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Proof by reductio ad absurdum: An experience with university students

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One technique that future mathematicians should dominate is proof by reductio ad absurdum. This paper presents part of a research study in which college students performed a task in which they needed to apply the knowledge gained using this technique. Small group discussions and a discussion led by the teacher were the methodology used in the classroom to solve the task. Both types of interactions were analyzed using the RBC-C model (Schwarz, Dreyfus, & Hershkowitz, 2009) to document how the construction process took place. It was found through consolidation of epistemic actions that although the students had to deal with difficulties associated with the proof, they were able to use the newly acquired knowledge.

Keywords: Proof, interaction, construction, knowledge.

INTRODUCTION

One of the main activities in mathematics is to prove. However, in recent years, some countries, such as Spain and Mexico have almost banished the practice of proofs from the school curriculum: “The idea of proving has changed over time; it depends on the context and cultural environment. Since the development of modern mathematics, which put too much emphasis on formal proofs, there has been a decline in their use in high school, this has strong implications for the transition to college” (Gómez-Chacón, 2009).

Possible reasons for the difficulties students face when having to develop a proof are a poor ability to formulate [and identify] mathematical statements, inappropriate concept images, a lack of training to generate and use their own examples and only an intuitive understanding of the concepts involved in the proof (Moore, 1994).

Everyday language is an obstacle for the learning of a mathematical proof because of the differences between this language and mathematical language (Epp, 2003). A conditional statement in every day language often admits various connotations of causality and temporality that makes its meaning quite different from the mathematical sense. Sometimes, ordinary language gives a different meaning to the statement caused by the tendency to deduce what is not said. In this sense, Epp (2003) suggests that the difference between everyday language and mathematical language can lead to committing the “reciprocal error” in accepting that “p only if q” is logically equivalent to “if p then q”; to difficulty in the interpretation of quantified propositions; and to the mistakes made in trying to deny the implications.

A lack of knowledge of proof techniques, how to choose the facts and theorems to be applied or when to use or not knowledge based solely on symbolic manipulation and the use of mathematical procedures are also perceived as problematic (Weber, 2001). Logic and proof are conceived separately; to avoid this, instruction must show the proof as a form of validation and the usefulness of language in developing and communicating proofs (Weber, 2001).

In this paper the process of solving a task using proof by reductio ad absurdum is analyzed in a study with university students in Mexico. This activity is part of a broader research study about the design of instructional tasks for teaching proofs and recording the advances made in the process of proving followed by the students as they solved these tasks. The solving process performed both in small groups and through guided interaction with the teacher are analyzed.

However, observation and detailed analysis of the process of construction of proofs in context can be

very complicated when the data are massive and confusing. Hershkowitz, Hadas, Dreyfus, & Schwarz (2007) provides an example of research in which the flow of knowledge from one student to another is analyzed until they arrive at a common knowledge base. This type of research focuses on the process of construction and on the constructs at a given point until consolidation is acquired. The authors consider knowledge to be shared if the common knowledge base allows the group of students to continue building and updating knowledge collaboratively on the same topic. The authors acknowledge that they have relied on the work by Cobb (1995) in regard to collaborative learning.

CONCEPTUAL FRAMEWORK

Since students are expected to develop abstract mathematical knowledge, we consider Abstraction in Context (AiC) by Schwarz, Dreyfus, & Hershkowitz (2009) as the a suitable framework for analysis of the interaction. In AiC, abstraction is defined as “a vertical activity for the reorganization of previous mathematical constructs within mathematics and by mathematical meanings so as to lead to a construct that is new to the learner” (Schwarz, Dreyfus, & Hershkowitz, 2009, p. 24).

An abstraction process has three stages: the need for a new construct, emergence and consolidation. The abstraction cannot occur without the need for a new construct; this need may arise from an intrinsic motivation to overcome contradictions, surprises, or uncertainty. The second stage is central and is where the new construct emerges. Three epistemic actions can be observed in this stage: R-actions (Recognizing), in which the learner recognizes that a specific prior construct is relevant to the situation at hand, B-actions (Building-with), with which the learner constructs recognized acts to achieve understanding of a situation or solve a problem; C-actions (Constructing), using B-actions and integrating previous actions to produce a new construct. The C-action refers to the first time that the learner uses or mentions a construct. In this process, R-actions are nested within B-actions and B-actions are nested within C-actions. C-actions can be nested in C-actions at a higher level. Finally, the third step, consolidation, is a long-term process which occurs when the construct is mentioned, constructed or used after a C-action. This stage is characterized by personal evidence, trust, immediacy, flexibility

and care when working with the construct (Dreyfus & Tsamir, 2004) and also when the language is becoming more precise (Hershkowitz, Schwarz, & Dreyfus, 2001), although Kidron (2008) and Gilboa, Dreyfus, & Kidron (2011) consider that the increase in language precision is characteristic of the construction stage itself and not just the consolidation.

In AiC the epistemic actions referred to are known the as RBC model (Recognizing, Building with, Constructing) and the RBC-C model, with the second C corresponding to the second stage of consolidation.

The aim of this paper is to describe the process of students' proof construction and how they transfer the knowledge already acquired to solve new situations. We analyze the interaction of a group of five students and with the teacher using the AiC model (Hershkowitz, Schwarz, & Dreyfus, 2001). Therefore the research questions to be answered are: what are the epistemic actions that arise in the course of an interaction in a group and with the teacher during the process of proving a statement by reductio ad absurdum? And, are the students capable of consolidation during a long process of teaching of the concept of proof through a collaborative work effort?

METHODOLOGY

The activity described below is part of a broader research study about introducing mathematical proofs to university students in their first semester of the Bachelor of Applied Mathematics program of Juarez University of Durango State, Mexico. The average age of the students was 18. Specifically, the task presented in this paper pertains to the eleventh session and their participation was voluntary. The students had previously worked with other tasks about proofs such as the generation of definitions (Alvarado & González, 2013a), which gave them some preliminary knowledge about proofs, the identification of the parts of a mathematical proposition (Alvarado & González, 2013b), the use of logical connectives, the generation of examples and counterexamples, the formulation of conjectures and the proof by direct demonstration (Alvarado & Gonzalez, 2013c). In this session, the students were introduced to the process of proof by contradiction. The teacher began by explaining this technique with the following example: *Show that prime numbers never finish, there is always one more*. Different tasks were proposed to the students to prove some state-

ments in the way shown by the teacher and once they had proved them, a challenge task, described in the next section, was proposed. The challenge task was to prove the following statement: *It is impossible to write numbers using each of the ten digits once so that their sum is 100.*

The classroom activity was conducted in a group with five students working on their own. Once they obtained an answer, an interaction with the teacher took place. Through collaborative dynamics the students were given the opportunity to develop knowledge together and express their ideas verbally. Planas & Morera (2011) argue that interaction is a skill that must be practiced by students and teachers in math class. This kind of an interaction is based on every participant's right to express their opinion and try to convince the others of the validity of their ideas.

The interactions can be effective if they feature a real and true exchange or communication. That is to say, if the participants: 1) undertake social interactions voluntarily with peers and with the teacher; 2) actively participate in interactions and engage with the task; 3) have developed the basis for sharing and receiving (taken-as-shared) in equal conditions; and 4) do not represent a mathematical authority in the course of the interaction. Cobb (1995) and Steffe & Wiefel (1992) consider points 3) and 4) necessary for small group interactions, for good communication and genuine collaborative learning in mathematics. In this sense it is expected that effective interaction will result in a product (definition, rationale, conjecture, method, argument, demonstration, illustration, etc.) agreed on by all participants.

During small group work, the teacher can see if students spend a long time looking for a way to approach the task. For this case the researchers had previously suggested that the teacher should interact with the students and give adequate support to their thinking. The teacher must take into account the four "teacher movements" (Jacobs & Ambrose, 2008) while the students continued working on the task. These movements are: a) ensure that students understand the task (what do they know about the problem?) and if necessary change the context to a more familiar one, b) change the problem to a similar one with simpler values, c) ask students what they have tried until that moment, and d) suggest using another strategy.

Discussions in the small group and with the teacher were videotaped and transcribed in their entirety. The analysis of interventions was made by identifying different units of analysis determined by the discussion of one aspect of the student's task. For each unit of analysis the epistemic actions evidenced were identified, and this identification was then triangulated between the researchers. To identify the epistemic actions, in this analysis R-actions were considered to be the epistemic actions that the students used to recognize the information assumed to be true, as well as the definitions and concepts involved, the B-actions were those that emerged from the statement by extraction of its meaning, or when calculations were performed to obtain deductions and to understand the meaning of the statement, to finally build the requested proof and organize it (C-action). The C-action occurred once the statement of the proposition and its proof were considered as a unit.

DATA ANALYSIS

Below we describe, characterize and analyze the process of solving this task which took place in the classroom during the interaction of a group with five students. We have differentiated two parts: the first one describes the work in the group and the second one the corresponding interaction with the teacher.

Group discussion

In the following dialogues only the students participated. We have not differentiated which student made each contribution because we are interested in the whole process as the overall production of the group. Each contribution is numbered indicating that it belongs to one of the students. The comments in brackets are comments made by the researchers.

The first *R-actions* served to recognize the assumptions or information available [2, 4], the conclusion [1 and 12] and the extraction of meaning from the data [5, 6]. As discussed below, students misinterpreted the information. They were considering a 10-digit number (all possible combinations) in which the digits 0 through 9 appear only once.

- | | |
|---|--|
| 1 | We must see that [a number] with the given conditions is impossible. |
| 2 | We have to use each of the 10 digits. |
| 3 | Yeah. It is impossible. |
| 4 | In P [the hypothesis] we have 10 digits. |

- 5 We can put that we form a total of however many numbers but the digits 0 through 9 always appear.
- 6 You are going to do all such combinations. One would be 1234567890.

Following this misinterpretation, we can identify some *B-actions* (advances in the proof from the *R-actions*) in their arguments [7–9 and 14] i.e. the commutative property was applied. The students showed flexible thinking [10] using another *B-action* linked to the process of deducting from the back to the front “The only way you have 100 would be ...” and as a result some [12 and 13] *B-actions* took place to extend their peers’ understanding.

- 7 [...] The sum will always be the same because the digits change places but not their value.
- 8 Good. P is that. Then P1 [the first deduction] which does not change the sum.
- 9 And by the property of the sum $0 + 1 + 2 + \dots + 9$
- 10 The only way you have 100 is with 10 places where the numbers were 10 and that’s not possible.
- 11 I don’t understand.
- 12 If the sum of all digits must give 100...
- 13 You are going to have a big number with 10 digits. Which you want without repeating. But their sum is not going to give 100.
- 14 [They read again] Well, the sum will always be equal using the commutative property.
- 15 But not different. It will always be the same.
- 16 See. It says that it is impossible.
- 17 How to build a contradiction?
- 18 Now, what must we prove?
- 19 That it’s impossible that the sum is 100. That is Q [the conclusion] //.
- 20 Here’s something missing. The sum will always be 45.

Another *B-action* occurred when formulating the negation of Q [21]. Finally [22 and 23] they found a contradiction and that was the step used to conclude that this was sufficient for proving the statement so this was the final *C-action*, i.e. the construction of a contradiction and thus the proof of the statement. It is important to mention that the statement did not appear in the conventional form “if P then Q”, and the students did not realize that the numbers could be formed with any number of digits, but each digit

from 0 to 9 could only appear once. This led them to prove a different proposition from the one requested.

- 21 First there are 0, 1, ..., 9. Then assume that their sum is 100.
- 22 Now I do the sum $0 + 1 + \dots + 9$ and that gives me 45 different from 100.
- 23 Since the digits always add up to 45 although we change positions, then the sum can never be that, and therefore Q is impossible, that the sum would be 100.

The figure below shows the written production made by the students, well-organized by steps in order to clearly communicate their ideas, even though they have not proved the adequate proposition.

Interaction with the teacher

The teacher reread the proof written by the group. First, he clarified the misunderstanding of the proposition as students had thought they had to consider a single number with 10 digits and with each of the digits from 0 through 9 only once. In [24] the teacher presented an example involving two numbers [25]. The teacher [25] reframed the situation and students in *R-actions* [26 and 27] recognised the hypothesis and the negation of the conclusion [27] and implicitly the conclusion. They recognized [29] that they should move from hypothesis and NO Q to construct a contradiction. The teacher considered it important to verify that the interpretation of the task was suitable and asked for some examples [30]. They showed *B-actions* [31–35] to build examples with the aim of understanding the nature of numbers and the negation of the conclusion.

- 24 Teacher: [...] seems to be a confusion [...] Of course, if I write a number with the digits 0 to 9 each written once, their sum does not give me 100. But look. The exercise is the sum of numbers written with these digits used only once [He writes two numbers: 12345, 6789] what is the sum?
- 25 Teacher: Well I can write it this way [two numbers] but the digits can appear once. Let’s see, what would be the way to reframe this situation? What do we know?
- 26 Students: The set of digits is 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Preguntas clave	Organizando la demostración
¿Qué información se supone cierta?	$P: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ No $Q: 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 100$
¿Qué deducimos de la información?	Que la suma de todos los dígitos es igual a 100
¿Cómo construir una contradicción?	$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45.$
¿Qué deducimos?	Dado que los dígitos siempre son los mismos y aunque los escribamos
¿Qué pretendemos probar?	Y como no se pueden repetir, la suma siempre será 45. Q Es imposible que la suma sea 100.

Figure 1: Students' written production

- 27 Students: The sum of them is equal to 100. That would be the negation [NO Q].
- 28 Teacher: How to build a contradiction?
- 29 Student: We must think of different numbers and their sum must be equal to 100.
- 30 Teacher: I have several numbers. I do not know how many. Their sum must be equal to 100. In addition, together they can contain no more than each digit once. See, what numbers would I use?
- 31 Student: So $a_1, a_2 \dots a_k$ should only have each digit once.
- 32 Teacher: That is, you may have one, two or more numbers, but if one digit is here in that number, you cannot use it in another number.
- 33 Student: Yes, [...] if I have 13, I cannot put 34 in another number [adding].
- 34 Student: But if in that case, we can obtain a sum even higher than 100.
- 35 Student: Yes, it may be higher, but we have assumed that it equals 100 [we must prove that it is impossible to add up to 100 and then its negation is that their sum is 100]
- 36 Teacher: Yes. It would be 100 [the sum], and what does that imply? For example, look at the numbers of units. What is needed to be 100?
- 37 Students: The units must give me a multiple of 10 [another student says]. Perhaps 10 or 20.
- 38 Teacher: What would it be? For example?
- 39-43 Students: 1 and 9/8 and 2/5, 2 and 3/7, 2 and 1/1, 4, 2 and 3.
- 44 Teacher: For example, in the case of 9 and 1, what happens? What about the other digits? Would they have a chance? If there were two numbers I would have

In the following excerpt, deductions were made from the hypothesis (they used numbers like 1245, 736 and 89) and the information derived from the negation of the conclusion. For the first deduction the teacher

- to accommodate the 8 digits left. Also if there were three-digit numbers the last digit of the third one would be 0.
- 45 Student: And I should accommodate 7 digits [2,3,4,5,6,7,8].
- 46 Teacher: So, at least on one of those three numbers we would have 3 locations to accommodate digits.
- 47 Students: Yes.
- 48 Teacher: But the sum is greater than 100, then that happens with numbers with more than three digits.
- 49 Students: One can do it if the number has two-digits, but I can't explain. Well, for example 5 two-digit numbers, using 0,1,2,3,4...
- 50 Teacher: For the units?
- 51 Student: Yeah, and then one must accommodate 5, 6, 7, 8, 9. We need to obtain in the tens, 9 [adding the units makes 10, therefore it accumulates 1 in the tens] and no ...
- 52 Teacher: Yes, well. I see.... Well, you can work on this idea. What matters to me is that you understand that the basic idea of reductio ad absurdum is to assume that the sum is 100 accompanied by the hypothesis. If I explore the possibilities and arrive at a contradiction, then we have proved the proposition.
- 53 Student: Well, if we think of a number with all digits the sum would not be 100 but rather greater.

the conclusion and from there, they proved the proposition although they made a misinterpretation of the statement. This is evidence of the consolidation of the knowledge acquired by the students.

They carefully managed the premises and constructed deductions from them, which is an indication of the consolidation of the new constructs as established in the RBC-C model and as shown by personal evidence. They also discussed the ideas they shared, showed confidence, immediacy, use of the construct when changing the context, working both in the group and with the teacher, seeking to clarify and refine their thinking, which may allow their language to increase in accuracy.

The RBC-C model allowed us to document and capture the complexity of group work for the construction of knowledge. Thanks to this model as a theoretical and methodological tool, we were able to identify the epistemic actions that occurred during the process. The group discussion with peers allowed students with the same level of knowledge to confront each other and identify some weaknesses, in this case comprehension of the task statement. The activity began with R-actions that progressed until a C-action. This activity sometimes required feedback to clarify to a classmate the reasoning performed, for example when a student says he does not understand. Throughout the task the students were persistent. They used examples in order to look at their structure, to understand their nature and to extract information in the construction of the contradiction.

Finally, the teacher stopped the discussion considering that the important thing is to understand the technique and the students must continue working later on the proof of the proposition on their own. As this took place in the last two days of school we were not able to collect the final evidence of the students' work on this proposition.

FINAL REMARKS

In this activity, a group discussion, the students' written production and the discussion led by the teacher to encourage knowledge construction from what they previously knew have allowed us to document the interactive generation of knowledge. The students recognized the hypothesis, the conclusion, and their role in the proof, they could construct the negation of

Interaction with the teacher was essential in this case to clarify the meaning of the proposition. In relation to comprehension and application of proof by reductio ad absurdum, the students managed the technique, handled it properly and they understood the function of negation of the conclusion in this proof.

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