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Analyzing the transition to epsilon-delta Calculus: A case study

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The purpose of this paper is to analyze and discuss a think-aloud production of a first year university student trying to prove an elementary theorem, concerning continuous functions, within the frame of the epsilon-delta Calculus. I will shortly consider current studies on the relationships between a visual justification, based on the graphical representation of a continuous function, and such epsilon-delta proving. Then I will try to show how a comprehensive theoretical framework (integrating different tools) based on Habermas' construct of rationality may account for the complexity of the whole process, for the difficulties met by the student in the transition from the visual-graphical to the epsilon-delta proving, and for the relevance of the visual-graphical reasoning to overcome them.

Keywords: Calculus proof, visual-graphical rationality, epsilon-delta rationality.

INTRODUCTION

The transition, in the first half of the nineteenth century, from Calculus based on the ideas of continuity and derivability of a function as continuity and smoothness of its graphical representation, to epsilon-delta Calculus (i.e. Calculus based on the epsilon-delta definitions of limit, and subsequently of continuity and derivative, usually also called the Cauchy-Weierstrass Calculus) has been dealt with by several authors in different disciplines (History of Mathematics, Epistemology of Mathematics, Psychology, Mathematics Education). In particular, the historical-epistemological analyses (see Grabiner, 1981; Jahnke, 2003) have shown the need of considering different reasons for that transition, like: the emergence of *monsters* and contradictions within the intuitive-visual treatment of continuous functions, when new types of functions not represented by ordinary formulas were considered; the nineteenth cen-

tury movement towards formal set-theoretic rigour, particularly when proving was concerned (see also Tall and Katz, 2014); the need of dealing in a rigorous way with Calculus in many variables (where reference to visual – graphical evidence may be lost, as concerns the validation of statements); and also the increasing needs arising from the applications of Calculus (particularly stressed in Jahnke, 2003). As concerns the cognitive side, the general agreement about the fact that transition to epsilon-delta Calculus results in many difficulties for students does not correspond to an unique interpretation of those difficulties. For most past Authors difficulties mainly originate in the passage from an intuitive, visual conceptualization to a formal treatment of the same notions. More recently, researchers belonging to the embodied cognition stream of research (cf. Nunez, Edwards, & Matos, 1999) put into evidence the fact that the *natural* notion of continuity of a function and the epsilon-delta notion of continuity refer to very different grounding metaphors: a dynamic metaphor, for *natural continuity*; and three static metaphors (among which the *Preservation of closeness*), in the second case, which result in a hard to overcome conflict with the notions developed according to the dynamic metaphor underlying *natural continuity*. Nowadays, by integrating historical-epistemological studies on the origins and development of Calculus and cognitive analyses, Tall and Katz (2014) provide us with reasons for

a re-evaluation of the relationship between the natural geometry and algebra of elementary calculus that continues to be used in applied mathematics, and the formal set theory of mathematical analysis that develops in pure mathematics and evolves into the logical development of non-standard analysis using infinitesimal concepts (p. 97).

During the nineties I had the opportunity, as teacher of an experimental course of calculus for first year university students in Mathematics, of collecting some interesting data (written texts, transcripts of oral think-aloud solving processes and notes taken by the observers concerning related gestures and attitudes). Students had been presented the intuitive, graphical notions of continuity and derivative for a real function defined on real numbers (or on an interval of real numbers), then the epsilon-delta definition of continuity at a point had been introduced. The derivative at a point had been introduced through the prolongation by continuity of the incremental ratio. Main theorems of calculus in one variable (in particular, the intermediate value theorem, IVT [1] in the following) had been presented with their epsilon-delta proofs; and some tasks demanding the epsilon-delta proof of easy properties by using those theorems had been proposed. Finally, the usual epsilon-delta notions of limit had been introduced (both in the case of finite and in the case of infinite limits) and applied to solve some exercises and to put into evidence how both continuity and derivative at a point were particular cases of the notion of limit.

Such an experimental approach somehow differed from the usual teaching of the epsilon-delta Calculus in Italy and in many other countries, based on the epsilon-delta definition of limit, because in our experiment the approach to the epsilon-delta reasoning referred to the intuitive notion of *regular* functions (smooth, continuous functions), according to the historical treatment of functions before the epsilon-delta revolution, and then moved to a formal description of *regularity* with the epsilon-delta language, and only at the end considered the epsilon-delta definition of limits.

Students were aware of the experimental character of that teaching of Calculus; they knew very well (thanks to elder students' experience) the difficulty of the subject matter, thus they were willing to engage in an alternative experience of teaching and learning, which possibly might have diminished the difficulties met in the ordinary learning of epsilon-delta calculus. Students volunteered in furnishing documents like private writings, informal drawings, oral reasoning in think-aloud situations.

I will not present the (modest) results of that teaching experiment, and I will not discuss the limitations

and the heavy consequences on students of the usual teaching of Calculus focusing on the epsilon-delta systematization (cf. Nunez et al, 1999; Tall & Katz, 2014) – in the reality, the teaching experiment did not represent a radical alternative to it, and probably it is not possible to do more at the university level. I only wanted to introduce the document analyzed in this paper by describing its context of production, in order to open the reflection on it. It is a written and oral document produced by a brilliant student (we will call him Ivan), during a think-aloud problem solving session; it concerns the epsilon-delta validation of a rather elementary statement of calculus. The interest of the document depends on the very detailed presentation offered in it of the mental path followed by Ivan to get the proof and on the fact that the difficulties met (and overcome) by him are very frequent (and frequently not overcome!) among students when they want to validate a statement according to the epsilon-delta criteria.

The aim of the research reported in this paper is to build a suitable theoretical framework to account for what happened during Ivan's problem solving, and to put into evidence the role played there by the visual – graphical treatment of the problem and its importance in the students' approach to epsilon-delta proving.

THE DOCUMENT

Myself as the teacher of the course and a Master Degree student of mine took notes about what we observed, and then compared and completed those field notes, finally integrating them in the transcript of the student's speech. The document is faithfully reported here. (...) for pauses of at least 5"; *italic* for sentences written by Ivan.

$f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function;

$$\lim_{x \rightarrow +\infty} f(x) = +\infty; \lim_{x \rightarrow -\infty} f(x) = -\infty.$$

The problem:

Prove that there is at least one point c such that $f(c)=0$

Ivan reads the text and after a few seconds draws the x and y axes, and two arrows, upwards on the right, downwards on the left; then he makes a gesture joining with a finger the two arrows and slowly and re-

peatedly crossing the x -axis; finally Ivan joins the two arrows by drawing an “oscillating” line.	18	<i>Thanks to continuity of f and IVT, $f(x'')$ $f(x') < 0$ implies that c exists in (x'', x') such that $f(c) = 0$.</i>
1 Well, it seems to me that (...) yes, it is a case of the IVT.	19	Not, it does not work! First, I need to find x' and x'' .
2 IVT says that I may find c such that $f(c) = 0$ (...)	20	<i>Thanks to hypotheses on limits, given $x' > 0$ I find $f(x') > 0$.</i>
3 yes, but how to find a and b ? First, I have to find a and b (...)	21	Not, still it does not work! I need to come back to how I could find $x' > 0$ such that! No, not how, why I could find it! Because of limits! How to write it well?
4 and to exploit continuity	22	<i>Thanks to the hypothesis on limits, given $M > 0$ I may find $x' > 0$ such that (...)</i>
Ivan comes back to look at the drawing; two index fingers are placed on the upwards and downward arrows.	23	Not: I may find x_M such that for every $x > x_M$ I have $f(x) > M$.
5 Yes, the limit perhaps says something.	Ivan puts a strikethrough over all what he had already written, and starts again to dictate to himself and to write down:	<i>Thanks to the hypothesis on limits, given $M > 0$ I may find x_M such that for every $x > x_M$ I have $f(x) > M$. I take one point $x' > x_M$, $f(x') > M > 0$.</i>
Ivan makes an horizontal movement with his right index finger and crosses the right upwards arrow.		24
6 If I take a value of y , the function must be over that value when x is enough big. But how to choose a value of x ? (...)	25	<i>In the same way I find x'' such that $f(x'') < 0$.</i>
7 Before I need to choose a value of y . But the value of y is whichever (...)	26	(...)
8 Whichever M , I take M and I may find T such that for every $x > T$ (...)	27	<i>There is a point c, $x'' < c < x'$, such that $f(c) = 0$ (...)</i>
Ivan makes again a horizontal movement with his right index finger and crosses the right upwards arrow.	28	Not, this is the conclusion, before it I must write the hypothesis and then the conclusion.
9 This might be the value of M , the value of y , then I find the value of x , the value of x which is b , such that if $x > b$ then $f(x) > M$	Ivan puts a strikethrough over the last line, named 27 above, and dictates to himself and writes down:	<i>Now I may apply the IVT to the function f in the interval (x'', x'): $f(x'')$ $f(x') < 0$, thus c exists in the interval (x'', x'), such that $f(c) = 0$</i>
10 But I have $f(x) > M$, not $f(x) = M$		29
11 Is it $f(b) = M$? (...)	When the document was analyzed, the complexity of the student’s behaviour and the analytical tools available in that moment (1995) did not allow to perform an exhaustive, in-depth description and interpretation of the difficulties met by the student and of how he was able to overcome them. At that time we were only able to find some elements that could qualify as mature Ivan’s mastery of the visual-graphical notion of continuity and infinite limits at the infinity, his identification of IVT as the theorem which might	
12 Not sure! (...)		
13 I might take one point x' such that $f(x') > M$; that point is the extremity of the interval on the right!		
14 But why $f(x') > 0$?		
15 Yes, I need to take $M > 0$! M is a number, whichever number!		
16 On the left it is the same: I find x'' such that $f(x'') < 0$.		
17 The interval for the IVT is (x'', x') . Let us write the proof:		
Ivan speaks and writes at the same time, by dictating to himself what he says		

have allowed him to perform a rigorous validation of the statement, the immediate reference to the problem of finding the extremities of the interval where to apply IVT, and the identification of the epsilon-delta definition of infinite limits at infinity as the crucial tool to solve the problem. Then Ivan enters the epsilon-delta reasoning and the situation seems to become more and more confuse: the search for the extremities of the interval where to apply IVT results in an apparently messy sequence of steps of reasoning to get the abscissa of a point where the function is positive. When the interval to apply IVT is constructed, a further problem concerns the organization of the text in order to satisfy the textual and logical constraints of a proof text. This description is a pure narration of Ivan problem solving process. It does not allow to account for:

- the functional relationships between Ivan inner questioning (see steps 3, 6, 11, 14, 21) and the different kinds of answers (sometimes a decision concerning how to go on, sometimes the control of a proposition or a chain of propositions);
- the nature of the difficulty met in the central part of the process (from 6 to 17), and what allowed to overcome them;
- the nature of the difficulties met in the last part of the process (from 18 to 29).

THE NEED FOR A COMPREHENSIVE FRAMEWORK: THE HABERMAS CONSTRUCT OF RATIONALITY

The document does not speak by itself; we need theoretical tools to make Ivan's voice understandable by us, and possibly identify an intentionality that might drive Ivan's towards the final result, in spite of the superficial impression of "*a random walk luckily resulting in a good conclusion*" (according to the evaluation of the document by a colleague of mine who teaches Calculus for first year university students). We need also to understand why apparently so obvious steps (like finding a point x where $f(x) > 0$, given the hypothesis of $+\infty$ limit at $+\infty$) become so difficult when the path moves through the epsilon-delta forest.

The first need suggested us to try and adapt Habermas' construct of rational behavior to Ivan's problem solving (as a process driven by intentionality to get a cor-

rect result by enchaining correct steps of reasoning, and to communicate it in an understandable way in a given community). The second need was satisfied by integrating, within the use of Habermas' construct, an analysis of some phases of Ivan problem solving process in terms of *mental dynamics* related to the treatment of propositions and the mastery of logical constraints, according to the Guala and Boero's elaboration on mental dynamics in problem solving (see Guala & Boero, 1999). They consider "mind times" that may be generated during the problem solving process: e.g. when imagining to move back from an hypothetically attained goal to a previous situation; or when going back and retrieving some information from memory, and projecting it in an imagined, future situation; etc.

Habermas' construct of rational behavior deals with the complexity of discursive practices according to three interrelated elements: knowledge at play (epistemic rationality); action and its goals (teleological rationality); communication and related choices (communicative rationality). Thus, it seems suitable for being applied to mathematical activities like proving and modeling that move along between epistemic validity, strategic choices and communicative requirements. The following aspects of Habermas' elaboration (1998, pp. 310–316) are relevant for us.

Concerning epistemic rationality

We know facts and have knowledge of them only when simultaneously know why the corresponding judgments are true. (...) Someone is irrational if she puts forward her beliefs dogmatically, clinging to them although she sees that she cannot justify them. In order to qualify a belief as rational, it is sufficient that it can be held to be true on the basis of good reasons in the relevant context of justification (...) The rationality of a judgment does not imply its truth but merely its justified acceptability in a given context. (p. 312)

The intentional character of rational behavior on the epistemic side emerges from these remarks in a perspective of progressive development of knowledge (the qualifying element being the tension towards knowing "*why the corresponding judgments are true*"). In Habermas' elaboration, the exercise of epistemic rationality is strictly intertwined with speech and with action (i.e. teleological rationality)— the latter resulting in the evolutionary character of knowledge:

Of course, the reflexive character of true judgments would not be possible if we could not represent our knowledge, that is, if we could not express it in sentences, and if we could not correct it and expand it; and this means: if we were not able also to learn from our practical dealings with a reality that resists us. To this extent, epistemic rationality is entwined with action and the use of language. (p. 312)

We will see how this way of conceiving the interplay between knowledge, speech, and action will account for some relevant aspects of Ivan's proving.

Concerning teleological rationality

Once again, the rationality of an action is proportionate not to whether the state actually occurring in the world as a result of the action coincides with the intended state and satisfies the corresponding conditions of success, but rather to whether the actor has achieved this result on the basis of the deliberately selected and implemented means (or, in accurately perceived circumstances, could normally have done so). (p. 313)

Let us consider problem solving in its widest meaning (including conjecturing, proving, modeling, finding counter-examples, generalizing, and so on): the above sentence highlights the quality of a process, which may be qualified as rational (on the teleological side) even if the original aim is not attained. The intentionality of action (including the choice and use of the means to achieve the goal) and the reflective attitude towards it are two relevant features of teleological rationality.

A successful actor has acted rationally only if he (i) knows why he was successful (or why he could have realized the set goal in normal circumstances) and if (ii) this knowledge motivates the actor (at least in part) in such a way that he carries out his action for reasons that can at the same time explain its possible success. (pp. 313–314)

Concerning communicative rationality

(...) communicative rationality is expressed in the unifying force of speech oriented toward reaching understanding, which secures for the participating speakers an intersubjectively shared lifeworld (...). (p. 314)

The above sentence illustrates an ideal practice of communicative rationality, and the related values. Then Habermas presents a condition that qualifies an actual individual behavior as rational on the communicative side:

(...) The rationality of the use of language oriented toward reaching understanding then depends on whether the speech acts are sufficiently comprehensible and acceptable for the speaker to achieve illocutionary success with them (or for him to be able to do so in normal circumstances). (p. 314)

Even in the above sentence the intentional, reflective character is pointed out (for the specific case of communicative rationality).

Habermas' construct offers a model to deal (after adaptation) with important aspects of mathematical activity, without capturing *all* the aspects (see Boero & Planas, 2014, pp. 207–208 for a brief presentation of some of its intrinsic limitations). It has been initially used as a tool to analyse students' rational behavior in proving activities according to the researchers' (and teachers') expectations (see Boero, 2006; Morselli & Boero, 2011). Its application to analyses that also use other constructs gradually resulted in a rich toolkit with various applications (see Boero & Planas, 2014).

ANALYSIS OF THE DOCUMENT [2]

In the perspective of the Habermas' construct of rationality, integrated with Guala and Boero's elaboration on mental dynamics, the document provides us with the opportunity to analyze and interpret Ivan's proving as a *rational* enterprise. It also offers an occasion (based on that analysis) to reflect on some crucial aspects of a successful approach to what we may call the *epsilon-delta rationality*. Indeed in the reported document we may identify:

- the continuously renewing, conscious interplay, driven by an inner questioning (steps 3, 6, 11, 14, 21), between the need of performing strategic choices (*teleological rationality*) aimed at getting the elements to move forth, and the epistemic control on how they fit (or do not fit) the requirements of *epistemic rationality*;
- some mental dynamics (cf. Guala & Boero, 2009) related to Ivan's strategic choices (*teleological rational-*

ity) performed to meet *epistemic* requirements. We may observe how in the first part of Ivan's work he follows (as the movements of his hands show – see description before step 1) the time ordering of the text of the task, moving from the hypothesis on limits to the visual-physical search of the points of intersection with the x axis. Then an abductive shift is made to the IVT, which should guarantee the same result within the theory: this means a shift to a different *epistemic rationality*. At that point, a reverse mental movement is made, from focusing on the existence of the intersection point (step 2), to the search for the interval to which the IVT should be applied (step 3). This movement means moving back from the time of the solved problem to the time of the hypotheses that guarantee the solution. A similar movement will be replicated later (step 7), and expressed through a temporal adverb (“before”, “prima” in Italian). In both cases an inner question related to how to validate (*epistemic rationality*) a partial result got during the development of the problem solving process suggests to go back to the condition that ensures its validity. Let us consider now what we might call the *quantifiers game*, played from step 7 to step 15 and then at least partially echoed in the writing phase (steps 18 to 24, where the *epistemic rationality* and the *communicative rationality* constraints are intertwined). From the satisfied condition of infinite limit at the infinity: *for every M there is x_M such that if $x > x_M$ then $f(x) > M$* it is necessary to move to: *in particular, I choose $M \geq 0$ and then I get x' such that $f(x') > M \geq 0$* , with a change of logical status: from general quantification and consequent existential quantification and subsequent universal quantification, to particularization of the generality of M in order to get a point where the function is positive. Focus must move from the general condition of limit to a specific implication of it. The condition to be satisfied becomes $f(x') > 0$, thus M must be chosen as $M \geq 0$. The Mind times toolkit, applied to a micro-analysis of this phase of the process, allows to interpret the mental weight inherent in this phase and its difficulty: a projection in the future time of the application of the IVT results in the choice of a *particular* value of M , at the beginning of the logical and mentally temporal chain of quantifiers, suitable to get the *appropriate* value of x' ;

- the full mastery of the logical and temporal structure of a theorem, interfaced with the above change of focus; this is evident in the steps 18 to 24;

- the need to satisfy the requirements of *communicative rationality*, consciously related to the *epistemic* requirements. *Communicative rationality* must account for satisfied *epistemic* requirements in the outer speech/writing, while epistemic requirements drive the inner dialogue towards consequent actions: this emerges in the steps 23–29, with communicative constraints particularly evident in the steps 27–29.

REFLECTIONS ON THE ANALYSIS OF THE DOCUMENT

The adaptation of the Habermas' construct to Mathematics Education offers the opportunity to qualify the visual-graphical Calculus as fully *rational*. Indeed that pre-Cauchy Calculus, when dealing with “ordinary” functions, has its own criteria of epistemic validity, its own strategies to solve problems, its own means of intentional communication. This is reflected in the first part of Ivan's elaboration: requirements of *epistemic validity* are of visual nature, and the chosen strategy to solve the problem consists in the translation of the verbal hypotheses of the statement into visual-graphical hypotheses, which will be connected to the thesis through the gesture of the finger that crosses the x axis in order to connect downwards and upwards arrows. Also inner and outer communication is mainly visual (through the signs on the sheet of paper and the gestures). This is not new (cf. some remarks by Nunez et al., 1999, and by Tall & Katz, 2014), but the use of Habermas' construct of rationality suggests us to reflect on how that system of thinking (with its specific epistemic, teleological and communicative characters), relevant in the history of mathematics, may work today not only as an intuitive first step when moving towards the epsilon-delta proving, but also as a resource during the epsilon-delta proving to support some of the most delicate student's actions (in the case of Ivan, this is particularly evident, thanks to his gestures, in the descriptions after the steps 4, 5 and 8).

In Ivan's elaboration, the statement of the IVT “represents” what is already clear in his first approach, and guaranteed by a gesture (see above); it also opens the way to its formal treatment by orienting the search for the interval in which it can be applied, based on the formal definition of infinite limit at infinity. We may interpret IVT as a pivot in Ivan's transition from the visual-graphical rationality to the epsilon-delta rationality in the treatment of the problem.

In terms of mental dynamics, we may better understand the complex relationships between the visual-graphical rationality and the epsilon-delta rationality, and the difficulties to manage the latter. On one side, the management of mind times in the epsilon-delta Calculus implies the necessity of changing the temporal and logical order of quantifiers, moving from $f(x) > M$ (whichever M) to the choice of a value of M such that we can get x' and $f(x') > M$. This suggests a perspective (alternative to Nunez, Edwards and Matos elaboration) to interpret why it is so difficult to move from visual-graphical to epsilon-delta calculus. On the other side, on the *teleological* dimension, the visual-graphical rationality provides Ivan with the visual and gestural support to bear the weight of the complex logical and temporal operations needed to construct the (x'', x') interval where to apply the IVT. Indeed the movement of his right index finger not only suggests the existence of a point where the function is positive, but also provides Ivan with the opportunity of “seeing” how to move from a general quantification on M , to the choice of a particular M in order to get a value x' such that $f(x') > 0$, thus orienting Ivan’s mental dynamics.

CONCLUSIONS

The case of Ivan’s think-aloud proving of a simple theorem of epsilon-delta Calculus was an occasion to search for a comprehensive framework to deal with the problem of analyzing the transition from visual-graphical proving, to epsilon-delta proving in Calculus. The Habermas’ construct of rationality, integrated with an analysis of the proving process in terms of mental dynamics, suggests a solution, which in the case of Ivan accounts for his intentional work and his difficulties. It also suggests to reconsider the visual-graphical treatment of proving not only as a heuristic starting point, but also as a consistent *rationality* and, as such, a permanent, sure reference when students move within the forest of the epsilon-delta proving during the approach to the epsilon-delta Calculus as taught in most universities.

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ENDNOTES

1. Precisely, by IVT we mean the theorem whose statement in Ivan’s textbook is: “If f is a continuous function in an interval (a, b) and $f(a) \cdot f(b) < 0$, then at least one point c exists in the interval (a, b) such that $f(c) = 0$.”
2. In the Turin international symposium (November, 21, 2014) on “Mathematics Education as a transversal discipline”, Ferdinando Arzarello presented an alternative analysis of the same protocol, based on an integration of the Habermas construct with analytical tools derived from Hintikka’s Logic of inquiry, with some points of contact with the analysis presented in this paper as concerns the teleological dimension of rationality.