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Didier Arnal, Wissem Bakbrahem, Mohamed Selmi

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# QUADRATIC AND PINCZON ALGEBRAS

DIDIER ARNAL, WISSEM BAKBRAHEM, AND MOHAMED SELMI

ABSTRACT. Given a symmetric non degenerated bilinear form  $b$  on a vector space  $V$ , G. Pinczon and R. Ushirobira defined a bracket  $\{ , \}$  on the space of multilinear skewsymmetric forms on  $V$ . With this bracket, the quadratic Lie algebra structure equation on  $(V, b)$  becomes simply  $\{\Omega, \Omega\} = 0$ .

We characterize similarly quadratic associative, commutative or pre-Lie structures on  $(V, b)$  by the same equation  $\{\Omega, \Omega\} = 0$ , but on different spaces of forms. These definitions extend to quadratic up to homotopy algebras and allows to describe the corresponding cohomologies.

## 1. INTRODUCTION

In [PU, DPU] (see also [D, MPU]), Georges Pinczon and Rosane Ushirobira introduced what they called a Poisson bracket on the space of skewsymmetric forms on a finite dimensional vector space  $V$ , equipped with a symmetric, non degenerated, bilinear form  $b$ . If  $(e_i)$  is a basis in  $V$  and  $(e'_j)$  the basis defined by the relations  $b(e'_j, e_i) = \delta_{ij}$ , the bracket is:

$$\{\alpha, \beta\} = \sum_i \iota_{e_i} \alpha \wedge \iota_{e'_i} \beta.$$

Especially, if  $\alpha$  is a  $(k + 1)$ -form, and  $\beta$  a  $(k' + 1)$ , then  $\{\alpha, \beta\}$  is a  $(k + k')$ -form. Shifting the degree on  $V$  by -1, we replace  $V$  by  $V[1]$ , the skew-symmetric bilinear forms on  $V$  becomes symmetric forms on  $V[1]$  and the bracket  $\{ , \}$  a Lie bracket.

In fact, the authors proved that a structure of quadratic Lie algebra  $(V, [ , ], b)$  on  $V$  is completely characterized by a 3-form  $I$ , such that  $\{I, I\} = 0$ . The relation between the Lie bracket and  $I$  is simply  $I(x, y, z) = b([x, y], z)$ , and the equation  $\{I, I\} = 0$  is the structure equation. A direct consequence of this construction is the existence of a cohomology on the space of forms, given by:

$$d\alpha = \{\alpha, I\}.$$

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This cohomology characterizes the problem of definition and deformation of quadratic Lie algebra structure on  $(V, b)$ .

In this paper, we first generalize this construction, defining the Pinczon bracket on the space  $\mathcal{C}(V)$  of cyclic multilinear forms on the shifted space  $V[1]$ . A Pinczon bracket on  $\mathcal{C}(V)$  is a Lie bracket  $(\Omega, \Omega') \mapsto \{\Omega, \Omega'\}$  such that  $\{\Omega, \Omega'\}$  is  $(k+k')$ -linear, if  $\Omega$  is  $(k+1)$ -linear, and  $\Omega'$   $(k'+1)$ -linear, and, for each linear form  $\alpha$ ,  $\{\alpha, \}$  is a derivation of the cyclic product.

As in [PU], there is a one-to-one correspondence between the set of Pinczon brackets on  $\mathcal{C}(V)$  and the set of non degenerated, bilinear symmetric forms  $b$  on  $V$ , and the correspondence is given through the Pinczon-Ushirobira formula:

$$\{\Omega, \Omega'\} = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega'.$$

On the other hand, it is well known that the space  $\bigotimes^+ V[1]$  is a cogebra for the comultiplication given by the deconcatenation map:

$$\Delta(x_1 \otimes \dots \otimes x_k) = \sum_{i=1}^{k-1} (x_1 \otimes \dots \otimes x_i) \bigotimes (x_{i+1} \otimes \dots \otimes x_k).$$

Moreover, the coderivations of  $\Delta$  are characterized by their Taylor series  $(Q_k)$  where  $Q_k : \bigotimes^k V[1] \rightarrow V[1]$ , and the bracket  $[Q, Q']$  of two such coderivation is still a coderivation.

Suppose now there is a symmetric non degenerated form  $b$  on  $V$ . Denotes  $B$  the corresponding form, but on  $V[1]$ . There is a bijective map between  $\mathcal{C}(V)$  and the space  $\mathcal{D}_B$  of  $B$ -quadratic coderivation  $Q$ , given by the formula:

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}).$$

This map is an isomorphism of Lie algebra:  $\{\Omega_Q, \Omega_{Q'}\} = \Omega_{[Q, Q']}$ .

With this construction, the notion of quadratic associative algebra, respectively quadratic associative algebra up to homotopy does coincide with the notion of Pinczon algebra structure on  $\mathcal{C}(V)$ . This gives also an explicit way to refine the Hochschild cohomology defined by the algebra structure.

Moreover, the subspace  $\mathcal{C}_{vsp}(V)$  of cyclic, vanishing on shuffle products forms is a Lie subalgebra of  $\mathcal{C}(V)$ . The restriction to this subalgebra of the above construction gives us the notion of quadratic commutative algebra (up to homotopy): it is a Pinczon algebra structure on  $\mathcal{C}_{vsp}(V)$ . Similarly, one refines the Harrison cohomology associated to commutative algebras.

A natural quotient of  $(\mathcal{C}(V), \{ , \})$  is the Lie algebra  $(\mathcal{S}, \{ , \})$  of totally symmetric multilinear forms on  $V[1]$ . This allows us to refine the notion of quadratic Lie algebra (up to homotopy), and the corresponding Chevalley cohomology.

Considering now bi-symmetric multilinear forms on  $V[1]$ : *i.e.* separately symmetric in their two last variables and in all their other variables, and extending canonically the Pinczon bracket to a bracket  $\{ , \}^+$  on this space of forms, we can define similarly quadratic pre-Lie algebra structures on  $(V, b)$ , and the corresponding pre-Lie cohomology.

Finally, we study the natural example of the space of  $n \times n$  matrices. An unpublished preprint ([A]) contains a part of these results.

## 2. CYCLIC FORMS

### 2.1. Koszul's rule.

In this paper,  $V$  is a finite dimensional graded vector space, on a characteristic 0 field  $\mathbb{K}$ . Denote  $|x|$  the degree of a vector  $x$  in  $V$ .

First recall the sign rule due to Koszul (see [Ko]). For any relation between quantities using letters representing homogeneous objects, in different orderings, it is always understood that there is an implicit  $+$  sign in front of the first term. For each other term, if  $\sigma$  is the permutation of the letters between the first quantity and the considered term, there is the implicit sign  $\varepsilon_{|letters|}(\sigma)$  (the sign of  $\sigma$ , taking into account only positions of the odd degree letters) in front of it.

As usual,  $V[1]$  is the space  $V$  with a shifted degree. If  $x$  is homogeneous, its degree in  $V[1]$  is  $\deg(x) = |x| - 1$ . Note simply  $x$  for  $\deg(x)$  when no confusion is possible. Very generally, we use a small letter for each mapping defined on  $V$ , and capital letter for the 'corresponding' mapping, defined on  $V[1]$ . Let us define now these corresponding mappings.

For any  $k$ , define a ' $k$ -cocycle'  $\eta$  by putting  $\eta(x_1, \dots, x_k) = (-1)^{\sum_{j \leq k} (k-j)x_j}$ . Then, if  $\varepsilon(\sigma)$  is the sign of the permutation  $\sigma$  in  $\mathfrak{S}_k$ ,

$$\eta(x_{\sigma(1)}, \dots, x_{\sigma(k)})\eta(x_1, \dots, x_k) = \varepsilon(\sigma)\varepsilon_{|x|}(\sigma)\varepsilon_x(\sigma).$$

If  $q$  is a  $k$ -linear mapping from  $V^k$  into a graded vector space  $W$ , define the associated  $k$ -linear mapping from  $V[1]^k$  into  $W[1]$ , with  $\deg(Q) = |q| + k - 1$ , by

$$Q(x_1, \dots, x_k) = \eta(x_1, \dots, x_k)q(x_1, \dots, x_k).$$

Preferring to keep the 0 degree for scalars, if  $\omega$  is a  $k$ -linear form on  $V$ , the same formula associates to  $\omega$  a form  $\Omega$ , on  $V[1]$ , with degree  $\deg(\Omega) = |\omega| + k$ .

This shift of degree modifies the symmetry properties of these mappings. For instance, if  $q$  is  $\sigma$ -invariant, then  $Q$  is  $\sigma$  skew-invariant.

## 2.2. Pinczon bracket.

It is defined on cyclic forms.

**Definition 2.1.** *Let  $\Omega$  be a  $(k+1)$ -linear form,  $\Omega$  is a cyclic form on  $V[1]$ , if it satisfies:*

$$\Omega(x_{k+1}, x_1, \dots, x_k) = \Omega(x_1, \dots, x_{k+1}).$$

Denote  $\mathcal{C}(V)$  the space of cyclic forms on  $V[1]$ .

Let  $\sigma \in \mathfrak{S}_k$  and  $\Omega$   $k$ -linear. Put  $(\Omega^\sigma)(x_1, \dots, x_k) = \Omega(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)})$ . Denote  $Cycl$  the subgroup of  $\mathfrak{S}_k$  generated by the cycle  $(1, 2, \dots, k)$ . Put

$$\Omega^{Cycl} = \sum_{\tau \in Cycl} \Omega^\tau, \quad \text{and} \quad A \odot B = (A \otimes B)^{Cycl}.$$

It is easy to prove that the so defined cyclic product is commutative, but non associative.

**Definition 2.2.** *A Pinczon bracket  $\{ , \}$  on the space  $\mathcal{C}(V)$  of cyclic multilinear forms on a graded space  $V[1]$  is a bilinear map such that*

1. *If  $\mathcal{C}^k$  is the space of  $k$ -linear cyclic forms,  $\{\mathcal{C}^{k+1}, \mathcal{C}^{k'+1}\} \subset \mathcal{C}^{k+k'}$ ,*
2.  *$\mathcal{C}(V)$ , equipped with  $\{ , \}$  is a graded Lie algebra, with center  $\mathcal{C}^0$ ,*
3. *for any linear form  $\alpha$ ,  $\{\alpha, \}$  is a derivation: if  $\beta_1, \dots, \beta_k$  are linear,*

$$\{\alpha, (\beta_1 \otimes \dots \otimes \beta_k)^{Cycl}\} = \sum_j \{\alpha, \beta_j\} (\beta_1 \otimes \dots \otimes \hat{\beta}_j \otimes \dots \otimes \beta_k)^{Cycl},$$

Now,

**Proposition 2.3.** 1. *There is a bijective map between the set  $\mathcal{P}$  of Pinczon bracket on  $\mathcal{C}(V)$  and the set  $\mathcal{B}$  of degree 0, symmetric, non degenerated bilinear forms  $b$  on  $V$ ,*

2. *Let  $b$  be in  $\mathcal{B}$ , and  $(e_i)$  a basis for  $V$ , if  $(e'_i)$  is the basis defined by  $b(e_i, e'_j) = \delta_{ij}$ , then the Pinczon bracket associated to  $b$  is:*

$$\{\Omega, \Omega'\} = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega'.$$

*Proof.* Let  $\{ , \}$  be a Pinczon bracket on  $\mathcal{C}(V)$ , then  $\{\mathcal{C}^0, \mathcal{C}(V)\} = 0$ , and  $\{\mathcal{C}^1, \mathcal{C}^1\} \subset \mathbb{K}$ . The bracket defines a degree 0 bilinear, antisymmetric form  $B^*$  on  $\mathcal{C}^1 = (V[1])^*$ . As above this defines a symmetric bilinear form  $b^*$  on the space  $V^* = (V[1])^*[1]$ .

Suppose that for some  $\alpha$  in  $\mathcal{C}^1$ ,  $B^*(\alpha, \mathcal{C}^1) = \{\alpha, \mathcal{C}^1\} = 0$ . Since  $\{\alpha, \cdot\}$  is a derivation,  $\{\alpha, \mathcal{C}^k\} = 0$ , and  $\alpha$  is a central element. Thus  $\alpha = 0$ ,  $b^*$  is non degenerated and allows to identify  $V^*$  and  $V$  and  $b^*$  to a non degenerated bilinear symmetric form  $b$  on  $V$ .

Let  $B^*$  be a skewsymmetric bilinear form on  $(V[1])^*$ . For any basis  $(e_i)$  of  $V[1]$  there are vectors  $e'_i$  such that:

$$B^*(\alpha, \beta) = \sum_i \iota_{e_i} \alpha \otimes \iota_{e'_i} \beta = \sum_i \alpha(e_i) \beta(e'_i) = \sum_i \iota_{e_i} \alpha \odot \iota_{e'_i} \beta.$$

Coming back to  $V^*$  this means, if all the objects are homogeneous,

$$b^*(\alpha, \beta) = \sum_i (-1)^{|\beta|} \alpha(e_i) \beta(e'_i).$$

Identify  $V^*$  to  $V$  by defining, for any  $\gamma$  in  $V^*$ , the vector  $x_\gamma$  in  $V$  such that  $\alpha(x_\gamma) = b^*(\alpha, \gamma)$ , for any  $\alpha$  in  $V^*$ . Then  $e'_i = x_{\epsilon_i}$ , where  $(\epsilon_i)$  is the dual basis of  $(e_i)$ . Therefore  $(e'_i)$  is a basis for  $V$ , and  $b(e'_j, e_i) = \delta_{ij}$ .

Consider now the bracket:

$$\{\Omega, \Omega'\}_P = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega',$$

where  $\Omega$  is a  $k+1$ -linear cyclic form and  $\Omega'$  a  $k'+1$ -linear one. In a following section, we shall prove this bracket defines a graded Lie algebra structure on  $\mathcal{C}(V)$ . It is clear that, for any  $\alpha$  and  $\beta_j$  linear,

$$\{\alpha, (\beta_1 \otimes \dots \otimes \beta_k)^{Cycl}\}_P = \sum_j \{\alpha, \beta_j\}_P (\beta_1 \otimes \dots \otimes \hat{\beta}_j \dots \otimes \beta_k)^{Cycl}.$$

Moreover the center of the Lie algebra  $(\mathcal{C}(V), \{\cdot, \cdot\}_P)$  is  $\mathcal{C}^0$ . In other word,  $\{\cdot, \cdot\}_P$  is a Pinczon bracket.

If  $k+k' \leq 0$ ,  $\{\Omega, \Omega'\} = \{\Omega, \Omega'\}_P$ . Suppose by induction this relation holds for  $k+k' < N$  and consider  $\Omega$  and  $\Omega'$  such that  $k+k' = N$ . For any  $i$ ,

$$\{\epsilon_i, \{\Omega, \Omega'\}\} = -\{\Omega, \{\Omega', \epsilon_i\}_P\}_P - \{\Omega', \{\epsilon_i, \Omega\}_P\}_P = \{\epsilon_i, \{\Omega, \Omega'\}_P\}_P = \iota_{e'_i} \{\Omega, \Omega'\}_P$$

On the other hand, if  $\{\Omega, \Omega'\} = \sum_\beta (\beta_1 \otimes \dots \otimes \beta_{k+k'})^{Cycl}$ ,

$$\{\epsilon_i, \{\Omega, \Omega'\}\} = \sum_{\beta, j} \beta_j(e'_i) (\beta_1 \otimes \dots \otimes \hat{\beta}_j \dots \otimes \beta_{k+k'})^{Cycl} = \iota_{e'_i} \{\Omega, \Omega'\}.$$

This proves the existence and unicity, and gives the form of the Pinczon bracket associated to the symmetric, non degenerated bilinear form  $b$  on  $V$ . □

### 3. CODIFFERENTIAL

### 3.1. General construction.

The deconcatenation  $\Delta$  is a natural comultiplication in the tensor algebra  $\bigotimes^+ V[1]$ :

$$\Delta(x_1 \otimes \dots \otimes x_k) = \sum_{r=1}^{k-1} (x_1 \otimes \dots \otimes x_r) \bigotimes (x_{r+1} \otimes \dots \otimes x_k).$$

The space  $\mathcal{D}$  of coderivations of  $\Delta$  is a natural graded Lie algebra for the commutator. It is well known (see for instance [K, LM]) that any multilinear mapping  $Q$  can be extended in an unique way into a coderivation  $D_Q$  of  $\Delta$ , and conversely any coderivation  $D$  has an unique form  $D = \sum_k D_{Q_k}$ . Thus the space of multilinear mappings is a graded Lie algebra for the bracket  $[Q, Q'] = Q \circ Q' - Q' \circ Q$ , where, if  $x_{[a,b]} = x_a \otimes x_{a+1} \otimes \dots \otimes x_b$ ,

$$Q \circ Q'(x_{[1, k+k'-1]}) = \sum_{r=0}^{k-1} Q(x_{[1,r]}, Q'(x_{[r+1, r+k']}, x_{[r+k'+1, k+k'-1]}).$$

### 3.2. Relation with the Pinczon bracket.

Consider a vector space  $V$  equipped with a symmetric, non degenerated bilinear form  $b$ , with degree 0, denote  $B$  the associated form on  $V[1]$ . For any linear map  $Q : V[1]^k \rightarrow V[1]$ , define the  $(k+1)$ -linear form:

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}),$$

and let us say that  $Q$  is  $B$ -quadratic if  $\Omega_Q$  is cyclic. Denote  $\mathcal{D}_B$  the space of  $B$ -quadratic multilinear maps.

The fundamental examples of cyclic maps are mappings associated to a Lie bracket or an associative multiplication on  $V$ . More precisely, if  $(x, y) \mapsto q(x, y)$  is any internal law, with degree 0, and  $Q(x, y) = (-1)^x q(x, y)$ , then  $\Omega_Q$  is cyclic if and only if:

$$b(q(x, y), z) = b(x, q(y, z)),$$

if and only if  $(V, q, b)$  is a quadratic algebra.

Remark that if  $q$  is a Lie bracket, then  $\Omega_Q$  is symmetric. Now, if  $q$  is a commutative (and associative) product, then  $\Omega_Q$  is vanishing on the image of the shuffle product on the 2 first variables:

$$sh_{(1,1)}(x_1 \otimes x_2) = (x_1 \otimes x_2) + (x_2 \otimes x_1).$$

**Proposition 3.1.** *The space  $\mathcal{D}_B$  of  $B$ -quadratic maps  $Q$  is a Lie subalgebra of  $\mathcal{D}$ .*

*The space  $\mathcal{C}(V)$  of cyclic forms, equipped with the Pinczon bracket is a graded Lie algebra, isomorphic to  $\mathcal{D}_B$ .*

*Proof.* For any sequence  $I = \{i_1, \dots, i_k\}$  of indices, denote  $x_I$  the tensor  $x_{i_1} \otimes \dots \otimes x_{i_k}$ .

Suppose  $Q$  is  $k$ -linear,  $Q'$   $k'$ -linear, both  $B$ -quadratic. Thus:

$$\begin{aligned} B(Q(x_{[1,k]}), Q'(x_{[k,k+k']})) &= -B(Q'(Q(x_{[1,k]}), x_{[k,k+k'-1]}), x_{k+k'}) \\ &= B(Q(Q'(x_{[k,k+k']}), x_{[1,k-1]}), x_k). \end{aligned}$$

Therefore:

$$\begin{aligned} B(Q \circ Q'(x_{[2,k+k']}), x_1) &= \sum_{1 \leq r} B(Q(x_{[2,r]}, Q'(x_{[r,r+k']}), x_{[r+k',k+k']}), x_1) \\ &= \sum_{1 \leq r < k} B(Q(x_{[1,r]}, Q'(x_{[r,r+k']}), x_{[r+k',k+k'-1]}), x_{k+k'}) - B(Q'(Q(x_{[1,k]}), x_{[k,k+k'-1]}), x_{k+k'}) \end{aligned}$$

Or

$$B([Q, Q'](x_{[2,k+k']}), x_1) = B([Q, Q'](x_{[1,k+k'-1]}), x_{k+k'})$$

This means  $[Q, Q']$  is  $B$ -quadratic,  $\mathcal{D}_B$  is a Lie subalgebra of  $\mathcal{D}$ .

Now  $\Omega_Q$  (resp.  $\Omega_{Q'}$ ) is a  $k+1$ -linear (resp.  $k'+1$ -linear) cyclic form, and

$$\begin{aligned} \{\Omega_Q, \Omega_{Q'}\}(x_1, \dots, x_{k+k'}) &= \left( \sum_i \iota_{e_i} \Omega_Q \otimes \iota_{e'_i} \Omega_{Q'} \right)^{Cycl} (x_1, \dots, x_{k+k'}) \\ &= \sum_{\sigma \in Cycl} \sum_i B(Q(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)}), e_i) B(Q'(x_{\sigma^{-1}(k+1)}, \dots, x_{\sigma^{-1}(k+k')}), e'_i) \\ &= \sum_{\sigma \in Cycl} B(Q(x_{\sigma^{-1}([1,k])}, Q'(x_{\sigma^{-1}([k+1,k+k'])})). \end{aligned}$$

Consider a term in this sum, such that  $k+k'$  belongs to  $\sigma^{-1}([1, k])$ . This term is:

$$B(Q(x_{[r+k'+1,k+k']}, x_{[1,r]}), Q'(x_{[r+1,r+k']})) = B(Q(x_{[1,r]}, Q'(x_{[r+1,r+k']}), x_{[r+k'+1,k+k'-1]}), x_{k+k'}).$$

The sum of all these terms is just  $B((Q \circ Q')(x_{[1,k+k'-1]}), x_{k+k'})$ .

Similarly, if  $k+k'$  is in  $\sigma^{-1}([k+1, k+k'])$ , we get:

$$B(Q(x_{[r+1,r+k']}), Q'(x_{[r+k+1,k+k']}, x_{[1,r]})) = -B(Q'(x_{[1,r]}, Q(x_{[r+1,r+k]}), x_{[r+k+1,k+k'-1]}), x_{k+k'}),$$

and the corresponding sum is  $-B((Q' \circ Q)(x_{[1,k+k'-1]}), x_{k+k'})$ .

This proves:

$$\{\Omega_Q, \Omega_{Q'}\} = \Omega_{[Q, Q']},$$

and the proposition, since  $Q \mapsto \Omega_Q$  is bijective. □

Let us now study in a more detailed way different cases, when  $Q$  corresponds to an associative, or a commutative or a Lie, or a pre-Lie structure, or to an up to homotopy such structure.



## 4. ASSOCIATIVE PINCZON ALGEBRAS

## 4.1. Associative quadratic algebras.

Suppose now  $q$  is a degree 0 associative product, and  $b$  is invariant, then the associated coderivation  $Q$  of  $\Delta$ , with degree 1 is the Bar resolution of the associative algebra  $(V, q)$ . The associativity of  $q$  is equivalent to the relation  $[Q, Q] = 0$ .

More generally, a structure of  $A_\infty$  algebra (or associative algebra up to homotopy) on the space  $V$  is a degree 1 coderivation  $Q$  of  $\Delta$  on  $\otimes^+ V[1]$ , such that  $[Q, Q] = 0$ . With this last relation, the Pinczon coboundary  $d_P : \Lambda \mapsto \{\Omega_Q, \Lambda\}$  is a degree 1 differential on the (graded) Lie algebra  $\mathcal{C}(V)$ . The corresponding cohomology is the Pinczon cohomology of cyclic forms. Write also  $\Omega_{d_P Q} = d_P \Omega_Q$ .

**Definition 4.1.** *An associative Pinczon algebra  $(\mathcal{C}(V), \{ , \}, \Omega)$  is a Pinczon bracket  $\{ , \}$  on  $\mathcal{C}(V)$ , and a degree 3 form  $\Omega \in \mathcal{C}(V)$ , such that  $\{\Omega, \Omega\} = 0$ .*

If  $\Omega$  is trilinear, then an associative Pinczon algebra is simply a quadratic associative algebra  $(V, b, q)$ , where  $b$  is the symmetric non degenerated form coming from the Pinczon bracket, and  $q$  the bilinear mapping associated to  $Q$  such that  $\Omega = \Omega_Q$ .

**Proposition 4.2.** *Let  $(\mathcal{C}(V), \{ , \}, \Omega)$  be an associative Pinczon algebra. Write  $\Omega = \Omega_Q$ ,  $Q = \sum_k Q_k$ , with  $Q_k : \otimes^k V[1] \rightarrow V[1]$ . Then  $Q$  is a structure of  $A_\infty$  algebra on  $V$ , and each  $Q_k$  is  $B$ -quadratic for the bilinear form  $B$  coming from the bracket.*

*Conversely, if  $(V, b)$  is a vector space with a non degenerated symmetric bilinear form, any  $B$ -quadratic structure  $Q$  of  $A_\infty$  algebra on  $V$  defines a unique structure of associative Pinczon algebra on  $V$ .*

## 4.2. Bimodules and Hochschild cohomology.

Suppose  $(V, q)$  is an associative algebra and  $M$  a bi-module. Then the Hochschild cohomology with value in  $M$  is a part of the Pinczon cohomology of a natural Pinczon algebra.

Consider first the semidirect product  $W = V \rtimes M$ , that is the vector space  $V \times M$ , equipped with the associative product  $q_W((x, a), (y, b)) = (q(x, y), (x \cdot b + a \cdot y))$ . The dual  $W^* = V^* \times M^*$  is now a  $(W, q_W)$ -bimodule, with:

$$((x, a) \cdot f)(z, c) = f((z, c)(x, a)), \quad (f \cdot (x, a))(z, c) = f((x, a)(z, c)),$$

or if  $f = (g, h) \in V^* \times M^*$ ,  $(x, a) \cdot (g, h) = (x \cdot g + a \cdot h, x \cdot h)$ ,  $(g, h) \cdot (x, a) = (g \cdot x + h \cdot a, h \cdot x)$ .

This defines a structure of algebra on the space  $\tilde{V} = W \times W^*$ , namely:  
 $\tilde{q}((x, a, g, h), (x', a', g', h')) = (xx', xa' + ax', x \cdot g' + g \cdot x' + a \cdot h' + h \cdot a', x \cdot h' + h \cdot x')$ ,  
 and a non degenerated symmetric bilinear form  $\tilde{b}$ ,

$$\tilde{b}((x, a, g, h), (x', a', g', h')) = g(x') + h(a') + g'(x) + h'(a).$$

Now  $(\tilde{V}, \tilde{b}, \tilde{q})$  is a quadratic associative algebra, it is the double semi-direct product of  $(V, q)$  by the bimodule  $M$ . As usual,  $\tilde{Q}$  is associated to  $\tilde{q}$ , after a shifting of degree.

Let now  $c : V^k \rightarrow M$ ,  $k$ -linear, with degree  $|c| = 2 - k$ , and identify  $C$  with  $\tilde{C} : \tilde{V}[1]^k \rightarrow \tilde{V}[1]$ , by putting:

$$\tilde{C}((x_1, a_1, g_1, h_1), \dots, (x_k, a_k, g_k, h_k)) = (0, C(x_1, \dots, x_k), \sum_j C_j(x_1, \dots, h_j, \dots, x_k), 0),$$

where  $C_j(x_1, \dots, h_j, \dots, x_k) = h_j(C(x_{j+1}, \dots, x_k, \cdot, x_1, \dots, x_{j-1})) \in V^*$ . A direct computation shows that  $\tilde{C}$  is  $\tilde{B}$ -quadratic. A direct computation gives:

$$d_P \tilde{Q} = [\tilde{Q}, \tilde{C}] = [\widetilde{Q, C}] = (\widetilde{d_H c})[1],$$

where  $d_H$  is the Hochschild coboundary operator on the bimodule  $M$ .

### Proposition 4.3.

*Let  $(V, q)$  be an associative algebra, and  $c \mapsto \Omega_{\tilde{C}}$  the map associating to any multilinear mapping  $c$  from  $V^k$  into  $M$ , with degree  $2 - k$ , the cyclic form  $\Omega_{\tilde{C}}$ . Then this map is a complex morphism between the Hochschild cohomology for the  $(V, q)$  bimodule  $M$  and the Pinczon cohomology of cyclic forms  $\mathcal{C}(\tilde{V})$  on  $\tilde{V}$ .*

## 5. COMMUTATIVE PINCZON ALGEBRAS

### 5.1. Commutative quadratic algebras.

Consider a quadratic associative algebra  $(V, b, q)$ , but suppose now  $q$  is commutative. Consider, as above the corresponding coderivation  $Q$ . It is now anticommutative, with degree 1, and seen as a map  $\underline{\otimes}^2 V[1] \rightarrow V[1]$ , where  $\underline{\otimes}^2 V[1]$  is the quotient of  $\otimes^2 V[1]$  by the 1, 1 shuffle product  $sh_{1,1}(x_1, x_2) = x_1 \otimes x_2 + x_2 \otimes x_1$ .

Recall that a  $p, q$  shuffle  $\sigma$  is a permutation  $\sigma \in \mathfrak{S}_{p+q}$  such that  $\sigma(1) < \dots < \sigma(p)$  and  $\sigma(p+1) < \dots < \sigma(p+q)$ . Denote  $Sh(p, q)$  the set of all such shuffles. Then the  $p, q$  shuffle product on  $\otimes^+ V[1]$  is

$$sh_{p,q}(x_{[1,p]}, x_{[p+1,p+q]}) = \sum_{\sigma \in Sh(p,q)} x_{\sigma^{-1}(1)} \otimes \dots \otimes x_{\sigma^{-1}(p+q)}.$$

Denote  $\underline{\otimes}^n(V[1])$  the quotient of  $\otimes^n V[1]$  by the sum of all the image of the maps  $sh_{p,n-p}$  ( $0 < p < n$ ), and  $\underline{x}_{[1,n]} = x_1 \underline{\otimes} \dots \underline{\otimes} x_n$  the class of  $x_1 \otimes \dots \otimes x_n$ , where the  $x_i$

belong to  $V[1]$ . Finally,  $\underline{\otimes}^+(V[1])$  is the sum of all  $\underline{\otimes}^n(V[1])$  ( $n > 0$ ).

On  $\underline{\otimes}^+V[1]$ , there is a Lie cobracket  $\delta$ , defined by

$$\delta(\underline{x}_{[1,n]}) = \sum_{j=1}^{n-1} \underline{x}_{[1,j]} \otimes \underline{x}_{[j+1,n]} - \underline{x}_{[j+1,n]} \otimes \underline{x}_{[1,j]}.$$

In fact  $\delta$  is well defined on the quotient and any coderivation  $Q$  of  $\delta$  is characterized by its Taylor expansion  $Q = \sum_k Q_k$  where each  $Q_k$  is a linear map from  $\underline{\otimes}^k V[1]$  into  $V[1]$  (see for instance [AAC1, BGHHW]).

**Definition 5.1.** *A structure of  $C_\infty$  algebra, or up to homotopy commutative algebra, on  $V$  is a degree 1 coderivation  $Q$  of  $\delta$ , on  $\underline{\otimes}^+V[1]$ , such that  $[Q, Q] = 0$ .*

Associated to the notion of vanishing on shuffle products mapping  $Q$ , there is a notion of vanishing on shuffle product forms  $\Omega$ .

**Definition 5.2.** *A  $(k+1)$ -linear cyclic form  $\Omega$  on the vector space  $V[1]$  is vanishing on shuffle product if, for any  $y, (x_1, \dots, x_k) \mapsto \Omega(x_1, \dots, x_k, y)$  is vanishing on shuffle product. Denote  $\mathcal{C}_{vsp}(V)$  the space of cyclic, vanishing on shuffle product multilinear forms on  $V[1]$ .*

**Proposition 5.3.** *Suppose  $\{ , \}$  is a Pinczon bracket on the space  $\mathcal{C}(V)$  of cyclic multilinear forms on  $V[1]$ . Then  $\mathcal{C}_{vsp}(V)$  is a Lie subalgebra of  $(\mathcal{C}(V), \{ , \})$ .*

*Proof.* In fact, the Pinczon bracket defines a non degenerate form  $b$  on  $V$ , thus  $B$  on  $V[1]$ , any form  $\Omega$  can be written as  $\Omega = \Omega_Q$ , with

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}).$$

Therefore  $\Omega$  is in  $\mathcal{C}_{vsp}(V)$  if and only if  $Q$  is vanishing on shuffle products. Now, in [AAC2] it is shown that if  $Q, Q'$  are vanishing on shuffle products, then  $[Q, Q']$  is also vanishing on shuffle products. This proves the proposition.  $\square$

**Definition 5.4.** *A commutative Pinczon algebra  $(\mathcal{C}(V), \{ , \}, \Omega)$  is a Pinczon bracket  $\{ , \}$  on  $\mathcal{C}(V)$ , and a degree 3 form  $\Omega \in \mathcal{C}_{vsp}(V)$ , such that  $\{\Omega, \Omega\} = 0$ .*

As for associative algebra, a commutative Pinczon algebra with  $\Omega$  trilinear is simply a quadratic commutative algebra  $(V, q, b)$ .

**Proposition 5.5.** *Let  $(\mathcal{C}(V), \{ \cdot, \cdot \}, \Omega)$  be an commutative Pinczon algebra. Write  $\Omega = \Omega_Q$ ,  $Q = \sum_k Q_k$ , with  $Q_k : \widehat{\otimes}^k V[1] \rightarrow V[1]$ . Then  $Q$  is a structure of  $C_\infty$  algebra on  $V$ , and each  $Q_k$  is  $B$ -quadratic for the bilinear form  $B$  coming from the bracket.*

*Conversely, if  $(V, b)$  is a vector space with a non degenerated symmetric bilinear form, any  $B$ -quadratic structure  $Q$  of  $C_\infty$  algebra on  $V$  defines an unique structure of commutative Pinczon algebra on  $V$ .*

### 5.2. (Bi)modules and Harrison cohomology.

Any module  $M$  on a commutative algebra  $(V, q)$  is a bimodule where right and left action are coinciding. Let now  $(V, q)$  be a commutative algebra, and  $M$  a  $(V, q)$ -module. Repeat the preceding construction of the quadratic associative algebra  $(\tilde{V}, \tilde{b}, \tilde{q})$ . Now  $(\tilde{V}, \tilde{b}, \tilde{q})$  is commutative.

As above, look now for a  $k$ -linear mapping  $c$  from  $V^k$  into  $M$ , with degree  $2 - k$  and vanishing on shuffle products, denote  $\tilde{C}$  the corresponding map  $\tilde{V}^k \rightarrow \tilde{V}$ , with degree 1, We saw that  $d_P \tilde{C} = [\tilde{Q}, \tilde{C}] = [\tilde{Q}, C] = \widetilde{d_{HC}[1]}$ . If we restrict this to vanishing on shuffle products map  $c$ , this is the Harrison coboundary  $\widetilde{d_{Ha}c}[1]$ .

**Proposition 5.6.** *Let  $(V, q)$  be a commutative algebra, and  $c \mapsto \Omega_{\tilde{C}}$  the map associating to any multilinear mapping  $c$  from  $V^k$  into  $M$ , with degree  $2 - k$ , the cyclic form  $\Omega_{\tilde{C}}$ . Then this map is a complex morphism between the Harrison cohomology for the  $(V, q)$  bimodule  $M$  and the Pinczon cohomology of cyclic forms  $\mathcal{C}(\tilde{V})$  on  $\tilde{V}$ .*

## 6. PINCZON LIE ALGEBRAS

### 6.1. Quadratic Lie algebras.

Suppose now  $(V, q)$  is a (graded) Lie algebra. Then the corresponding Bar resolution consists in replacing the space  $\widehat{\otimes}^+ V[1]$  of tensors by the subspace  $S^+(V[1])$  of symmetric tensors, spanned by the symmetric products  $x_1 \cdot \dots \cdot x_k = \sum_{\sigma \in \mathfrak{S}_k} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)}$ . Then, the natural comultiplication  $\Delta$  on  $S^+(V[1])$  is:

$$\Delta(x_1 \cdot \dots \cdot x_k) = \sum_{\substack{I \sqcup J = [1, k] \\ 0 < \#I < k}} x_{.I} \otimes x_{.J},$$

where, if  $I = \{i_1 < \dots < i_r\}$ ,  $x_{.I}$  means  $x_{i_1} \cdot \dots \cdot x_{i_r}$ .

As above, any coderivation  $Q$  of the comultiplication  $\Delta$  is characterized by its Taylor coefficients  $Q_k : S^k(V[1]) \rightarrow V[1]$ . The bracket of two such coderivations  $Q$ ,

$Q'$  becomes:

$$[Q, Q'](x_1 \cdots x_{k+k'-1}) = \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#J = k'}} Q(Q'(x_{.J}) \cdot x_{.I}) - \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#I = k}} Q'(Q(x_{.I}) \cdot x_{.J}),$$

Now to the Lie bracket  $q$  on  $V$  is associated a map  $Q : S^2(V[1]) \rightarrow V[1]$ , thus a degree 1 coderivation, still denoted  $Q$ , of  $\Delta$  and the Jacobi identity for  $q$  is equivalent to the relation  $[Q, Q] = 0$ . The corresponding cohomology is the Chevalley cohomology.

For any  $k$ , consider  $S^k(V[1])$  as a subspace of  $\otimes^k V[1]$ , with a projection  $Sym : \otimes^k V[1] \rightarrow S^k(V[1])$ :

$$Sym(x_1 \otimes \dots \otimes x_k) = x_1 \cdots x_k.$$

Denote  $L(\otimes^k V[1], V[1])$  the of linear maps  $Q : \otimes^k V[1] \rightarrow V[1]$ . By restriction to  $S^k(V[1])$ ,  $Q$  defines a symmetric map, denote it  $Q^{Sym}$ . Then  $Q^{Sym} = Q \circ Sym$ , and the space  $L(S^k(V[1]), V[1])$  of symmetric maps is a quotient of  $L(\otimes^k V[1], V[1])$ .

Similarly, the restriction of  $\Omega \in \mathcal{C}^{k+1}$  is a symmetric form, denote it  $\Omega^{Sym}$ :  $\Omega^{Sym} = \Omega \circ Sym$ . as above, the space  $\mathcal{C}|_S(V)$  of the restrictions of cyclic multilinear forms to  $S^+(V[1])$  is a quotient of  $\mathcal{C}(V)$ .

Suppose there is a Pinczon bracket  $\{ , \}$  on  $\mathcal{C}(V)$ , then any  $\Omega$  can be written  $\Omega = \Omega_Q$ , with  $Q$   $k$ -linear and  $B$ -quadratic. But, any element  $\sigma$  in  $\mathfrak{S}_{k+1}$  can be written in an unique way as a product  $\tau \circ \rho$  where  $\tau$  is in  $\mathfrak{S}_k$ , viewed as a subgroup of  $\mathfrak{S}_{k+1}$  and  $\rho$  in  $Cycl$ . Therefore, with our notation,

$$\Omega^{Sym} = (\Omega_Q)^{Sym} = (\Omega_{Q^{Sym}})^{Cycl} = (k+1)\Omega_{Q^{Sym}}.$$

**Proposition 6.1.** *The bracket defined on  $L(\otimes^+ V[1], V[1])$  by the commutator of coderivations in  $(\otimes^+ V[1], \Delta)$ , induces a well defined bracket on  $L(S^+(V[1]), V[1])$ , this bracket is the above commutator of coderivations in  $(S^+(V[1]), \Delta)$ :*

$$[Q^{Sym}, (Q')^{Sym}] = [Q, Q']^{Sym}.$$

Any Pinczon bracket  $\{ , \}$  on  $\mathcal{C}(V)$  induces a well defined bracket on the quotient  $\mathcal{C}|_S(V)$ , this bracket denoted  $\{ , \}$  is:

$$\{\Omega^{Sym}, (\Omega')^{Sym}\} | = (\{\Omega, \Omega'\})^{Sym} = \frac{k+k'}{(k+1)!(k'+1)!} \sum_i \iota_{e_i}(\Omega^{Sym}) \cdot \iota_{e'_i}((\Omega')^{Sym}).$$

*Proof.* The first assertion is a simple computation:

$$(Q \circ Q')(x_{[1, k+k'-1]}) = \sum_{r=1}^{k-1} Q(x_{[1, r-1]} \otimes Q'(x_{[r, r+k'-1]}) \otimes x_{[r+k', k+k'-1]}).$$

Then,

$$(Q \circ Q')^{Sym}(x_{.[1, k+k'-1]}) = \sum_{\substack{r \\ \sigma \in \mathfrak{S}_{k+k'-1}}} Q(x_{\sigma^{-1}([1, r-1])}, Q'(x_{\sigma^{-1}([r, r+k'-1])}), x_{\sigma^{-1}([r+k', k+k'-1])})$$

Fix  $r$  and  $\sigma$ , put  $J = \sigma^{-1}([r, r+k'-1]) = \{j_1 < \dots < j_{k'}\}$ , and  $j_{\tau^{-1}(t)} = \sigma^{-1}(r+t-1)$  ( $1 \leq t \leq k'$ ). This define  $\sigma \in \mathfrak{S}_{k'}$ . Similarly, put  $I = [1, k+k'-1] \setminus J$ ,  $\{0\} \cup I = \{i_1 < \dots < i_k\}$  and define  $\rho \in \mathfrak{S}_k$  by:

$$i_{\rho^{-1}(t)} = \sigma^{-1}(t) \quad (1 \leq t \leq r-1), \quad i_{\rho^{-1}(r)} = 0, \quad i_{\rho^{-1}(t)} = \sigma^{-1}(t-1) \quad (r+1 \leq t \leq k).$$

The correspondence  $(r, \sigma) \mapsto (r, \tau, \rho)$  is one-to-one and, suming up, we get:

$$(Q \circ Q')^{Sym}(x_{.[1, k+k'-1]}) = \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#J = k'}} Q^{Sym}((Q')^{Sym}(x_{.J}) \cdot x_{.I}).$$

The commutator of coderivation of  $(S^+(V[1]), \Delta)$  is the quotient bracket:

$$[Q, Q']^{Sym} = [Q^{Sym}, (Q')^{Sym}].$$

Let now  $\{, \}$  be a Pinczon bracket on the space  $\mathcal{C}(V)$ . Thus any cyclic form  $\Omega$  is written as  $\Omega_Q$ . Then:

$$\{\Omega, \Omega'\}^{Sym} = \{\Omega_Q, \Omega_{Q'}\}^{Sym} = \Omega_{[Q, Q']}^{Sym} = (k+k')\Omega_{[Q, Q']^{Sym}} = (k+k')\Omega_{[Q^{Sym}, (Q')^{Sym}]}.$$

The last bracket is the commutator of coderivations in  $S^+(V[1])$ , thus

$$\{\Omega, \Omega'\}^{Sym}(x_{.[1, k+k']}) = (k+k') \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#J = k'}} B(Q^{Sym}(x_{.I}), (Q')^{Sym}(x_{.J})).$$

On the other hand,

$$\iota_{e_i} \left( \Omega_Q^{Sym} \right) (x_{.[1, k]}) = (k+1)B(Q^{Sym}(x_{.[1, k]}), e_i) = (k+1) (\iota_{e_i} \Omega)^{Sym} (x_{.[1, k]}).$$

Therefore

$$\begin{aligned} \sum_i \iota_{e_i} \left( \Omega_Q^{Sym} \right) \cdot \iota_{e'_i} \left( \Omega_{Q'}^{Sym} \right) (x_{.[1, k+k']}) &= \sum_i \left( \iota_{e_i} \left( \Omega_Q^{Sym} \right) \otimes \iota_{e'_i} \left( \Omega_{Q'}^{Sym} \right) \right)^{Sym} (x_{.[1, k+k']}) \\ &= (k+1)!(k'+1)! \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#J = k'}} B(Q^{Sym}(x_{.I}), (Q')^{Sym}(x_{.J})) \end{aligned}$$

This achieves the proof.  $\square$

Explicitely, if the forms  $\Omega$  and  $\Omega'$  are symmetric,  $\Omega^{Sym} = (k+1)\Omega$  and the Pinczon bracket on the quotient becomes the bracket defined in [PU, DPU]:

$$\{\Omega, \Omega'\} = (k+k') \sum_i \iota_{e_i} \Omega \cdot \iota_{e'_i} \Omega'.$$

## 6.2. Quadratic $L_\infty$ algebras.

As above,

**Definition 6.2.** *A Pinczon Lie algebra  $(\mathcal{C}(V), \{ , \}, \Omega)$  is a Pinczon bracket  $\{ , \}$  on  $\mathcal{C}(V)$ , and a degree 3 symmetric form  $\Omega \in \mathcal{C}(V)^{Sym}$ , such that  $\{\Omega, \Omega\} = 0$ .*

Since a structure of  $L_\infty$  algebra (or Lie algebra up to homotopy) on  $V$  is a degree 1 coderivation  $Q$  of  $(S^+(V[1]), \Delta)$ , such that the commutator  $[Q, Q]$  vanishes. With the preceding computations, a Pinczon Lie algebra is in fact a quadratic  $L_\infty$  algebra.

**Proposition 6.3.** *Let  $(\mathcal{C}(V), \{ , \}, \Omega)$  be a Pinczon Lie algebra. Write  $\Omega = \Omega_Q$ ,  $Q = \sum_k Q_k$ , with  $Q_k : S^k(V[1]) \rightarrow V[1]$ . Then  $Q$  is a structure of  $L_\infty$  algebra on  $V$ , and each  $Q_k$  is  $B$ -quadratic for the bilinear form  $B$  coming from the bracket.*

*Conversely, if  $(V, b)$  is a vector space with a non degenerated symmetric bilinear form, any  $B$ -quadratic structure  $Q$  of  $L_\infty$  algebra on  $V$  defines an unique structure of commutative Pinczon algebra on  $V$ .*

## 6.3. Modules and Chevalley cohomology.

Let  $(V, q)$  be a Lie algebra, and  $M$  a  $(V, q)$ -module. To refine the corresponding Chevalley coboundary operator  $d_{Ch}$ , build, as above, the double semidirect product of  $V$  by  $M$ .

First, consider the semidirect product  $W = V \rtimes M$ , that is the vector space  $V \oplus M$ , equipped with the Lie bracket  $q_W((x, a), (y, b)) = ([x, y], x \cdot b - y \cdot a)$ ; its dual  $W^*$  is also a  $(W, q_W)$ -module.

Then  $\tilde{V} = W \times V \times W^*$  is a quadratic Lie algebra for the bracket

$$\tilde{q}((x, a, g, h), (x', a', g', h')) = ([x, x'], x \cdot a' - x' \cdot a, x \cdot g' - x' \cdot g + a \cdot h' - a' \cdot h, x \cdot h' - x' \cdot h),$$

and the non degenerated symmetric bilinear form  $\tilde{b}$

$$\tilde{b}((x, a, g, h), (x', a', g', h')) = g(x') + h(a') + g'(x) + h'(a).$$

A direct computation (see [BB, MR]) shows that  $(\tilde{V}, \tilde{b}, \tilde{q})$  is a quadratic Lie algebra, the double semidirect product of  $(V, q)$  by its module  $M$ .

Look now for a skew-symmetric  $k$ -linear mapping  $c$  from  $V^k$  into  $M$ , with degree  $|c| = 2 - k$ . Associate to it the mapping:

$$\tilde{C}((x_1, a_1, g_1, h_1), \dots, (x_k, a_k, f_k, g_k)) = (0, C(x_1, \dots, x_k), \sum_{j=1}^k C_j(x_1, \dots, h_j, \dots, x_k), 0).$$

Clearly,  $\tilde{C}$  is totally symmetric from  $\tilde{V}[1]^k$  into  $\tilde{V}[1]$ . More precisely  $\widetilde{C^{Sym}} = \tilde{C}^{Sym}$ .

Then,

$$[\tilde{Q}, \tilde{C}]^{Sym} = \widetilde{[Q, C]^{Sym}} = [\widetilde{Q}, \widetilde{C}]^{Sym} = \widetilde{d_{Ch}c[1]}.$$

Now, if  $d_P$  is the Pinczon coboundary operator, defined by:

$$\{\Omega_{\tilde{Q}}, \Omega_{\tilde{C}}\} = \Omega_{d_P \tilde{C}},$$

this can be written  $d_P \tilde{C} = (2 + k)\widetilde{d_{Ch}c[1]}$ .

**Proposition 6.4.** *Let  $(V, q)$  be a Lie algebra, and  $c \mapsto \Omega_{\tilde{C}}$  the map associating to any multilinear skewsymmetric mapping  $c$  from  $V^k$  into  $M$ , with degree  $2 - k$ , the symmetric form  $\Omega_{\tilde{C}}$ . Then this map is a complex morphism between the Chevalley cohomology for the  $(V, q)$  module  $M$  and the Pinczon cohomology of symmetric forms  $\mathcal{C}(\tilde{V})|_S$  on  $\tilde{V}$ .*

## 7. PINCZON PRE-LIE ALGEBRAS

### 7.1. Quadratic pre-Lie algebras.

A left pre-Lie algebra  $(V, q)$  is a (graded) vector space with a product  $q$  such that:

$$q(x, q(y, z)) - q(q(x, y), z) = q(y, q(x, z)) - q(q(y, x), z).$$

Then the bracket  $[x, y] = q(x, y) - q(y, x)$  is a Lie bracket. Remark that any associative algebra  $(V, q)$  is a pre-Lie algebra.

A vector space  $M$  is a left  $(V, q)$ -module for the linear map  $x \otimes a \mapsto x \cdot a$ , if

$$q(x, y) \cdot a - x \cdot (y \cdot a) = q(y, x) \cdot a - y \cdot (x \cdot a) \quad (a \in M, x \in V).$$

A left module is a bi-module, if there is a linear map  $a \otimes x \mapsto a * x$  such that:

$$(a * x) * y - a * q(x, y) = (x \cdot a) * y - x \cdot (a * y) \quad (a \in M, x, y \in V).$$

Then a direct computation proves

**Lemma 7.1.** *Let  $(V, q)$  a left pre-Lie algebra, then*

1. *The dual  $V^*$  of  $V$  is a left  $(V, q)$ -module, for:  $(x \cdot f)(y) = -f(q(x, y))$ .*
2. *Let  $M$  be a  $(V, q)$  bi-module then  $W = V \rtimes M = (V \oplus M, q_W)$ , where:*

$$q_W(x + a, y + b) = q(x, y) + x \cdot b + a * y$$

*is a left pre-Lie algebra, the semi-direct product of  $V$  by  $M$ .*



Chapoton and Livernet define in [CL] the notion of pre- $L_\infty$ -algebra. If  $V$  is a graded vector space, we consider the space  $S(V[1]) \otimes V([1])$ , generated by the tensors

$$x_{[1,k]} \otimes y = x_1 \cdots x_k \otimes y.$$

Put  $P^k = S^k(V[1]) \otimes V([1])$ , and  $P = \sum_{k \geq 0} P^k$ . On  $P$ , the coproduct  $\Delta$  is defined by  $\Delta(1 \otimes y) = 0$  and:

$$\Delta(x_{[1,k]} \otimes y) = \sum_{\substack{1 \leq j \leq k \\ I \sqcup J = [1,k] \setminus \{j\}}} (x_I \otimes x_j) \otimes (x_J \otimes y).$$

A linear map  $Q : P^k \rightarrow P^0$  extends to a coderivation, still denoted  $Q$  by:

$$Q(x_{[1,n]} \otimes y) = \sum_{\substack{I \sqcup J = [1,n] \\ \#J=k}} x_I \otimes Q(x_J \otimes y) + \sum_{\substack{1 \leq j \leq n \\ I \sqcup J = [1,n] \setminus \{j\} \\ \#J=k}} x_I \cdot Q(x_J \otimes x_j) \otimes y.$$

The commutator of two coderivations is a coderivation, and a map  $q : V \otimes V \rightarrow V$  is a left pre-Lie product if and only if the structure equation  $[Q, Q] = 0$  holds for the corresponding map  $Q : P^1 \rightarrow P^0$ . Thus:

**Definition 7.2.** *A structure of pre- $L_\infty$  algebra on  $V$  is a degree 1 coderivation  $Q$  of  $(P, \Delta)$ , such that  $[Q, Q] = 0$ .*

Now, a quadratic pre-Lie algebra  $(V, b, q)$  is a pre-Lie algebra  $(V, q)$  equipped with a symmetric, non degenerated, degree 0, bilinear form  $b$  such that

$$b(q(x, y), z) + b(y, q(x, z)) = 0, \quad \text{or} \quad B(Q(x, y), z) = B(Q(x, z), y).$$

**Example 7.3.** *Let  $(V, q)$  be a left pre-Lie algebra, consider the pre-Lie algebra, semidirect product  $W = V \rtimes V^*$ , with*

$$q_W(x + f, y + g) = q(x, y) + x \cdot g = q(x, y) - g(q(x, \cdot)).$$

*It is a quadratic pre-Lie algebra if we endow  $W$  by the canonical symmetric, non degenerated form  $b(x + f, y + g) = f(y) + g(x)$ .*

Generalizing, let us say that a coderivation  $Q = Q_0 + Q_1 + \dots$  of  $\Delta$  is  $B$ -quadratic if, for any  $k$ ,

$$B(Q_k(x_{[1,k]} \otimes y_1), y_2) = B(Q_k(x_{[1,k]} \otimes y_2), y_1).$$

Then a direct computation gives:

**Lemma 7.4.** *Let  $Q$  and  $Q'$  be two  $B$ -quadratic coderivations of  $\Delta$ . Then  $[Q, Q']$  is  $B$ -quadratic.*

A structure of quadratic pre- $L_\infty$  algebra on  $(V, b)$  is thus a  $B$ -quadratic coderivation  $Q$  of  $\Delta$  such that  $[Q, Q] = 0$ .

## 7.2. Pinczon bracket for pre-Lie algebras.

To a  $k + 1$ -linear coderivation  $Q$ , let us associate the form  $\Omega_Q$ :

$$\Omega_Q(x_{[1,k]} \otimes y_1 \cdot y_2) = B(Q(x_{[1,k]} \otimes y_1), y_2) + B(Q(x_{[1,k]} \otimes y_2), y_1).$$

This form is separately symmetric in its  $k$  first variables, and its 2 last variables. Define thus  $\mathcal{P}_k$  the space of such forms, and  $\mathcal{P}(V) = \sum_{k \geq 0} \mathcal{P}_k$ . An element of  $\mathcal{P}(V)$  is called a bi-symmetric form. Now it is possible to extend to  $\mathcal{P}(V)$  the Pinczon bracket  $\{ , \}$  associated to  $B$ . Indeed, a direct computation shows:

**Lemma 7.5.** *Let  $\mathfrak{g}$  be a Lie algebra and  $A$  a commutative algebra which is a right  $\mathfrak{g}$ -module such that, for any  $x \in \mathfrak{g}$ ,  $f \mapsto f \cdot x$  is a derivation of  $A$ . Then the formula:*

$$[f \otimes x, g \otimes y] = fg \otimes [x, y] + (f \cdot y)g \otimes x - f(g \cdot x) \otimes y$$

*defines a Lie bracket on  $A \otimes \mathfrak{g}$ .*

Denote  $\mathcal{S}(V) = (\mathcal{C}(V))^{\text{Sym}}$  the symmetric algebra of  $V$ , it is a commutative algebra for the symmetric product  $\cdot$ . On the other hand, by construction,  $\mathcal{C}^2$  is a Lie algebra for the Pinczon bracket  $\{ , \}$ , acting on  $\mathcal{S}$  through  $(\Omega, \alpha) \mapsto \{\Omega, \alpha\}$ . Then the properties of the Pinczon bracket assure that  $\mathcal{S}$  is a  $(\mathcal{S}^2, \{ , \})$ -module and the action is a derivation of  $\mathcal{S}$ . Finally remark that  $\mathcal{P}(V) = \mathcal{S}(V) \otimes \mathcal{C}^2$ , therefore:

**Corollary 7.6.** *Let  $(V, b)$  be a vector space with a symmetric, non degenerated bilinear form, then the space  $\mathcal{P}(V)$  of bi-symmetric forms on  $V$  is a Lie algebra for the Pinczon bracket:*

$$\{\Omega \otimes \alpha, \Omega' \otimes \alpha'\} = \Omega \cdot \{\alpha, \Omega'\} \otimes \alpha' + \Omega' \cdot \{\Omega, \alpha'\} \otimes \alpha + \Omega \cdot \Omega' \otimes \{\alpha, \alpha'\}.$$

This bracket is related to the commutator of  $B$ -preserving coderivations, since:

**Proposition 7.7.** *Suppose  $b$  is a symmetric non degenerated form on  $V$ , and  $Q, Q'$  two  $B$ -quadratic coderivations of  $\Delta$ . Consider the forms  $\Omega_Q, \Omega_{Q'}$  in  $\mathcal{P}(V)$ , then*

$$\{\Omega_Q, \Omega_{Q'}\} = 2\Omega_{[Q, Q']}.$$

*Proof.* Suppose  $\Omega_Q = \beta \otimes \alpha \in \mathcal{P}^k$ , and  $\Omega_{Q'} = \beta' \otimes \alpha' \in \mathcal{P}^{k'}$ , then

$$\begin{aligned} & \beta \cdot \beta' \otimes \{\alpha, \alpha'\}(x_{[1, k+k']}, y_{[1, 2]}) = \\ & = \sum_{\substack{i, I \sqcup J = [1, k+k'] \\ \#I = k}} (\Omega_Q(x_I \otimes e_i \cdot y_1) \Omega_{Q'}(x_J \otimes e'_i \cdot y_2) + \Omega_Q(x_I \otimes e_i \cdot y_2) \Omega_{Q'}(x_J \otimes e'_i \cdot y_1)) \\ & = 4 \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#I = k}} -B(Q'(x_J \otimes Q(x_I \otimes y_1)), y_2) + B(Q(x_I \otimes Q'(x_J \otimes y_1)), y_2) \end{aligned}$$

On the other hand,

$$\begin{aligned}
& \beta \cdot \{\alpha, \beta'\} \otimes \alpha'(x_{[1, k+k']}, y_{[1, 2]}) = \\
& = \sum_{\substack{j, I \sqcup J = [1, k+k'] \setminus \{j\} \\ \#I=k}} \sum_i \Omega_Q(x_I \otimes e_i \cdot x_j) \Omega_{Q'}(e'_i \cdot x_J \otimes y_1 \cdot y_2) \\
& = -4 \sum_{\substack{j, I \sqcup J = [1, k+k'] \setminus \{j\} \\ \#I=k}} B(Q'(Q(x_I \otimes x_j) \cdot x_J \otimes y_1), y_2).
\end{aligned}$$

And similarly for the last term  $\{\beta, \alpha'\} \cdot \beta' \otimes \alpha$  in the Pinczon bracket. Suming up, we get:

$$\begin{aligned}
\{\Omega_Q, \Omega_{Q'}\}(x_{[1, k+k']} \otimes y_{[1, 2]}) & \stackrel{\text{deg}}{=} 2B([Q, Q'](x_{[1, k+k']} \otimes y_1), y_2) + 2B([Q, Q'](x_{[1, k+k']} \otimes y_2), y_1) \\
& \stackrel{\text{deg}}{=} 2\Omega_{[Q, Q']}(x_{[1, k+k']} \otimes y_{[1, 2]}).
\end{aligned}$$

□

As for the other sort of algebras,

**Definition 7.8.** *A Pinczon pre-Lie algebra  $(\mathcal{P}(V), \{ , \}, \Omega)$  is a Pinczon bracket  $\{ , \}$  on  $\mathcal{C}(V)$ , extended to  $\mathcal{P}(V)$ , and a degree 3 bi-symmetric form  $\Omega \in \mathcal{P}(V)$ , such that  $\{\Omega, \Omega\} = 0$ .*

Since a structure of pre- $L_\infty$  algebra on  $V$  is a degree 1 coderivation  $Q$  of  $(P, \Delta)$ , such that the commutator  $[Q, Q]$  vanishes. With the preceding computations, a Pinczon pre-Lie algebra is in fact a quadratic pre- $L_\infty$  algebra.

**Proposition 7.9.** *Let  $(\mathcal{P}(V), \{ , \}, \Omega)$  be a Pinczon pre-Lie algebra. Write  $\Omega = \Omega_Q$ , and  $Q = \sum_k Q_k$ , with  $Q_k : P^k \rightarrow P^0$ . Then  $Q$  is a pre- $L_\infty$  algebra structure on  $V$ , and each  $Q_k$  is  $B$ -quadratic.*

*Conversely, if  $(V, b)$  is a vector space with a non degenerated symmetric bilinear form, any  $B$ -quadratic structure  $Q$  of pre- $L_\infty$  algebra on  $V$  defines an unique structure of Pinczon pre-Lie algebra on  $V$ .*

### 7.3. Pinczon and pre-Lie algebra cohomologies.

Suppose  $(V, q)$  is a left pre-Lie algebra and let  $Q$  be the coderivation of  $\Delta$  associated to  $q$ . Since  $[Q, Q] = 0$ ,  $d : C \mapsto [Q, C]$  is a coboundary operator.

Now, let  $M$  be a  $(V, q)$  bi-module. Let  $W = V \rtimes M$  be the semi-direct product of  $V$  by  $M$ . Any map  $c : \wedge^k V \otimes V \rightarrow M$  can be naturally extended to a map, still denoted  $c$ , from  $\wedge^k W \otimes W$  to  $W$ . If  $C$  is the corresponding coderivation, the map  $dc$ , where the coderivation corresponding to  $dc : \wedge^{k+1} W \otimes W \rightarrow W$  is  $[Q, C]$ , is the extension

of a map  $d_{pLC} : \wedge^{k+1}V \otimes V \rightarrow M$ . The operator  $d$  is the pre-Lie coboundary operator. In an unpublished work, Ridha Chatbouri computed explicitly this operator:

**Proposition 7.10.** *The cohomology of the pre-Lie algebra  $(V, q)$  is defined as follows. Let  $M$  be a  $(V, q)$  bi-module, the cohomology with value in  $M$  is given by the following operator: If  $c : \wedge^k V \otimes V \rightarrow M$  is a  $(k+1)$ -cochain, with degree  $|c|$ , then  $dc$  is explicitly:*

$$\begin{aligned} (-1)^{|c|} d_{pLC}(x_0 \wedge \dots \wedge x_k \otimes y) = & \\ & \sum_{i=0}^k (-1)^i c(x_0 \wedge \dots \hat{i} \dots \wedge x_k \otimes x_i) * y - \sum_{i=0}^k (-1)^i c(x_0 \wedge \dots \hat{i} \dots \wedge x_k \otimes q(x_i, y)) \\ & + \sum_{i < j} (-1)^{i+j} c([x_i, x_j] \wedge x_0 \wedge \dots \hat{i} \hat{j} \dots \wedge x_k \otimes y) + \sum_{i=0}^k (-1)^i x_i \cdot c(x_0 \wedge \dots \hat{i} \dots \wedge x_k \otimes y). \end{aligned}$$

In [Dz], A. Dzhumaldil'daev defined a coboundary operator  $d$  for right pre-Lie algebra, which is the same as the operator computed in the preceding proposition, modulo the change of side for pre-Lie axioms. Then he used this operator to compute a corresponding homology. The proof of the proposition is the inverse of the Dzhumaldil'daev proof.

**Remark 7.11.** *A left  $(V, q)$ -module is nothing else than a  $(V, [ , ], q)$ -module. However the symmetry of a pre-Lie cochain differs of the symmetry of a Lie cochain. Thus the cocycles are not the same. For instance we consider  $V = C_c^\infty(\mathbb{R})$ , with  $q(f, g) = fg'$ . Choose  $M = V$  and  $f \cdot g = q(f, g)$ . Then  $M$  is a left module. Put  $c(f, g) = fg$ . It is easy to verify it is a cocycle, but it is not skewsymmetric. In fact it is the coboundary of  $f \mapsto b(f)$ , with  $(b(f))(t) = tf(t)$ .*

More generally, if  $Q$  is a coderivation of  $\Delta$ , which is a pre- $L_\infty$  structure, then the operator  $C \mapsto [Q, C]$  is a coboundary operator. Let us call the corresponding cohomology the  $(V, Q)$  cohomology. Then

**Corollary 7.12.** *Suppose  $(V, b, q)$  is a quadratic pre-Lie algebra or, more generally,  $(\mathcal{P}(V), \{ , \}, \Omega)$  a Pinczon pre-Lie algebra. Then the operator  $d : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$  defined by  $d\Omega = \{\Omega_Q, \Omega\}$  is a coboundary operator and the corresponding cohomology coincides with the  $(V, Q)$ -cohomology.*

## 8. A NATURAL EXAMPLE

Recall that the infinitesimal deformations of an associative (resp. Lie, resp. pre-Lie) algebra  $(V, q)$  are described by the corresponding second cohomology group

(Hochschild, pre-Lie or Chevalley) of  $V$  (see [G, NR]). Indeed, putting  $q_t = q + tc$ , with  $t^2 = 0$ , the associativity, Jacobi or pre-Lie relation are respectively equivalent to  $[Q, C] = 0$ . Therefore, if  $t^2 = 0$ ,  $(V + tV, q_t)$ , is an associative, Lie or pre-Lie algebra if and only if  $c$  is respectively a Hochschild, a Chevalley or a pre-Lie cocycle.

Such a deformation is trivial if there is a linear map  $a$  such that  $\varphi_t(x) = x + ta(x)$  satisfies  $q_t(\varphi_t(x), \varphi_t(y)) = \varphi_t(q(x, y))$ . With  $t^2 = 0$ , these conditions are equivalent to  $c = d_H a$ , resp.  $c = d_{Ch} a$ ,  $c = d_{pL} a$ . If the only infinitesimal deformations are the trivial ones, that is if the second Hochschild cohomology group  $H^2((V, q))$  vanishes, we say that the corresponding structure is infinitesimally rigid.

Suppose now  $(V, q)$  is an associative algebra. Therefore it is a pre-Lie algebra for the multiplication  $q$ , and a Lie algebra for the bracket  $[x, y] = q(x, y) - q(y, x)$ . Some of these structures can be rigid, and other can be not rigid. Let us study here the natural associative algebra  $M_n(\mathbb{K})$  of  $n \times n$  matrices on a characteristic zero field  $\mathbb{K}$ . Denote  $\mathfrak{gl}_n(\mathbb{K})$  the corresponding Lie algebra, and  $(M_n, q)$  the pre-Lie algebra. Remark that  $\mathfrak{gl}_n(\mathbb{K})$  is a direct product:  $\mathfrak{sl}_n(\mathbb{K}) \oplus \mathbb{K}id$ , where  $\mathfrak{sl}_n(\mathbb{K})$  is the space of traceless matrices. Put  $f(x) = \frac{1}{n}\text{tr}(x)id$ . Recall first some well-known results (see [CH] Chap IX.7, and use Hochschild-Serre sequence and Whitehead Lemmas [J]):

**Proposition 8.1.** *Let  $M$  be a  $M_n(\mathbb{K})$  bi-module, then  $H^k(M_n(\mathbb{K}), M) = 0$  for  $k > 0$ . Especially, if  $M = M_n(\mathbb{K})$ ,*

$$H^0(M_n(\mathbb{K})) = \mathbb{K}id, \quad H^k(M_n(\mathbb{K})) = 0 \quad \text{for } k > 0.$$

*Let  $M$  be a  $\mathfrak{gl}_n(\mathbb{K})$ -module, and  $k > 0$ , then:*

$$H^k(\mathfrak{gl}_n(\mathbb{K}), M) \simeq H^k(\mathfrak{sl}_n(\mathbb{K}), \mathbb{K}) \otimes M^{\mathfrak{gl}_n(\mathbb{K})} \oplus H^{k-1}(\mathfrak{sl}_n(\mathbb{K}), \mathbb{K}) \otimes (H^1(\mathbb{K}id, M))^{\mathfrak{gl}_n(\mathbb{K})}.$$

*Especially, if  $M = \mathfrak{gl}_n(\mathbb{K})$ , and  $\tilde{f}$  is the class of the projection  $f$ ,*

$$H^0(\mathfrak{gl}_n(\mathbb{K})) = \mathbb{K}id, \quad H^1(\mathfrak{gl}_n(\mathbb{K})) = \mathbb{K}\tilde{f}, \quad H^2(\mathfrak{gl}_n(\mathbb{K})) = 0.$$

Especially,  $M_n(\mathbb{K})$  and  $\mathfrak{gl}_n(\mathbb{K})$  are rigid. However the pre-Lie algebra  $V$  is not infinitesimally rigid:

**Lemma 8.2.** *The second cohomology group  $H^2(M_n, q)$  of the pre-Lie algebra  $(M_n, q)$  is not vanishing.*

*Proof.* Following [Dz], for any matrix  $a$ , consider  $c_a(x, y) = \frac{1}{n}\text{tr}(x)[y, a]$ . Then:

$$d_H c_a(x, y, z) = \frac{1}{n}(\text{tr}(y)x - \text{tr}(xy)id + \text{tr}(x)y)[z, a] = (d_H f)(x, y)[z, a].$$

Then  $d_{pL} c_a(x, y, z) = d_H c_a(x, y, z) - d_H c_a(y, x, z) = 0$ . But remark that  $Z^2(M_n(\mathbb{K})) = B^2(M_n(\mathbb{K})) = B^2(M_n, q)$ . Thus, if  $a$  is not a scalar matrix, there is  $z$  such that

$[z, a] \neq 0$ , and  $d_H c_a(e_{12}, e_{21}, z) = -[z, a] \neq 0$ ,  $c_a$  is not in  $B^2(M_n, q)$ . □

Let  $(x, y) \mapsto q(x, y)$  any algebraic structure on a space  $V$  and  $c_k : V \otimes V \rightarrow V$  ( $k \geq 1$ ). If for any  $t$ ,  $q_t = q + \sum_{k>0} t^k c_k$  endows  $V \otimes \mathbb{K}[[t]]$  with the same sort of structure, we say that  $q_t$  is a formal deformation of  $q$ . If there is  $\varphi_t = id + \sum_{k>0} t^k b_k$ , such that  $q_t(\varphi_t(x), \varphi_t(y)) = \varphi_t(q(x, y))$ , we say that  $q_t$  is trivial. If there is a non trivial deformation  $q_t$ , we say that  $(V, q)$  is not rigid. If moreover  $c_k = 0$  for any  $k > 1$ , and  $c_1 \notin B^2(V, q)$ , we say that  $q_t = q + tc_1$  is a true deformation at order 1 of  $(V, q)$ .

Using induction on  $k$ , it is easy to prove that an infinitesimally rigid structure is rigid. Of course the existence of a true deformation at order 1 implies that  $(V, q)$  is not rigid.

**Corollary 8.3.** *The algebras  $M_n(\mathbb{K})$ ,  $\mathfrak{gl}_n(\mathbb{K})$  are rigid. The pre-Lie algebra  $(M_n, q)$  admits true deformations at order 1.*

*Proof.* The first points are the previous results. Consider now the pre-Lie algebra  $(M_n, q)$ . A direct computation shows that  $q_t = q + tc_a$  is a true deformation at order 1 of  $(V, q)$  (see also [Dz]). □

Suppose now  $\mathbb{K} = \mathbb{C}$ . Then  $x \mapsto \text{tr}(x^2)$ , and  $x \mapsto (\text{tr}(x))^2$  generate the space of  $\mathfrak{gl}_n(\mathbb{C})$ -invariant degree 2 polynomial functions on  $\mathfrak{gl}_n(\mathbb{C})$  (see [W]). Any symmetric invariant bilinear form  $b$  on  $\mathfrak{gl}_n(\mathbb{C})$  can be written:

$$b(x, y) = \alpha \text{tr}(xy) + \beta \text{tr}(x)\text{tr}(y) = \alpha \text{tr}(x(y - f(y))) + \left(\frac{1}{n}\alpha + \beta\right)\text{tr}(x)\text{tr}(y).$$

Thus,  $b$  is non degenerate, if and only if  $\alpha(\alpha + n\beta) \neq 0$ .

Observe now that  $(x, y) \mapsto \text{tr}(xy)$  is  $M_n(\mathbb{C})$ -invariant, but  $(x, y) \mapsto \text{tr}(x)\text{tr}(y)$  is not  $M_n(\mathbb{C})$ -invariant. Thus the non degenerated bilinear form  $(x, y) \mapsto \text{tr}(xy)$  generates the cone of  $M_n(\mathbb{C})$ -invariant symmetric bilinear forms.

Similarly, if  $b$  is  $(M_n, q)$ -invariant, it is  $\mathfrak{gl}_n(\mathbb{C})$ -invariant, thus  $b(x, y) = \alpha \text{tr}(xy) + \beta \text{tr}(x)\text{tr}(y)$ , and the invariance relation reads:

$$\alpha \text{tr}(xyz + yxz) + \beta(\text{tr}(xy)\text{tr}(z) + \text{tr}(y)\text{tr}(xz)) = 0.$$

Choosing for instance  $x = y = z = e_{11}$ , we get  $2(\alpha + \beta) = 0$ , then  $x = y = z = id$  gives  $2n(\alpha + n\beta) = 0$ , the unique  $(M_n, q)$ -invariant symmetric bilinear form is identically zero. Summarizing,

**Lemma 8.4.** 1. *The associative algebra  $M_n(\mathbb{C})$  is a quadratic algebra for the 1-dimensional cone of symmetric bilinear forms  $\alpha \text{tr}(xy)$ ,*

2. The Lie algebra  $\mathfrak{gl}_n(\mathbb{C})$  is a quadratic Lie algebra for the 2-dimensional cone of symmetric bilinear forms  $\alpha \text{tr}(xy) + \beta \text{tr}(x)\text{tr}(y)$  ( $\alpha(\alpha + n\beta) \neq 0$ ),
3. The pre-Lie algebra  $(M_n, q)$  is not a quadratic pre-Lie algebra.

Recall that in Example 7.3, we saw that the space  $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$  is a quadratic pre-lie algebra for the product and the bilinear form:

$$q_W(x_1 + x_2, y_1 + y_2) = x_1y_1 - y_2x_1, \quad b(x_1 + x_2, y_1 + y_2) = \text{tr}(x_2y_1 + x_1y_2).$$

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INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UMR CNRS 5584, UNIVERSITÉ DE BOURGOGNE-FRANCHE COMTÉ, U.F.R. SCIENCES ET TECHNIQUES B.P. 47870, F-21078 DIJON CEDEX, FRANCE

*E-mail address:* `Didier.Arnal@u-bourgogne.fr`

UNIVERSITÉ DE SOUSSE, LABORATOIRE DE MATHÉMATIQUE PHYSIQUE, FONCTIONS SPÉCIALES ET APPLICATIONS, ECOLE SUPÉRIEURE DES SCIENCES ET DE TECHNOLOGIE DE HAMMAM SOUSSE, RUE LAMINE ABASSI, 4011 H.SOUSSE, TUNISIE

*E-mail address:* `wissem-essths@hotmail.fr`

UNIVERSITÉ DE SOUSSE, LABORATOIRE DE MATHÉMATIQUE PHYSIQUE, FONCTIONS SPÉCIALES ET APPLICATIONS, ECOLE SUPÉRIEURE DES SCIENCES ET DE TECHNOLOGIE DE HAMMAM SOUSSE, RUE LAMINE ABASSI, 4011 H.SOUSSE, TUNISIE

*E-mail address:* `mohamed.selmi@fss.rnu.tn`