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# Proof writing at undergraduate level

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*This paper deals with the writing of a proof text as the final step of the proving process. In particular, students' difficulties to get a satisfactory product, which frequently result in an unclear text in a disorganised form, are analysed. Differently from other difficulties related to proof and proving, we noticed this phenomenon particularly when third year university students were answering some open questions where the process has to be built up, differently from traditional questions of calculation or direct use of a mathematical result where the steps of the process are known, regardless the correctness of the proof. We'll try to identify the reasons behind writing an unclear, messy draft instead of a clear readable proof text; its consequences on students' making proofs at the university level will be considered.*

**Keywords:** Proof, university, proof text, open tasks, meta-mathematics knowledge.

## INTRODUCTION

The study reported in this paper is a part of a PhD research on proof and proving at the undergraduate level, focused on students' difficulties when making proofs within a course of Complex Analysis. Research concerns third year university students' productions while responding to tests during the academic year. These students are prepared to become engineers in a high level selective university in Algeria; the mathematics programme for them is the same as for mathematics majors in any other university, but with more time and more complete exercise activities. Each test contained three questions, chosen in order to investigate difficulties of students' proving. During the analysis of collected data (consisting of students' written productions) of the three tests, for a few students an expected difficulty emerged related to the production of the final written proof text; a fourth test was designed to explore that competence. In the first three tests, the questions were closed (like 'calculate', 'prove that'), and most students' proof texts were quite clear,

well organized and legible, regardless the correctness of the solution. In the fourth test, the questions were open (like 'why...', 'is it possible to have ...') and students were required to justify their answers: we realized that in most cases the final proof text was written like a draft, in a messy form that makes it difficult to read, even though, most of the time the idea behind seemed to be correct.

We then decided to investigate the problem of writing a proof at the undergraduate level, because we think that it's a crucial skill and one of the factors that prevent students from making successful proofs or/and completing correctly a proof already began. When the first sentences are insufficiently developed and written like a draft in a disorganised style, they cannot be a strong base to develop a complete well organized proof and even to check the correctness of reasoning; moreover the students may misread what they wrote, thus they are misguided in their way through.

In a previous study (Azrou, 2013), performed with first year university students attending an algebra course, the findings indicated that most of the proofs given by the students in closed-type tasks, even the incorrect ones, were clearly written and could be read easily. The purpose of the study reported in this paper is to try to identify the reasons why many students produced those disorganised unreadable proof texts while answering open questions.

My hypothesis is that reasons might have been inherent: in the didactic contract that shapes the teacher-learners relationships; in the lack of meta-mathematical knowledge (in particular, as concerns proof); and in the cognitive difficulties inherent in moving from intuitions, knowledge and evidences elaborated at the inner or oral level, to a well-organized and clear written proof text, with an appropriate and correct use of mathematical language. Based on this working hypothesis, in the next section we will consider some theoretical elaborations concerning the above

possible reasons, in order to address and deepen our analyses.

## THEORETICAL BACKGROUND

The final proof text is a result of a process involving various components and is influenced by several factors. First of all, we will consider the relationship with the teaching and how proof texts are written by teachers; this will be explored using the construct of didactic contract (Brousseau, 1988) that concerns, in particular, the relationships between what teachers do and expect from the students, regarding knowledge that teachers have and intend to teach, and what students think that their teachers expect from them. Several teachers' expectations (concerning what students should do and learn) are not made explicit, or, when explicit, are not understood by students in the right way. On the contrary, the efficiency of a didactic relation assumes students having some intelligence about the intentions of the teacher (Mercier, 2010). In the case of proof, different teachers write different kinds of proofs at the blackboard according to their conceptions about what is relevant in a mathematical proof, and according to the needs of the moment. Some teachers write all the details, while others write only what seem to be important for them in that moment. Some teachers add special comments on proving, others write only a chain of symbolic statements and comment them orally. Students "by contract" might tend to provide proofs that are as close as possible to their teachers' ones.

At a meta-mathematical level, most students do not know what doing mathematics is, which means knowledge about mathematics as a science. By teaching only its contents, several teachers (especially school teachers) assume that over time, students will be able one day to acquire knowledge about that science and find out how it works. But it has been shown (see Morselli, 2007) that some university students' difficulties are still caused by lack of meta-mathematical knowledge, particularly as concerns proving, like: how to exploit a known theorem while proving another theorem; what is the difference between a definition and a theorem; what is a counter-example; thus, they ignore some important rules of the game. Moreover, as pointed out by Morselli (2007), frequently students confuse exploratory argumentation and proof; they produce written arguments that are different from a proof by writing their "proofs" when they are engaged in the

exploratory phase. The problem is that teachers do not know that students ignore what exactly a proof and a complete proving process are, and thus they might tend to stick to models derived from their teachers' presentations of proof-models, which may be different from one teacher to another. We say briefly that, concerning proof, several students lack knowledge about it (which we consider as meta-knowledge about proof); and we hypothesize that this fact may result in a stronger influence of the didactic contract, which works as the only factor orienting their behaviour.

As concerns the writing of proof, it is also close to language and expression issues. The nature of mathematical language, with attention paid to the case of proof, was described in (Boero, Douek, & Ferrari, 2008), in particular as concerns the specific use of current verbal expressions in mathematics (with change of meanings, in comparison with ordinary language), and the integration of mathematical symbols within a verbal written text. In that paper, the authors also presented some results concerning university students engaged in problem-solving in an Algebra course; they found out that almost all those who, in an entrance test demanding verbal explanations, had produced answers with no verbal comment, or with rambling words and poorly organized sentences, failed the final exam. That paper is a general reference for our work: it offers a frame which provides a definition of mathematical language (pp. 265–266), a rich set of reflections and a wide perspective to deal with students' difficulties concerning mathematical language.

Finally, we'll also consider the logical structure of students' proof texts and how they link their written propositions together. For this, we'll make reference to Durand Guerrier's research work on the logical aspect of proving. In particular, students' use of '*then*' should be considered: it is currently used in mathematics to introduce a conclusion, to end a proof or set the desired result, and within the "*if...then...*" construct. Students may misuse it or not use it when they should, which makes a proof sound strange as there is no connection between the different propositions. In general students need to validate their different steps to carry on the proof process; if they are not able to do so because of difficulties related to logical symbols and connectives (cf. Boero et al., 2008; Durand Guerrier et al., 2012), their proof are likely to be vague and unclear, or mistaken.

## METHODOLOGY

To study proof text, it's important to analyze students' proofs written individually. We have designed a test with open questions (thus not in the form: "prove that"), whose answers should not result from a standard procedure involving a mathematical result or a theorem, but rather engage students in a creative process. The test, administered to 98 students, proposes three simple short questions about a definition, a property or a result well known by the students, to be answered during thirty minutes. The degree of difficulty (complexity, connections to be established between known properties) of the questions was lower than in the case of some questions already tackled by the students during the exercises sessions. However, the questions were asked in a way unfamiliar to them, which put the students in a new situation that engaged them in gathering and linking their information and organizing them in order to build up a proof clearly written. This choice was aimed at identifying specific reasons influencing writing, and resulting in confused, unreadable proof texts.

The construct of didactic contract will be used to identify the relationships between students' productions, and how proofs were usually presented to the students by the two teachers (I'm one of them) during the course and the exercises sessions. Teaching the course consisted in two lecture sessions and one exercises session each week, all during a period of twelve weeks. There was no special focus on proofs even though all results are proved in lectures. The focus was rather on contents, statements of theorems, results and definitions and how students understand and use them; most of the exercises were about calculations, only some ones were about reasoning and links about concepts; time was not enough to develop in details every concept. There was no check of students' answers to questions (additional or not completely answered questions) posed to them, or of proofs written by them during the exercise sessions. Most of what students wrote was copying down what was already written by the teacher at the blackboard. The analysis will also consider students' oral expression ability during the academic year and how they managed to talk mathematically: as I know the students, I can tell about that for the students whose productions are analyzed.

The study will be focused on the organisation of the students' steps of reasoning in their proof texts; in

order to better classify them, we will distinguish between three phases of proof production (Arzarello, 2006):

Step1: exploration and production of reasons for validating the statement.

Step2: organisation of reasoning into a cogent argumentation.

Step3: production of a standard deductive text.

Moreover, students' productions will be examined from a logical aspect, by exploring how they link the different steps and whether and how they deduce any written statement from the previous one, and what are the means used for that: are they, in particular, logical connectives, or symbols like the arrow implication, or other transition words like 'then', 'thus' or 'finally'?

## A-PRIORI ANALYSIS OF THE FOURTH TEST

The questions of the fourth test were of open type and needed to be answered by producing the answer and building up a proof to validate it in an autonomous way (as concerns the whole process). The three questions were as follows:

- 1) *Is it possible to find a holomorphic function that admits 0 as a simple pole such that Residue of  $f$  at 0 is 0 ( $\text{Res}(f, 0) = 0$ )?*
- 2) *May the residue of a holomorphic function at the infinity be zero? Justify.*
- 3) *Why is the residue of a removable singularity zero?*

Like in the other tests, questions were related to properties and results that should have been well known by the students (and indeed other tests demonstrated that it was so). We aimed at ascertaining if the students were able to write down the argument, based on known definitions and theorems, in a well organised, clear mathematical form. Even if this kind of open questions was familiar for students in an oral form, it was quite new for them in a written form: they had never seen the written answers to such kind of question; they had never been engaged in producing written answers to them. Thus questions could have revealed the students' competence of *autono-*



*mous* proof writing (proof writing not induced by a stereotyped request) and the *consciousness* about its functions (related to the very nature of proof: shortly, as an unchaining of propositions aimed at validating a statement). The most interesting data have been collected through the first question.

The question is about some results concerning simple poles and residues ( $P \Rightarrow Q$ ) and its negation ( $P$  and  $\bar{Q}$ ) at the same time, which results in a contradiction: if a holomorphic function has a point 0 as a simple pole (P), it means that the residue of the function at this point (which is the coefficient of  $1/z$  in its Laurent development) cannot be zero (Q); the reason is that, in the case at stake, the residue is calculated by the formula  $\lim_{z \rightarrow 0} zf(z)$  when  $z \rightarrow 0$ , which is exactly the coefficient of  $1/z$  already taken not zero. Logically speaking: the fact that the residue at a point is not 0 is a direct consequence of that point being a simple pole for the function. We have chosen to refer to the point 0 to simplify the formula. There was no doubt that students knew all these concepts because they had used them many times before, but always when performing calculations (to calculate the residue for a given point, to establish whether given points were simple poles, multiple poles or other singularities). However no request of identifying and exploring the links between concepts had been made especially in a written way.

## ANALYSIS OF SOME STUDENTS' PRODUCTIONS

We'll present and analyze two examples of students' productions (French is the ordinary used language) that consist of a disorganised, unreadable proof text. This will exemplify the work done on the collected productions, and how we got the conclusions reported in the next sections.

### Student 1:

The student starts by rewriting the hypothesis of the question: 'a function admitting 0 as a simple pole'. She

continues at the beginning of the line by putting an implication arrow, followed by crossing out ( $x=0$ ); then she writes  $f/x$ , adding the Laurent series of the function - not complete, with some blanks in it. Another implication arrow follows by stating the definition of the residue (that is the  $C_k$  coefficient). The student goes on by marking a slash, 'if the' and a comma; then she calculates the formula of the residue of  $f$  at 0 which is, according to the student, the limit when some non declared 'a' goes to 0 of  $xf(z)$ , which is zero. With a last implication arrow, the student ends her proof by stating that  $f(z)$  is not a simple pole.

The proof in this production is not totally wrong but is not written correctly and not complete. This doesn't tell clearly if the student's reasoning is correct or is a result of a mere coincidence. The first thing to notice is that two variables are used:  $x$  and  $z$ . According to the student a function that admits 0 as a simple pole has a denominator with  $x$ , which is correct. She certainly meant to write  $f(x)=g(x)/x$ , but the way she expresses it is so odd, she writes  $f/x$ ; she wanted probably to say that the principal part of the Laurent development contains only  $1/x$ . This shows how the student struggles to translate her intuitive vague idea about the formula into a mathematical expression with mathematical symbols. The writing of the student is confusing; she gives flashes of what seems important to her without taking into account how the proof should be presented to the reader and how the different steps and formulas should be linked, which indicates poor meta-knowledge about proof. Many variables like  $z_0$ ,  $c_k$ ,  $c_n$  and 'a' are used without defining and specifying their meanings; we are not sure that their meanings are clear for her. At the end, she deduces, by an implication arrow, that  $f(z)$  cannot be a simple pole, and certainly she meant the point 0. She writes down briefly how she remembers the related definitions and shows the main ingredients of the proof (one denominator with  $z$ , one coefficient is the residue and a non zero limit is for a simple pole).

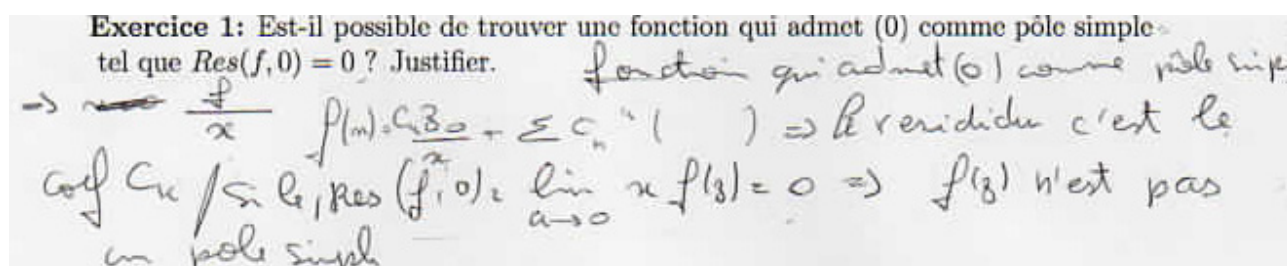


Figure 1: Student 1

She works within the step 1, the exploratory phase and production of the reasons for validating the result.

We can see how the student's effort, influenced by the didactic contract, is aimed at showing to the teacher that she got the most important ideas related to the question, no matter if several details are mistaken or neglected. The text also allows to guess her own proof conception, which seems to be shaped by doing it for the teacher, who is supposed to understand the missing details and the unsaid words, according to the student. The logical links are not clear, particularly the implication used is not the logical one: it is more about shifting to another idea not necessarily and clearly connected to the previous one; moreover the student doesn't tell about her proof technique, which seems to be a proof by contradiction. From a language point of view, the student is one of those who showed real difficulty orally when expressing any question or a comment involving a mathematical idea. Her natural spoken language is confusing and poor, and lacks flexibility to express mathematical meanings (Boero et al., 2008).

## Student 2

In this production, it's hard, first, to decipher the handwriting of the student and to follow the lines that are broken, so it's not clear where a line starts and where it continues and ends. He starts by explaining what means that a function admits 0 as a simple pole by putting one arrow, which is meant to be the implication arrow, oriented towards  $f(z)$  expressed as  $\varphi(z)/z$ ,  $\varphi$  being holomorphic. Then, the reader is puzzled where to continue reading, is it on the same line or beneath? At the same line, the student calculates the residue value by the limit formula, but it's indicated with  $z$  going to  $\infty$  instead to 0, even though the result is with  $z$  going to 0 which is  $\varphi(0)$ ; this last result is declared to be different from 0, without any justification, as an obvious fact. At the next line, he states that  $\varphi(z)$  is

$c_0 z$  with a hesitation on the power of  $z$ ; at the next line, he adds that  $f(z)$  is  $c_0/x$  plus a sum of  $c_n$  and something not clear; at the end, he concludes by writing 'then' *it is not pole*, without explicitly indicating what is 'it'. With an unreadable handwriting presenting the proof like a sketch, the organisation fails to clarify what the student is exactly doing. However, after reading this text many times, the first part appears to be not totally incorrect. The student gives the definition of 0 being a simple pole and calculates the residue of the function, which is not 0, a valid step of reasoning, but this would need a conclusion: that is a contradiction to the question. The student gives the main idea: the formula of the function  $f$  is given by a holomorphic function ( $\varphi(z)$ ) with  $z$  at the denominator in the case of having 0 as a simple pole, and then calculate the first element of the Laurent development of  $f$ , which should correspond to  $c_0$ , but fails to justify and make explicit the other details taken as obvious, like: why  $c_0 \neq 0$ , which is an important point in the proof. In doing so, the student is about showing to the teacher (or doing like the teacher) that he got the main idea of the proof, which seems more important than clarifying explicitly the "small details" which are clear for him. Leaving some blanks in the proving process makes inevitably the statements logically disconnected, especially because the student doesn't use logical connectors (except one 'then' at the end) to link a statement to another. I remember this student as one of the brilliant ones, but he is weak both orally and at writing. When he talks, no one understands him: he swallows his words, bubbles and repeats the same expression to say different meanings. On a cognitive level, he is very smart and most of the times he finds the good idea where all other students are stuck; it is rare that his answers are wrong, which makes his proof's idea possibly correct. For him, writing a proof text is writing a set of partial arguments presented in a disorganised way with incomplete formulas, far from how a proof should be.

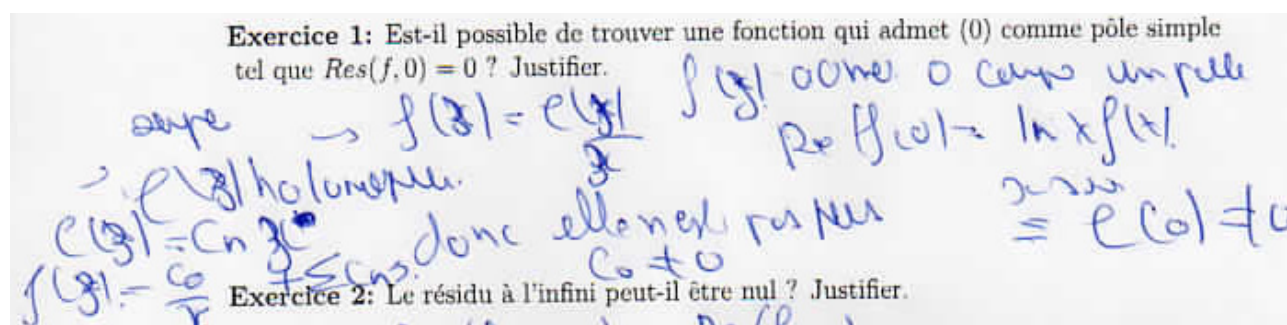


Figure 2: Student 2

## RESULTS

Analysis indicates that students' difficulties concerning writing an organised and understandable proof text are originated in the didactic contract that substitutes the mathematically relevant aim of structuring and writing a clear proof text with the aim of imitating teacher' incomplete proofs. This results in a disorganised, poor text somehow similar to some sketches of proofs produced by teachers at the blackboard. Moreover, writing a proof is strongly influenced by students' engagement in showing to the teacher what they know (the relevant idea to achieve the proof), underestimating the importance of writing a complete, well organised proof text. The previous texts show that students fail to go through the three steps of the proving process; they only develop the first exploratory phase and write it down by presenting the main ideas and missing many details. Failing to engage in the second phase and reach the third phase of writing the proof text, logical links between statements remain unclear (student 1) and even missing (student 2); with misused logical connectors (the implication arrow for student 1 and 'then' for student 2), the logical structure of the proof text is totally disconnected. Such students' behaviors might be contrasted by a good meta-knowledge about proof, but as their meta-knowledge about proof is poor, what is suggested by the didactic contract becomes dominant. The poor proof texts show difficulties inherent in the mastery of natural language (for both students) in the mathematical register (Boero et al., 2008), particularly as concerns its logical features, which contributes to students' difficulties of logically connecting the steps of reasoning in an explicit and appropriate way, and to their using incomplete definitions and formulas (student 1 and 2). In some cases, disorganised proof texts have been produced by students with difficulties in mastering of natural language: students 1 and 2 are examples of them. Thus a flexible mastery of natural language (which cannot be achieved by means of every day-life experience alone, and requires specific interventions also concerning scientific communication) appears to be a necessary condition for mathematical proficiency (in agreement with (Boero et al., 2008), even for a course at the undergraduate level). The above presentation of the results of the analysis of students' productions suggests that the four components used to interpret the poor quality of the proof texts are not independent, but rather overlapping and intertwining.

## DISCUSSION

Our hypothesis on difficulties of writing proofs to open tasks that require a creative construction of a proving process and writing clearly the logic structure of how links are created between different arguments and concepts were set on: lack of flexible mastery of natural language; lack of knowledge about proof at a meta-level; influence of the didactic contract; and weak consideration of logical proof structure. This led to the question of identifying the possible reasons for these results and suggesting some possible didactical implications to cope with these problems.

According to the didactic contract, we may hypothesize some links between the identified difficulties and our teaching of mathematics at the undergraduate level, which does not introduce learning of proofs and less writing a proof text. Moreover, during the grading of exams copies, we base our positive evaluation on correct ideas in students' proofs and give partial credit, even if they are not clearly and rigorously presented. As far as I'm concerned as a teacher, it happens that I write some of the proofs at the blackboard by giving the plan and the main idea, and devote the short available time to the full explanation of the concepts to be used, and the links between the ideas of the proof. I sometimes leave the final text to the initiative of the students, who feel satisfied by getting the main idea and the main details. Both students and I (as a teacher) have always held the assumption that when the main idea of the proof is clearly explained and understood, writing the proof details and organizing the proof text is a simple thing! However, the fourth test provides strong evidence that this is wrong. Most of the meta-mathematical knowledge about proof is absent in our teaching, in particular as concerns the relationships between the proving process and the proof text. The lack of this knowledge might explain why students write directly their final text at the same time when they explore the question: they have no mean to interpret and situate what their teachers do when they write a proof at the blackboard. We (teachers) generally write proofs in a direct linear way; knowing already the steps of the proof, we write them one after another, till the conclusion. We do not show the exploratory phase and how the partial arguments produced in that phase are re-considered and arranged to produce the final text (similar situation with mathematics textbooks). Students learn to do



the same: when they first set some ideas about how to solve a problem, they write their first exploratory draft as a final text because they were never shown how to go further till the written proof text. Another component of the meta-knowledge about proof that would have helped to cope with the problem of writing the proof text is about the difference between argumentation and proof, which is not clear for the students especially from a structural perspective. The logical structure of an argumentation differs from the logical structure of a proof: while in a proof all steps are deductive, in an argumentation the steps may be of different nature: abductive steps or inductive steps (Peirce, 1960; Polya, 1962 cited in Pedemonte, 2007). In this case, the construction of a deductive proof requires a structural change: from abductive or inductive to deductive steps. This change is not always straightforward for students, but is usually necessary (Pedemonte, 2007). The results support the intrinsic cognitive difference between the open and the closed tasks, which implies different roles of verbal language. With closed tasks, proof text plays a narrative ritual function, while in the open ones, the text plays, in the first phase, a constructive creative function. With proofs for closed tasks, students go through a secure process, behaving in a conventional ritual way and showing a coherent reasoning guided by the didactic contract, according to teachers' and textbooks' models. The result is a clear, well organisation proof text (regardless the correctness of the proof). The use and control of meta-mathematics knowledge is needless in this case. On the contrary, in proofs for open tasks, a creative organisation of the arguments is needed, but lack of meta-mathematics knowledge about proof and some uncompleted proof text models offered by teachers lead students to write their proofs when reasoning (which is a nonlinear process). Failing to go through the two following phases, the text would look more like a draft. Furthermore, an in-depth analysis of students' mastery of their native language and of written French is needed to complete our study.

What instructional interventions can be effective in overcoming these difficulties? Teaching meta-mathematical knowledge and especially meta-knowledge about proof seems to be extremely important and effective for introducing the learning of proof. It should regard the differences between deductive, abductive and inductive reasoning, the logical structure of proof, and how to organize proof text. Introducing some open tasks, among the activities, would break the

ritual pattern of proofs for closed tasks and encourage creativity. Explaining proofs and writing them directly fails to show students how really they are constructed in different phases; while some writing proof exercises proposed to students with hints and main idea of proof would enhance writing proof text.

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