\min\{\mathbf{i}, \mathbf{j}\}}]. \quad (4.1.12)$$

We define $\mathcal{G}_{\mathbf{i}} := \sigma(g \circ T^{\mathbf{u}}, \mathbf{u} \preceq \mathbf{i})$ and apply Proposition 2 in [WW13] to the mutually independent σ -algebras

$$\mathcal{F} := \sigma(g \circ T^{\mathbf{u}}, \mathbf{u} \preceq \mathbf{i}, \exists q \in \langle d \rangle \mid u_q > j_q) \quad (4.1.13)$$

$$\mathcal{G} := \mathcal{G}_{\min\{\mathbf{i}, \mathbf{j}\}} \vee \mathcal{C} \quad (4.1.14)$$

$$\mathcal{H} := \sigma(g \circ T^{\mathbf{u}}, \mathbf{u} \preceq \mathbf{j}, \exists q \in \langle d \rangle \mid u_q > i_q) \quad (4.1.15)$$

in order to get (4.1.12).

In order to prove the second part of Proposition 4.2, notice that for each $q \in \langle d \rangle$, we have by (4.1.10) and (4.1.11) that

$$\mathbb{E}[m | T_q \mathcal{M}] = \mathbb{E}[fg | \sigma(g \circ T^{\mathbf{l}}, \mathbf{l} \preceq -\mathbf{e}_q) \vee \mathcal{C}]. \quad (4.1.16)$$

Since the function f is \mathcal{C} -measurable and the function g is independent of $\sigma(g \circ T^{\mathbf{l}}, \mathbf{l} \preceq -\mathbf{e}_q) \vee \mathcal{C}$, the terms in (4.1.16) are zero. This concludes the proof of Proposition 4.2. \square

Proposition 4.3. *The function m satisfies $\lim_{t \rightarrow +\infty} t^{p(\alpha)} \mu\{|m| > t\} = 0$.*

Proof. We first estimate $\sup_{t > 0} t^{p(\alpha)} \mu\{|f_l| > t\}$ for a fixed $l \geq 1$. Using Lemma 1.5 of [Gir16] with

$$A_j := \bigcup_{i_1=2^{l_1+j+1}}^{2^{l_1+j+1}} \bigcup_{0 \leq i_q \leq l, q \geq 2} T_1^{N_l - i_1} \prod_{q=2}^d T_q^{l - i_q} C_l \text{ and} \quad (4.1.17)$$

$$a_j := \frac{1}{L_l} \left(\frac{N_l}{2^{l_1}} \right)^{1/p(\alpha)} l^{d-1} 2^{-j/p(\alpha)}, \quad (4.1.18)$$

we have the estimate

$$\sup_{t > 0} t^{p(\alpha)} \mu\{|f_l| > t\} \leq \max_{1 \leq j \leq J_l - 1} \left(\frac{1}{L_l} \left(\frac{N_l}{2^{l_1}} \right)^{1/p(\alpha)} l^{d-1} \right)^{p(\alpha)} 2^{-j} \sum_{i=1}^j \mu(A_i), \quad (4.1.19)$$

and since $\mu(A_i) \leq 2^{l_1+i} l^{d-1} 2^{-(l_1+J_l+K_l)} / l^{d-1} = 2^{i-J_l-K_l}$, we obtain

$$\sup_{t > 0} t^{p(\alpha)} \mu\{|f_l| > t\} \leq 2^{1-J_l-K_l} \left(\frac{1}{L_l} \left(\frac{N_l}{2^{l_1}} \right)^{1/p(\alpha)} l^{d-1} \right)^{p(\alpha)} = \left(\frac{l^{d-1}}{L_l} \right)^{p(\alpha)}. \quad (4.1.20)$$

Let us fix $l_0 \geq 1$. Since $\sum_{l=1}^{l_0-1} f_l$ is bounded, we have

$$\limsup_{t \rightarrow +\infty} t^{p(\alpha)} \mu\{f > 2t\} \leq \limsup_{t \rightarrow +\infty} t^{p(\alpha)} \mu\left\{ \sum_{l=l_0+1}^{+\infty} f_l > t \right\}, \quad (4.1.21)$$

and since $\sum_{l=1}^{+\infty} l^{-2} \leq 1$, we have, in view of (4.1.20)

$$\limsup_{t \rightarrow +\infty} t^{p(\alpha)} \mu\{f > 2t\} \leq \sum_{l=l_0+1}^{+\infty} \left(\frac{l^{d-1}}{L_l} \right)^{p(\alpha)} l^{2p(\alpha)}, \quad (4.1.22)$$

and since l_0 is arbitrary and g is bounded by 1, we may conclude the proof of Proposition 4.3 by (4.1.2). \square

Now, we have to prove that the sequence $\left((N_l l^{d-1})^{-1/2} S_{N_l, l, \dots, l}(m, \cdot) \right)$ is not tight in $\mathcal{H}_\alpha([0, 1]^d)$. The first step in this direction is

Proposition 4.4. *For l large enough, we have*

$$\mu \left\{ \omega_\alpha \left(\frac{1}{\sqrt{2^{I_l+J_l+K_l} l^{d-1}}} S_{(2^{I_l+J_l+K_l}, l, \dots, l)}(g \cdot f_l), 2^{-K_l} \right) \geq 1 \right\} \geq \frac{1}{16}. \quad (4.1.23)$$

Proof. We fix $l \geq 1$.

Let $\mathbf{u} = (u_1, \dots, u_d)$ be such that $0 \leq u_1 \leq N_l - 2^{I_l+J_l}$ and $1 \leq u_q \leq l$ for $2 \leq q \leq d$. Denoting $\mathbf{u}' := (u_2, \dots, u_d)$, we observe that for each $j \in \{1, \dots, J_l - 1\}$, the following equality holds:

$$\begin{aligned} & \mathbf{1}(T^{\mathbf{u}}C_l) \left[S_{(N_l-u_1-2^{I_l+j}, \mathbf{u}')} (g f_l) - S_{(N_l-u_1-2^{I_l+j+1}, \mathbf{u}')} (g f_l) \right] \\ &= \frac{1}{L_l} \left(\frac{N_l}{2^{I_l}} \right)^{1/p(\alpha)} l^{(d-1)/2} 2^{-j/p(\alpha)} \mathbf{1}(T^{\mathbf{u}}C_l) \left[S_{(N_l-u_1-2^{I_l+j}, \mathbf{u}')} (g) - S_{(N_l-u_1-2^{I_l+j+1}, \mathbf{u}')} (g) \right]. \end{aligned} \quad (4.1.24)$$

Indeed, if ω belongs to $T^{\mathbf{u}}C_l$, then for each $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{N}^d$ such that $N_l - u_1 - 2^{I_l+j+1} \leq v_1 \leq N_l - u_1 - 2^{I_l+j} - 1$ and $1 \leq v_q \leq l$ for $2 \leq q \leq d$, we have

$$f_l \circ T^{\mathbf{v}}(\omega) = \frac{1}{L_l} \left(\frac{N_l}{2^{I_l}} \right)^{1/p(\alpha)} l^{(d-1)/2} 2^{-j/p(\alpha)}. \quad (4.1.25)$$

Since $2^{I_l+j}/2^{I_l+J_l+K_l} \leq 2^{-K_l}$, the inequality

$$\begin{aligned} & \omega_\alpha \left(\frac{1}{\sqrt{2^{I_l+J_l+K_l} l^{d-1}}} S_{(2^{I_l+J_l+K_l}, l, \dots, l)}(g \cdot f_l), 2^{-K_l} \right) \geq \\ & \max_{0 \leq u_1 \leq 2^{I_l+J_l+K_l} - 2^{I_l+J_l}} \max_{\substack{1 \leq u_q \leq l \\ 2 \leq q \leq d}} \frac{\left| S_{(N_l-u_1-2^{I_l+j}, \mathbf{u}')} (g f_l) - S_{(N_l-u_1-2^{I_l+j+1}, \mathbf{u}')} (g f_l) \right|}{\sqrt{2^{I_l+J_l+K_l} l^{d-1}} \left(\frac{2^{I_l+j}}{2^{I_l+J_l+K_l}} \right)^\alpha} \end{aligned} \quad (4.1.26)$$

holds. In view of equality (4.1.24), it suffices to prove that $p_l \geq 1/16$, where

$$\begin{aligned} p_l := \mu \left(\right. & \bigcup_{0 \leq u_1 \leq 2^{I_l+J_l} (2^{K_l} - 1)} \bigcup_{\substack{1 \leq u_q \leq l \\ 2 \leq q \leq d}} [T^{\mathbf{u}}C_l \cap \\ & \left. \left\{ \max_{1 \leq j \leq J_l - 1} \frac{\left| S_{(N_l-u_1-2^{I_l+j}, \mathbf{u}')} (g) - S_{(N_l-u_1-2^{I_l+j+1}, \mathbf{u}')} (g) \right|}{\sqrt{2^{I_l+j}}} \geq L_l \right\} \right]. \end{aligned} \quad (4.1.27)$$

By (4.1.7) and the fact that the action T is measure preserving, we have

$$p_l \geq (2^{I_l+J_l} (2^{K_l} - 1) + 1) l^{d-1} \mu \left(C_l \cap \left\{ \max_{1 \leq j \leq J_l - 1} \frac{\left| S_{(N_l-2^{I_l+j}, \mathbf{1})} (g) - S_{(N_l-2^{I_l+j+1}, \mathbf{1})} (g) \right|}{\sqrt{2^{I_l+j}}} \geq L_l \right\} \right),$$

The events $\left\{ \max_{1 \leq j \leq J_l-1} \left| S_{(N_l-2^{I_l+j}, \mathbf{1})}(g) - S_{(N_l-2^{I_l+j+1}, \mathbf{1})}(g) \right| \sqrt{2^{-(I_l+j)}} \geq L_l \right\}$ and C_l belonging to \mathcal{B} and \mathcal{C} respectively, they are independent. Consequently,

$$p_l \geq (2^{I_l+J_l} (2^{K_l} - 1) + 1) l^{d-1} \mu(C_l) \mu \left\{ \max_{1 \leq j \leq J_l-1} \frac{\left| S_{(N_l-2^{I_l+j}, \mathbf{1})}(g) - S_{(N_l-2^{I_l+j+1}, \mathbf{1})}(g) \right|}{\sqrt{2^{I_l+j}}} \geq L_l \right\},$$

which entails, by (4.1.7) and (4.1.8),

$$p_l \geq \frac{1 - 2^{-K_l}}{2} \mu \left(\bigcup_{1 \leq j \leq J_l-1} A_j \right), \quad (4.1.28)$$

where

$$A_j := \left\{ \frac{\left| S_{(N_l-2^{I_l+j}, \mathbf{1})}(g) - S_{(N_l-2^{I_l+j+1}, \mathbf{1})}(g) \right|}{\sqrt{2^{I_l+j}}} \geq L_l \right\}. \quad (4.1.29)$$

Using Bonferroni's inequality and independence of the family $(A_j)_{1 \leq j \leq J_l-1}$, we derive

$$\begin{aligned} \mu \left(\bigcup_{1 \leq j \leq J_l-1} A_j \right) &\geq \sum_{j=1}^{J_l-1} \mu(A_j) - \sum_{1 \leq i < j \leq J_l-1} \mu(A_i) \mu(A_j) \\ &= \sum_{j=1}^{J_l-1} \mu(A_j) - \frac{1}{2} \left(\sum_{j=1}^{J_l-1} \mu(A_j) \right)^2 + 2 \sum_{j=1}^{J_l-1} \mu(A_j)^2 \\ &\geq \sum_{j=1}^{J_l-1} \mu(A_j) - \frac{1}{2} \left(\sum_{j=1}^{J_l-1} \mu(A_j) \right)^2, \end{aligned} \quad (4.1.30)$$

hence we have to prove that for l large enough,

$$\sum_{j=1}^{J_l-1} \mu(A_j) - \frac{1}{2} \left(\sum_{j=1}^{J_l-1} \mu(A_j) \right)^2 \geq \frac{1}{4}. \quad (4.1.31)$$

By the Berry-Esseen theorem, the estimate

$$\left| \mu(A_j) - \mu \{ |N| \|g\|_2 > L_l \} \right| \leq \frac{C}{\|g\|_2^3} 2^{-(I_l+j)/2} \quad (4.1.32)$$

holds for any $j \in \{1, \dots, J_l-1\}$, where C is independent of l and j . Consequently,

$$\left| \sum_{j=1}^{J_l-1} (\mu(A_j) - \mu \{ |N| \|g\|_2 > L_l \}) \right| \leq \frac{C}{\|g\|_2^3} 2^{-I_l/2} \sum_{j=1}^{+\infty} 2^{-j/2} \quad (4.1.33)$$

and by (4.1.3), (4.1.4) and (4.1.33), we have

$$\lim_{l \rightarrow +\infty} \sum_{j=1}^{J_l-1} \mu(A_j) - \frac{1}{2} \left(\sum_{j=1}^{J_l-1} \mu(A_j) \right)^2 = \frac{1}{2}, \quad (4.1.34)$$

which gives (4.1.23) in view of (4.1.28) and (4.1.31). This concludes the proof of Proposition 4.4. \square

Corollary 4.5. *For l large enough,*

$$\mu \left\{ \omega_\alpha \left(\frac{1}{\sqrt{2^{I_l+J_l+K_l}l^{d-1}}} S_{(2^{I_l+J_l+K_l}, l, \dots, l)}(m), 2^{-K_l} \right) \geq \frac{1}{2} \right\} \geq \frac{1}{32}. \quad (4.1.35)$$

Proof. Let l be such that (4.1.23) holds. Since $m = gf_l + \sum_{i \neq l} gf_i$, we have to show that the contribution of the term $\sum_{i \neq l} gf_i$ is negligible in (4.1.35). To this aim, we shall show the following two things:

$$\left\| \frac{1}{\sqrt{2^{I_l+J_l+K_l}l^{d-1}}} S_{(2^{I_l+J_l+K_l}, l, \dots, l)} \left(\sum_{i=0}^{l-1} gf_i, \cdot \right) \right\|_{\mathcal{H}_\alpha} \leq \frac{1}{2}; \quad (4.1.36)$$

$$\lim_{l \rightarrow +\infty} \mu \left(\bigcup_{i \geq l+1} \left\{ \omega_\alpha \left(\frac{1}{\sqrt{2^{I_l+J_l+K_l}l^{d-1}}} S_{(2^{I_l+J_l+K_l}, l, \dots, l)}(g \cdot f_i), 2^{-K_l} \right) \neq 0 \right\} \right) = 0. \quad (4.1.37)$$

Let us prove (4.1.36). If \mathbf{s} and \mathbf{t} are two distinct elements of $[0, 1]^d$ and $\mathbf{n} \in (\mathbb{N}^*)^d$, then for a function h ,

$$\begin{aligned} |S_{\mathbf{n}}(h, \mathbf{t}) - S_{\mathbf{n}}(h, \mathbf{s})| &= \left| \sum_{\mathbf{i} \in \mathbb{Z}^d} (\lambda([0, \mathbf{n} \cdot \mathbf{t}] \cap R_{\mathbf{i}}) - \lambda([0, \mathbf{n} \cdot \mathbf{s}] \cap R_{\mathbf{i}})) h \circ T^{\mathbf{i}} \right| \\ &\leq \max_{\mathbf{u} \in [0, \mathbf{n}]} |h \circ T^{\mathbf{u}}| \cdot \sum_{\mathbf{i} \in \mathbb{Z}^d} \lambda(R_{\mathbf{i}} \cap ([0, \mathbf{n}\mathbf{t}] \Delta [0, \mathbf{n}\mathbf{s}])) \\ &\leq |\mathbf{n}| \max_{\mathbf{u} \in [0, \mathbf{n}]} |h \circ T^{\mathbf{u}}| \|\mathbf{t} - \mathbf{s}\|, \end{aligned}$$

which gives for $\|\mathbf{t} - \mathbf{s}\| \leq |\mathbf{n}|^{-1}$,

$$\frac{1}{\sqrt{|\mathbf{n}|} \|\mathbf{t} - \mathbf{s}\|^\alpha} |S_{\mathbf{n}}(h, \mathbf{t}) - S_{\mathbf{n}}(h, \mathbf{s})| \leq |\mathbf{n}|^{-1/2+1-(1-\alpha)} \max_{\mathbf{u} \in [0, \mathbf{n}]} |h \circ T^{\mathbf{u}}|, \quad (4.1.38)$$

and if $0 < \|\mathbf{t} - \mathbf{s}\| > |\mathbf{n}|^{-1}$, we have

$$\frac{1}{\sqrt{|\mathbf{n}|} \|\mathbf{t} - \mathbf{s}\|^\alpha} |S_{\mathbf{n}}(h, \mathbf{t}) - S_{\mathbf{n}}(h, \mathbf{s})| \leq \frac{1}{|\mathbf{n}|^{1/2-\alpha}} |\mathbf{n}|^{-1/2+1-(1-\alpha)} \max_{\mathbf{u} \in [0, \mathbf{n}]} |h \circ T^{\mathbf{u}}| \quad (4.1.39)$$

As a consequence, we obtain

$$\left\| \frac{1}{\sqrt{2^{I_l+J_l+K_l}l^{d-1}}} S_{(2^{I_l+J_l+K_l}, l, \dots, l)} \left(\sum_{i=0}^{l-1} gf_i, \cdot \right) \right\|_{\mathcal{H}_\alpha} \quad (4.1.40)$$

$$\leq \left(2^{I_l+J_l+K_l} l^{(d-1)} \right)^{-(1/2-\alpha)} \max_{0 \leq u_1 \leq 2^{I_l+J_l+K_l}} \max_{\substack{1 \leq u_q \leq l \\ 2 \leq q \leq d}} \left| \left(\sum_{i=0}^{l-1} gf_i \right) \circ T^{\mathbf{u}} \right|, \quad (4.1.41)$$

and since $|gf_i|$ is bounded by $L_i^{-1}(n_i 2^{-I_i})^{1/p(\alpha)} i^{d-1}$, we infer that

$$\begin{aligned} &\left\| \frac{1}{\sqrt{2^{I_l+J_l+K_l}l^{d-1}}} S_{(2^{I_l+J_l+K_l}, l, \dots, l)} \left(\sum_{i=0}^{l-1} gf_i, \cdot \right) \right\|_{\mathcal{H}_\alpha} \\ &\leq \left(2^{I_l+J_l+K_l} l^{(d-1)} \right)^{-(1/2-\alpha)} \sum_{i=1}^{l-1} L_i^{-1} (n_i 2^{-I_i})^{1/2-\alpha} i^{d-1}, \quad (4.1.42) \end{aligned}$$

which in turn does not exceed 1/2 by (4.1.6). This proves (4.1.36).

We now show (4.1.37). Let us fix $l \geq 1$ and $i > l$. The inclusion

$$\left\{ \omega_\alpha \left(\frac{1}{\sqrt{2^{I_l+J_l+K_l}l^{d-1}}} S_{(2^{I_l+J_l+K_l}, l, \dots, l)}(g \cdot f_i), 2^{-K_l} \right) \neq 0 \right\} \\ \subset \bigcup_{0 \leq u_1 \leq 2^{I_l+J_l+K_l}} \bigcup_{\substack{1 \leq u_q \leq l \\ 2 \leq q \leq d}} \{(g \cdot f_i) \circ T^{\mathbf{u}} \neq 0\} \quad (4.1.43)$$

holds. We thus have

$$\mu \left\{ \omega_\alpha \left(\frac{1}{\sqrt{2^{I_l+J_l+K_l}l^{d-1}}} S_{(2^{I_l+J_l+K_l}, l, \dots, l)}(g \cdot f_i), 2^{-K_l} \right) \neq 0 \right\} \\ \leq (2^{I_l+J_l+K_l} + 1)l^{d-1} \mu \{f_i \neq 0\}. \quad (4.1.44)$$

By (4.1.9), the inclusion

$$\{f_i \neq 0\} \subset \bigcup_{2^{I_l+J_l+K_l} - 2^{I_l+J_l} \leq u_1 \leq 2^{I_l+J_l+K_l}} \bigcup_{\substack{0 \leq u_q \leq l \\ 2 \leq q \leq d}} T^{\mathbf{u}}(C_i) \quad (4.1.45)$$

holds, which entails, by (4.1.7) and (4.1.8),

$$\mu \{f_i \neq 0\} \leq (2^{I_l+J_l} + 1) i^{d-1} 2^{-(I_l+J_l+K_l)} / i^{d-1} = 2^{1-K_l} \quad (4.1.46)$$

Now (4.1.37) appears as a consequence of (4.1.44), (4.1.46) and (4.1.5). This concludes the proof of Corollary 4.5 and that of Theorem 2.1. \square

Remark 4.6. Since for each $\mathbf{k} \in \mathbb{Z}^d$, $\sigma(f \circ T^{\mathbf{i}}, \mathbf{i} \neq \mathbf{k}) \subset \sigma(g \circ T^{\mathbf{i}}, \mathbf{i} \neq \mathbf{k}) \vee \mathcal{C}$, the random field $(U^{\mathbf{i}}f)_{\mathbf{i} \in \mathbb{Z}^d}$ is a strong martingale random field in the sense of Nahapetian and Petrosian [NP92]. This shows that even with this choice of martingale random fields, the condition on the law of m has to be reinforced compared to the i.i.d. case.

4.1.2. *Proof of Theorem 2.2.* We already know by [Vol15] that the finite dimensional distributions of $|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(m, \cdot)$ converge to those of $\|m\|_2 W$ as $\min \mathbf{n}$ goes to infinity. The rest of the proof is thus devoted to asymptotic tightness in $\mathcal{H}_\alpha^o([0, 1]^d)$ of the net $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(m, \cdot))_{\mathbf{n} \geq \mathbf{1}}$. It suffices to check that (3.1.3) holds. To this aim, we fix $\varepsilon > 0$ and consider $J \geq 1$ and $\mathbf{n} \in \mathbb{N}^d$ such that $\log \min \mathbf{n} \geq J$. Let $i \in \{1, \dots, d\}$. An application of (3.2.23) with $q > (1/2 - \alpha)^{-1}$, \mathbf{n}' defined by $n'_a = n_a$ if $a \neq i$ and $n'_i = n_i 2^{-j}$ instead of \mathbf{n} and $t = 2^{-\alpha j} C\varepsilon$ yields

$$\sum_{j=J}^{n_i} 2^j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \leq \mathbf{s} \leq \mathbf{1}} U^{\mathbf{s}} f \right| > C\varepsilon \prod_{q=1}^d \sqrt{n_q} \right\} \leq \\ c(q, d) n_i \sum_{j=J}^{\log n_i} \int_0^{+\infty} \mu \{ |m| > \sqrt{n_i} C\varepsilon v 2^{-\alpha j} \} (1 + |\log v|)^{d-1} \min \{ v, v^{q-1} \} dv \\ + c(q, d) \sum_{j=J}^{\log n_i} 2^j \int_0^{+\infty} \mu \left\{ (\mathbb{E}[m^2 | T_i \mathcal{M}])^{1/2} > v 2^{(1/2-\alpha)j} C\varepsilon \right\} (1 + |\log v|)^{d-1} \min \{ v, v^{q-1} \} dv \\ =: A(J, \mathbf{n}) + B(J, \mathbf{n}). \quad (4.1.47)$$

Notice that for each $j \in \{J, \dots, \lfloor \log n_i \rfloor\}$, we have $\sqrt{n_i} 2^{-\alpha j} \geq n_i^{1/p(\alpha)}$ hence each positive v , the following bound holds:

$$\begin{aligned} \mu \{ |m| > \sqrt{n_i} C \varepsilon v 2^{-\alpha j} \} \\ \leq (\sqrt{n_i} C \varepsilon v 2^{-\alpha j})^{-p(\alpha)} \cdot \sup \left\{ t^{p(\alpha)} \mu \{ |m| > t \}, t \geq C \varepsilon v n_i^{1/p(\alpha)} \right\}. \end{aligned} \quad (4.1.48)$$

Since $p(\alpha) \cdot \alpha = p(\alpha)/2 - 1$ and $\sum_{j=J}^{\log n_i} 2^{j(p(\alpha)/2-1)} \leq \kappa(\alpha) n_i^{p(\alpha)/2-1}$, we derive the estimate

$$A(J, \mathbf{n}) \leq \kappa(\alpha, \varepsilon) \int_0^{+\infty} I_{\mathbf{n}}(v) (1 + |\log v|)^{d-1} v^{-p(\alpha)} \min \{v, v^{q-1}\} dv, \quad (4.1.49)$$

where

$$I_{\mathbf{n}}(v) = \sup \left\{ t^{p(\alpha)} \mu \{ |m| > t \}, t \geq C \varepsilon v (\min \mathbf{n})^{1/p(\alpha)} \right\}. \quad (4.1.50)$$

In order to control $B(J, \mathbf{n})$, we use the following elementary inequality: for a non-negative function f ,

$$\sum_{j \geq J} 2^j \mu \{ f \geq 2^j \} \leq 2 \mathbb{E} [f \mathbf{1} \{ f \geq 2^J \}]. \quad (4.1.51)$$

Applying this to $f := (\mathbb{E}[m^2 | T_i \mathcal{M}])^{p/2}$, we get

$$\begin{aligned} B(J, \mathbf{n}) \leq \kappa(C, \varepsilon) \int_0^{+\infty} \mathbb{E} \left[(\mathbb{E}[m^2 | T_i \mathcal{M}])^{p/2} \mathbf{1} \left\{ (\mathbb{E}[m^2 | T_i \mathcal{M}])^{p/2} \geq 2^J v C \varepsilon \right\} \right] \\ v^{-p(\alpha)} (1 + |\log v|)^{d-1} \min \{v, v^{q-1}\} dv. \end{aligned} \quad (4.1.52)$$

Combining inequalities (4.1.47), (4.1.49) and (4.1.52), we get

$$\begin{aligned} \sum_{i=1}^d \sum_{j=J}^{n_i} 2^j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \leq \mathbf{s} \leq \mathbf{1}} U^{\mathbf{s}} f \right| > C \varepsilon \prod_{q=1}^d \sqrt{n_q} \right\} \leq \\ \kappa(q, \alpha, d) \varepsilon^{-p} \int_0^{+\infty} I_{\mathbf{n}}(v) (1 + |\log v|)^{d-1} v^{-p(\alpha)} \min \{v, v^{q-1}\} dv \\ + \kappa(q, \alpha, d) \varepsilon^{-p} \sum_{i=1}^d \int_0^{+\infty} \mathbb{E} \left[(\mathbb{E}[m^2 | T_i \mathcal{M}])^{p/2} \mathbf{1} \left\{ (\mathbb{E}[m^2 | T_i \mathcal{M}])^{p/2} \geq 2^J v C \varepsilon \right\} \right] \\ v^{-p(\alpha)} (1 + |\log v|)^{d-1} \min \{v, v^{q-1}\} dv, \end{aligned} \quad (4.1.53)$$

from which we deduce that (3.1.3) holds by monotone convergence. This concludes the proof of Theorem 2.2.

4.2. Hannan type condition. The convergence of the finite dimensional distributions of $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot))$ to those of $\sigma^2 W$ where σ is given by (2.2.3) is implied by the result of [VW14] and the Remark 1 of [Vol15]. Therefore, it remains to prove the asymptotic tightness of $(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot))_{\mathbf{n} \geq \mathbf{1}}$. The main idea is to use the decomposition $f = \sum_{i \in \mathbb{Z}^d} P_i(f)$ and to split the sum into two parts: the sum on the box $[-R\mathbf{1}, R\mathbf{1}]$ and its complement for some large R . The sum $\sum_{i \in [-R\mathbf{1}, R\mathbf{1}]} P_i(f)$ admits an orthomartingale-coboundary decomposition, while the partial sum process associated to $P_i(f)$ has a negligible contribution.

Let us write the presented idea in a precise way. The assumptions of Theorem 2.5 imply that

$$f = \sum_{i \in \mathbb{Z}^d} P_i(f), \quad (4.2.1)$$

where the summation converges in \mathbb{L}^p . For a fixed positive integer R , let us introduce the following functions:

$$f^{(R)} := \sum_{-R \cdot \mathbf{1} \preceq i \preceq R \cdot \mathbf{1}} P_i(f). \quad (4.2.2)$$

We shall prove the asymptotic tightness of the net $\left(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot) \right)_{\mathbf{n} \succeq \mathbf{1}}$ by using the following relationship:

$$f = f - f^{(R)} + f^{(R)}. \quad (4.2.3)$$

4.2.1. *The orthomartingale-coboundary-case.* The goal of this subsection is to prove the

Proposition 4.7. *For each $R \geq 1$, the net $\left(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f^{(R)}, \dots) \right)_{\mathbf{n} \succeq \mathbf{1}}$ is asymptotically tight in $\mathcal{H}_\alpha([0, 1]^d)$.*

As noted in Theorem 4.1. of [VW14], for each integer R , the function $f^{(R)}$ admits an orthomartingale-coboundary decomposition, namely,

$$f^{(R)} = \sum_{E \subseteq \{1, \dots, d\}} \prod_{a \in E} (I - U_a) m_E, \quad (4.2.4)$$

where for each $E \subseteq \{1, \dots, d\}$, the random field $(U^i m_E)_{\substack{i \in \mathbb{Z}^d \\ i_q = 0, q \in E}}$ is an orthomartingale differences random field with respect to a completely commuting filtration (and an element of $\mathbb{L}^{p(\alpha)}$). This was done when the filtration is generated by an i.i.d. random field, but the proof works for a general completely commuting filtration.

Proof of Proposition 4.7. We have to show that the net of normalized partial sum processes associated to each element in (4.2.4) is asymptotically tight. This is already done for m_\emptyset (by Theorem 2.2). Let us fix a non-empty subset E of $\{1, \dots, d\}$. We have to check that for each $i \in \{1, \dots, d\}$ and each positive ε ,

$$\lim_{J \rightarrow +\infty} \limsup_{\mathbf{n} \rightarrow +\infty} \sum_{j=J}^{n_i} 2^j \mu \left\{ 2^{\alpha j} \max_{1 \leq l_i \leq n_i 2^{-j}} \max_{\substack{1 \leq l_k \leq n_k \\ k \neq i}} \left| \sum_{\mathbf{1} \preceq \mathbf{s} \preceq \mathbf{1}} U^{\mathbf{s}} \prod_{a \in E} (I - U_a) m_J \right| > \varepsilon \sqrt{|\mathbf{n}|} \right\} = 0. \quad (4.2.5)$$

We shall bound the involved probability by distinguishing the cases $i \in E$ and $i \notin E$.

Assume that i belongs to E . Using telescoping properties, we reduce the proof of (4.2.5) to

$$\lim_{J \rightarrow +\infty} \limsup_{\mathbf{n} \rightarrow +\infty} \sum_{j=J}^{n_i} \prod_{a \in E} n_a \mu \left\{ 2^{\alpha j} \max_{\substack{1 \leq l_k \leq n_k \\ k \notin E}} \left| \sum_{\substack{1 \leq s_q \leq l_q \\ q \notin E}} \prod_{q \notin E} U_q^{s_q} m_E \right| > \varepsilon \sqrt{|\mathbf{n}|} \right\} = 0. \quad (4.2.6)$$

Since $(U^i m_E)_{\substack{\mathbf{i} \in \mathbb{Z}^d \\ i_q=0, q \in E}}$ is an orthomartingale differences random field with respect to a completely commuting filtration, we can apply inequality (3.2.24) with an exponent $q > p(\alpha)$. This gives, accounting the relation $p(\alpha)\alpha = p/2 - 1$,

$$\begin{aligned}
& \prod_{a \in E} n_a \sum_{j=J}^{n_i} \mu \left\{ 2^{\alpha j} \max_{\substack{1 \leq l_k \leq n_k \\ k \notin E}} \left| \sum_{\substack{1 \leq s_q \leq l_q \\ q \notin E}} \prod_{q \notin E} U_q^{s_q} m_E \right| > \varepsilon \sqrt{|\mathbf{n}|} \right\} \\
& \leq c(q, d) \prod_{a \in E} n_a \sum_{j=J}^{n_i} \int_0^{+\infty} \mu \left\{ |m_E| > 2^{-\alpha j} u \varepsilon \prod_{a \in E} \sqrt{n_a} \right\} \min \{u, u^{q-1}\} du \\
& \leq c(q, d) \prod_{a \in E} n_a^{1-p(\alpha)/2} \sum_{j=J}^{n_i} 2^{j\alpha p(\alpha)} \int_0^{+\infty} \mathbb{E} \left[|m_E|^{p(\alpha)} \mathbf{1} \left\{ |m_E| \geq 2^{-\alpha j} u \prod_{a \in E} \sqrt{n_a} \varepsilon \right\} \right] \min \{u, u^{q-1}\} du \\
& \leq c(q, d) \int_0^{+\infty} \mathbb{E} \left[|m_E|^{p(\alpha)} \mathbf{1} \left\{ |m_E| \geq n_i^{1/2-\alpha} \varepsilon \right\} \right] \min \{u, u^{q-1}\} du, \quad (4.2.7)
\end{aligned}$$

and (4.2.6) is a consequence of the monotone convergence theorem.

Assume now that $i \notin E$. Using again telescoping properties, we have to show that

$$\lim_{J \rightarrow +\infty} \limsup_{\mathbf{n} \rightarrow +\infty} \sum_{j=J}^{n_i} 2^j \prod_{a \in E} n_a \mu \left\{ 2^{\alpha j} \max_{\substack{1 \leq l_i \leq 2n_i 2^{-j} \\ k \notin E \cup \{i\}}} \max_{\substack{1 \leq l_k \leq n_k \\ k \notin E \cup \{i\}}} \left| \sum_{\substack{1 \leq s_q \leq l_q \\ q \notin E}} \prod_{q \notin E} U_q^{s_q} m_E \right| > \varepsilon \sqrt{|\mathbf{n}|} \right\} = 0. \quad (4.2.8)$$

By (3.2.24) applied with $q > p(\alpha)$, and using (4.1.51) we obtain

$$\begin{aligned}
& \sum_{j=J}^{n_i} 2^j \prod_{a \in E} n_a \mu \left\{ 2^{\alpha j} \max_{\substack{1 \leq l_i \leq 2n_i 2^{-j} \\ k \notin E \cup \{i\}}} \max_{\substack{1 \leq l_k \leq n_k \\ k \notin E \cup \{i\}}} \left| \sum_{\substack{1 \leq s_q \leq l_q \\ q \notin E}} \prod_{q \notin E} U_q^{s_q} m_E \right| > \varepsilon \sqrt{|\mathbf{n}|} \right\} \\
& \leq \sum_{j=J}^{n_i} 2^j \prod_{a \in E} n_a \int_0^{+\infty} \mu \left\{ |m_E| > 2^{j/p(\alpha)} \prod_{a \in E} \sqrt{n_a} u \varepsilon \right\} \min \{u, u^{q-1}\} (1 + |\log u|)^{d-1} du \\
& \leq 2 \prod_{a \in E} \varepsilon^{-p(\alpha)} \mathbb{E} [|m_E|^p] n_a^{1-p(\alpha)/2} \int_0^{+\infty} u^{-p(\alpha)} \min \{u, u^{q-1}\} (1 + |\log u|)^{d-1} du,
\end{aligned}$$

which shows (4.2.8) (since $p(\alpha) > 2$) hence (4.2.5). This concludes the proof of Proposition 4.7. \square

4.2.2. *General case.* Once we proved that the net $\left(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f^{(R)}, \cdot) \right)_{\mathbf{n} \geq \mathbf{1}}$ is asymptotically tight in $\mathcal{H}_\alpha([0, 1]^d)$, we have to control the Hölderian norm of the partial sum process associated to $f - f^{(R)}$. To this aim, we shall use the following inequality:

$$\left\| \left\| \frac{1}{\sqrt{|\mathbf{n}|}} S_{\mathbf{n}}(P_{\mathbf{i}}(f), \cdot) \right\| \right\|_{\mathcal{H}_\alpha} \leq C(p, d) \|P_{\mathbf{i}}(f)\|_p, \quad \mathbf{i} \in \mathbb{Z}^d. \quad (4.2.9)$$

This can be proved combining inequality (3.1.2) with $J = 0$ and (3.2.24), since $(U^j P_{\mathbf{i}}(f))_{\mathbf{j} \in \mathbb{Z}^d}$ is an orthomartingale differences random field.

Using (4.2.9), we derive that

$$\mu \left\{ \left\| \frac{1}{\sqrt{|\mathbf{n}|}} S_{\mathbf{n}} \left(f - f^{(R)}, \cdot \right) \right\|_{\mathcal{H}_{\alpha}} > \varepsilon \right\} \leq \frac{C(p, d)}{\varepsilon} \sum_{\mathbf{i} \in E_R} \|P_{\mathbf{i}}(f)\|_p, \quad (4.2.10)$$

where $E_R := \{\mathbf{i} \in \mathbb{Z}^d, |i_q| > R \text{ for some } q \in \{1, \dots, d\}\}$. By Proposition 4.7 and (4.2.10), we derive that for each R ,

$$\lim_{J \rightarrow +\infty} \limsup_{\min \mathbf{n} \rightarrow +\infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f, \cdot))| > \varepsilon \right\} \leq \frac{2C(p, d)}{\varepsilon} \sum_{\mathbf{i} \in E_R} \|P_{\mathbf{i}}(f)\|_p. \quad (4.2.11)$$

By (2.2.1), we obtain that

$$\lim_{J \rightarrow +\infty} \limsup_{\min \mathbf{n} \rightarrow +\infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{\mathbf{v} \in V_j} |\lambda_{j, \mathbf{v}}(S_{\mathbf{n}}(f, \cdot))| > \sqrt{|\mathbf{n}|} \varepsilon \right\}, \quad (4.2.12)$$

which shows, by Theorem 3.1, that the net $\left(|\mathbf{n}|^{-1/2} S_{\mathbf{n}}(f, \cdot) \right)_{\mathbf{n} \geq 1}$ is asymptotically tight in $\mathcal{H}_{\alpha}([0, 1]^d)$. This ends the proof of Theorem 2.5.

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