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## ► To cite this version:

Orly Buchbinder, Alice Cook. Pre-service teachers' construction of algebraic proof through exploration of math-tricks. CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education, Charles University in Prague, Faculty of Education; ERME, Feb 2015, Prague, Czech Republic. pp.100-106. hal-01280547

**HAL Id: hal-01280547**

**<https://hal.science/hal-01280547>**

Submitted on 1 Mar 2016

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# Pre-service teachers' construction of algebraic proof through exploration of math-tricks

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*This paper contributes to the ongoing effort to create rich learning opportunities for prospective teachers to engage with reasoning and proving. Twenty elementary and middle school pre-service teachers completed individual projects in which they explored “math-tricks” – unconventional computational algorithms – as a part of an undergraduate proof course. Our findings suggest that the task evoked uncertainty with respect to why the tricks work and motivation to resolve the uncertainty by means of algebraic proof. We discuss the potential of this task to create rich opportunities for prospective teachers to conduct explorations, construct algebraic proofs and reflect on their experience from learner’s perspective.*

**Key words:** Proof, pre-service teachers, uncertainty, algebraic reasoning.

## INTRODUCTION

The central role of reasoning and proof in teaching and learning mathematics has been long recognized by mathematics education community and by policy makers. In recent years there has been a movement in the United States towards making proof an integral part of mathematics curriculum not just for students in high-school geometry but across all grade levels (NCTM, 2000; CCSSO, 2010). This trend is also associated with growing demand for deeper conceptual knowledge of number sense and algebraic reasoning (CCSSO, 2010).

In order to provide students with learning environments that emphasize reasoning and proof, teachers themselves need to have strong subject matter knowledge and a solid understanding of proof (e.g., Knuth, 2002, Stylianides & Ball, 2008). However, research studies consistently show that pre-service teachers (PSTs) have inadequate conceptions of proof, and

experience difficulties in distinguishing between empirical and deductive arguments and in understanding the different roles of proof in mathematics (e.g., Martin & Harel, 1989; Ko, 2010).

In recognizing the impact of teachers’ knowledge of proof on students’ experiences with proof, many teacher education programs have designed courses and instructional activities oriented towards developing pre-service teachers’ conceptions of proof, especially at the elementary and middle school levels (e.g. Stylianides & Stylianides, 2009). According to Ball, Hill and Bass (2005) teachers need to experience mathematics from the learner’s perspective in order to gain appreciation of mathematical ideas, develop better understanding of the ways students interact with them and become aware of the difficulties students encounter. With respect to proving, this might entail engaging PSTs in mathematical exploration, conjecturing, and proof construction. However, designing such tasks for elementary and middle school teachers is a pedagogical challenge for teacher educators. Teachers’ prior knowledge of mathematical content might hinder their ability to grasp the complexity of underlying mathematical ideas or their ability to consider students’ perspective. It also might reinforce an inadequate view of proof as a routine exercise of justifying well-known and prior established facts (Knuth, 2002).

Several approaches to address this issue have been suggested. For example, Barkai and colleagues (2002) asked elementary PSTs to analyse students’ arguments in elementary number theory. Stylianides and Stylianides (2009) suggested a “construction-evaluation” model in which elementary teacher candidates were asked to write a proof of a given statement and then to evaluate the validity and generality of their arguments. Despite the reported success of these

approaches, Ko (2010) points to the need for more research on strategies for developing PSTs' conceptions of proof and adequate knowledge of proving.

With this paper we aim to contribute to the ongoing effort of the field to create learning opportunities for teacher candidates to engage with reasoning and proving. We report on one task that was developed and implemented in our undergraduate course titled "Reasoning, Justification and Proof for elementary and middle school teachers". The task aimed at promoting PSTs' understanding of algebraic proof through analysis of unconventional computational algorithms (math-tricks) and through reflection on their proving experiences. Qualitative methods were used to analyse the types of student-generated proofs, their spontaneous use of algebraic proof and/or of algebraic notation in it, and to explore cognitive and affective aspects of the ways in which prospective teachers' coped with the task from the students' perspective.

## THEORETICAL CONSIDERATIONS

Our approach for designing the task "exploration of math-tricks" is grounded in several theoretical notions. First, we build on Harel's (2007) premise that instruction should appeal to and foster students' *intellectual need to prove*. Implementation of this principle entails creating learning situations in which the need for proof arises intrinsically. One possible way to achieve this is by creating uncertainty regarding whether a certain mathematical phenomenon is true or false. Research studies have shown that such tasks generated a need to resolve the evoked uncertainty by means of argumentation, explanation, convincing and proving (e.g. Buchbinder & Zaslavsky, 2011; Hadass, Hershkowitz, & Schwarz 2000).

Zazkis (1999) suggests that uncertainty can be evoked by exploration of "non-conventional" mathematical objects such as number systems other than base 10 or non-Cartesian coordinate systems. Building on this approach, we created a task that invited PSTs to evaluate non-conventional computational algorithms, or math-tricks, and either prove or refute them. Thus, prospective teachers could explore unfamiliar mathematical phenomena embedded within familiar content.

Finally, our work is grounded in the research that highlights the importance of teachers' reflection on their own thinking from the learner's perspective. According to Zaslavsky and Sullivan (2011) the process of resolving uncertainty combined with reflection on personal experience can lead teachers to reevaluate and refine their understanding of mathematical content of the task, and promote teachers' awareness of difficulties students might experience while engaging with this content.

## THE SETTING

Twenty elementary and middle school PSTs participated in the course on mathematical reasoning and proof which most students took during the 3<sup>rd</sup> or the 4<sup>th</sup> year of their program. The task "Exploration of math-tricks" was given to PSTs as individual project and asked them to (a) watch, understand and describe a math-trick presented in a video in their own words; (b) analyse the math-trick and either prove that it works or disprove it by a counterexample; (c) compare the math-trick to the corresponding convention algorithm and discuss similarities and differences between them; and finally (d) write a reflection on the exploration and proving process in this task. The students were given four weeks to complete the project. They were encouraged to cooperate with each other, but required to submit their original work.

### Task analysis

The content to which we refer as "math-tricks" is a collection of short videos which we found on the web<sup>1</sup>. Each video presents an unconventional, albeit correct under limitations, algorithm for one of: multiplication, division, calculation of square or cube roots, solving systems of linear equations and many others. Due to space constraints we present here a short description of two such math-tricks (Figure 1).

These algorithms can be regarded as unconventional since they differ substantially from standard algorithms found in mathematical textbooks both in content and in presentation style. In the video, the presenter used several numeric examples to illustrate each algorithm, describing them as "faster, easier and smarter ways", "math-tricks", and "pure magic", without any expectation on the part of the

1 It is not our intent to either criticize or promote this online content for any purpose beyond described in this paper.

**Trick #1: Special trick for multiplication**

Suppose you want to multiply two two-digit numbers that have the same tens digits and the sum of their unit digits is 10. For example, consider:  $36 \times 34$ . First, multiply the tens digit of a multiplicand by next counting number:  $3 \times (3 + 1) = 3 \times 4 = 12$ . These are the first two digits of the product. To obtain the last two digits of the product simply multiply the unit digits, e.g.  $6 \times 4 = 24$ . Thus, the product is:  $36 \times 34 = 1224$ .

**Trick #2: Trick for squaring numbers 10 – 19**

Suppose you want to square a number between 10 and 19, e.g.  $17^2$ . To find the first two digits of the product add the unit digit to the original number:  $(17 + 7) = 24$ . To find the last digit, square the unit digit of the original number  $7^2 = 49$  and add it to the number obtained before, carrying forward the 4 and ensuring that 9 remains the units digit:  $24\Box + 49 = 289$ . Thus,  $17^2 = 289$ .

**Figure 1:** Description of two “math-tricks” as presented in the video

viewer to understand why these algorithms produce correct results. However, as shown in Figure 2, when represented algebraically, the “tricks” appear to be just special cases of multiplication of two two-digit numbers. Thus, the math tricks could be analysed with algebraic techniques accessible to pre-service elementary teachers.

a proper justification. For example, students in this category merely restated the algorithm or wrote that it works because of its similarity to conventional algorithm. Two other categories were *Partial general argument* - unfinished general argument or missing details, and *Valid proof*. For the second criterion, use of algebraic notation, we identified the following categories: *No algebraic notation* – students used only nu-

Conventional multiplication of two-digit numbers	Special trick for multiplication	Trick for squaring numbers 11-19
$a, b, c, d$ digits	$a = c, b + d = 10$	$a = c = 1, b = d$
$(10a + b)(10c + d) = 100ac + 10(ad + bc) + bd$	$(10a + b)(10a + d) =$ $100a^2 + 10a(b + d) + bd =$ $100a^2 + 100a + bd =$ $100a(a + 1) + bd$	$(10 + b)(10 + b) =$ $100 + 10(b + b) + b^2 =$ $10[(10 + b) + b] + b^2$

**Figure 2:** Algebraic representation of the conventional multiplication and two “math-tricks”<sup>2</sup>

### Data and analytic techniques

The data for this paper come from 20 student project papers on 13 different math-tricks. In order to explore the students' proof production, their mathematical arguments were analysed through open coding (Staruss, 1987). Two major criteria emerged, along with corresponding categories focusing on algebraic notation and proof justification. For the first criterion – proof production - the following categories emerged: *Invalid justification* – students either relied solely on empirical evidence or did not produce

mer examples or described their thinking in words; *Semi-algebraic notation* – although some variables and algebraic symbols were present in student work, they were not used meaningfully or were not functional for proof construction; and *Correct algebraic notation*. The latter category was further divided into two sub-categories: *Standard algebraic notation*, such as using  $(10a + b)$  to represent a two-digit number; and *Self-invented notation*, meaning that students came up with their own correct algebraic representations.

When analysing students' written reflections we looked for evidence of uncertainty evoked by the task and students' approaches to resolve it. We also identified and examined instances of students reflecting on their proof experiences as learners, their mathemati-

<sup>2</sup> Note that the constraints on variables in both tricks account for the place value of the digits in the product in such way that adding the unit digits can be described as “placing” them in the units place.

*Representing the trick with algebraic notation:*  $\prod x \bullet y = a \bullet (a+1), b_1 \bullet b_2$

Where  $x$  is the multiplicand,  $y$  is the multiplier,  $a$  is the first digit of the multiplicand,  $b_1$  is the last digit of the multiplicand and  $b_2$  is the last digit of the multiplier. The comma is to separate portions of final product.

*Why it works:* The math trick works because the requirements are extremely narrow. Since both beginning portions of the multiplicand and multiplier must be the same, it allows for consistency between the parts of the algorithm. The last digits being multiplied by each other happens in both the conventional algorithm and the math trick. By adding the following consecutive integer to the first digit of the multiplicand, it covers the part of the conventional algorithm that incorporates the addition of the problem.

**Figure 4:** Melanie's analysis and justification of *Special trick for multiplication*

cal struggle, and connections to their future teaching. Due to the space constraints not all categories will be reported herein.

## RESULTS

### Students' mathematical arguments

First we report on the kinds of PSTs' mathematical arguments and their use of algebraic notation. This section illustrates some of the categories described above with examples of students' arguments and excerpts from their reflections. Figure 3 below summarizes distribution of students' arguments across all categories.

Our data suggests that success with proof production was strongly related to use of algebraic representation. PSTs who used correct algebraic notation were able to either produce valid proofs or partial general arguments that could potentially be turned into proofs. Interestingly, only 3 PSTs used standard algebraic notation, while 8 students invented their own representations (see examples below). Five students used, what we call, semi-algebraic notation, meaning that their use of variables was insufficient or not functional and did not result in production of valid proofs. Nevertheless, 3 students provided additional written explanations and produced partial general

arguments. Four students whose arguments were categorized as invalid either relied solely on empirical evidence or used semi-algebraic notation. Only one of them seemed to be aware of limitations of such a line of reasoning.

### Examples of students' work

Melanie's work on the *Special trick for multiplication* (Figure 4) illustrates the category of arguments that used semi-algebraic notation and produced invalid justification.

Melanie recognised the similarity between the conventional algorithm and the math-trick, and tried convey it in her explanation. She described her observations eloquently, but provided little insight on *why* the trick works. For example, it is not clear from her explanation *how* the similarity of "both beginning portions" of the two factors "allows for consistency between the parts of the algorithm". In her reflection, Melanie described the challenges she encountered in representing the trick algebraically and in explaining why it works. She also referred to her experiences as student and difficulties in understanding algorithms in general:

.... creating an algebraic notation was very challenging [...] I found it very challenging to under-

Type of argument \ Use of algebraic notation	No algebraic notation	Semi-algebraic notation	Correct algebraic notation	
			Self-invented	Standard
Invalid justification	2	2	--	--
Partial general argument	2	3	2	--
Valid proof	--	--	6	3

**Figure 3:** Distribution of teacher candidates' mathematical arguments (N=20)

*Why it works?*

Let's look at this algebraically: If we look at  $ab \times ac$ , we can see that  $a$  is the tens digit, so if your numbers were 22 and 28 then  $a$  would equal 20. Since we are adding 1 to the tens digit, we are essentially adding 10 to 20. This gives us the  $a(a+10)$ . I needed to incorporate the rule that the sum of  $b$  and  $c$  is 10, so I made  $b = 10 - c$ .

$$(a+b)(a+c) = a(a+10) + bc$$

$$(a+10-c)(a+c) = a(a+10) + (10-c)c$$

$$a^2 + 10a - ac + ac + 10c - c^2 = a^2 + 10a + 10c - c^2$$

$$a^2 + 10a + 10c - c^2 = a^2 + 10a + 10c - c^2$$

**Figure 5:** Thomas's analysis and justification of *Special trick for multiplication*

*stand why the trick works as a whole, although I can understand how to do it and its limitations. I have personally never been one very strong in understanding the functioning of algorithms at the core so it did not surprise me that I had difficulty understanding how the trick worked.*

*am not sure why it clicked, but I realized  $a$  was the tens digit, while  $b$  and  $c$  were the ones digits [...] It has been quite some time since I have been as excited as I was when all of this clicked. It took me analyzing and breaking down both the trick and the conventional algorithm to create my equation.*

Another student, Thomas, who analysed the same trick, came up with his own way to represent two-digit numbers and used it to produce a valid proof (Figure 5).

Thomas represented a two-digit number using notation somewhat similar to the standard, as  $(a+b)$ , where  $a$  represents tens. He then used the distributive property to show that the product of two two-digit numbers yields the same algebraic expression as described in the math-trick. One disadvantage to Thomas's notation is the need to remember that the variables  $a$  and  $c$  in the final expression represent different things: tens and units respectively—a detail imperative to account for place value of the product's digits. Thomas's reflection is a detailed account of his thinking process, his initial frustration and feeling of excitement when the proof was completed:

*I was impressed with this trick from the beginning, but I struggled expressing it algebraically. It seems simple now, but at the time it was very frustrating. During the process of writing my equation I kept treating  $a$  like it was a single digit number instead of treating it like the place value that it held. [...] I*

Another student, Cindy, also came up with her own version of base 10 notation to analyse the *Trick for squaring numbers 10–19* (Figure 1) and was able to produce a valid proof for this trick. Cindy used  $n$  to denote the entire two-digit number between 10 and 19, and used  $(n-10)$  to represent the units digit (Figure 5).

Cindy's reflection reveals both cognitive and affective sides of the process she went through in her exploration. Although the exploration of math-trick was not an easy task for Cindy she persevered and was able to solve it correctly. She wrote:

*This task was a little time consuming and confusing at first. [...] I did this by trial and error, until I realized a key element I was missing. My "aha" moment was when writing the general rule. [...] I kept trying to manipulate the equation in different ways, and finally realized that I need to account that we are solving for the first 2 digits in a 3-digit number. Therefore, I need to multiply the first portion of the equation by 10 to account for the 10's and 100's digits.*

To find the first two digits of  $n^2$ :  $10[n + (n-10)]$

To find the last digit of  $n^2$ :  $(n-10)^2$

Conjecture: For all whole numbers from 10 to 19:  $n^2 = 10[n + (n-10)] + (n-10)^2$

Check the conjecture using appropriate algebraic notation:

$$n^2 = 10[n + (n-10)] + (n-10)^2$$

$$n^2 = 20n - 100 + n^2 - 20n + 100 \longrightarrow \boxed{n^2} = 20n - 100 + \boxed{n^2} - 20n + 100$$

$$n^2 = n^2$$

**Figure 5:** Cindy's analysis and justification of "*Trick for squaring numbers 10–19*"

Cindy and Thomas's reflection shows that the analysis of an unconventional algorithm led them to consider more carefully the standard multiplication algorithm and place value. This was a recurring theme in most of students' reflections, including students who eventually were not successful in producing valid proofs.

### Students' reflections on the exploration process

In our analysis of PSTs' reflections we identified evidence for the uncertainty evoked by the task, expressions of students' interest in the math-trick project and their consideration of the task from the learner's perspective. This analysis was overlaid onto an analysis of students' mathematical arguments. Not surprisingly, students who produced valid proofs or partial general arguments described feelings of accomplishment and satisfaction with the project (including the two students who produced invalid arguments but were unaware of this). Although the degree of uncertainty evoked by the task varied within the group, almost all PSTs indicated that they were surprised by the math-tricks. PSTs distinguished between challenges involved in understanding the steps of the trick, *why* it works, and constructing a proof. Some PSTs who understood the trick relatively easily, but struggled with proving, tended to describe the task as frustrating, or hard. For example, Lisa wrote:

*It was easy for me to see how the trick was connected to the conventional method and to see that it would always work. The hard part was explaining it because I cannot think of a way to put this trick in algebraic notation and it is very hard to visualize the math using words.*

Other students felt initially perplexed by the algorithms, but felt more comfortable once they represented them algebraically. For instance, Helen wrote:

*Understanding how to complete the trick wasn't too complicated, but understanding why it works was a difficult task. [...] for me, the "Aha" moment was once I thought of using variables. Seeing the steps in variable form, instead of using numbers made it very clear to me how the process worked.*

For the majority of PSTs, transitioning from empirical exploration to algebraic representation was not a straightforward process. Aside from 3 students who felt comfortable with it, the group reported on dif-

ficulties they encountered, the strategies employed to resolve the impasse and their "Aha!" moments. Students like Melanie or Lisa, who felt that they did not resolve the uncertainty, shared feelings of frustration and lesser competence in their mathematical ability. This might have been avoided or reduced by encouraging greater collaboration and sharing ideas among students.

Several students reflected on their exploration of math-tricks from both learner and future teacher perspectives. Following is an excerpt from Natalie's project paper, with the original emphasis:

*This process was different for me because my role was to be both a student and a teacher. When I was a student in this process I had to figure out how the trick worked, but when I was a teacher in this process, I had to explain why the trick worked. As a student, this involved noticing patterns, and as a teacher, this involved synthesizing and explaining the real math behind the trick.*

### CONCLUSION

In this paper we described the design and implementation of the task "Exploration of math-tricks" with elementary and middle school pre-service teachers. The task aimed to enhance PSTs' appreciation of proof and highlight its exploratory function through investigation of unconventional computational algorithms – "math-tricks". Our data show that, as anticipated, the task evoked uncertainty regarding how and why these algorithms function, which most of the students resolved through proving. The majority of students in our course used standard or self-invented algebraic representations to produce proofs or partial general arguments. Not surprisingly, prospective teachers who felt less proficient with algebra were less successful in producing proofs and in resolving uncertainty as evident from their comments. Nevertheless, exploration of unconventional algorithms allowed teacher candidates to review and refine their knowledge of standard computational algorithms and algebraic techniques. Our findings concur with theoretical notions underlying the role of uncertainty in fostering a need for proving and with empirical studies which utilized exploration of unconventional mathematical objects in instructional tasks with PSTs (e.g., Buchbinder & Zaslavsky, 2011; Zaslavsky, 2005; Zazkis, 1999).

Our analysis of PSTs' reflections revealed a complex account of cognitive and affective aspects of engagement with exploration of math-tricks; while some PSTs described difficulties in transitioning from informal argument to algebraic proof, others felt that algebraic notation aided in expressing their mathematical ideas. PSTs reported on mixed feelings of initial challenge and struggle with the task; but also on their excitement after successful (in their view) production of proof, or frustration when failed to produce one. We conclude that exploration of math-tricks combined with reflection on their own proving process allowed PSTs to evaluate this experience from learners' perspective and, in some cases, consider difficulties students might encounter with the concept of place value in multi-digit arithmetic. Furthermore, it seems that engaging in exploration and proving of math-tricks can provide pre-service teachers with the valuable opportunity to consider the role of both student and teacher.

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