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Harald Andr  s Helfgott, Anne de Roton

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IMPROVING ROTH'S THEOREM IN THE PRIMES

HARALD ANDRÉS HELFGOTT AND ANNE DE ROTON

ABSTRACT. Let A be a subset of the primes. Let

$$\delta_P(N) = \frac{|\{n \in A : n \leq N\}|}{|\{n \text{ prime} : n \leq N\}|}.$$

We prove that, if

$$\delta_P(N) \geq C \frac{\log \log \log N}{(\log \log N)^{1/3}}$$

for $N \geq N_0$, where C and N_0 are absolute constants, then $A \cap [1, N]$ contains a non-trivial three-term arithmetic progression.

This improves on Green's result [Gr], which needs

$$\delta_P(N) \geq C' \sqrt{\frac{\log \log \log \log \log N}{\log \log \log N}}.$$

1. INTRODUCTION

1.1. History and statement. In 1953, K. Roth [Ro] proved that any subset of positive integers of positive density contains infinitely many non trivial three-term arithmetic progressions. More precisely, his result is as follows. Given a set $A \subset \mathbb{Z}^+$, we define the *density* of $A \cap [1, N]$ by $\delta(N) = \frac{1}{N} |\{n \in A : n \leq N\}|$. (We write $|S|$ for the number of elements of a set S .) Roth proved that, given any set of integers $A \subset \mathbb{Z}^+$ such that $\delta(N) \geq C/\log \log N$ for some $N \geq N_0$ (where C and N_0 are absolute constants) there must be at least one non-trivial three-term arithmetic progression in $A \cap [1, N]$. (By a *non-trivial* arithmetic progression we mean one with non-zero modulus, i.e., $(a, a + d, a + 2d)$ with $d \neq 0$.)

Much later, Heath-Brown [HB] (1987) and Szemerédi [Sz] (1990) improved this result by showing that it is enough to require that $\delta(N) \geq C(\log N)^{-c}$ for some small positive c . By considering Bohr sets where previous arguments had used arithmetic progressions, Bourgain relaxed the condition to $\delta(N) \geq C\sqrt{\log \log N / \log N}$ in [Bo2] (1999) and to $\delta(N) \geq C(\log \log N)^2(\log N)^{-2/3}$ in [Bo3] (2006).

Van der Corput proved [vdC] that the primes contain infinitely many non trivial 3-term arithmetic progressions. The question then arises – is Roth's theorem true in the primes? That is – must a subset of primes of positive relative density¹ contain a non-trivial 3-term arithmetic progression?

¹Given a subset A of the set P of all primes, we define the relative density $\delta_P(N)$ of A to be $\delta_P(N) = |\{n \in A : n \leq N\}|/|\{n \text{ prime} : n \leq N\}|$. We are asking whether, given $A \subset P$ such that $\delta_P(N) > \delta_0$ ($\delta_0 > 0$) for some sufficiently large N , the set A contains a non-trivial 3-term arithmetic progression.

In [Gr], B. Green showed that the answer is “yes”. He proved that, given any subset A of the primes such that $A \cap [1, N]$ has relative density $\delta_P(N) \geq C(\log \log \log \log \log N / \log \log \log \log N)^{1/2}$ for some $N \geq N_0$, where C and N_0 are absolute constants, there exists a 3-term arithmetic progression in A .

We prove the following result.

Theorem 1.1. *Let A be a subset of the primes. Assume that $A \cap [1, N]$ is of relative density*

$$\delta_P(N) \geq C \frac{\log \log \log N}{(\log \log N)^{1/3}}$$

for some $N \geq N_0$, where C and N_0 are absolute constants. Then A contains a non-trivial 3-term arithmetic progression.

In other words, we gain two logs over what was previously known. One of the two logs gained is ultimately due to an enveloping use of a sieve; this idea is by now familiar to the specialists, and, indeed, it will come into our proof via a restriction theorem from [GT] (based partially on work on sieves in [Ra]). The other gain of a log stems from a more essential change in approach.

Our overall procedure is as follows. The first step is to replace the characteristic function a of A by a smoothed-out version a_1 whose Fourier transform is close to that a (and thus, as can be easily shown, a_1 behaves like a does when it comes to the number of 3-term progressions). This is much the same as in [Gr, §6]; it is in accord with the general strategy (the “uniformity strategy”) described in [Ta, §6]. We then show that the ℓ_2 -norm of a_1 is actually small enough that one can find a set A' of large density in the integers such that a_1 is large on A' . This reduces the problem over the primes to the problem over the integers.

1.2. Notation. Let N' be a positive integer. Let $f : \mathbb{Z}/N'\mathbb{Z} \rightarrow \mathbb{C}$ be a function in $\ell^1(\mathbb{Z}/N'\mathbb{Z})$. We define the Fourier transform of f as the function

$$\begin{aligned} \hat{f} : \mathbb{Z}/N'\mathbb{Z} &\rightarrow \mathbb{C} \\ b &\mapsto \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} f(n) e(-nb/N'), \end{aligned}$$

where we write $e(x)$ for $e^{2\pi i x}$. We write π for the reduction map $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/N'\mathbb{Z}$. Given $x \in \mathbb{R}$, we define $\{x\}$ to be the distance of x to the nearest integer. We define $\|n\| = \{n/N'\}$; this works because $\{x\}$ depends only on $x \bmod 1$.

Given a finite or countable set S , a function $f : S \rightarrow \mathbb{C}$ and a parameter $0 < r < \infty$, we define the ℓ_r -norm $|f|_r$ of f by $|f|_r = (\sum_{x \in S} |f(x)|^r)^{1/r}$.

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2. FROM THE PRIMES TO THE INTEGERS

2.1. From the primes to the set $\{n : b + nM \text{ is prime}\}$. Let us first show that we can focus on the intersection of the primes with an arithmetic progression of large modulus, rather than work on all the primes.

Lemma 2.1. *Let α, z be positive real numbers and N be a large integer. We define $M = \prod_{p \leq z} p$. Let A be a subset of the primes less than N such that $|A| \geq \alpha N / \log N$. Then there exists some arithmetic progression $P(b) = \{b + nM : 1 \leq n \leq N/M\}$ such that*

$$|P(b) \cap A| \gg \alpha \frac{\log z}{\log N} \frac{N}{M} - \log z,$$

where the implied constant is absolute.

Proof. If $(b, M) \neq 1$, the set $\{m \in P(b) : m \text{ prime}\}$ is empty. Since the progressions $P(b)$ with $(b, M) = 1$ are distinct, we have

$$\sum_{b: (b, M)=1} |A \cap P(b)| = |A| - |A \cap [1, M-1]| \geq \alpha \frac{N}{\log N} - M.$$

But $|\{b \leq M : (b, M) = 1\}| \sim M / \log z \sim e^z / \log z$. Therefore there exists some progression $P(b)$ such that

$$|A \cap P(b)| \gg \left(\alpha \frac{N}{\log N} - M \right) \frac{\log z}{M} \gg \alpha \frac{\log z}{\log N} \frac{N}{M} - \log z.$$

□

The passage to an arithmetic progression $b + nM$ of large modulus is exactly what Green and Tao [GT2, p. 484] call the “ W -trick” (due to Green’s use of the letter W for M in [Gr]). Green uses the fact that such a passage removes all but the largest peaks in the Fourier transform of the primes, whereas we simply use in a more direct way the fact that the elements of $\{n : b + nM \text{ prime}\}$ are not forbidden from having small divisors. Of course, these are two manifestations of the same idea.

Now, we fix $z = \frac{1}{3} \log N$, $M = \prod_{p \leq z} p$, and let N' be the least prime larger than $\lceil 2N/M \rceil$. (The requirement $N' > \lceil 2N/M \rceil$ will ensure that no new three-term arithmetic progressions are created when we apply the reduction map $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/N'\mathbb{Z}$ to a set contained in $[1, N/M]$.) By Bertrand’s postulate, $N' \ll N/M$. Let A be a subset of the primes less than N such that $|A| \geq \alpha N / \log N$. We assume $\alpha \geq (\log N)N^{-1/2}$ (say) and obtain from Lemma 2.1 that there is an arithmetic progression $P(b)$ such that $|P(b) \cap A| \gg \alpha (\log z / \log N) N'$. We define A_0 to be

$$(2.1) \quad A_0 = \pi \left(\left\{ n = \frac{m-b}{M} : m \in P(b) \cap A \right\} \right).$$

This is a subset of $\pi(\{n \in [1, N'] : b + nM \text{ is prime}\})$ satisfying $|A_0| \gg \alpha (\log z / \log N) N'$. Our task is to show that there is a non-trivial three-term arithmetic progression in $A_0 \subset \mathbb{Z}/N'\mathbb{Z}$. It will follow immediately that there is a non-trivial three-term arithmetic progression in $A \subset \mathbb{Z}$.

2.2. From the set $\{n : b + nM \text{ is prime}\}$ to the set of integers. Let a be the normalised characteristic function of A_0 , i.e., $a = (\log N / (N' \log z)) \mathbf{1}_{A_0}$. Fixing $\delta > 0$ and $\varepsilon \in (0, 1/4)$ to be chosen later, define $R := \{x \in \mathbb{Z}/N'\mathbb{Z} : |\hat{a}(x)| \geq \delta\} \cup \{1\}$ and the Bohr set

$$B := \{n \in \mathbb{Z}/N'\mathbb{Z} : \forall x \in R, \|nx\| \leq \varepsilon\}.$$

We also define on $\mathbb{Z}/N'\mathbb{Z}$ the functions $\sigma = \frac{1}{|B|} \mathbf{1}_B$ and $a_1 = a * \sigma$.

To begin with, we remark that $|a|_1 = (\log N / (N' \log z)) |A_0| \gg \alpha$ and $|a_1|_1 = |a|_1 |\sigma|_1 = |a|_1$. Thus $|a_1|_1 \gg \alpha$, i.e., a_1 is large in ℓ_1 -norm. We will later show that a_1 is small in ℓ_2 -norm. These bounds on the ℓ_1 -norm and the ℓ_2 -norm will enable us to find a large set of integers on which a_1 is $\gg \frac{1}{N}$. This will enable us to reduce the problem for large subsets of the primes to Roth's theorem for large subsets of the integers.

We first have to show that a_1 is “close” to a in the sense that we care about, namely – we must show that a_1 is large on all three terms of many three-term arithmetic progressions if and only if the same is true of a (i.e., if and only if A contains many three-term arithmetic progressions). More precisely, our aim is to bound from above the quantity

$$(2.2) \quad \Delta = N' \cdot \left| \sum_{n_1, n_2, n_3 \text{ in AP}} a(n_1)a(n_2)a(n_3) - \sum_{n_1, n_2, n_3 \text{ in AP}} a_1(n_1)a_1(n_2)a_1(n_3) \right|$$

where the sums $\sum_{n_1, n_2, n_3 \text{ in AP}}$ are over all triples (n_1, n_2, n_3) of elements of $\mathbb{Z}/N'\mathbb{Z}$ in arithmetic progression. Since (n_1, n_2, n_3) is an arithmetic progression if and only if $n_1 + n_3 = 2n_2$,

$$\Delta = \left| \sum_m \hat{a}(-2m)(\hat{a}(m))^2 - \sum_m \hat{a}_1(-2m)(\hat{a}_1(m))^2 \right|,$$

as we can see simply by replacing all Fourier transforms by their definitions and using the fact that $\sum_m e((n_1 + n_3 - 2n_2)m/N') = 0$ when $n_1 + n_3 - 2n_2 \neq 0$.

We will show that Δ is small, namely, $\Delta \ll \epsilon + \delta$. First note that, since $a_1 = a * \sigma$ and so $\hat{a}_1 = \hat{a} \hat{\sigma}$,

$$\Delta \leq \sum_m |\hat{a}(-2m) - (\hat{a}(m))^2| |1 - \hat{\sigma}(-2m)(\hat{\sigma}(m))^2|.$$

For $x \in R$, since σ is supported on B and $\sum_n \sigma(n) = 1$, we have

$$\begin{aligned} |\hat{\sigma}(x) - 1| &= \left| \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \sigma(n) e(nx/N') - 1 \right| \\ &= \left| \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \sigma(n) - 1 + \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \sigma(n) (e(nx/N') - 1) \right| \\ &\leq \sum_{n \in B} \sigma(n) |e(nx/N') - 1| \ll \sum_{n \in B} \sigma(n) \|nx\| \ll \varepsilon. \end{aligned}$$

Similarly, for $x \in R$,

$$|\hat{\sigma}(-2x) - 1| \ll \sum_{n \in B} \sigma(n) \| -2nx \| \ll \sum_{n \in B} \sigma(n) \varepsilon = \varepsilon$$

and so

$$(2.3) \quad |1 - \hat{\sigma}(-2x)\hat{\sigma}(x)^2| \ll \varepsilon$$

for $x \in R$, i.e., when $|\hat{a}(x)| \geq \delta$.

Before we proceed further, we need to bound $\hat{a}$ in an average sense.

Lemma 2.2. *For $p > 2$,*

$$(2.4) \quad \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^p \ll_p 1.$$

This is the same as [Gr, Lemma 6.6]; the only difference is that our function a was defined with a much larger modulus M than in [Gr], and thus we must use a restriction theorem for an upper-bound sieve, rather than a restriction theorem for the primes (such as [Bo, (4.39)]).

Proof. Applying [GT, Prop. 4.2] with $F(n) = b + nM$ and $R = N'^{1/10}$, we obtain that, for $p > 2$ and any complex sequence $(b_n)_n$,

$$(2.5) \quad \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} \left| \frac{1}{N'} \sum_{n=1}^{N'} b_n \beta(n) e(-mn/N') \right|^p \ll_p \left(\frac{1}{N'} \sum_{n=1}^{N'} |b_n|^2 \beta(n) \right)^{p/2},$$

where β is an enveloping sieve function with $R = N'^{1/10}$. This means that, according to [GT, Prop. 3.1], $\beta : \mathbb{Z}^+ \rightarrow \mathbb{R}$ is a non-negative function satisfying the majorant property

$$\beta(n) \gg \mathfrak{S}_F^{-1} \cdot \log R \cdot \mathbf{1}_{X_{R!}}(n)$$

with

$$\mathfrak{S}_F = \prod_p \frac{\gamma(p)}{1 - 1/p},$$

$$\gamma(p) = \frac{1}{p} |\{n \in \mathbb{Z}/p\mathbb{Z}, (p, b + nM) = 1\}| = \begin{cases} (1 - 1/p) & \text{if } p > z \\ 1 & \text{if } p \leq z \end{cases}$$

and

$$X_{R!} = \{n \in \mathbb{Z} : \forall d \leq R \ (b + nM, d) = 1\}.$$

In particular, for any integer $n \in A_0$, we have $n \in X_{R!}$ and

$$\beta(n) \gg (\log R) \prod_{p \leq z} (1 - 1/p)^{-1} \gg \frac{\log N}{\log z}.$$

We apply (2.5) to the sequence $(b_n)_n$ defined by

$$b_n = \begin{cases} \frac{1}{\beta(n)} a(n) & \text{if } n \in A_0 \\ 0 & \text{otherwise} \end{cases}$$

and get

$$\begin{aligned}
\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^p &= (N')^p \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} \left| \frac{1}{N'} \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} a(n) e(-mn/N) \right|^p \\
&\ll_p (N')^{p/2} \left(\sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \frac{1}{\beta(n)} a(n)^2 \right)^{p/2} \\
&\ll_p (N')^{p/2} \left(\sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \frac{\log z}{\log N} \left(\frac{\log N}{N' \log z} \right) a(n) \right)^{p/2} \\
&\ll_p \left(\sum_{n \in \mathbb{Z}/N'\mathbb{Z}} a(n) \right)^{p/2} \ll_p 1,
\end{aligned}$$

since $a(n)$ was normalised so that $\sum_n a(n) \ll 1$. □

By Hölder's inequality and Lemma 2.2, we have

$$\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(-2m)\hat{a}(m)^2| \leq \left(\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{5/2} \right)^{2/5} \left(\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{10/3} \right)^{3/5} \ll 1.$$

Hence, by (2.3),

$$\sum_{m: |\hat{a}(m)| \geq \delta} |\hat{a}(-2m)\hat{a}(m)^2| |1 - \hat{\sigma}(-2m)\hat{\sigma}(m)^2| \ll \varepsilon.$$

On the other hand (again by Hölder, and again by Lemma 2.2),

$$\begin{aligned}
\sum_{m: |\hat{a}(m)| < \delta} |\hat{a}(-2m)\hat{a}(m)^2| |1 - \hat{\sigma}(-2m)\hat{\sigma}(m)^2| &\leq 2 \sum_{m: |\hat{a}(m)| < \delta} |\hat{a}(-2m)\hat{a}(m)^2| \\
&\leq 2 \left(\sum_{m: |\hat{a}(m)| < \delta} |\hat{a}(m)|^{5/2} \right)^{2/5} \left(\sum_{m: |\hat{a}(m)| < \delta} |\hat{a}(m)|^{10/3} \right)^{3/5} \\
&\leq 2 \left(\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{5/2} \right)^{2/5} \left(\delta^{5/3} \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{5/3} \right)^{3/5} \ll \delta.
\end{aligned}$$

Thus

$$(2.6) \quad \Delta \ll (\varepsilon + \delta).$$

2.3. An upper bound for the ℓ_2 -norm of a_1 . Our aim in this subsection is to bound from above the ℓ_2 -norm of the function $a_1 = a * \sigma$. (This will later enable us to show that a_1 is in some sense close to the characteristic function of a set of large density in the integers.) We will prove that $|a_1|_2 \ll 1/\sqrt{N'}$, where the implied constant is absolute.

Recall that we write π for the reduction map $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/N'\mathbb{Z}$. Given a function $f : \mathbb{Z}/N'\mathbb{Z} \rightarrow \mathbb{C}$, we can lift it to a function $\tilde{f} : \mathbb{Z} \rightarrow \mathbb{C}$ supported on $[-(N'-1)/2, (N'-1)/2]$:

$$\tilde{f}(n) = \begin{cases} f(n \bmod N') & \text{if } n \in [-\frac{N'-1}{2}, \frac{N'-1}{2}], \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of A_0 and a , we see that $A_0 \subset \pi([1, (N'-1)/2])$, and thus a is supported on $\pi([1, (N'-1)/2])$. By the definition of R , B and σ and the assumption $\varepsilon < 1/4$, we see that σ is supported on $\pi([-N'/4, N'/4])$. Thus $|a * \sigma|_2 = |\tilde{a} * \tilde{\sigma}|_2$.

By the definition of a , we have $0 \leq \tilde{a}(n) \leq \lambda(n)$, where $\lambda : \mathbb{Z} \rightarrow \mathbb{R}$ is defined by

$$(2.7) \quad \lambda(n) = \begin{cases} \frac{\log N}{N' \log z} & \text{if } 1 \leq n \leq N' \text{ and } b + nM \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

Recall that σ is non-negative, and thus $\tilde{\sigma}$ is non-negative. Hence $|\tilde{a} * \tilde{\sigma}|_2 \leq |\lambda * \tilde{\sigma}|_2$. We conclude that

$$|a_1|_2 = |a * \sigma|_2 = |\tilde{a} * \tilde{\sigma}|_2 \leq |\lambda * \tilde{\sigma}|_2.$$

It is thus our task to prove that $|\lambda * \tilde{\sigma}|_2 \ll 1/\sqrt{N'}$.

We proceed as follows:

$$(2.8) \quad \begin{aligned} \sum_n |\tilde{\sigma} * \lambda(n)|^2 &= \sum_n \left| \sum_m \tilde{\sigma}(m) \lambda(n-m) \right|^2 \\ &= \sum_{m_1} \sum_{m_2} \tilde{\sigma}(m_1) \tilde{\sigma}(m_2) \sum_n \lambda(n+m_1) \lambda(n+m_2), \end{aligned}$$

where we recall that $\sigma(m) = \sigma(-m)$ (by the definition of B and σ).

Lemma 2.3. *Let λ be as in (2.7). Then, for any integers m_1, m_2 ,*

$$(2.9) \quad \sum_n \lambda(n+m_1) \lambda(n+m_2) \ll \begin{cases} \log N / (N' \log z) & \text{if } m_1 = m_2, \\ \frac{1}{N'} \prod_{p|(m_1-m_2), p > z} \frac{p}{p-1} & \text{if } m_1 \neq m_2, \end{cases}$$

where the implied constant is absolute.

Proof. The case $m_1 = m_2$ follows from Brun-Titchmarsh:

$$\begin{aligned} \sum_n \lambda^2(n+m) &= \left(\frac{\log N}{N' \log z} \right)^2 |\{m \leq n \leq N' + m : b + (n-m)M \text{ is prime}\}| \\ &\ll \left(\frac{\log N}{N' \log z} \right)^2 \frac{N'M}{\varphi(M) \log N'} \\ &\ll \left(\frac{\log N}{N' \log z} \right)^2 \frac{N'}{\log N'} \prod_{p|M} (1 - 1/p)^{-1} \\ &\ll \frac{\log N}{N' \log z}. \end{aligned}$$

To obtain the case $m_1 \neq m_2$, we will use a result based on Selberg's sieve. (This is a familiar type of application of upper-bound sieves, similar to the proof that the

number of twin primes up to N is at most a constant times its conjectured value.) It is clear that $\sum_n \lambda(n + m_1) \lambda(n + m_2)$ equals $(\log N / (N' \log z))^2$ times

$$(2.10) \quad \left| \{1 \leq n \leq N' : b + nM \text{ and } b + (n + m_2 - m_1)M \text{ are primes}\} \right|.$$

By [HR, Thm. 5.7],

$$(2.10) \ll \prod_p \left(1 - \frac{\rho(p) - 1}{p - 1}\right) \left(1 - \frac{1}{p}\right)^{-1} \frac{N'}{(\log N')^2},$$

where the implied constant is absolute. (We are implicitly using the fact that $\log M \ll \log N'$, and thus the term in the third line of [HR, (8.3)] is $= 1 + o(1)$.) Here $\rho(p)$ is the number of solutions $x \in \mathbb{Z}/p\mathbb{Z}$ to

$$(b + xM)(b + (x + m_2 - m_1)M) \equiv 0 \pmod{p}$$

for p prime. It is easy to see that $\rho(p) = 0$ if $p|M$ (i.e., if $p \leq z$), $\rho(p) = 1$ if $p > z$ and $p|(m_2 - m_1)$, and $\rho(p) = 2$ if $p > z$ and $p \nmid (m_2 - m_1)$. Hence

$$\begin{aligned} (2.10) &\ll \prod_{p \leq z} \left(1 - \frac{1}{p}\right)^{-2} \prod_{\substack{p > z \\ p|m_1 - m_2}} \left(1 - \frac{1}{p}\right)^{-1} \frac{N'}{(\log N')^2} \\ &\ll \prod_{\substack{p > z \\ p|m_1 - m_2}} \frac{p}{p - 1} \frac{N'(\log z)^2}{(\log N')^2}. \end{aligned}$$

The statement follows. $\square$

Let us now evaluate the last line of (2.8), with Lemma 2.3 in hand. The contribution of the diagonal terms ($m_1 = m_2$) in (2.8) is $\ll \log N / (|B|N' \log z)$. The contribution of the non-diagonal terms ($m_1 \neq m_2$) is

$$(2.11) \quad \ll \frac{1}{N'} \sum_{\substack{m_1 \\ m_2 \neq m_1}} \sum_{m_2} \tilde{\sigma}(m_1) \tilde{\sigma}(m_2) \prod_{\substack{p > z \\ p|m_1 - m_2}} \frac{p}{p - 1}.$$

Recall that $\tilde{\sigma}$ is supported on $[-N'/4, N'/4]$, and thus $|m_2 - m_1| \leq N'/2 < N'$ whenever $\tilde{\sigma}(m_1) \tilde{\sigma}(m_2) \neq 0$.

Now, a non-zero integer m with $|m| \leq N'$ cannot have more than $\log N' / \log z$ prime factors $p > z$. Since $x/(x - 1)$ is decreasing on x , this means that

$$\prod_{\substack{p > z \\ p|m}} \frac{p}{p - 1} \leq \left(\frac{z}{z - 1}\right)^{(\log N')/(\log z)}.$$

Now $(z/(z - 1))^z \ll 1$ (because $\lim_n (1 + 1/n)^n = e$) and

$$\frac{\log N'}{\log z} \ll \frac{\log N}{\log \log N} < \log N \ll z.$$

Hence

$$\prod_{\substack{p > z \\ p|m}} \frac{p}{p-1} \ll 1$$

for any $m \neq 0$ with $|m| \leq N'$. Thus

$$(2.11) \ll \frac{1}{N'} \sum_{\substack{m_1 \\ m_2 \neq m_1}} \sum_{m_2} \tilde{\sigma}(m_1) \tilde{\sigma}(m_2) \ll \frac{1}{N'}.$$

Putting everything together, we conclude that

$$\sum_n |\tilde{\sigma} * \lambda(n)|^2 \ll \frac{1}{N'} \left(\frac{\log N}{|B| \log z} + 1 \right).$$

The right side is $\ll 1/N'$ as long as $|B| \gg \log N / \log z$.

Now, as is well-known (see, e.g., [TV, Lem. 4.20]),

$$|B| \gg \varepsilon^r N',$$

where $r = |R|$. (The proof of this is a simple pigeonhole argument.) Since by (2.4) we have $\sum_m |\hat{a}(m)|^{5/2} \ll 1$, we know that the set of $x \in \mathbb{Z}/N'\mathbb{Z}$ with $|\hat{a}(x)| \geq \delta$ has at most $\ll \delta^{-5/2}$ elements. Thus, $r \ll \delta^{-5/2}$.

Hence all that we need for $|B| \geq \log N / \log z$ to hold is that $\varepsilon^{\delta^{-5/2}} \geq N^{-1/2}$ (say). In other words, we need $|\log \varepsilon| \cdot \delta^{-5/2} \leq \frac{1}{2} \log N$. We will recall that we need to satisfy this condition at the end.

2.4. Extracting a dense set from a_1 . We now have a function $a_1 : \mathbb{Z}/N'\mathbb{Z} \rightarrow \mathbb{R}_0^+$ of ℓ_2 norm $\ll 1/\sqrt{N'}$. Its ℓ_1 norm is $\gg \alpha$, where α is the density of our original set A on the primes. We must show that there is a large set on which a_1 is large.

Lemma 2.4. *Let S be a set with N' elements. Let $a : S \rightarrow \mathbb{R}_0^+$ and $0 < \alpha < 1$ be such that*

- (a) $\|a\|_1 \geq \alpha$;
- (b) $\|a\|_2^2 \leq c/N'$.

Then there exists a subset A' of S such that

- (a) $|A'| \geq \alpha^2 N' / (4c)$;
- (b) $\forall n \in A', a(n) \geq \alpha / (2N')$.

Proof. If $A' = \{n : a(n) \geq \alpha / (2N')\}$, then

$$\begin{aligned} \alpha &\leq \sum_n a(n) \leq \frac{\alpha}{2N'} (N' - |A'|) + \sum_{n \in A'} a(n) \\ &\leq \frac{\alpha}{2N'} (N' - |A'|) + \sqrt{|A'|} \sqrt{\frac{c}{N'}}, \end{aligned}$$

by $\|a\|_2 \leq c/N'$ and Cauchy's inequality. In other words, $f(\sqrt{|A'|}) \leq 0$, where $f(x) = \frac{\alpha}{2N'} \left(x^2 - 2\frac{\sqrt{cN'}}{\alpha} x + N' \right)$. Completing the square, we see that $f(x) \leq 0$

implies $x \geq \frac{\sqrt{cN'}}{\alpha} - \sqrt{\left(\frac{c}{\alpha^2} - 1\right)N'}$. Hence

$$|A'| \geq N' \cdot \left(\frac{\sqrt{c}}{\alpha} - \sqrt{\frac{c}{\alpha^2} - 1} \right)^2 \geq \frac{\alpha^2 N'}{4c}.$$

□

We apply Lemma 2 to a_1 with the bound $\|a_1\|_2^2 \leq c/N'$ being provided by our work in §2.3. We get a subset A' of $\mathbb{Z}/N'\mathbb{Z}$ such that $|A'| \geq \alpha^2 N'/(4c) \gg \alpha^2 N'$ and $a_1(n) \geq \alpha/(2N')$ for every $n \in A'$. Hence

$$(2.12) \quad \sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) \geq \frac{\alpha^3 Z}{8N'^3},$$

where Z is the number of 3-term arithmetic progressions in A' .

Lemma 2.5. *Let $A' \subset \mathbb{Z}/N'\mathbb{Z}$, where N' is a prime. Assume $|A'| \geq \eta N'$, $\eta > 0$. The number of 3-term arithmetic progressions in A' is then at least*

$$\frac{\eta N'^2}{c_0 \exp(c_1 \eta^{-3/2} (\log(1/\eta))^3)},$$

where c_0 and c_1 are absolute constants.

Proof. We will proceed much as in [Gr, Lem. 6.8]; the basic argument goes back to Varnavides [Va]. Bourgain's best result on three-term arithmetic progressions in the integers [Bo3, Thm. 1] states that, for given L and $\eta \gg (\log \log L)^2 (\log L)^{-2/3}$, every subset of $\{1, 2, \dots, L\}$ with $\geq \eta L$ elements contains at least one non-trivial three-term arithmetic progression. This can be rephrased as follows: there are constants c_0 and c_1 such that, if $L \geq c_0 \exp(c_1 \eta^{-3/2} (\log(1/\eta))^3)$, then any subset of $\{1, \dots, L\}$ of density at least $\eta/2$ contains a non-trivial three-term arithmetic progression. (Here we are simply expressing L in terms of the density, rather than the density in terms of L .)

It follows that, given an arithmetic progression $S_{a,d} = \{a+d, a+2d, a+3d, \dots, a+Ld\}$ in $\mathbb{Z}/N'\mathbb{Z}$ ($a, d \in \mathbb{Z}/N'\mathbb{Z}$, $d \neq 0$, $L \leq N'$) whose intersection with A' has at least $(\eta/2)L$ elements, there is at least one non-trivial three-term arithmetic progression in $A' \cap S \subset \mathbb{Z}/N'\mathbb{Z}$. (Note that there is no need for the progression S to be the reduction mod N' of a progression in the integers $\{1, 2, \dots, N'\}$; the argument works regardless of this.) If we consider all arithmetic progressions of length L and given modulus $d \neq 0$ in $\mathbb{Z}/N'\mathbb{Z}$, we see that each element of A' is contained in exactly L of them. Hence, $\sum_a |S_{a,d} \cap A'| = L|A'| \geq \eta N' L$, and so (for $d \neq 0$ fixed) $|S_{a,d} \cap A'| \geq (\eta/2)L$ for at least $(\eta/2)N'$ values of a . Varying d , we get that $|S_{a,d} \cap A'| \geq (\eta/2)L$ for at least $(\eta/2)N'(N'-1)$ arithmetic progressions $S_{a,d}$. By the above, each such intersection $S_{a,d} \cap A'$ contains at least one non-trivial three-term arithmetic progression.

Each non-trivial three-term arithmetic progression a_1, a_2, a_3 in $\mathbb{Z}/N'\mathbb{Z}$ can be contained in at most $L(L-1)$ arithmetic progressions $\{a+d, a+2d, \dots, a+Ld\}$ of length L (the indices of a_1 and a_2 in the progression of length L determine the progression). Hence, when we count the three-term arithmetic progressions coming from the intersections $S_{a,d} \cap A'$, we are counting each such progression at most $L(L-1)$

times. Thus we have shown that A' contains at least $(\eta/2)(N'(N'-1))/(L(L-1)) \geq (\eta/2)(N'/L)^2$ distinct non-trivial three-term arithmetic progressions for $L = \lceil c_0 \exp(c_1 \eta^{-3/2} (\log(1/\delta))^3) \rceil$, provided that $L \leq N'$. If $L > N'$, the bound in the statement of the lemma is trivially true (as there is always at least one trivial three-term arithmetic progression in A' .) $\square$

From (2.12) and Lemma 2.5, we conclude that

$$(2.13) \quad \begin{aligned} \sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) &\geq \frac{\alpha^3}{8N'^3} \frac{\alpha^2}{8c} \frac{(N')^2}{c_0 \exp(c_1 (\alpha^2/4c)^{-3/2} (\log(4c/\alpha^2))^3)} \\ &\geq \frac{1}{N'} \frac{1}{c_2 \exp(c_3 \alpha^{-3} (\log(1/\alpha))^3)}, \end{aligned}$$

where $c_2, c_3 > 0$ are absolute constants.

3. CONCLUSION

Assume that A contains no non-trivial three-term arithmetic progressions. Then A_0 (defined in (2.1)) contains no non-trivial three term arithmetic progressions, and so

$$\sum_{m,d} a(m)a(m+d)a(m+2d) = \sum_{m,d} a(m)^3 \ll \left(\frac{\log N}{N' \log z} \right)^2.$$

We also have

$$\Delta = N' \left| \sum_{m,d} a(m)a(m+d)a(m+2d) - \sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) \right| \ll (\varepsilon + \delta),$$

by the definition (2.2) and (2.6). Lastly, we have just shown that

$$\sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) \geq \frac{1}{N'} \frac{1}{c_2 \exp(c_3 \alpha^{-3} (\log(1/\alpha))^3)}$$

(see (2.13)). We conclude that

$$(3.1) \quad \frac{1}{c_2 \exp(c_3 \alpha^{-3} (\log(1/\alpha))^3)} \ll \varepsilon + \delta + \frac{1}{N'} \left(\frac{\log N}{\log z} \right)^2.$$

Recall that $z = (\log N)/3$. There are constants c_4, c_5 such that, for

$$\delta = \varepsilon = \frac{1}{c_4} \exp(-c_5 \alpha^{-3} \log^3(1/\alpha)),$$

we get a contradiction with (3.1), provided that N is larger than an absolute constant and $\alpha \geq (\log N)^{-1/4}$, say. These values of δ and ε satisfy $|\log \varepsilon| \delta^{-2.5} \leq (\log N)/2$ as long as

$$(\log(c_4) + c_5 \alpha^{-3} \log^3(1/\alpha)) \cdot c_4^{2.5} \exp(2.5 c_5 \alpha^{-3} \log^3(1/\alpha)) \leq \frac{1}{2} \log N.$$

Therefore we have a contradiction if $\alpha \geq C \log \log \log N (\log \log N)^{-1/3}$, where C is a large enough constant and N is larger than an absolute constant. Theorem 1.1 is thereby proven.

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H. A. HELFGOTT, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRISTOL, BRISTOL BS8 1TW, UNITED KINGDOM

E-mail address: `h.andres.helfgott@bristol.ac.uk`

A. DE ROTON, INSTITUT ELIE CARTAN, UMR 7502, NANCY-UNIVERSITÉ, CNRS, INRIA, B.P. 239, 54506 VANDOEUVRE-LÈS-NANCY CEDEX, FRANCE; A. DE ROTON, PACIFIC INSTITUTE FOR THE MATHEMATICAL SCIENCES (CNRS UMI PIMS) UNIVERSITY OF BRITISH COLUMBIA VANCOUVER BC V6T 1Z2 CANADA

E-mail address: `deroton@iecn.u-nancy.fr`