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# IMPROVING ROTH'S THEOREM IN THE PRIMES

HARALD ANDRÉS HELFGOTT AND ANNE DE ROTON

ABSTRACT. Let  $A$  be a subset of the primes. Let

$$\delta_P(N) = \frac{|\{n \in A : n \leq N\}|}{|\{n \text{ prime} : n \leq N\}|}.$$

We prove that, if

$$\delta_P(N) \geq C \frac{\log \log \log N}{(\log \log N)^{1/3}}$$

for  $N \geq N_0$ , where  $C$  and  $N_0$  are absolute constants, then  $A \cap [1, N]$  contains a non-trivial three-term arithmetic progression.

This improves on Green's result [Gr], which needs

$$\delta_P(N) \geq C' \sqrt{\frac{\log \log \log \log \log N}{\log \log \log \log N}}.$$

## 1. INTRODUCTION

**1.1. History and statement.** In 1953, K. Roth [Ro] proved that any subset of positive integers of positive density contains infinitely many non trivial three-term arithmetic progressions. More precisely, his result is as follows. Given a set  $A \subset \mathbb{Z}^+$ , we define the *density* of  $A \cap [1, N]$  by  $\delta(N) = \frac{1}{N} |\{n \in A : n \leq N\}|$ . (We write  $|S|$  for the number of elements of a set  $S$ .) Roth proved that, given any set of integers  $A \subset \mathbb{Z}^+$  such that  $\delta(N) \geq C/\log \log N$  for some  $N \geq N_0$  (where  $C$  and  $N_0$  are absolute constants) there must be at least one non-trivial three-term arithmetic progression in  $A \cap [1, N]$ . (By a *non-trivial* arithmetic progression we mean one with non-zero modulus, i.e.,  $(a, a + d, a + 2d)$  with  $d \neq 0$ .)

Much later, Heath-Brown [HB] (1987) and Szemerédi [Sz] (1990) improved this result by showing that it is enough to require that  $\delta(N) \geq C(\log N)^{-c}$  for some small positive  $c$ . By considering Bohr sets where previous arguments had used arithmetic progressions, Bourgain relaxed the condition to  $\delta(N) \geq C\sqrt{\log \log N/\log N}$  in [Bo2] (1999) and to  $\delta(N) \geq C(\log \log N)^2(\log N)^{-2/3}$  in [Bo3] (2006).

Van der Corput proved [vdC] that the primes contain infinitely many non trivial 3-term arithmetic progressions. The question then arises – is Roth's theorem true in the primes? That is – must a subset of primes of positive relative density<sup>1</sup> contain a non-trivial 3-term arithmetic progression?

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<sup>1</sup>Given a subset  $A$  of the set  $P$  of all primes, we define the relative density  $\delta_P(N)$  of  $A$  to be  $\delta_P(N) = |\{n \in A : n \leq N\}|/|\{n \text{ prime} : n \leq N\}|$ . We are asking whether, given  $A \subset P$  such that  $\delta_P(N) > \delta_0$  ( $\delta_0 > 0$ ) for some sufficiently large  $N$ , the set  $A$  contains a non-trivial 3-term arithmetic progression.

In [Gr], B. Green showed that the answer is “yes”. He proved that, given any subset  $A$  of the primes such that  $A \cap [1, N]$  has relative density  $\delta_P(N) \geq C(\log \log \log \log \log N / \log \log \log \log N)^{1/2}$  for some  $N \geq N_0$ , where  $C$  and  $N_0$  are absolute constants, there exists a 3-term arithmetic progression in  $A$ .

We prove the following result.

**Theorem 1.1.** *Let  $A$  be a subset of the primes. Assume that  $A \cap [1, N]$  is of relative density*

$$\delta_P(N) \geq C \frac{\log \log \log N}{(\log \log N)^{1/3}}$$

*for some  $N \geq N_0$ , where  $C$  and  $N_0$  are absolute constants. Then  $A$  contains a non-trivial 3-term arithmetic progression.*

In other words, we gain two logs over what was previously known. One of the two logs gained is ultimately due to an enveloping use of a sieve; this idea is by now familiar to the specialists, and, indeed, it will come into our proof via a restriction theorem from [GT] (based partially on work on sieves in [Ra]). The other gain of a log stems from a more essential change in approach.

Our overall procedure is as follows. The first step is to replace the characteristic function  $a$  of  $A$  by a smoothed-out version  $a_1$  whose Fourier transform is close to that of  $a$  (and thus, as can be easily shown,  $a_1$  behaves like  $a$  does when it comes to the number of 3-term progressions). This is much the same as in [Gr, §6]; it is in accord with the general strategy (the “uniformity strategy”) described in [Ta, §6]. We then show that the  $\ell_2$ -norm of  $a_1$  is actually small enough that one can find a set  $A'$  of large density in the integers such that  $a_1$  is large on  $A'$ . This reduces the problem over the primes to the problem over the integers.

**1.2. Notation.** Let  $N'$  be a positive integer. Let  $f : \mathbb{Z}/N'\mathbb{Z} \rightarrow \mathbb{C}$  be a function in  $\ell^1(\mathbb{Z}/N'\mathbb{Z})$ . We define the Fourier transform of  $f$  as the function

$$\begin{aligned} \hat{f} : \mathbb{Z}/N'\mathbb{Z} &\rightarrow \mathbb{C} \\ b &\mapsto \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} f(n) e(-nb/N'), \end{aligned}$$

where we write  $e(x)$  for  $e^{2\pi i x}$ . We write  $\pi$  for the reduction map  $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/N'\mathbb{Z}$ . Given  $x \in \mathbb{R}$ , we define  $\{x\}$  to be the distance of  $x$  to the nearest integer. We define  $\|n\| = \{n/N'\}$ ; this works because  $\{x\}$  depends only on  $x \bmod 1$ .

Given a finite or countable set  $S$ , a function  $f : S \rightarrow \mathbb{C}$  and a parameter  $0 < r < \infty$ , we define the  $\ell_r$ -norm  $\|f\|_r$  of  $f$  by  $\|f\|_r = (\sum_{x \in S} |f(x)|^r)^{1/r}$ .

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## 2. FROM THE PRIMES TO THE INTEGERS

**2.1. From the primes to the set  $\{n : b + nM \text{ is prime}\}$ .** Let us first show that we can focus on the intersection of the primes with an arithmetic progression of large modulus, rather than work on all the primes.

**Lemma 2.1.** *Let  $\alpha, z$  be positive real numbers and  $N$  be a large integer. We define  $M = \prod_{p \leq z} p$ . Let  $A$  be a subset of the primes less than  $N$  such that  $|A| \geq \alpha N / \log N$ . Then there exists some arithmetic progression  $P(b) = \{b + nM : 1 \leq n \leq N/M\}$  such that*

$$|P(b) \cap A| \gg \alpha \frac{\log z}{\log N} \frac{N}{M} - \log z,$$

where the implied constant is absolute.

*Proof.* If  $(b, M) \neq 1$ , the set  $\{m \in P(b) : m \text{ prime}\}$  is empty. Since the progressions  $P(b)$  with  $(b, M) = 1$  are distinct, we have

$$\sum_{b:(b,M)=1} |A \cap P(b)| = |A| - |A \cap [1, M-1]| \geq \alpha \frac{N}{\log N} - M.$$

But  $|\{b \leq M : (b, M) = 1\}| \sim M / \log z \sim e^z / \log z$ . Therefore there exists some progression  $P(b)$  such that

$$|A \cap P(b)| \gg \left( \alpha \frac{N}{\log N} - M \right) \frac{\log z}{M} \gg \alpha \frac{\log z}{\log N} \frac{N}{M} - \log z.$$

□

The passage to an arithmetic progression  $b + nM$  of large modulus is exactly what Green and Tao [GT2, p. 484] call the “ $W$ -trick” (due to Green’s use of the letter  $W$  for  $M$  in [Gr]). Green uses the fact that such a passage removes all but the largest peaks in the Fourier transform of the primes, whereas we simply use in a more direct way the fact that the elements of  $\{n : b + nM \text{ prime}\}$  are not forbidden from having small divisors. Of course, these are two manifestations of the same idea.

Now, we fix  $z = \frac{1}{3} \log N$ ,  $M = \prod_{p \leq z} p$ , and let  $N'$  be the least prime larger than  $\lceil 2N/M \rceil$ . (The requirement  $N' > \lceil 2N/M \rceil$  will ensure that no new three-term arithmetic progressions are created when we apply the reduction map  $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/N'\mathbb{Z}$  to a set contained in  $[1, N/M]$ .) By Bertrand’s postulate,  $N' \ll N/M$ . Let  $A$  be a subset of the primes less than  $N$  such that  $|A| \geq \alpha N / \log N$ . We assume  $\alpha \geq (\log N)N^{-1/2}$  (say) and obtain from Lemma 2.1 that there is an arithmetic progression  $P(b)$  such that  $|P(b) \cap A| \gg \alpha (\log z / \log N) N'$ . We define  $A_0$  to be

$$(2.1) \quad A_0 = \pi \left( \left\{ n = \frac{m-b}{M} : m \in P(b) \cap A \right\} \right).$$

This is a subset of  $\pi(\{n \in [1, N'] : b + nM \text{ is prime}\})$  satisfying  $|A_0| \gg \alpha (\log z / \log N) N'$ . Our task is to show that there is a non-trivial three-term arithmetic progression in  $A_0 \subset \mathbb{Z}/N'\mathbb{Z}$ . It will follow immediately that there is a non-trivial three-term arithmetic progression in  $A \subset \mathbb{Z}$ .

**2.2. From the set  $\{n : b + nM \text{ is prime}\}$  to the set of integers.** Let  $a$  be the normalised characteristic function of  $A_0$ , i.e.,  $a = (\log N / (N' \log z)) \mathbf{1}_{A_0}$ . Fixing  $\delta > 0$  and  $\varepsilon \in (0, 1/4)$  to be chosen later, define  $R := \{x \in \mathbb{Z}/N'\mathbb{Z} : |\hat{a}(x)| \geq \delta\} \cup \{1\}$  and the Bohr set

$$B := \{n \in \mathbb{Z}/N'\mathbb{Z} : \forall x \in R, \|nx\| \leq \varepsilon\}.$$

We also define on  $\mathbb{Z}/N'\mathbb{Z}$  the functions  $\sigma = \frac{1}{|B|} \mathbf{1}_B$  and  $a_1 = a * \sigma$ .

To begin with, we remark that  $|a|_1 = (\log N / (N' \log z)) |A_0| \gg \alpha$  and  $|a_1|_1 = |a|_1 |\sigma|_1 = |a|_1$ . Thus  $|a_1|_1 \gg \alpha$ , i.e.,  $a_1$  is large in  $\ell_1$ -norm. We will later show that  $a_1$  is small in  $\ell_2$ -norm. These bounds on the  $\ell_1$ -norm and the  $\ell_2$ -norm will enable us to find a large set of integers on which  $a_1$  is  $\gg \frac{1}{N}$ . This will enable us to reduce the problem for large subsets of the primes to Roth's theorem for large subsets of the integers.

We first have to show that  $a_1$  is “close” to  $a$  in the sense that we care about, namely – we must show that  $a_1$  is large on all three terms of many three-term arithmetic progressions if and only if the same is true of  $a$  (i.e., if and only if  $A$  contains many three-term arithmetic progressions). More precisely, our aim is to bound from above the quantity

$$(2.2) \quad \Delta = N' \cdot \left| \sum_{n_1, n_2, n_3 \text{ in AP}} a(n_1)a(n_2)a(n_3) - \sum_{n_1, n_2, n_3 \text{ in AP}} a_1(n_1)a_1(n_2)a_1(n_3) \right|$$

where the sums  $\sum_{n_1, n_2, n_3 \text{ in AP}}$  are over all triples  $(n_1, n_2, n_3)$  of elements of  $\mathbb{Z}/N'\mathbb{Z}$  in arithmetic progression. Since  $(n_1, n_2, n_3)$  is an arithmetic progression if and only if  $n_1 + n_3 = 2n_2$ ,

$$\Delta = \left| \sum_m \hat{a}(-2m)(\hat{a}(m))^2 - \sum_m \hat{a}_1(-2m)(\hat{a}_1(m))^2 \right|,$$

as we can see simply by replacing all Fourier transforms by their definitions and using the fact that  $\sum_m e((n_1 + n_3 - 2n_2)m/N') = 0$  when  $n_1 + n_3 - 2n_2 \neq 0$ .

We will show that  $\Delta$  is small, namely,  $\Delta \ll \varepsilon + \delta$ . First note that, since  $a_1 = a * \sigma$  and so  $\hat{a}_1 = \hat{a} \hat{\sigma}$ ,

$$\Delta \leq \sum_m |\hat{a}(-2m) - (\hat{a}(m))^2| |1 - \hat{\sigma}(-2m)(\hat{\sigma}(m))^2|.$$

For  $x \in R$ , since  $\sigma$  is supported on  $B$  and  $\sum_n \sigma(n) = 1$ , we have

$$\begin{aligned} |\hat{\sigma}(x) - 1| &= \left| \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \sigma(n) e(nx/N') - 1 \right| \\ &= \left| \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \sigma(n) - 1 + \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \sigma(n) (e(nx/N') - 1) \right| \\ &\leq \sum_{n \in B} \sigma(n) |e(nx/N') - 1| \ll \sum_{n \in B} \sigma(n) \|nx\| \ll \varepsilon. \end{aligned}$$

Similarly, for  $x \in R$ ,

$$|\hat{\sigma}(-2x) - 1| \ll \sum_{n \in B} \sigma(n) \|\cdot - 2nx\| \ll \sum_{n \in B} \sigma(n) \varepsilon = \varepsilon$$

and so

$$(2.3) \quad |1 - \hat{\sigma}(-2x)\hat{\sigma}(x)^2| \ll \varepsilon$$

for  $x \in R$ , i.e., when  $|\hat{a}(x)| \geq \delta$ .

Before we proceed further, we need to bound  $\hat{a}$  in an average sense.

**Lemma 2.2.** *For  $p > 2$ ,*

$$(2.4) \quad \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^p \ll_p 1.$$

This is the same as [Gr, Lemma 6.6]; the only difference is that our function  $a$  was defined with a much larger modulus  $M$  than in [Gr], and thus we must use a restriction theorem for an upper-bound sieve, rather than a restriction theorem for the primes (such as [Bo, (4.39)]).

*Proof.* Applying [GT, Prop. 4.2] with  $F(n) = b + nM$  and  $R = N'^{1/10}$ , we obtain that, for  $p > 2$  and any complex sequence  $(b_n)_n$ ,

$$(2.5) \quad \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} \left| \frac{1}{N'} \sum_{n=1}^{N'} b_n \beta(n) e(-mn/N') \right|^p \ll_p \left( \frac{1}{N'} \sum_{n=1}^{N'} |b_n|^2 \beta(n) \right)^{p/2},$$

where  $\beta$  is an enveloping sieve function with  $R = N'^{1/10}$ . This means that, according to [GT, Prop. 3.1],  $\beta : \mathbb{Z}^+ \rightarrow \mathbb{R}$  is a non-negative function satisfying the majorant property

$$\beta(n) \gg \mathfrak{S}_F^{-1} \cdot \log R \cdot \mathbf{1}_{X_{R!}}(n)$$

with

$$\mathfrak{S}_F = \prod_p \frac{\gamma(p)}{1 - 1/p},$$

$$\gamma(p) = \frac{1}{p} |\{n \in \mathbb{Z}/p\mathbb{Z}, (p, b + nM) = 1\}| = \begin{cases} (1 - 1/p) & \text{if } p > z \\ 1 & \text{if } p \leq z \end{cases}$$

and

$$X_{R!} = \{n \in \mathbb{Z} : \forall d \leq R \ (b + nM, d) = 1\}.$$

In particular, for any integer  $n \in A_0$ , we have  $n \in X_{R!}$  and

$$\beta(n) \gg (\log R) \prod_{p \leq z} (1 - 1/p)^{-1} \gg \frac{\log N}{\log z}.$$

We apply (2.5) to the sequence  $(b_n)_n$  defined by

$$b_n = \begin{cases} \frac{1}{\beta(n)} a(n) & \text{if } n \in A_0 \\ 0 & \text{otherwise} \end{cases}$$

and get

$$\begin{aligned}
\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^p &= (N')^p \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} \left| \frac{1}{N'} \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} a(n) e(-mn/N) \right|^p \\
&\ll_p (N')^{p/2} \left( \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \frac{1}{\beta(n)} a(n)^2 \right)^{p/2} \\
&\ll_p (N')^{p/2} \left( \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \frac{\log z}{\log N} \left( \frac{\log N}{N' \log z} \right) a(n) \right)^{p/2} \\
&\ll_p \left( \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} a(n) \right)^{p/2} \ll_p 1,
\end{aligned}$$

since  $a(n)$  was normalised so that  $\sum_n a(n) \ll 1$ .  $\square$

By Hölder's inequality and Lemma 2.2, we have

$$\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(-2m)\hat{a}(m)^2| \leq \left( \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{5/2} \right)^{2/5} \left( \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{10/3} \right)^{3/5} \ll 1.$$

Hence, by (2.3),

$$\sum_{m: |\hat{a}(m)| \geq \delta} |\hat{a}(-2m)\hat{a}(m)^2| |1 - \hat{\sigma}(-2m)\hat{\sigma}(m)^2| \ll \varepsilon.$$

On the other hand (again by Hölder, and again by Lemma 2.2),

$$\begin{aligned}
\sum_{m: |\hat{a}(m)| < \delta} |\hat{a}(-2m)\hat{a}(m)^2| |1 - \hat{\sigma}(-2m)\hat{\sigma}(m)^2| &\leq 2 \sum_{m: |\hat{a}(m)| < \delta} |\hat{a}(-2m)\hat{a}(m)^2| \\
&\leq 2 \left( \sum_{m: |\hat{a}(m)| < \delta} |\hat{a}(m)|^{5/2} \right)^{2/5} \left( \sum_{m: |\hat{a}(m)| < \delta} |\hat{a}(m)|^{10/3} \right)^{3/5} \\
&\leq 2 \left( \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{5/2} \right)^{2/5} \left( \delta^{5/3} \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{5/3} \right)^{3/5} \ll \delta.
\end{aligned}$$

Thus

$$(2.6) \quad \Delta \ll (\varepsilon + \delta).$$

**2.3. An upper bound for the  $\ell_2$ -norm of  $a_1$ .** Our aim in this subsection is to bound from above the  $\ell_2$ -norm of the function  $a_1 = a * \sigma$ . (This will later enable us to show that  $a_1$  is in some sense close to the characteristic function of a set of large density in the integers.) We will prove that  $|a_1|_2 \ll 1/\sqrt{N'}$ , where the implied constant is absolute.

Recall that we write  $\pi$  for the reduction map  $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/N'\mathbb{Z}$ . Given a function  $f : \mathbb{Z}/N'\mathbb{Z} \rightarrow \mathbb{C}$ , we can lift it to a function  $\tilde{f} : \mathbb{Z} \rightarrow \mathbb{C}$  supported on  $[-(N'-1)/2, (N'-1)/2]$ :

$$\tilde{f}(n) = \begin{cases} f(n \bmod N') & \text{if } n \in [-\frac{N'-1}{2}, \frac{N'-1}{2}], \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of  $A_0$  and  $a$ , we see that  $A_0 \subset \pi([1, (N'-1)/2])$ , and thus  $a$  is supported on  $\pi([1, (N'-1)/2])$ . By the definition of  $R$ ,  $B$  and  $\sigma$  and the assumption  $\varepsilon < 1/4$ , we see that  $\sigma$  is supported on  $\pi([-N'/4, N'/4])$ . Thus  $|a * \sigma|_2 = |\tilde{a} * \tilde{\sigma}|_2$ .

By the definition of  $a$ , we have  $0 \leq \tilde{a}(n) \leq \lambda(n)$ , where  $\lambda : \mathbb{Z} \rightarrow \mathbb{R}$  is defined by

$$(2.7) \quad \lambda(n) = \begin{cases} \frac{\log N}{N' \log z} & \text{if } 1 \leq n \leq N' \text{ and } b + nM \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

Recall that  $\sigma$  is non-negative, and thus  $\tilde{\sigma}$  is non-negative. Hence  $|\tilde{a} * \tilde{\sigma}|_2 \leq |\lambda * \tilde{\sigma}|_2$ . We conclude that

$$|a_1|_2 = |a * \sigma|_2 = |\tilde{a} * \tilde{\sigma}|_2 \leq |\lambda * \tilde{\sigma}|_2.$$

It is thus our task to prove that  $|\lambda * \tilde{\sigma}|_2 \ll 1/\sqrt{N'}$ .

We proceed as follows:

$$(2.8) \quad \begin{aligned} \sum_n |\tilde{\sigma} * \lambda(n)|^2 &= \sum_n \left| \sum_m \tilde{\sigma}(m) \lambda(n-m) \right|^2 \\ &= \sum_{m_1} \sum_{m_2} \tilde{\sigma}(m_1) \tilde{\sigma}(m_2) \sum_n \lambda(n+m_1) \lambda(n+m_2), \end{aligned}$$

where we recall that  $\sigma(m) = \sigma(-m)$  (by the definition of  $B$  and  $\sigma$ ).

**Lemma 2.3.** *Let  $\lambda$  be as in (2.7). Then, for any integers  $m_1, m_2$ ,*

$$(2.9) \quad \sum_n \lambda(n+m_1) \lambda(n+m_2) \ll \begin{cases} \frac{\log N / (N' \log z)}{1} & \text{if } m_1 = m_2, \\ \frac{1}{N'} \prod_{p|(m_1-m_2), p > z} \frac{p}{p-1} & \text{if } m_1 \neq m_2, \end{cases}$$

where the implied constant is absolute.

*Proof.* The case  $m_1 = m_2$  follows from Brun-Titchmarsh:

$$\begin{aligned} \sum_n \lambda^2(n+m) &= \left( \frac{\log N}{N' \log z} \right)^2 |\{m \leq n \leq N' + m : b + (n-m)M \text{ is prime}\}| \\ &\ll \left( \frac{\log N}{N' \log z} \right)^2 \frac{N'M}{\varphi(M) \log N'} \\ &\ll \left( \frac{\log N}{N' \log z} \right)^2 \frac{N'}{\log N'} \prod_{p|M} (1-1/p)^{-1} \\ &\ll \frac{\log N}{N' \log z}. \end{aligned}$$

To obtain the case  $m_1 \neq m_2$ , we will use a result based on Selberg's sieve. (This is a familiar type of application of upper-bound sieves, similar to the proof that the



number of twin primes up to  $N$  is at most a constant times its conjectured value.) It is clear that  $\sum_n \lambda(n + m_1)\lambda(n + m_2)$  equals  $(\log N/(N' \log z))^2$  times

$$(2.10) \quad \left| \{1 \leq n \leq N' : b + nM \text{ and } b + (n + m_2 - m_1)M \text{ are primes}\} \right|.$$

By [HR, Thm. 5.7],

$$(2.10) \ll \prod_p \left(1 - \frac{\rho(p) - 1}{p - 1}\right) \left(1 - \frac{1}{p}\right)^{-1} \frac{N'}{(\log N')^2},$$

where the implied constant is absolute. (We are implicitly using the fact that  $\log M \ll \log N'$ , and thus the term in the third line of [HR, (8.3)] is  $= 1 + o(1)$ .) Here  $\rho(p)$  is the number of solutions  $x \in \mathbb{Z}/p\mathbb{Z}$  to

$$(b + xM)(b + (x + m_2 - m_1)M) \equiv 0 \pmod{p}$$

for  $p$  prime. It is easy to see that  $\rho(p) = 0$  if  $p|M$  (i.e., if  $p \leq z$ ),  $\rho(p) = 1$  if  $p > z$  and  $p|(m_2 - m_1)$ , and  $\rho(p) = 2$  if  $p > z$  and  $p \nmid (m_2 - m_1)$ . Hence

$$(2.10) \ll \prod_{p \leq z} \left(1 - \frac{1}{p}\right)^{-2} \prod_{\substack{p > z \\ p|m_1 - m_2}} \left(1 - \frac{1}{p}\right)^{-1} \frac{N'}{(\log N')^2} \\ \ll \prod_{\substack{p > z \\ p|m_1 - m_2}} \frac{p}{p - 1} \frac{N'(\log z)^2}{(\log N')^2}.$$

The statement follows.  $\square$

Let us now evaluate the last line of (2.8), with Lemma 2.3 in hand. The contribution of the diagonal terms ( $m_1 = m_2$ ) in (2.8) is  $\ll \log N/(|B|N' \log z)$ . The contribution of the non-diagonal terms ( $m_1 \neq m_2$ ) is

$$(2.11) \quad \ll \frac{1}{N'} \sum_{\substack{m_1 \\ m_2 \neq m_1}} \sum_{m_2} \tilde{\sigma}(m_1)\tilde{\sigma}(m_2) \prod_{\substack{p > z \\ p|m_1 - m_2}} \frac{p}{p - 1}.$$

Recall that  $\tilde{\sigma}$  is supported on  $[-N'/4, N'/4]$ , and thus  $|m_2 - m_1| \leq N'/2 < N'$  whenever  $\tilde{\sigma}(m_1)\tilde{\sigma}(m_2) \neq 0$ .

Now, a non-zero integer  $m$  with  $|m| \leq N'$  cannot have more than  $\log N'/\log z$  prime factors  $p > z$ . Since  $x/(x - 1)$  is decreasing on  $x$ , this means that

$$\prod_{\substack{p > z \\ p|m}} \frac{p}{p - 1} \leq \left(\frac{z}{z - 1}\right)^{(\log N')/(\log z)}.$$

Now  $(z/(z - 1))^z \ll 1$  (because  $\lim_n (1 + 1/n)^n = e$ ) and

$$\frac{\log N'}{\log z} \ll \frac{\log N}{\log \log N} < \log N \ll z.$$

Hence

$$\prod_{\substack{p > z \\ p|m}} \frac{p}{p-1} \ll 1$$

for any  $m \neq 0$  with  $|m| \leq N'$ . Thus

$$(2.11) \ll \frac{1}{N'} \sum_{\substack{m_1 \\ m_2 \neq m_1}} \sum_{m_2} \tilde{\sigma}(m_1) \tilde{\sigma}(m_2) \ll \frac{1}{N'}.$$

Putting everything together, we conclude that

$$\sum_n |\tilde{\sigma} * \lambda(n)|^2 \ll \frac{1}{N'} \left( \frac{\log N}{|B| \log z} + 1 \right).$$

The right side is  $\ll 1/N'$  as long as  $|B| \gg \log N / \log z$ .

Now, as is well-known (see, e.g., [TV, Lem. 4.20]),

$$|B| \gg \varepsilon^r N',$$

where  $r = |R|$ . (The proof of this is a simple pigeonhole argument.) Since by (2.4) we have  $\sum_m |\hat{a}(m)|^{5/2} \ll 1$ , we know that the set of  $x \in \mathbb{Z}/N'\mathbb{Z}$  with  $|\hat{a}(x)| \geq \delta$  has at most  $\ll \delta^{-5/2}$  elements. Thus,  $r \ll \delta^{-5/2}$ .

Hence all that we need for  $|B| \geq \log N / \log z$  to hold is that  $\varepsilon^{\delta^{-5/2}} \geq N^{-1/2}$  (say). In other words, we need  $|\log \varepsilon| \cdot \delta^{-5/2} \leq \frac{1}{2} \log N$ . We will recall that we need to satisfy this condition at the end.

**2.4. Extracting a dense set from  $a_1$ .** We now have a function  $a_1 : \mathbb{Z}/N'\mathbb{Z} \rightarrow \mathbb{R}_0^+$  of  $\ell_2$  norm  $\ll 1/\sqrt{N'}$ . Its  $\ell_1$  norm is  $\gg \alpha$ , where  $\alpha$  is the density of our original set  $A$  on the primes. We must show that there is a large set on which  $a_1$  is large.

**Lemma 2.4.** *Let  $S$  be a set with  $N'$  elements. Let  $a : S \rightarrow \mathbb{R}_0^+$  and  $0 < \alpha < 1$  be such that*

- (a)  $\|a\|_1 \geq \alpha$ ;
- (b)  $\|a\|_2^2 \leq c/N'$ .

*Then there exists a subset  $A'$  of  $S$  such that*

- (a)  $|A'| \geq \alpha^2 N' / (4c)$ ;
- (b)  $\forall n \in A', a(n) \geq \alpha / (2N')$ .

*Proof.* If  $A' = \{n : a(n) \geq \alpha / (2N')\}$ , then

$$\begin{aligned} \alpha &\leq \sum_n a(n) \leq \frac{\alpha}{2N'} (N' - |A'|) + \sum_{n \in A'} a(n) \\ &\leq \frac{\alpha}{2N'} (N' - |A'|) + \sqrt{|A'|} \sqrt{\frac{c}{N'}}, \end{aligned}$$

by  $\|a\|_2 \leq c/N'$  and Cauchy's inequality. In other words,  $f(\sqrt{|A'|}) \leq 0$ , where  $f(x) = \frac{\alpha}{2N'} \left( x^2 - 2\frac{\sqrt{cN'}}{\alpha} x + N' \right)$ . Completing the square, we see that  $f(x) \leq 0$

implies  $x \geq \frac{\sqrt{cN'}}{\alpha} - \sqrt{\left(\frac{c}{\alpha^2} - 1\right)N'}$ . Hence

$$|A'| \geq N' \cdot \left( \frac{\sqrt{c}}{\alpha} - \sqrt{\frac{c}{\alpha^2} - 1} \right)^2 \geq \frac{\alpha^2 N'}{4c}.$$

□

We apply Lemma 2 to  $a_1$  with the bound  $\|a_1\|_2^2 \leq c/N'$  being provided by our work in §2.3. We get a subset  $A'$  of  $\mathbb{Z}/N'\mathbb{Z}$  such that  $|A'| \geq \alpha^2 N'/(4c) \gg \alpha^2 N'$  and  $a_1(n) \geq \alpha/(2N')$  for every  $n \in A'$ . Hence

$$(2.12) \quad \sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) \geq \frac{\alpha^3 Z}{8N'^3},$$

where  $Z$  is the number of 3-term arithmetic progressions in  $A'$ .

**Lemma 2.5.** *Let  $A' \subset \mathbb{Z}/N'\mathbb{Z}$ , where  $N'$  is a prime. Assume  $|A'| \geq \eta N'$ ,  $\eta > 0$ . The number of 3-term arithmetic progressions in  $A'$  is then at least*

$$\frac{\eta N'^2}{c_0 \exp(c_1 \eta^{-3/2} (\log(1/\eta))^3)},$$

where  $c_0$  and  $c_1$  are absolute constants.

*Proof.* We will proceed much as in [Gr, Lem. 6.8]; the basic argument goes back to Varnavides [Va]. Bourgain's best result on three-term arithmetic progressions in the integers [Bo3, Thm. 1] states that, for given  $L$  and  $\eta \gg (\log \log L)^2 (\log L)^{-2/3}$ , every subset of  $\{1, 2, \dots, L\}$  with  $\geq \eta L$  elements contains at least one non-trivial three-term arithmetic progression. This can be rephrased as follows: there are constants  $c_0$  and  $c_1$  such that, if  $L \geq c_0 \exp(c_1 \eta^{-3/2} (\log(1/\eta))^3)$ , then any subset of  $\{1, \dots, L\}$  of density at least  $\eta/2$  contains a non-trivial three-term arithmetic progression. (Here we are simply expressing  $L$  in terms of the density, rather than the density in terms of  $L$ .)

It follows that, given an arithmetic progression  $S_{a,d} = \{a+d, a+2d, a+3d, \dots, a+Ld\}$  in  $\mathbb{Z}/N'\mathbb{Z}$  ( $a, d \in \mathbb{Z}/N'\mathbb{Z}$ ,  $d \neq 0$ ,  $L \leq N'$ ) whose intersection with  $A'$  has at least  $(\eta/2)L$  elements, there is at least one non-trivial three-term arithmetic progression in  $A' \cap S \subset \mathbb{Z}/N'\mathbb{Z}$ . (Note that there is no need for the progression  $S$  to be the reduction mod  $N'$  of a progression in the integers  $\{1, 2, \dots, N'\}$ ; the argument works regardless of this.) If we consider all arithmetic progressions of length  $L$  and given modulus  $d \neq 0$  in  $\mathbb{Z}/N'\mathbb{Z}$ , we see that each element of  $A'$  is contained in exactly  $L$  of them. Hence,  $\sum_a |S_{a,d} \cap A'| = L|A'| \geq \eta N' L$ , and so (for  $d \neq 0$  fixed)  $|S_{a,d} \cap A'| \geq (\eta/2)L$  for at least  $(\eta/2)N'$  values of  $a$ . Varying  $d$ , we get that  $|S_{a,d} \cap A'| \geq (\eta/2)L$  for at least  $(\eta/2)N'(N'-1)$  arithmetic progressions  $S_{a,d}$ . By the above, each such intersection  $S_{a,d} \cap A'$  contains at least one non-trivial three-term arithmetic progression.

Each non-trivial three-term arithmetic progression  $a_1, a_2, a_3$  in  $\mathbb{Z}/N'\mathbb{Z}$  can be contained in at most  $L(L-1)$  arithmetic progressions  $\{a+d, a+2d, \dots, a+Ld\}$  of length  $L$  (the indices of  $a_1$  and  $a_2$  in the progression of length  $L$  determine the progression). Hence, when we count the three-term arithmetic progressions coming from the intersections  $S_{a,d} \cap A'$ , we are counting each such progression at most  $L(L-1)$

times. Thus we have shown that  $A'$  contains at least  $(\eta/2)(N'(N'-1))/(L(L-1)) \geq (\eta/2)(N'/L)^2$  distinct non-trivial three-term arithmetic progressions for  $L = \lceil c_0 \exp(c_1 \eta^{-3/2} (\log(1/\delta))^3) \rceil$ , provided that  $L \leq N'$ . If  $L > N'$ , the bound in the statement of the lemma is trivially true (as there is always at least one trivial three-term arithmetic progression in  $A'$ .)  $\square$

From (2.12) and Lemma 2.5, we conclude that

$$(2.13) \quad \sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) \geq \frac{\alpha^3}{8N'^3} \frac{\alpha^2}{8c} \frac{(N')^2}{c_0 \exp(c_1 (\alpha^2/4c)^{-3/2} (\log(4c/\alpha^2))^3)} \\ \geq \frac{1}{N'} \frac{1}{c_2 \exp(c_3 \alpha^{-3} (\log(1/\alpha))^3)},$$

where  $c_2, c_3 > 0$  are absolute constants.

### 3. CONCLUSION

Assume that  $A$  contains no non-trivial three-term arithmetic progressions. Then  $A_0$  (defined in (2.1)) contains no non-trivial three term arithmetic progressions, and so

$$\sum_{m,d} a(m)a(m+d)a(m+2d) = \sum_{m,d} a(m)^3 \ll \left( \frac{\log N}{N' \log z} \right)^2.$$

We also have

$$\Delta = N' \left| \sum_{m,d} a(m)a(m+d)a(m+2d) - \sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) \right| \ll (\varepsilon + \delta),$$

by the definition (2.2) and (2.6). Lastly, we have just shown that

$$\sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) \geq \frac{1}{N'} \frac{1}{c_2 \exp(c_3 \alpha^{-3} (\log(1/\alpha))^3)}$$

(see (2.13)). We conclude that

$$(3.1) \quad \frac{1}{c_2 \exp(c_3 \alpha^{-3} (\log(1/\alpha))^3)} \ll \varepsilon + \delta + \frac{1}{N'} \left( \frac{\log N}{\log z} \right)^2.$$

Recall that  $z = (\log N)/3$ . There are constants  $c_4, c_5$  such that, for

$$\delta = \varepsilon = \frac{1}{c_4} \exp(-c_5 \alpha^{-3} \log^3(1/\alpha)),$$

we get a contradiction with (3.1), provided that  $N$  is larger than an absolute constant and  $\alpha \geq (\log N)^{-1/4}$ , say. These values of  $\delta$  and  $\varepsilon$  satisfy  $|\log \varepsilon| \delta^{-2.5} \leq (\log N)/2$  as long as

$$(\log(c_4) + c_5 \alpha^{-3} \log^3(1/\alpha)) \cdot c_4^{2.5} \exp(2.5 c_5 \alpha^{-3} \log^3(1/\alpha)) \leq \frac{1}{2} \log N.$$

Therefore we have a contradiction if  $\alpha \geq C \log \log \log N (\log \log N)^{-1/3}$ , where  $C$  is a large enough constant and  $N$  is larger than an absolute constant. Theorem 1.1 is thereby proven.

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