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A B S T R A C T

This paper presents the extension of results on reasoning with totally ordered belief bases to the partially ordered case. The idea is to reason from logical bases equipped with a partial order expressing relative certainty and to construct a partially ordered deductive closure. The difficult point lies in the fact that equivalent definitions in the totally ordered case are no longer equivalent in the partially ordered one. At the syntactic level we can either use a language expressing pairs of related formulas and axioms describing the properties of the ordering, or use formulas with partially ordered symbolic weights attached to them in the spirit of possibilistic logic. A possible semantics consists in assuming the partial order on formulas stems from a partial order on interpretations. It requires the capability of inducing a partial order on subsets of a set from a partial order on its elements so as to extend possibility theory functions. Among different possible definitions of induced partial order relations, we select the one generalizing necessity orderings (closely related to epistemic entrenchments). We study such a semantic approach inspired from possibilistic logic, and show its limitations when relying on a unique partial order on interpretations. We propose a more general sound and complete approach to relative certainty, inspired by conditional modal logics, in order to get a partial order on the whole propositional language. Some links between several inference systems, namely conditional logic, modal epistemic logic and non-monotonic preferential inference are established. Possibilistic logic with partially ordered symbolic weights is also revisited and a comparison with the relative certainty approach is made via mutual translations.

1. Introduction

Reasoning with ordered knowledge bases expressing the relative strength of formulas has been extensively studied for more than twenty years in Artificial Intelligence. This concept goes back to Rescher's work on plausible reasoning [38]. But the idea of reasoning from formulas of various strengths is even older, since it goes back to antiquity with the texts of Theophrastus, a disciple of Aristotle, who claimed that the validity of a chain of reasoning is the validity of its weakest link.

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Possibilistic logic [21] is an approach to reason under uncertainty using totally ordered propositional bases. Each formula is assigned a degree, often encoded by a weight belonging to $(0, 1]$, seen as a lower bound on the certainty level of the formula. Such degrees of certainty obey graded versions of the principles that found the notions of belief or knowledge in epistemic logic, namely the conjunction of two formulas is not believed less than the least believed of their conjuncts. This is the basic axiom of degrees of necessity in possibility theory [18].

Deduction in possibilistic logic follows the rule of the weakest link: the strength of an inference chain is that of the least certain formula involved in this chain. The weight of a formula in the deductive closure is the weight of the strongest path leading from the base to the formula. The deductive closure of a base in possibilistic logic corresponds to a total preorder obeying the properties of an epistemic entrenchment [28] on formulas of the classical closure of the base without weights [16]. It is the dual relation to the comparative possibility originally introduced by Lewis [33], and independently retrieved as a counterpart to comparative probability by one of the authors [15], in the context of decision theory.

Possibilistic logic has developed techniques for knowledge representation and reasoning in various areas, such as non-monotonic reasoning, belief revision and belief merging (especially relevant for this paper is [17]); see references in [19,20].

In the last 10 years, only a few authors were interested in extending the possibility theory framework to the partially ordered case, and different approaches have been proposed [30,8,39,4]. In particular, the approach based on partially ordered symbolic weights [4] appears a natural extension of possibilistic logic and looks convenient to implement. It is also worth mentioning the early extension of belief revision theory to partial epistemic entrenchments by Lindström and Rabinowicz [34].

Independently, after the work of Lewis [33] on comparative possibility, conditional logics have been proposed to reason with pairs of formulas linked by a connective expressing relative certainty (or possibility), in a totally ordered setting. Halpern [30] extended relations of comparative possibility to the partially ordered case, studying several ways to extend a partial order on a set to a partial order on its subsets. Moreover, Halpern [30] proposed a conditional logic of partially ordered formulas deriving from a partial order on models.

Following the path opened by Halpern, this paper proposes a simple language, semantics and a proof method for reasoning with partially ordered belief bases, and moreover we compare this approach to possibilistic logic with partially ordered symbolic weights [4]. Before envisaging to reason with partially ordered belief bases, one may wonder where this partial order comes from, and what it means. There are two ways of understanding the lack of completeness of the relation in a partially ordered base:

- *Incomparability*: It reflects the failure to conclude on a preference between two propositions ϕ and ψ, because, according to one point of view, ϕ is preferred to ψ, and from another point of view, the opposite holds. This kind of situation is usual in multiple-criteria decision analysis. We cannot solve this type of incomparability except by modifying the data. See [14] for discussions on the meaning and relevance of the notion of incomparability in decision sciences.
- *Lack of information*: we only know that $\phi > \psi$ is true, but nothing is known for other formulas. In this case the partial order accounts for all total orders that extend it, assuming that only one of them will be correct. This view looks natural if we consider that only partial information about relative strength, for instance of belief, is available, due to lack of time to collect the whole information: the agent expresses only partial knowledge on a subset of propositions he or she finds meaningful.

In the introduction of his paper, Halpern [30] clearly adopts the first view. However, if the relationship $>$ expresses relative certainty, as in our case, one can argue that the second approach is the most natural.

This paper is the follow-up of a previous one [10] that had systematically reviewed the techniques for moving from a partial order on the elements of a set to a partial order on its parts, and systematically studied the properties of the obtained partial orders. The most promising approach in the scope of modeling relative plausibility (called "weak optimistic dominance") is a generalization of comparative possibility. It is adapted here to inference from a set of propositional logic formulas partially ordered in terms of relative certainty.

We focus on a non-modal approach, we call *relative certainty approach*, with a simplified conditional logic syntax that can only express strict dominance between formulas. There are two ways to define the semantics of a propositional partially ordered base.

- Either see each interpretation of the propositional language as the subset of formulas it violates in the base, and apply weak optimistic dominance to build a partial order on the set of interpretations. This is the "best-out" relation on models formerly introduced in [5]. It penalizes models that violate the most certain formulas.
- Or directly interpret the partial order on the set of formulas as a constraint on the possible partial orders between subsets of models.

In the case of a total preorder on formulas consisting of a fragment of a comparative possibility ordering, both approaches yield the same results. However this is no longer the case in the partially ordered case, and the first approach turns out to be faulty. We compare our approach with preferential inference of Lehmann and colleagues in non-monotonic reasoning [31], with a simplified form of epistemic logic [1] as well as some conditional logics.

It is also interesting to compare the above relative certainty approach with a variant of possibilistic logic where the weights attached to formulas are partially ordered symbols representing ill-known certainty degrees, as proposed by Benferhat and Prade [4]. This partial order of weights is interpreted as partial knowledge of a total order (symbolic weights are variables with values on the unit interval). Even in the totally preordered case, the relative certainty and possibilistic logics do not fully match because

- The relative certainty approach views the partial order on formulas in the base as constraints to be respected in the partially ordered deductive closure. In contrast, the degrees attached to formulas in possibilistic logic are but lower bounds on certainty levels, which only loosely constrain the final ranking of formulas in the deductive closure.
- Symbolic possibilistic logic uses a principle of least commitment, which tends to assign certainty levels as small as possible. It helps selecting a best preordering of interpretations. This principle is not at all assumed in the relative certainty approach.

The paper is organized as follows. First, Section 2 provides a refresher on standard possibilistic logic, namely syntax, semantics, and the proof method. In Section 3, we recall what becomes of comparative possibility and necessity relations in the partially ordered setting. A first attempt to reason from a partially ordered base using a natural semantic approach inspired from possibilistic logic is considered in Section 4, based on finding a partial order on interpretations. We show the weaknesses of this semantic approach. Then, Section 5 presents a sound and complete logic which fixes the flaw of the previous approach. This logic is close to a work by Halpern [30], albeit using a simpler language. Section 5 also extends to partial orders an approach to reason in possibilistic logic based on cuts and shows that while this approach is sound, it is not complete. Section 6 revisits possibilistic logic with partially ordered weights and proposes a variant thereof that is more in agreement with the relative certainty logic. We provide a detailed comparison between the two approaches. Finally we discuss related works. Proofs of main results can be found in Appendix A. As we use many relation symbols, a nomenclature is provided in Appendix B.

2. Background on standard possibilistic logic

We consider a propositional language $\mathcal{L}$ where formulas are denoted by $\phi, \psi, \ldots$, and Ω is the set of its interpretations. $[\phi]$ denotes the set of models of ϕ, a subset of Ω. We denote by $\vdash$ the classical syntactic inference and by $\models$ the classical semantic inference. Possibilistic logic is an extension of classical logic and it encodes conjunctions of weighted formulas of the form (ϕ_j, p_j) where ϕ_j is a propositional formula and $p_j \in]0, 1]$. The weight p_j is interpreted as a lower bound on the certainty level of ϕ_j, namely $N(\phi_j) \geq p_j > 0$, where N is a necessity measure in the sense of possibility theory [19].

Let us recall basic tools of possibility theory [18]. A possibility distribution is a mapping $\pi : \Omega \to [0, 1]$ expressing to what extent a situation, encoded as an interpretation is plausible. At least one situation must be fully plausible ($\pi(\omega) = 1$ for some ω), and $\pi(\omega) = 0$ means that this situation is impossible. A possibility measure can be defined on subsets of Ω from a possibility distribution as $\Pi(A) = \max_{\omega \in A} \pi(\omega)$ expressing the plausibility of any event A. Note that $\Pi(\varnothing) = 0$ and $\Pi(\Omega) = 1$. The necessity measure expressing certainty levels is defined by conjugacy: $N(A) = 1 - \Pi(\overline{A})$ where $\overline{A}$ denotes the complement of A. Note that $N(\varnothing) = 0$ and $N(\Omega) = 1$.

In the following, we attach possibility and necessity degrees to propositional formulas, letting $\Pi(\phi) = \Pi([\phi])$. In particular:

$$N(\phi_j) = \min_{\omega \notin [\phi_j]} (1 - \pi(\omega))$$

The basic axiom of necessity measures can then be written as $N(\phi \wedge \psi) = \min(N(\phi), N(\psi))$. The reader will notice the similarity between this axiom and one of the basic properties of belief in modal epistemic logics [25], where the equivalence between $\Box(\phi \wedge \psi)$ and $\Box\phi \wedge \Box\psi$ is the law of accepted beliefs (if you believe in ϕ and in ψ, then you believe in their conjunction). In the scope of possibilistic logic, $\Box\phi$ stands for $(\phi, 1)$ [1], but the basic form of possibilistic logic does not consider negations and disjunctions of weighted formulae (see [24] for generalized possibilistic logic, where these connectives make sense).

2.1. *Semantics of possibilistic logic bases using possibility distributions*

A possibilistic logic base (PL-base, for short) is a set of weighted formulas $\Sigma = \{(\phi_j, p_j) : j = 1, \cdots, m\}$. The (fuzzy) set of models of a PL-base is defined by a possibility distribution π_Σ on Ω defined as follows.

First, each weighted formula (ϕ_j, p_j) can be associated with a possibility distribution π_j on Ω defined by [21]:

$$\pi_j(\omega) = \begin{cases} 1 & \text{if } \omega \in [\phi_j], \\ 1 - p_j & \text{if } \omega \notin [\phi_j]. \end{cases} \tag{1}$$

and then:

$$\pi_\Sigma(\omega) = \min_j (\pi_j(\omega)). \tag{2}$$

The rationale is based on a minimal commitment principle called *the principle of minimal specificity*. It presupposes that any ω remains possible unless explicitly ruled out [18]. A possibility distribution π is less specific (less informative) than π' if $\pi \geq \pi'$ (π leaves more possibilities than π'). The principle of minimal specificity tends to maximize possibility degrees.

We can define the semantics of possibilistic logic in terms of the satisfaction of a PL-base Σ by a possibility distribution π on Ω as $\pi \models \Sigma$ if and only if $N(\phi_j) \geq p_j,\ j = 1, \cdots, m$ where $N(\phi_j)$ is the degree of necessity of ϕ_j w.r.t. π. Then, we can show that $\pi \models \Sigma$ if and only of $\pi \leq \pi_\Sigma$ [21]. It indicates that the possibilistic logic semantics is based on the selection of the least informative possibility distribution that satisfies Σ.

Let N_Σ be the necessity measure induced by π_Σ:

Definition 1. The deductive (semantic) closure of Σ is defined as follows:

$$\mathcal{C}_\pi(\Sigma) = \{(\phi, N_\Sigma(\phi)) : \phi \in \mathcal{L},\ N_\Sigma(\phi) > 0\}$$

Note that the initial order of the possibilistic base can be modified according to the logical dependencies between formulas. We can have $N_\Sigma(\phi_j) > p_j$. It is the case for example if $\exists i,\ \phi_i \models \phi_j$ and $p_i > p_j$.

Since the weights are only lower bounds, they never add inconsistency to the base. The only reason for inconsistency comes from the classical inconsistency of $\Sigma^* = \{\phi_1, \cdots, \phi_m\}$, we call the *skeleton* of Σ. If Σ^* is inconsistent, one may have that both $N_\Sigma(\phi) > 0$ and $N_\Sigma(\neg\phi) > 0, \forall \phi \in \mathcal{L}$. In this case, $Cons(\Sigma) = \max_\omega \pi_\Sigma(\omega) < 1$ represents the *degree of consistency* of the possibilistic base Σ.

Proposition 1. *(See [21].)* $\min_{\phi \in \Sigma} N_\Sigma(\phi) = 1 - Cons(\Sigma)$.

Then, we can define the set of non-trivial consequences of Σ as:

$$\mathcal{C}_\pi^{nt}(\Sigma) = \{(\phi, N_\Sigma(\phi)) : \phi \in \mathcal{L},\ N_\Sigma(\phi) > 1 - Cons(\Sigma)\}$$

which coincides with $\mathcal{C}_\pi(\Sigma)$ if $Cons(\Sigma) = 1$. In the latter case, it holds that $\min(N_\Sigma(\phi), N_\Sigma(\neg\phi)) = 0, \forall \phi \in \mathcal{L}$, so that the skeleton of $\mathcal{C}_\pi(\Sigma)$ is consistent.

In the general case, ϕ is called a *plausible* consequence of Σ if and only $N_\Sigma(\phi) > 1 - Cons(\Sigma)$. The meaning of plausible consequences is the following: ϕ is a plausible consequence of Σ if and only if ϕ is satisfied in all preferred models according to π_Σ [6]. If we define the strict cut at level p of Σ as $\Sigma_p^> = \{(\phi_j, p_j) : p_j > p\}$, it is easy to see that $\mathcal{C}_\pi^{nt}(\Sigma) = \mathcal{C}_\pi^{nt}(\Sigma^>_{1-Cons(\Sigma)})$.

2.2. *Ordinal semantics of possibilistic bases*

Let $\Sigma(\omega)$ denote the formulas in Σ^* satisfied by the interpretation ω. Its complement $\overline{\Sigma(\omega)}$ denotes the set of formulas in Σ^* falsified by ω. Note that

$$1 - \pi_\Sigma(\omega) = \max_{j:\phi_j \notin \Sigma(\omega)} p_j$$

which corresponds to the so-called "best-out" ordering [17]. We can then express $N_\Sigma(\phi)$ directly as:

$$N_\Sigma(\phi) = \min_{\omega \not\models \phi} \max_{j:\phi_j \notin \Sigma(\omega)} p_j$$

without explicitly referring to the possibility distribution π_Σ on models. Let $\trianglerighteq_\Sigma$ be the total pre-order on Ω defined by:

$$\omega \trianglerighteq_\Sigma \omega' \text{ if and only if } \forall \phi_j \in \overline{\Sigma(\omega)}, \exists \psi_i \in \overline{\Sigma(\omega')} \text{ such that } p_i \geq p_j \tag{3}$$

Proposition 2. *(See [17].)* $\pi_\Sigma(\omega) \geq \pi_\Sigma(\omega')$ *if and only if* $\omega \trianglerighteq_\Sigma \omega'$.

The total pre-order $\trianglerighteq_\Sigma$ allows us to build a totally pre-ordered deductive closure based on a preordering $\succeq$ as follows:

$$\phi \succeq \psi \text{ if and only if } \forall \omega \in \overline{[\phi]},\ \exists \omega' \in \overline{[\psi]},\ \omega' \trianglerighteq_{\Sigma} \omega \tag{4}$$

expressing the relative certainty of propositions in agreement with necessity measures. In particular, we have

$$\phi \succ \psi \text{ if and only if } N_{\Sigma}(\phi) > N_{\Sigma}(\psi).$$

It is important to notice that the preorder $\succeq$ does not depend on the precise values of the weights p_i of formulas, if the priority order indicated by the p_i's remains unchanged. In this sense possibilistic logic is not a genuinely numerical uncertainty logic.

2.3. Syntactic inference in possibilistic logic

A sound and complete syntactic inference $\vdash_{\pi}$ for possibilistic logic can be defined with the following axioms and inference rules [21]:

Axioms

- $(\phi \to (\psi \to \phi), 1)$
- $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)), 1)$
- $(((\neg\phi) \to (\neg\psi)) \to (\psi \to \phi), 1)$

Inference rules:

- Weakening rule: If $p_i > p_j$ then $(\phi, p_i) \vdash_{\pi} (\phi, p_j)$
- Modus Ponens: $\{(\phi \to \psi, p), (\phi, p)\} \vdash_{\pi} (\psi, p)$

The axioms are those of propositional logic with weight 1. The Modus Ponens rule embodies the law of accepted beliefs at any level, assuming they form a deductively closed set [22]. It is related to axiom K of modal epistemic logic [25]. The soundness and completeness of possibilistic logic for the above proof theory can be translated by the following equality [21]:

$$N_{\Sigma}(\phi) = \max\{p : \Sigma \vdash_{\pi} (\phi, p)\}$$

Note that we can also express inference in possibilistic logic by classical inference on cuts [21]:

$$N_{\Sigma}(\phi) = \max\{p : (\Sigma_{p}^{\geq})^{*} \vdash \phi\} \tag{5}$$

where $\Sigma_{p}^{\geq} = \{(\phi_j, p_j) : p_j \geq p\}$ is the weak cut at level p of Σ, and $(\Sigma_{p}^{\geq})^{*}$ its skeleton.

We can compute the degree of inconsistency $Inc(\Sigma)$ of a possibilistic base Σ syntactically as follows:

$$Inc(\Sigma) = \max\{p : \Sigma \vdash_{\pi} (\bot, p)\}.$$

We can prove that [21,6]:

- $Inc(\Sigma) = 1 - Cons(\Sigma) = 1 - \max_{\omega \in \Omega} \pi_{\Sigma}(\omega)$
- $N_{\Sigma}(\phi) = Inc(\Sigma \cup (\neg\phi, 1))$

So, in standard possibilistic logic, there are four ways of defining the deductive closure of a totally ordered base: the semantic approach based on a ranking of interpretations, the syntactic approach based on Modus Ponens and Weakening, the classical approach based on cuts, and reasoning by refutation. They are equivalent and yield the same deductive closure. However, we will see in the next sections that this is no longer true in the partially ordered case.

3. Possibility theory in the partially ordered setting

Let $(S, >)$ be a partially ordered set, where $>$ is a strict partial order (i.e., an asymmetric and transitive relation). There are various possible definitions for building a relation from $>$ that compares subsets A and B of S and extend comparative possibility [33] and necessity [15] to strict partial possibility orderings. These relations have been studied by Halpern [30], Benferhat et al. [8], and more extensively in [10]. There are two extensions of strict comparative possibility:

1. Weak optimistic strict dominance:
   $A >_{wos} B$ if and only if $A \neq \varnothing$ and $\forall b \in B,\ \exists a \in A,\ a > b$.
2. Strong optimistic strict dominance:
   $A >_{Sos} B$ if and only if $\exists a \in A,\ \forall b \in B,\ a > b$

It is clear that $>_{wos}$ is asymmetric and that if $(S, >)$ is the strict part of a complete preordering $\geq$ encoded by a possibility distribution π, $A >_{wos} B$ if and only if $A >_{Sos} B$ if and only if $\Pi(A) > \Pi(B)$ (where Π is the possibility measure defined from π, see Section 2), which we can denote by $A >_{\Pi} B$. This is the comparative possibility of Lewis [33]. The following property (along with obvious non-triviality properties such as $S >_{\Pi} \varnothing$ and $A \geq_{\Pi} \varnothing$)

Stability for Union (SU): If $A \geq_{\Pi} B$ then $A \cup C \geq_{\Pi} B \cup C$

is enough to characterize these complete transitive relations, which can be represented only by possibility measures [15]. The following additional properties are obviously valid for the relation $>_{\Pi}$ and its weak form $\geq_{\Pi}$ between subsets [22]:

- **Compatibility with Inclusion (CI)** If $B \subseteq A$ then $A \geq_{\Pi} B$
- **Orderliness (O)** If $A >_{\Pi} B$, $A \subseteq A'$ and $B' \subseteq B$ then $A' >_{\Pi} B'$
- **Qualitativeness (Q)** If $A \cup B >_{\Pi} C$ and $A \cup C >_{\Pi} B$ then $A >_{\Pi} B \cup C$
- **Negligibility (N)** If $A >_{\Pi} B$ and $A >_{\Pi} C$ then $A >_{\Pi} B \cup C$
- **Conditional Closure by Implication (CCI)** If $A \subseteq B$ and $A \cap C >_{\Pi} \overline{A} \cap C$ then $B \cap C >_{\Pi} \overline{B} \cap C$
- **Conditional Closure by Conjunction (CCC)** If $C \cap A >_{\Pi} C \cap \overline{A}$ and $C \cap B >_{\Pi} C \cap \overline{B}$ then $C \cap (A \cap B) >_{\Pi} C \cap \overline{A \cap B}$
- **Left Disjunction (OR)** If $A \cap C >_{\Pi} A \cap \overline{C}$ and $B \cap C >_{\Pi} B \cap \overline{C}$ then $(A \cup B) \cap C >_{\Pi} (A \cup B) \cap \overline{C}$
- **Cut (CUT)** If $A \cap B >_{\Pi} A \cap \overline{B}$ and $A \cap B \cap C >_{\Pi} A \cap B \cap \overline{C}$ then $A \cap C >_{\Pi} A \cap \overline{C}$
- **Cautious Monotony (CM)**: If $A \cap B >_{\Pi} A \cap \overline{B}$ and $A \cap C >_{\Pi} A \cap \overline{C}$ then $A \cap B \cap C >_{\Pi} A \cap B \cap \overline{C}$

Remark 1. Relation $\geq_{\Pi}$ is one of many ones extending a total order on a set to a relation between its subsets. This question is the topic of a large literature in social and decision sciences; see for instance the survey by Barberà et al. [2]. However, the problem they address is different from ours and leads to specific axioms that do not always make sense in the scope of representing comparative uncertainty (see [29] for an impressive list of impossibility theorems based on such axioms). One such axiom asserts that if an element s is worse than all elements in a set A, then adding this element to this set results in a worse set $A \cup \{s\}$. It clearly contradicts a basic axiom of comparative uncertainty, that is, Compatibility with Inclusion CI.

But this axiom is natural if sets represent ranges of possible outcomes of actions, or groups of objects from which to choose, instead of events.

In the case of partial orders, strong and weak optimistic dominance do not coincide. Clearly, $>_{Sos}$ is more demanding than $>_{wos}$. The following properties have been established for the strict orderings $>_{wos}$ and $>_{Sos}$ [30,10,11]:

Proposition 3. *The weak optimistic strict dominance $>_{wos}$ is a partial order that satisfies Qualitativeness (Q) and Orderliness (O).*

If a relation between subsets of S satisfies Qualitativeness (Q) and Orderliness (O), then it is transitive and satisfies Negligibility (N), Conditional Closure by Implication (CCI), Conditional Closure by Conjunction (CCC), Left Disjunction (OR), (CUT), (CM), and the converse of (SU) in the form: if $A \cup C >_{wos} B \cup C$ then $A >_{wos} B$.

The strong optimistic strict dominance $>_{Sos}$ is a strict order satisfying Orderliness (O) and Cautious Monotony (CM). However it fails to satisfy Negligibility, Qualitativeness, CUT and Left Disjunction (OR).

It is clear that the weak optimistic strict dominance $>_{wos}$ is the most promising extension of the comparative possibility relation to partial plausibility orders on S. In view of Proposition 3, we can introduce the following definition that generalizes comparative possibilities of [15]:

Definition 2. A partial possibility relation is an asymmetric relation $\succ_\Pi$ on 2^S that satisfies Q and O.

Due to Proposition 3, $\succ_\Pi$ is a strict partial order on the power set of S.

The properties of the relation $\succ_\Pi$, obtained by weak optimistic dominance, can be used to obtain the properties satisfied by the dual necessity relation $\succ_N$ defined by

$$A \succ_N B \text{ if and only if } \overline{B} \succ_\Pi \overline{A}.$$

Let P be a property of the relation $\succ_\Pi$. The dual property P^d is a property of the relation $\succ_N$ such that $\succ_\Pi$ satisfies P if and only if $\succ_N$ satisfies P^d.

Proposition 4. *The relation $\succ_N$ satisfies the following properties:*

$\mathbf{Q}^d$: *If $C \succ_N A \cap B$ and $B \succ_N A \cap C$ then $B \cap C \succ_N A$*
$\mathbf{O} = \mathbf{O}^d$: *If $A \succ_N B$, $A \subseteq A'$ and $B' \subseteq B$ then $A' \succ_N B'$*
Converse of Stability for intersection ($\mathbf{SI}^c$) *If $A \cap C \succ_N B \cap C$ then $A \succ_N B$*
$\mathbf{N}^d$ (Adjunction): *If $B \succ_N A$ and $C \succ_N A$ then $B \cap C \succ_N A$*
$\mathbf{OR}^d$: *If $\overline{A} \cup C \succ_N \overline{A} \cup \overline{C}$ and $\overline{B} \cup C \succ_N \overline{B} \cup \overline{C}$ then $(\overline{A \cup B}) \cup C \succ_N (\overline{A \cup B}) \cup \overline{C}$*
$\mathbf{CCC}^d$: *If $\overline{C} \cup A \succ_N \overline{C} \cup \overline{A}$ and $\overline{C} \cup B \succ_N \overline{C} \cup \overline{B}$ then $\overline{C} \cup (A \cap B) \succ_N \overline{C} \cup (\overline{A \cap B})$.*
$\mathbf{CUT}^d$: *If $\overline{A} \cup B \succ_N \overline{A} \cup \overline{B}$ and $(\overline{A \cap B}) \cup C \succ_N (\overline{A \cap B}) \cup \overline{C}$ then $\overline{A} \cup C \succ_N \overline{A} \cup \overline{C}$*
$\mathbf{CM}^d$: *If $\overline{A} \cup B \succ_N \overline{A} \cup \overline{B}$ and $\overline{A} \cup C \succ_N \overline{A} \cup \overline{C}$ then $(\overline{A \cap B}) \cup C \succ_N (\overline{A \cap B}) \cup \overline{C}$*

The adjunction property is instrumental to ensure the deductive closure of the set of propositions $\{\phi : \phi \succ_N \psi\}$ more believed than a given one, ψ. In view of Proposition 3 and Definition 2, we can generalize comparative certainty relations:

Definition 3. A partial certainty relation is an asymmetric relation $\succ_N$ on 2^S that satisfies Q^d and O.

Such a relation is adjunctive and transitive. It is closely related to the partial epistemic entrenchment relations on a logical language in [34]. Viewed as relations between sets of possible worlds, the latter are transitive, reflexive, satisfy CI and Adjunction.

4. Partially ordered deductive closure via partially ordered interpretations

Consider a propositional language $\mathcal{L}$, and a finite subset $\mathcal{K}$ of $\mathcal{L}$ equipped with a strict partial order. We call $(\mathcal{K}, >)$ a *partially ordered belief base* (*po-base*, for short), where $\phi_i > \psi_i$ is supposed to express that ϕ_i is more certainly true than ψ_i. In this section, we will study how to infer from $(\mathcal{K}, >)$ a partial certainty relation between formulas in the whole language $\mathcal{L}$. As recalled in [10], there already exist some approaches addressing this problem:

1. Viewing the strict partial order $>$ as a family of total orders that extend the partially ordered base $(\mathcal{K}, >)$, we can interpret a partially ordered base as a set of totally ordered bases, and apply possibilistic logic to all of them [39].
2. Viewing $(\mathcal{K}, >)$ as a finite set of assertions of the form $\phi_i > \psi_i$ encoded in a modal logic format (in the tradition of conditional logic), we can directly infer formulas such as $\phi > \psi$ [30].
3. We can attach a symbolic weight to each formula in $\mathcal{K}$ and generalize possibilistic logic to certainty degrees forming a partially ordered set [4].

The first approach looks difficult to implement in practice, because there are numerous total orders that extend a given partial order. In this section, we adopt the syntax used in the second point of view, albeit with the concern of sticking to a simple language amenable to an implementation in practice. In the following, we will propose a definition of deductive closure of $(\mathcal{K}, >)$ from a semantic point of view, by first building a partial order $\rhd$ over interpretations of $\mathcal{L}$, in the style of possibilistic logic, then reconstructing a partial certainty relation on $\mathcal{L}$ from $\rhd$.

Note that the partially ordered logic of [8] relies on this approach, but it aims at inferring preferred formulas only. On the contrary, our aim is to extend the partial certainty ordering from the set $\mathcal{K}$ to the whole language, in the spirit of possibilistic logic. However, this approach, that mimics the semantics of possibilistic logic, is here shown to be not as satisfactory as expected.

4.1. Extending possibilistic logic semantics

Suppose an epistemic state is modeled by a partial preorder $\rhd$ on Ω. If ω and ω' are two elements of Ω, the assertion $\omega' \rhd \omega$ is interpreted as: "ω' is more plausible than ω". In the knowledge representation literature (like in [8]), the main concern is often to extract the closed set of accepted beliefs (or belief set) associated with $(\Omega, \rhd)$. The idea is to build $\rhd$ from the ordering $>$ on $\mathcal{K}$. The set of accepted beliefs is often defined as the deductively closed set of formulas whose models form the set $M(\Omega, \rhd)$ of most plausible models. Our aim is to go further and to define a partially ordered deductive closure induced by $(\Omega, \rhd)$ on the language. The question is thus to go from $(\mathcal{K}, >)$ to $(\Omega, \rhd)$ and back. In possibilistic logic, two interpretations are compared by considering their most certainly falsified formulas, and two formulas are then compared by considering their most plausible counter-models. This ordinal construction can be generalized to arbitrary po-bases for building a deductive closure:

From $(\mathcal{K}, >)$ *to* $(\Omega, \rhd)$ Starting from a po-base, $(\mathcal{K}, >)$ a natural approach is to compare two interpretations ω and ω' by comparing subsets of formulas of $\mathcal{K}$ defined via these interpretations. As we are modeling relative certainty, we can generalize the definition of $\unrhd_\Sigma$ in possibilistic logic (see Equation (3)), and define $\omega' \rhd \omega$

by comparing the two subsets of formulas of $\mathcal{K}$ respectively *falsified* by each of these interpretations. Indeed, an interpretation ω is all the less plausible as it violates more certain propositions.

From $(\Omega, \rhd)$ *to* $(\mathcal{L}, \succ)$ Starting from a partial preorder on Ω, the problem is to build a partial preorder on the set of the formulas of the language $\mathcal{L}$. To this end, it is natural to compare two formulas ϕ and ψ by comparing subsets of interpretations built from these formulas. Since $(\mathcal{K}, >)$ is interpreted in terms of relative certainty as in possibilistic logic, it is natural to compare their sets of counter-models, that is the models of $\neg\psi$ and $\neg\phi$, in agreement with Equation (4).

The above construction requires the capability of inducing a partial order on subsets of a set from a partial order on its elements, as done in the previous section. Let $\mathcal{K}(\omega)$ denote the subset of formulas of $\mathcal{K}$ satisfied by ω. So its complement $\overline{\mathcal{K}(\omega)}$ denotes the subset of formulas of $\mathcal{K}$ falsified by ω.

A partially ordered deductive closure $(\mathcal{L}, \succ)$ reflecting a partial certainty ordering, already suggested in [10], can be constructed in two steps, applying twice the extension of a partial order on the elements to a partial order on subsets, using the weak optimistic relation $>_{wos}$. The partial order on interpretations can be derived from $(\mathcal{K}, >)$ by extending Equation (3) as follows [8]:

Definition 4. $\forall \omega,\ \omega' \in \Omega,\ \omega \rhd \omega'$ if and only if $\overline{\mathcal{K}(\omega')} >_{wos} \overline{\mathcal{K}(\omega)}$.

In the spirit of possibilistic logic, it defines the dominance on interpretations in terms of violation of the more certain formulas (best-out order) [17]. We obtain a strict partial plausibility over the interpretations. However, in the partially ordered case, some interpretations may be incomparable.

The partial certainty order induced on the language $\mathcal{L}$ by $(\mathcal{K}, >)$ is then defined as in Equation (4):

Definition 5. $\forall \phi, \psi \in \mathcal{L}, \phi \succ \psi$ if and only if $\overline{[\psi]} \rhd_{wos} \overline{[\phi]}$.

In other words, $\phi \succ \psi$ if and only if $\forall \omega \in \overline{[\phi]}, \exists \omega' \in \overline{[\psi]},\ \omega' \rhd \omega$. In the case of a total order $\rhd$ on Ω, this amounts to defining a necessity-based ranking on the language [19]. The relation defined by $[\phi] \succ [\psi]$ if and only if $\phi \succ \psi$ is a partial certainty relation.

Now, the deductive closure can be defined as follows:

Definition 6. The possibilistic deductive closure $\mathcal{C}_{wos}(\mathcal{K}, >)$ of the partially ordered set $(\mathcal{K}, >)$ is the partial order induced on $\mathcal{L}$ by the relation $\succ$, i.e., we define $\mathcal{C}_{wos}(\mathcal{K}, >) = (\mathcal{L}, \succ)$. We write $(\mathcal{K}, >) \models \phi \succ \psi$ whenever $\phi \succ \psi$ holds in $\mathcal{C}_{wos}(\mathcal{K}, >)$.

Remark 2. In agreement with [22], one may extract from $\mathcal{C}_{wos}(\mathcal{K}, >)$ the deductively closed set of accepted beliefs when ϕ is known to be true as:

$$\mathcal{A}_\phi(\mathcal{K}, >) = \{\psi : \phi \to \psi \succ \phi \to \neg\psi\}.$$

This is a kind of conditioning, and a form of revision by ϕ [34]. In fact, if $\psi \in \mathcal{A}_\phi(\mathcal{K}, >)$ it means that, in the context where we only know that ϕ is true, ψ is more plausible than its negation. The relative certainty logic in [8] essentially computes $\mathcal{A}_\top(\mathcal{K}, >)$.

The above notion of semantic consequence directly generalizes the one of possibilistic logic, but it turns out to be problematic as shown now.

4.2. Limitations of the possibilistic semantic closure

We start our discussion by giving a counterexample showing that the above construction has flaws. Suppose the language $\mathcal{L}$ has atomic variables x, y, z.

Example 1. Let $\mathcal{K} = \{\neg x \vee y, x \wedge y, x, \neg x\}$ be a base equipped with the strict partial order $\neg x \vee y > x \wedge y, x \wedge y > \neg x$ and $x > \neg x$.

- From $(\mathcal{K}, >)$ to $(\Omega, \rhd)$: we obtain $xy \rhd \overline{x}y$, $xy \rhd x\overline{y}$, $xy \rhd \overline{x}\ \overline{y}$. For instance, $\mathcal{K}(xy) = \{\neg x \vee y, x \wedge y, x\}, \mathcal{K}(\overline{x}y) = \{\neg x \vee y, \neg x\}$. Their complements are respectively $\{\neg x\}$ and $\{x \wedge y, x\}$, and both $x > \neg x$, $x \wedge y > \neg x$ hold.
- From $(\Omega, \rhd)$ to $(\mathcal{L}, \succ)$: we obtain $x \succ \neg x$, $x \wedge y \succ \neg x$ and $\neg x \vee y \succ \neg x$ but not $\neg x \vee y \succ x \wedge y$. Indeed, for the latter, we must check that $[\neg(x \wedge y)] \rhd_{wos} [\neg(\neg x \vee y)]$, that is, check if $\{\overline{x}y, \overline{x}\ \overline{y}, x\overline{y}\} \rhd_{wos} \{x\overline{y}\}$. However, these elements are incomparable using $\rhd_{wos}$.

So, in the above example, $(\mathcal{K}, >)$ is not a fragment of $\mathcal{C}_{wos}(\mathcal{K}, >)$, thus violating one of Tarski's axioms (reflexivity: $A \subseteq \mathcal{C}(A)$). The reason is that some information has been lost when going from $(\mathcal{K}, >)$ to $(\Omega, \rhd)$. Indeed, if the strict partial order $>$ on the base $\mathcal{K}$ is interpreted as part of a strict relation $\succ$ on the language,[1] induced by some ill-known partial order $\rhd'$ on interpretations according to Definition 5, the following constraints must be satisfied:

- due to $\neg x \vee y \succ x \wedge y$ we must have $(\overline{x}y \rhd' x\overline{y}$ or $\overline{x}\ \overline{y} \rhd' x\overline{y})$
- due to $x \wedge y \succ \neg x$ we must have $(xy \rhd' x\overline{y})$ and $(xy \rhd' \overline{x}y$ or $x\overline{y} \rhd' \overline{x}y)$ and $(xy \rhd' \overline{x}\ \overline{y}$ or $x\overline{y} \rhd' \overline{x}\ \overline{y})$
- due to $x \succ \neg x$ we must have $(xy \rhd' \overline{x}y$ or $x\overline{y} \rhd' \overline{x}y)$ and $(xy \rhd' \overline{x}\ \overline{y}$ or $x\overline{y} \rhd' \overline{x}\ \overline{y})$

It is easy to see that these constraints enforce a partial order such that $xy \rhd'\ \omega$, $\forall \omega \in \{\overline{x}y, x\overline{y}, \overline{x}\ \overline{y}\}$ (this is the partial order $\rhd$) and also $(\overline{x}y \rhd' x\overline{y}$ or $\overline{x}\ \overline{y} \rhd' x\overline{y})$. The latter is an additional condition on top of the one defining the partial order $\rhd$, and obtained by applying Definition 4 to the example; these two conditions cannot be represented by a single partial order on Ω.

So, the partial order $\rhd$ on interpretations in Definition 4 cannot capture the semantic content of $(\mathcal{K}, >)$ in terms of relative certainty. This is because, contrary to the totally ordered case, a partial order over $\mathcal{K}$ cannot be characterized by a unique partial order on interpretations. When using only the partial order $\rhd$, we drop the additional constraint $(\overline{x}y \rhd' x\overline{y}$ or $\overline{x}\ \overline{y} \rhd' x\overline{y})$, hence we lose some knowledge from $(\mathcal{K}, >)$ on the way.

In addition, the relation $\succ$ possesses properties that do not appear explicitly in $(\mathcal{K}, >)$, and the relation $\rhd$ induced on models will ignore statements in $(\mathcal{K}, >)$ in conflict with such properties. For example if ϕ and ψ are logically equivalent consistent formulas, i.e. $[\phi] = [\psi] = A \neq \varnothing$, and $(\mathcal{K}, >) = \{\phi > \psi\}$, it is straightforward to see that $\omega \rhd \omega'$ if and only if $\omega \models \phi$ and $\omega' \models \neg\phi$. So we shall not get $\phi \succ \psi$ in the deductive closure (since $\succ$ is irreflexive).

In contrast, even if $(\mathcal{K}, >)$ expresses properties of $\succ$, they may be lost by the possibilistic semantic closure. For instance, let $\mathcal{K} = \{x, y, x \wedge z, y \wedge z\}$ with $x > y, x \wedge z > y \wedge z$, then $\rhd$ only contains $xyz \rhd \omega, \forall \omega \neq xyz$, $x\overline{y}z \rhd \overline{x}yz$, and $x\overline{y}z \rhd \overline{x}y\overline{z}$. It allows us to retrieve neither $x > y$ nor $x \wedge z > y \wedge z$. However the po-base syntactically expresses a property (SI^c, see Proposition 4) of the certainty relation $\succ$ between the respective sets of models.

5. An inference system for relative certainty

As shown in Section 4, the deductive closure built from constructing a partial order on interpretations from a partial order on formulas does not always preserve the initial ordering of the base. We propose to use a stronger semantics in terms of subsets of interpretations, and an inference system that is faithful to this semantics. The idea is to express the po-base $(\mathcal{K}, >)$ in a higher order language $\mathcal{L}_>$, and to use the

[1] Enforcing reflexivity of the consequence relation $\mathcal{C}$.

properties of the relation $\succ_N$ listed in Proposition 4 as inference rules in the spirit of [30]. We formally define the syntax of $\mathcal{L}_>$, and the alternative semantics, before proposing an inference system $\mathcal{S}$ which is sound and complete for that semantics.

5.1. Syntax

The main idea of the proposed syntax is to encapsulate the language $\mathcal{L}$ inside a language equipped with a binary connective $>$ (interpreted as a partial order relation). Formally, a positive literal $\Phi \in \mathcal{L}_>$ is of the form $\phi > \psi$ where ϕ and ψ are classical propositional formulas in $\mathcal{L}$. A formula of $\mathcal{L}_>$ is either a positive literal Φ of $\mathcal{L}_>$, or a conjunction of positive literals, that is, $\Psi \wedge \Gamma \in \mathcal{L}_>$ if $\Psi, \Gamma \in \mathcal{L}_>$ or the formula $\perp_>$ which stands for the contradiction in $\mathcal{L}_>$ (we exclude negations and disjunctions of atomic formulas just like in basic possibilistic logic, where we do not use negations nor disjunctions of weighted formulae).

A *relative certainty base* (*rc-base*, for short) $\mathcal{B}$ is a finite subset of $\mathcal{L}_>$. We associate to a po-base $(\mathcal{K}, >)$ the set of formulas of the form $\phi > \psi$ encoding the strict partial order, and forming a base $\mathcal{B}_{(\mathcal{K},>)} \subset \mathcal{L}_>$. Note that if $\phi > \psi$ and $\psi > \eta$ are in $\mathcal{B}_{(\mathcal{K},>)}$, so is $\phi > \eta \in \mathcal{B}_{(\mathcal{K},>)}$, because the partial order $>$ over $\mathcal{K}$ is transitive.

For each po-base $(\mathcal{K}, >)$, $(\mathcal{K}, >) \vdash_X \Phi$ denotes that $\Phi \in \mathcal{L}_>$ is a consequence of the base $\mathcal{B}_{(\mathcal{K},>)}$ in an inference system X.

5.2. A semantics based on partial certainty relations

We consider a semantics of a po-base defined by a relation between *sets of* interpretations (instead of interpretations). The idea is to interpret the formula $\phi > \psi$ on 2^Ω by $[\phi] \succ_N [\psi]$ for a partial certainty relation $\succ_N$ as per Definition 3. A partial relative certainty model is a structure $\mathcal{N} = (2^\Omega, \succ_N)$ where $\succ_N$ is a partial certainty relation on 2^Ω.

We define the satisfiability of a formula $\phi > \psi \in \mathcal{L}_>$ as $\mathcal{N} \vDash (\phi > \psi)$ if and only if $[\phi] \succ_N [\psi]$. Then the satisfiability of the set of formulas $\mathcal{B}_{(\mathcal{K},>)}$ is defined as $\mathcal{N} \vDash \mathcal{B}_{(\mathcal{K},>)}$ if and only if $\mathcal{N} \vDash (\phi_i > \psi_i), \forall \phi_i > \psi_i \in \mathcal{B}_{(\mathcal{K},>)}$. The associated semantic consequence $\vDash_N$ can then be defined in the usual way:

$$(\mathcal{K}, >) \vDash_N \phi > \psi \text{ if and only if } \forall \mathcal{N}, \text{ if } \mathcal{N} \vDash \mathcal{B}_{(\mathcal{K},>)} \text{ then } \mathcal{N} \vDash \phi > \psi. \tag{6}$$

Note that we cannot always interpret $\phi > \psi$ in $(\mathcal{K}, >)$ as $[\phi] \succ_N [\psi]$ for a partial relative certainty relation. For instance, suppose $\phi \vDash \psi$ and $\phi > \psi \in \mathcal{B}(\mathcal{K}, >)$; then, it is impossible to have $[\phi] \succ_N [\psi]$ since the relation $\succ_N$ satisfies O. This comes down to saying that no model of this formula in $\mathcal{L}_>$ exists for this semantics. We then say that $(\mathcal{K}, >)$ is *inconsistent with respect to the relative certainty semantics*, in short, *rc-inconsistent*.

5.3. Axioms and inference rules

Our inference system, denoted by $\mathcal{S}$, directly interprets the atomic formulas $\phi > \psi$ in $\mathcal{L}_>$ by means of the partial certainty relation $\succ_N$ having properties in Proposition 4 for comparing the sets of models $[\phi]$ and $[\psi]$. The idea behind the proof system is to use the characteristic properties of relation $\succ_N$, expressed in terms of formulas, as inference rules. We need one axiom and three inference rules in the language $\mathcal{L}_>$:

Axiom

ax_1: $\phi > \perp$ whenever ϕ is a tautology.

Inference rules

RI_1:	If $\chi > \phi \wedge \psi$ and $\psi > \phi \wedge \chi$ then $\psi \wedge \chi > \phi$	(Q^d).
RI_2:	If $\phi > \psi$, $\phi \vDash \phi'$ and $\psi' \vDash \psi$ then $\phi' > \psi'$	(O).
RI_3:	If $\phi > \psi$ and $\psi > \phi$ then $\perp_>$	(AS).

The axiom says that the order relation is not trivial.[2] Rules RI_1 and RI_2 correspond to the properties of Qualitativeness and Orderliness. Rule RI_3 expresses the asymmetry of the relation $>$. We denote by $\mathcal{S}$ the inference system composed of the axiom ax_1 and the three inference rules RI_1, RI_2, RI_3.

Remark 3. The order relation $>$ does not contradict classical inference. Indeed, if we have $\psi \vDash \phi$ and $\psi > \phi$, by RI_2 we prove that $\phi > \phi$ and by RI_3 the contradiction.

Other inference rules can be derived from the above rules, due to Proposition 4:

RI_4:	If $\phi > \psi$ and $\psi > \chi$ then $\phi > \chi$	(T).
RI_5:	If $\psi > \phi$ and $\chi > \phi$ then $\psi \wedge \chi > \phi$	(ADJunction).
RI_6:	If $\phi \to \chi > \phi \to \neg\chi$ and $\psi \to \chi > \psi \to \neg\chi$ then $(\phi \vee \psi) \to \chi > (\phi \vee \psi) \to \neg\chi$	(OR^d).
RI_7:	If $\chi \to \phi > \chi \to \neg\phi$ and $\chi \to \psi > \chi \to \neg\psi$ then $\chi \to (\phi \wedge \psi) > \chi \to \neg(\phi \wedge \psi)$	(CCC^d).
RI_8:	If $\phi \to \psi > \phi \to \neg\psi$ and $(\phi \wedge \psi) \to \chi > (\phi \wedge \psi) \to \neg\chi$ then $\phi \to \chi > \phi \to \neg\chi$	(CUT^d).
RI_9:	If $\phi \to \psi > \phi \to \neg\psi$ and $\phi \to \chi > \phi \to \neg\chi$ then $(\phi \wedge \psi) \to \chi > (\phi \wedge \psi) \to \neg\chi$	(CM^d).
RI_{10}:	If $\phi > \perp$ then $\phi > \neg\phi$	

Proposition 5. *The inference rules $RI_4, RI_5, RI_6, RI_7, RI_8, RI_9, RI_{10}$ are valid in $\mathcal{S}$.*

Proof of Proposition 5. This is because properties Q and O are enough to derive the other properties of $>_{wos}$, as recalled in Proposition 3 and proven in [30,10,11], hence Q^d and O are enough to retrieve the properties listed in Proposition 4 for the relation modeled by the rules of the inference system $\mathcal{S}$. For RI_{10}, apply RI_1 to $\varphi > \perp$ written twice as $\varphi > \varphi \wedge \neg\varphi$, letting $\chi = \psi = \varphi, \phi = \neg\varphi$, and get $\varphi \wedge \varphi > \neg\varphi$. $\square$

In the following, $(\mathcal{K}, >) \vdash_{\mathcal{S}} \Phi$ denotes that Φ is a consequence of the partially ordered set $(\mathcal{K}, >)$ in the inference system $\mathcal{S}$.

Using the semantics and semantic consequence outlined in the previous subsection, we can prove:

Proposition 6. *Let $(\mathcal{K}, >)$ be a po-base, $\phi, \psi \in \mathcal{L}$.*

- ***Soundness:***
  If $(\mathcal{K}, >) \vdash_{\mathcal{S}} \phi > \psi$ then $(\mathcal{K}, >) \vDash_N \phi > \psi$.
- ***Completeness:***
  If $(\mathcal{K}, >)$ is rc-consistent and $(\mathcal{K}, >) \vDash_N \phi > \psi$ then $(\mathcal{K}, >) \vdash_{\mathcal{S}} \phi > \psi$.
  If $(\mathcal{K}, >)$ is rc-inconsistent then $(\mathcal{K}, >) \vdash_{\mathcal{S}} \perp_>$.

[2] This axiom could be replaced by $\phi \ \vee \neg\phi > \psi \wedge \neg\psi$, in the presence of the inference rule RI_2.

Note that the only possible form of inconsistency that can be syntactically detected in $(\mathcal{K}, >)$ is when $(\mathcal{K}, >) \vdash_{\mathcal{S}} \phi > \psi$ and $(\mathcal{K}, >) \vdash_{\mathcal{S}} \psi > \phi$.

By construction, the inference relation $\vdash_{\mathcal{S}}$ does not lose any statement $\phi > \psi$ originally present in the base on the way, contrary to the inference using a partial order $\rhd$ on Ω in Example 1.

Example 2. Let $\mathcal{K} = \{x, \neg x, y, \neg y\}$ be equipped with the strict partial order $x > \neg x$ and $y > \neg y$. Using inference rule RI_7 by considering $x > \neg x$ as $\top \to x > \top \to \neg x$ and $y > \neg y$ as $\top \to y > \top \to \neg y$, we have $x \wedge y > \neg x \vee \neg y$. Then by RI_2 we obtain $y > \neg x$. And similarly we obtain $x > \neg y$. The inference using $\rhd$ on Ω yields none of these results.

5.4. *Partially ordered deductive closure based on level cuts*

Another approach to inferring from a po-base is to adapt the cut-based methodology of possibilistic logic using a counterpart to Equation (5). Namely, we apply classical logic to classical bases formed by considering formulas with certainty level higher than a prescribed threshold, then moving this threshold. However we shall see that, contrary to the case of possibilistic logic, this approach, although sound, is not powerful enough. Let $(\mathcal{K}, >)$ be a po-base. Starting from a logical base equipped with a strict partial order $>$, the idea is to conclude $\phi > \psi$ whenever ϕ is classically deducible from a consistent set of formulas $\{\chi_i \in \mathcal{K}\}$ such that $\forall i, \chi_i > \psi$ is in $(\mathcal{K}, >)$. This intuition makes sense in the scope of relative certainty relations, that satisfy Adjunction. Let $\psi \in \mathcal{K}$, we define:

$$\mathcal{K}_{\psi}^{>} = \{\chi : \chi \in \mathcal{K} \text{ and } \chi > \psi\}.$$

Definition 7. The formula $\phi > \psi$ in $\mathcal{L}_{>}$ is cut-inferred from a po-base $(\mathcal{K}, >)$, what we denote by $(\mathcal{K}, >) \vdash_c \phi > \psi$, if and only if $\mathcal{K}_{\psi}^{>} \vdash \phi$.

Let $\mathcal{C}_c(\mathcal{K}, >) = \{\phi > \psi : (\mathcal{K}, >) \vdash_c \phi > \psi\}$ be the cut-based deductive closure of $(\mathcal{K}, >)$.

This inference overcomes some shortcomings of the semantic approach based on a single partial ordering of models of Section 4.

Example 1 (continued). $\mathcal{K}_{\neg x}^{>} = \{x,\ x \wedge y,\ \neg x \vee y\}$ and $\mathcal{K}_{\neg x}^{>} \vdash y$, so $(\mathcal{K}, >) \vdash_c y > \neg x$. $\mathcal{K}_{x \wedge y}^{>} = \{\neg x \vee y\}$ and $\mathcal{K}_{x \wedge y}^{>} \vdash \neg x \vee y$, so $(\mathcal{K}, >) \vdash_c \neg x \vee y > x \wedge y$. The comparison $\neg x \vee y > x \wedge y$ has been preserved, which is not the case in the approach of Section 4.1.

It is obvious that this inference relation is reflexive, i.e., $\mathcal{B}_{(\mathcal{K},>)} \subset \mathcal{C}_c(\mathcal{K}, >)$, since by construction, if $\phi > \psi \in (\mathcal{K}, >), \phi \in \mathcal{K}_{\psi}^{>}$. Note that if the set of formulas $\mathcal{K}_{\psi}^{>}$ is classically inconsistent for some $\psi \in \mathcal{K}$, then $\mathcal{C}_c(\mathcal{K}, >)$ will contain $\chi > \psi, \forall \chi \in \mathcal{L}$, which trivializes this inference.

Example 3. Consider the base $\mathcal{K} = \{x, \neg x, y\}$ equipped with the strict partial order $\neg x > y$ and $x > y$. Then $\mathcal{K}_{y}^{>}$ is inconsistent and so $(\mathcal{K}, >) \vdash_c \phi > y, \forall \phi$.

Let us consider the relation $>_c$ on $\mathcal{L}_{>}$ defined by $\phi >_c \psi$ whenever $\phi > \psi \in \mathcal{C}_c(\mathcal{K}, >)$. Relation $>_c$ partially satisfies the property of Orderliness (O): if $\phi_1 > \psi \in \mathcal{B}_{(\mathcal{K},>)}$ and $\phi_1 \vDash \phi_2$ then $\mathcal{K}_{\psi}^{>} \vdash \phi_2$ and so $\phi_2 > \psi \in \mathcal{C}_c(\mathcal{K}, >)$.

Proposition 7. *The relation $>_c$ satisfies Adjunction N^d.*

Proof of Proposition 7. Assume that $\phi_1 >_c \psi$ and $\phi_2 >_c \psi$. It means that $\phi_1 > \psi$ and $\phi_2 > \psi$ belong to $\mathcal{C}_c(\mathcal{K}, >)$. So, we have $\mathcal{K}_{\psi}^{>} \vdash \phi_1$ and $\mathcal{K}_{\psi}^{>} \vdash \phi_2$. Then $\mathcal{K}_{\psi}^{>} \vdash \phi_1 \wedge \phi_2$ and so $\phi_1 \wedge \phi_2 >_c \psi$ can be inferred. □

However, this inference relation has drawbacks:

- if $(\mathcal{K}, >)$ contains $\phi > \psi_2$ only and $\psi_1 \vDash \psi_2$ we cannot infer $\phi > \psi_1$ whenever ψ_1 does not appear in $\mathcal{K}$. So the other side of Property O fails.
- It is assumed that the relation $>$ is irreflexive and transitive (it is a partial order on $\mathcal{K}$). But, $>_c$ on 2^Ω may fail to be so. Irreflexivity of $>_c$ is lost in case of inconsistency of cuts as shown in Example 3, where $(\mathcal{K}, >) \vdash_c y > y$.
- $>_c$ is generally not transitive.

The following example shows the loss of transitivity.

Example 4. Let $\mathcal{K} = \{\delta, \xi, \gamma, \psi\}$ be equipped with the strict partial order defined by $\delta > \xi$ and $\gamma > \psi$. Assume that $\delta \vDash \psi$ and $\gamma \vDash \phi$. So we have $\psi >_c \xi$ and $\phi >_c \psi$. But we do not have $\phi >_c \xi$, because $\mathcal{K}_\xi = \{\delta\}$ and it is not assumed that $\delta \vDash \phi$. For instance, we can take $\delta = x \wedge y, \xi = \neg x, \gamma = \neg y, \psi = x, \phi = \neg x \vee \neg y$.

So the lack of transitivity of the closure by cuts indicates that it lacks inferential power. In fact, it is clear that whenever $\phi > \psi \in \mathcal{C}_c(\mathcal{K}, >)$, it holds that $\psi \in \mathcal{K}$. That is, if $\psi \notin \mathcal{K}$, it is not possible to infer anything of the form $\phi > \psi$.

Finally, one anomaly of this inference relation is pointed by the next example:

Example 5. Let $\mathcal{K} = \{\neg x, \neg y, \neg x \vee y\}$ be equipped with the strict partial order $>$ given by $\neg x > \neg x \vee y$ and $\neg y > \neg x \vee y$. We have $\mathcal{K}^>_{\neg x \vee y} = \{\neg x, \neg y\} \vdash \neg x$. So, $(\mathcal{K}, >) \vdash_c \neg x >_c \neg x \vee y$. However, the base $(\mathcal{K}, >)$ does not have a model in the sense of Section 5.2. In fact, there is no partial certainty relation $\succ_N$ on 2^Ω such that $[\neg x] \succ_N [\neg x \vee y]$, because it implies $[\neg x \vee y] \succ_N [\neg x \vee y]$ by axiom O, hence violates irreflexivity. The base $(\mathcal{K}, >)$ is rc-inconsistent. So the inference $\mathcal{K}^>_{\neg x \vee y} \vdash \neg x$ is questionable.

In fact, the above defect can be solved if we systematically complete $(\mathcal{K}, >)$ by forbidding impossible non-contradictory propositions, i.e., adding $\phi > \bot, \forall \phi \in \mathcal{K}$ to $(\mathcal{K}, >)$. In this case, $\mathcal{K}^>_\bot$ is inconsistent and $\mathcal{C}_c(\mathcal{K}, >)$ is trivialized as expected.

The following proposition states that the syntactic inference based on level cuts $\vdash_c$ is sound for the semantics defined in Section 5.2.

Proposition 8. *Let $(\mathcal{K}, >)$ be a po-base.*[3] *For any formula ψ in $\mathcal{K}$, if $(\mathcal{K}, >) \vdash_c \phi > \psi$ then $(\mathcal{K}, >) \vDash_N \phi > \psi$. The converse is false.*

Here is a counter-example for the converse:

Example 2 (continued). Let $(\mathcal{K}, >)$ be the po-base of Example 2, that is, $\{x > \neg x, y > \neg y\}$. Interpret $>$ by $\succ_N$ on sets of models. We have $(\mathcal{K}, >) \vDash_N x > \neg y$ as shown previously in the first part of this example (using completeness of $\vdash_\mathcal{S}$). However, we do not have $\mathcal{K}^>_{\neg y} \vdash x$ because $\mathcal{K}^>_{\neg y} = \{y\}$.

The above result shows that adapting the inference by cuts of possibilistic logic to the partially ordered case is sound with respect to the semantics in terms of partial certainty relations, but, contrary to the case of possibilistic logic, this inference mode is too weak to account for them.

[3] Note that the result is trivial if the base $(\mathcal{K}, >)$ does not have a model for the semantics of relative certainty.

6. Comparison with symbolic possibilistic logic

In this section, our purpose is to compare the inference system $\mathcal{S}$ for partial certainty relations with inference in possibilistic bases with partially ordered symbolic weights, initiated by Benferhat and Prade [4]. We first present a version of symbolic possibilistic logic. Then, we investigate possible ways of encoding a symbolic possibilistic base with a po-base and conversely.

6.1. Symbolic possibilistic logic

In symbolic possibilistic logic (SPL), only partial knowledge is available on the relative strength of weights attached to formulas. So, weights are symbolic expressions taking values on a totally ordered necessity scale (such as $[0, 1]$), and there is a set of constraints over these weights, describing what is known of their relative strength.

The set $\mathcal{P}$ of symbolic weights p_j is recursively obtained using a finite set of variables (called elementary weights) $H = \{a_1, \cdots, a_k, \cdots\}$ taking values on the scale $]0, 1]$ and max / min expressions built on H: $H \subseteq \mathcal{P}$, $0, 1 \in \mathcal{P}$, and if $p_i, p_j \in \mathcal{P}$, so $\max(p_i, p_j) \in \mathcal{P}$, $\min(p_i, p_j) \in \mathcal{P}$. We suppose also $1 \geq a_i > 0, \forall i$.

Let $\Sigma = \{(\phi_j, p_j), j = 1, \cdots, m\}$ be a symbolic possibilistic base where p_j is a max / min expression built on H. A formula (ϕ_j, p_j) is still interpreted as $N(\phi_j) \geq p_j$ [4]. The knowledge about weights is encoded by a finite set $C = \{p_i > p_j\}$ of strict constraints between max / min expressions, a partial ordering on $\{p_j, j = 1, \cdots, m\}$. Every finite set of constraints can be put in canonical form as:

$$\max_{k=1\cdots n} a_{ik} > \min_{l=1\cdots m} a_{jl}, \qquad \text{where} \;\; a_{ik}, a_{jl} \in H$$

which means $\exists k \in \{1 \cdots n\}, \exists l \in \{1 \cdots m\}, a_{ik} > a_{jl}$. Define $C \vDash p > q$ if and only if every valuation of symbols appearing in p, q (on $]0, 1]$) which satisfies the constraints in C also satisfies $p > q$.

Remark 4. In [4], the authors assume a partial preorder on weights, i.e. constraints of the form $p_i \geq p_j$, and they define $C \vDash p > q$ by $C \vDash p \geq q$ and $C \nvDash q \geq p$ (analogously to the strict Pareto order between vectors). With this definition, from $p \geq q$ we could infer $\max(p, q) > p$. This is problematic because it amounts to interpreting strict inequality as the impossibility of proving a weak one. In the present paper, $C \vDash p > q$ holds provided that $p > q$ holds for all instantiations of p, q in accordance with the constraints. Only such strict constraints appear in C. Likewise, we define $C \vDash p \geq q$ in the same way; for instance, $\max(p, q) \geq p$ holds if C is empty.

6.1.1. Semantics of symbolic possibilistic bases

The semantics of an SPL base can be defined in two ways, just as in standard possibilistic logic. The first definition is based on the construction of a possibility distribution associated with the possibilistic base as per Equations (1) and (2). Here, this possibility distribution will attach a symbolic expression to each interpretation. However, the presence of terms of the form $1 - .$ prevents $\pi_\Sigma(\omega)$ from lying in $\mathcal{P}$. So in the case of SPL, it is more convenient to represent an *impossibility* distribution $\iota_\Sigma = 1 - \pi_\Sigma$, namely $\forall \omega \in \Omega$:

$$\iota_\Sigma(\omega) = 1 - \pi_\Sigma(\omega) = \begin{cases} \max_{j:\phi_j \notin \Sigma(\omega)} p_j, (\in \mathcal{P}) \\ 0 \text{ if } \Sigma(\omega) = \Sigma^* \end{cases} \tag{7}$$

Let Σ be an SPL-base and $\omega, \omega' \in \Omega$ be two interpretations. We can again define two (here partial) orderings on Ω, induced by Σ, the possibilistic ordering and the best-out one, respectively:

$$\omega >_\Sigma \omega' \;\text{ if and only if }\; C \vDash \iota_\Sigma(\omega) < \iota_\Sigma(\omega') \tag{8}$$

$$\omega \rhd_\Sigma \omega' \text{ if and only if } \forall(\phi_j, p_j) \in \Sigma \text{ such that } \phi_j \in \overline{\Sigma(\omega)},\ \exists(\phi_i, p_i) \in \Sigma \text{ such that } \phi_i \in \overline{\Sigma(\omega')} \text{ and } C \vDash p_i > p_j \tag{9}$$

They coincide in standard possibilistic logic. However, if weights are variables, it is easy to see that the second semantics (best-out) is more demanding. For instance, suppose $\iota_\Sigma(\omega) = c_j$ and $\iota_\Sigma(\omega') = \max(a_i, a_k)$. The first semantics requires that $c_j < \max(a_i, a_k)$ for all instantiations of a_i, a_k, c_j in agreement with constraints in C, while the second semantics comes down to requiring that one of a_i, a_k is greater than c_j for all instantiations in agreement with constraints in C. However, while for instance, $c_j < a_i$ for all instantiations of a_i, c_j implies that $c_j < \max(a_i, a_k)$ for all instantiations of a_i, a_k, c_j, the converse is false: $c_j < \max(a_i, a_k)$ for all instantiations of a_i, a_k, c_j means that for each such instantiation, we have either $c_j < a_i$ or $c_j < a_k$.

Proposition 9. *Let Σ be a symbolic possibilistic base with strict constraints, we have*

- $\omega \rhd_\Sigma \omega'$ *implies* $\omega >_\Sigma \omega'$.
- $\forall \omega \in \overline{[\phi]},\ \exists \omega' \in \overline{[\psi]},\ \omega' \rhd_\Sigma \omega$ *implies* $N_\Sigma(\phi) > N_\Sigma(\psi)$

where $N_\Sigma(\phi)$ *is the* max / min *expression*

$$N_\Sigma(\phi) = \min_{\omega \nvDash \phi} \max_{j:\phi_j \notin \Sigma(\omega)} p_j \tag{10}$$

Note that in the above expression we can restrict to $\Sigma(\omega)$ that are maximal for inclusion.[4] In the following, we adopt the less demanding semantics of SPL based on comparing symbolic expressions $N_\Sigma(\phi)$.

The semantics of possibilistic logic allows us to replace weighted conjunctions $(\bigwedge_i \phi_i, p)$ by a set of formulas (ϕ_i, p) without altering the underlying possibility distribution, since $N(\phi \wedge \psi) = \min(N(\phi), N(\psi))$: from the minimal specificity principle, we can associate the same weight to each sub-formula in the conjunction. Therefore, we can turn any SPL base into a semantically equivalent weighted clausal base, and restrict to such bases.

6.1.2. Inference in symbolic possibilistic bases

As in standard possibilistic logic, in order to get the deductive closure, we must compute $N_\Sigma(\phi), \forall \phi \in \mathcal{L}$. However, to do it by inference at the syntax level, one must slightly reformulate the inference rules of possibilistic logic:

- Weakening rule: If $p_i > p_j$ then $(\phi, p_i) \vdash_\pi (\phi, p_j)$
- Fusion rule: $\{(\phi, p), (\phi, p')\} \vdash_\pi (\phi, \max(p, p'))$
- Weighted Modus Ponens: $\{(\phi \to \psi, p), (\phi, p')\} \vdash_\pi (\psi, \min(p, p'))$

involving a symbolic handling of weights using max and min. This inference system is denoted by $\mathcal{S}_{SPL}$. If B is a subset of the skeleton Σ^* of Σ that implies ϕ, it is clear that $(\Sigma, C) \vdash_\pi (\phi, \min_{\phi_j \in B} p_j)$. Using syntactic inference, we can compute the expression representing the strength of deduction of ϕ from Σ:

$$N_\Sigma^\vdash(\phi) = \max_{B \subseteq \Sigma^*, B \vdash \phi} \min_{\phi_j \in B} p_j.$$

Note that in the above expression, it suffices to take max on subsets B minimal for the inclusion that imply ϕ.[5]

[4] This is due to the fact that the symbolic weights take values in a totally ordered set, so $\max_{j \in A} p_j \leq \max_{j \in B} p_j$, whenever $A \subset B$. This claim is false if we consider C as an abstract strict partial order on H.

[5] For the same reason as for the semantic degree of necessity.

Proposition 10. *The symbolic possibilistic logic SPL is sound and complete for the inference system $\mathcal{S}_{SPL}$.*

Proof of Proposition 10. We only give an outline of the proof which is a bit out of the scope of the paper (see [12]). We have to prove that $N_\Sigma(\phi) = N_\Sigma^\vdash(\phi)$. First it is easy to see that the result holds if Σ^* is a minimal base that implies ϕ. We easily conclude that $N_\Sigma(\phi) \geq N_\Sigma^\vdash(\phi)$. The rest is a matter of applying distributivity of max over min. Using distributivity, we can rewrite the syntactic necessity degree $N_\Sigma^\vdash(\phi)$ in terms of the minimal hitting sets [37] $H_s, s \in \mathcal{S}$ of the set $\{B_1, \ldots, B_n\}$ of minimal sub-bases that entail ϕ. It is known from the literature that the complement of each minimal hitting set H_s of $\{B_1, \ldots B_n\}$ is a maximal sub-base of Σ^* consistent with $\neg\phi$ ([35], Section 4.3). From this result, the converse inequality holds, since in fact there is an exact correspondence between the set of maximal sub-bases of Σ^* consistent with $\neg\phi$ and the set of minimal hitting sets $H_s = \{\phi_1, \ldots, \phi_n\}$ of $\{B_1, \ldots, B_n\}$. $\square$

We have $N_\Sigma^\vdash(\phi) = N_{\Sigma\cup(\neg\phi,1)}^\vdash(\bot)$, where the degree of inconsistency $Inc(\Sigma)$ is formalized by the expression $N_\Sigma^\vdash(\bot)$. As in standard possibilistic logic, we can define the plausible inference with symbolic weights.

Definition 8. ϕ is a plausible consequence of (Σ, C), denoted by $(\Sigma, C) \vdash_{PL} \phi$ if and only if:

$$C \vDash N_\Sigma(\phi) > Inc(\Sigma) = N_\Sigma^\vdash(\bot)$$

Example 6. Let $\Sigma = \{(\phi, a), (\neg\phi \vee \psi, b), (\neg\psi, c)\}$ with $C = \{a > c, b > c\}$. $Inc(\Sigma) = \min(a, b, c) = c$. Now, $N_\Sigma^\vdash(\psi) = N_{\Sigma\cup(\neg\psi,1)}^\vdash(\bot) = \max(\min(a, b), c) = \min(a, b)$. So, $C \vDash N_\Sigma(\psi) > c$ and $(\Sigma, C) \vdash_{PL} \psi$.

Note that in SPL, the inference of non-trivial consequences from SPL bases with inconsistent skeletons in the above sense requires that the set of constraints C be not empty. Otherwise, no strict inequalities can be inferred between formula weights.

Under possibilistic inference rules, we can compare the strength degrees of two formulas via their resulting weights (thus generalizing Definition 8):

Definition 9. $(\Sigma, C) \vdash_\pi \phi > \psi$ if and only if $C \vDash N_\Sigma(\phi) > N_\Sigma(\psi)$.

Example 7. Let $\Sigma = \{(\phi, a), (\neg\phi \vee \psi, b), (\neg\phi, c), (\neg\psi, d)\}$ and $C = \{a > b, b > c, b > d\}$. We have $N_\Sigma(\psi) = \min(a, b) = b$ and $N_\Sigma(\phi) = a$. So we have $(\Sigma, C) \vdash_\pi \phi > \psi$.

Therefore, we can view the weighted deductive closure of a symbolic possibilistic logic base as a partial order on the language $\mathcal{L}$. It is possible to compare SPL with the inference system $\mathcal{S}$ over po-bases, as long as the latter are translated into SPL. So, the next step is to define how to translate a possibilistic base into a po-base and conversely.

6.2. Encoding a po-base as a symbolic possibilistic base

First, we define how to translate a po-base into a symbolic PL-base. A partially ordered base $(\mathcal{K}, >)$ is encoded by a set of pairs of the form $\phi > \psi \in \mathcal{L}_>$. So, we must attach symbolic weights to formulas then write constraints over these symbolic weights induced by the partial order $>$.

Formally, let $\eta : \mathcal{K} \longrightarrow H = L_\mathcal{K}$ be a function that associates to each formula ϕ of $\mathcal{K}$ the elementary symbolic weight $\eta(\phi)$. Then we build a set of constraints C such that $a > b$ if and only if $a = \eta(\phi), b = \eta(\psi)$ and $\phi > \psi$ according to $(\mathcal{K}, >)$.

Definition 10. Let $(\mathcal{K}, >)$ be a partially ordered base. It is translated by the base $(\Sigma_\mathcal{K}, C_\mathcal{K})$[6]

[6] We should actually denote the obtained possibilistic base by $(\Sigma_\mathcal{K}, C_{\mathcal{B}_{(\mathcal{K},>)}})$.

- $\Sigma_{\mathcal{K}} = \{(\phi, \eta(\phi)), \phi \in \mathcal{K}\}$
- $C_{\mathcal{K}} = \{a > b : (\phi, a), (\psi, b) \in \Sigma_{\mathcal{K}} \text{ and } \phi > \psi \in (\mathcal{K}, >)\}$.

The following example illustrates that $(\mathcal{K}, >) \vdash_{\mathcal{S}} \phi > \psi \Rightarrow (\Sigma_{\mathcal{K}}, C_{\mathcal{K}}) \vdash_{\pi} \phi > \psi$ may not hold if $(\mathcal{K}, >)$ is rc-inconsistent:

Example 8. Let $\mathcal{K} = \{\neg x, \neg y, \neg x \vee y\}$ be equipped with the strict partial order $\neg x > \neg x \vee y$ and $\neg y > \neg x \vee y$. $(\mathcal{K}, >)$ is encoded with $\Sigma_{\mathcal{K}} = \{(\neg x \vee y, a),\ (\neg x, b),\ (\neg y, c)\}$ and $C_{\mathcal{K}} = \{c > a, b > a\}$. We can notice that, by definition, $(\mathcal{K}, >) \vdash_{\mathcal{S}} \neg x > \neg x \vee y$. However, the base $(\mathcal{K}, >)$ is rc-inconsistent (for the relative certainty semantics), so $(\mathcal{K}, >) \vdash_{\mathcal{S}} \phi > \psi$ always holds. In contrast, $\Sigma_{\mathcal{K}}$ is consistent in SPL. Possibilistic logic will correct the weight of formula $\neg x \vee y$, because $N_{\Sigma}(\neg x \vee y) = b$. So we do not have that $(\Sigma, C) \vdash_{\pi} \neg x > \neg x \vee y$.

However, this kind of encoding may also add some unwanted information, as shown by the following example.

Example 9. Let $\mathcal{K} = \{\neg x \vee y, x \wedge y\}$ be equipped with the partial order $\neg x \vee y > x \wedge y$. Let $a = \eta(\neg x \vee y)$ and $b = \eta(x \wedge y)$. We obtain $\Sigma_{\mathcal{K}} = \{(\neg x \vee y, a), (x \wedge y, b)\}$ and $C_{\mathcal{K}} = \{a > b\}$. In possibilistic logic, believing each of two formulas to the same degree is equivalent to believing their conjunction to that degree. So, we can replace $(x \wedge y, b)$ by $(x, b), (y, b)$. Thus, we obtain $\Sigma = \{(\neg x \vee y, a), (x, b), (y, b)\}$ which is semantically equivalent to $\Sigma_{\mathcal{K}}$ (in the sense that $N_{\Sigma_{\mathcal{K}}} = N_{\Sigma}$, with symbolic weights).

However, from $\neg x \vee y > x \wedge y$, we can deduce neither $\neg x \vee y > x$ nor $\neg x \vee y > y$ using inference system $\mathcal{S}$.

In fact, possibilistic logic relies on the minimal specificity principle, it interprets $\Sigma_{\mathcal{K}}$ and Σ with the same possibility distribution such that $N_{\Sigma_{\mathcal{K}}}(x \wedge y) = N_{\Sigma_{\mathcal{K}}}(x) = N_{\Sigma_{\mathcal{K}}}(y)$. SPL deduces $\neg x \vee y > x$ because by default, it supposes that x and y have the same certainty degree as $x \wedge y$. But the inference system $\mathcal{S}$ is weaker and does not use this principle.

In order to get a faithful translation of the inference system $\mathcal{S}$ into possibilistic logic, we should restrict to particular cases of rc-consistent partially ordered bases $(\mathcal{K}, >)$, where we do not have dominated conjunctions (conjunctions on the right of $\mathcal{L}_>$ formulas). In contrast, under relative certainty semantics, we have that $\phi \wedge \psi > \chi \Leftrightarrow \phi > \chi$ and $\psi > \chi$ (properties O and N^d). So we can always come down to constraints $\phi > \psi$ where ϕ is a clause. In addition, in this semantics, we can replace every rc-consistent formula of the form $\phi > \phi \wedge \psi$ by an equivalent formula $\phi > \psi$, which allows us to eliminate some conjunctions on the right of $\mathcal{L}_>$ formulas.

Note also that, for any consistent formula ϕ, if $(\phi, a) \in \Sigma_{\mathcal{K}}$, then $a > 0$, which means $\phi > \bot$. Such requirement is not compulsory in po-bases. So for the sake of the translation, it means that any formula appearing in $(\mathcal{K}, >)$ should be interpreted as a belief, no matter how little entrenched it may be. If $\phi > \psi$ is stated, it means that ψ is present in the belief base of the agent. The above remarks lead us to consider the following restrictions:

1. We consider only bases $(\mathcal{K}, >)$ that are rc-consistent.
2. We suppose that $\mathcal{K}$ only contains clauses.
3. We assume that for each formula $\phi \in \mathcal{K}$, $\phi > \bot$ (denoted by $\inf(\mathcal{K}) = \{\bot\}$).

Under such restrictions, we can prove:

Proposition 11. *Let $(\mathcal{K}, >)$ be an rc-consistent partially ordered base, containing only clauses and such that $\inf(\mathcal{K}) = \{\bot\}$:*

$$(\mathcal{K}, >) \vdash_{\mathcal{S}} \phi > \psi \Rightarrow (\Sigma_{\mathcal{K}}, C_{\mathcal{K}}) \vdash_{\pi} \phi > \psi$$

The converse is not true as shown by the following example:

Example 10. Let $\mathcal{K} = \{x, y, z\}$ be made of atoms, with $\{x > \bot, y > \bot, z > \bot, x > y, x > z\}$. The translation into possibilistic logic gives the SPL base $\Sigma_{\mathcal{K}} = \{(x, a), (y, b), (z, c)\}$ with atomic symbolic weights a, b, c and $C_{\mathcal{K}} = \{a > b, a > c\}$. We have $N_{\Sigma_{\mathcal{K}}}(y \vee z) = \max(b, c)$ by the minimal specificity principle. So we have $(\Sigma_{\mathcal{K}}, C_{\mathcal{K}}) \vdash_{\pi} (x > y \vee z)$. But we cannot deduce it from $(\mathcal{K}, >)$ using the inference system $\mathcal{S}$.

Examples 9 and 10 show that possibilistic logic infers more conclusions than the inference system $\mathcal{S}$. Nevertheless it is easy to see that if ϕ, ψ are clauses in $\mathcal{K}$, then $(\phi, a), (\psi, b) \in \Sigma_{\mathcal{K}}$ with $a > b$ only if $\phi > \psi \in \mathcal{B}_{(\mathcal{K},>)}$.

If in a statement $\phi > \psi$ appearing in $(\mathcal{K}, >)$, ψ is a conjunction instead of a clause, a systematic translation of $\phi > \psi$ into $\{(\phi, a), (\psi, b)\}$ with $a > b$ may prevent valid consequences of the inference system $\mathcal{S}$, using rules such as RI_1, from being deducible in SPL from the possibilistic translation of the po-base.

Example 11. Let $\mathcal{K} = \{x, y, x \wedge z, y \wedge z\}$ with $\{x > y \wedge z > \bot, y > x \wedge z > \bot\}$. By $RI1$ we deduce $x \wedge y > z$.

Let $\Sigma_{\mathcal{K}} = \{(x, a), (y, c), (y \wedge z, b), (x \wedge z, d)\}$ and $C_{\mathcal{K}} = \{a > b, c > d\}$ be the translation of $(\mathcal{K}, >)$ into possibilistic logic. We can check that:

- $N_{\Sigma_{\mathcal{K}}}(x) = \max(a, d)$; $N_{\Sigma_{\mathcal{K}}}(y) = \max(c, b)$.
- $N_{\Sigma_{\mathcal{K}}}(x \wedge y) = \min(\max(a, d), \max(c, b))$ and $N_{\Sigma_{\mathcal{K}}}(z) = \max(b, d)$

But we do not have $\min(\max(a, d), \max(c, b)) > \max(b, d)$, so we do not have $(\Sigma_{\mathcal{K}}, C_{\mathcal{K}}) \vdash_{\pi} (x \wedge y > z)$.

What happens is that the use of 4 weights a, c, b, d in the translation introduces additional degrees of freedom in the PL-base, not present in $(\mathcal{K}, >)$. Suppose we introduce a symbolic weight ε for z, i.e. $N_{\Sigma_K}(z) = \varepsilon$. Then, we can let $b = \min(c, \varepsilon)$, and $d = \min(a, \varepsilon)$ due to the axiom of necessity measures, which restores the constraint existing between $y \wedge z$ and $x \wedge z$. If we put in C constraints $a > \min(c, \varepsilon)$ and $c > \min(a, \varepsilon)$, instead of $a > b$ and $c > d$, we will have:

- $N_{\Sigma_{\mathcal{K}}}(x \wedge y) = \min(a, c)$
- $N_{\Sigma_{\mathcal{K}}}(z) = \max(b, d) = \max(\min(a, \varepsilon), \min(c, \varepsilon)) = \varepsilon < N_{\Sigma_{\mathcal{K}}}(x \wedge y)$

since $a > \min(c, \varepsilon)$ and $c > \min(a, \varepsilon)$ imply $\min(a, c) > \varepsilon$. It allows us to retrieve the conclusion of the inference system $\mathcal{S}$ in possibilistic logic.

The above example shows how we could extend the translation of $(\mathcal{K}, >)$ to the general case:

- Put ϕ and ψ in conjunctive normal form: $\bigwedge_i \phi_i$, and $\bigwedge_j \psi_j$, where ϕ_i and ψ are clauses.
- Replace $\phi > \psi$ by $\{\phi_i > \psi\}_i$.
- Translate each $\phi_i > \psi$ into possibilistic logic as $\{(\phi_i, a_i), (\psi_j, b_j), \forall j\}$ and enforce a constraint $a_i > \min_j b_j$. But then we should allow non-elementary weights in C (or disjunction of elementary constraints).

It is an open problem whether this would be enough to ensure the converse of Proposition 11. It is unlikely though, due to the presence of the minimal specificity principle at work in SPL.

6.3. Encoding a symbolic possibilistic base as a partially ordered base

Conversely, we can try to translate a PL-base with symbolic weights into a partially ordered base. Given (ϕ, a) and (ψ, b) in Σ, with $a > b \in C$, a natural idea is to state that $\phi > \psi$. However, we know that a possibilistic formula (ϕ, a) is interpreted as $N(\phi) \geq a$ where N is a necessity measure. So, it may occur that in the deductive closure, $N_\Sigma(\psi) > b$, as discussed in Section 2. As a consequence, it must be ensured that the formulas of Σ are assigned their maximum weight prior to making the translation.

Definition 11. Let (Σ, C) be a symbolic possibilistic base containing only clauses, and C the set of constraints on symbolic weights. The SPL base induced by (Σ, C) defined as

$$\widehat{\Sigma} = \{(\phi, N_\Sigma(\phi)) : \phi \in \Sigma^*\},$$

(where $N_\Sigma(\phi)$ is computed as in Section 2.2 and in Equation (10)) is called the *coherent completion*[7] of Σ, where Σ^* is its skeleton.

Now, we derive partially ordered formulas in $(\mathcal{K}, >)$ by comparing weights of formulas in $\widehat{\Sigma}$, using constraints in C. There are two cases, according to whether Σ^* is consistent or not. If it is consistent, we add to $(\mathcal{K}, >)$ the constraints of strict dominance between formulas $\phi \in \Sigma^*$, as well as the constraints $\phi > \bot$ for each possibilistic formula $(\phi, p) \in \widehat{\Sigma}$. If Σ^* is inconsistent, we only put in $\mathcal{K}$ formulas ϕ such that $(\phi, p) \in \widehat{\Sigma}$, for some max / min expression p, and $C \vDash p > \mathit{Inc}(\Sigma)$.

Definition 12. Let (Σ, C) be a symbolic possibilistic base containing only clauses. We build $(\mathcal{K}, >)_\Sigma$ as follows:

- $\mathcal{K} = \{\phi \in \Sigma^* : N_\Sigma(\phi) > \mathit{Inc}(\Sigma)\} \cup \{\bot\}$
- The strict partial order on $\mathcal{K}$ is defined by $\{\phi > \psi : (\phi, p) \in \widehat{\Sigma}, (\psi, q) \in \widehat{\Sigma}, C \vDash p > q > \mathit{Inc}(\Sigma)\} \cup \{\phi > \bot : \phi \in \Sigma^* \text{ and } C \vDash (N_\Sigma(\phi) > \mathit{Inc}(\Sigma))\}$.

Note that po-bases $(\mathcal{K}, >)$ do not always contain statements of the form $\phi > \bot$, while po-bases of the form $(\mathcal{K}, >)_\Sigma$ always will.

Example 12. Let $\Sigma = \{(x, a), (\neg y, c), (\neg x, d), (y, e), (\neg x \vee y, b)\}$ with $C = \{a > b, b > d, d > e, a > c, c > e\}$.

- $\mathit{Inc}(\Sigma) = N_\Sigma^\vdash(\bot) = \max(d, \min(b, c))$
- $N_\Sigma(y) = \max(e, \min(a, b)) = b$. So $(y, b) \in \widehat{\Sigma}$
- $N_\Sigma(\neg x) = \max(d, \min(b, c))$. So $(\neg x, \max(d, \min(b, c))) \in \widehat{\Sigma}$.

So $\widehat{\Sigma} = \{(\neg x, \max(d, \min(b, c))), (x, a), (\neg y, c), (\neg x \vee y, b), (y, b)\}$ with $C \vDash a > \mathit{Inc}(\Sigma)$. Only x escapes inconsistency. So, the translation reduces to $(\mathcal{K}, >)_\Sigma = \{(x > \bot)\}$.

This shows that the presence of inconsistencies in the SPL base (Σ, C) will forbid a faithful translation of (Σ, C) into $(\mathcal{K}, >)_\Sigma$.

In the following we consider only bases (Σ, C) where Σ^* is consistent. An important point to be noticed is that since Σ^* is assumed to be consistent, the base $\mathcal{K}$ obtained from (Σ, C) is Σ^* and so $\mathcal{K}$ is consistent. Moreover, as N_Σ defines a partial order on the language compatible with the axioms of $\mathcal{S}$, the translation $(\mathcal{K}, >)_\Sigma$ will be rc-consistent.

Starting from a possibilistic base with symbolic weights, we will compare the partial ordering on $\mathcal{L}$ induced by comparing the weights $N_\Sigma(\phi)$ between formulas in $\mathcal{L}$ (Definition 9) and the partially ordered deductive closure of the associated po-base. We will consider successively the deductive closure induced by level cuts and the closure by the inference system $\mathcal{S}$.

[7] The term "coherent" is here borrowed, by analogy, from Walley's imprecise probability theory.

Proposition 12. *Let (Σ, C) be a consistent symbolic possibilistic base ($Inc(\Sigma) = 0$), and $(\mathcal{K}, >)$*[8] *its translation (according to Definition 12). Let $\psi \in \mathcal{K}$, we have:*

$$(\Sigma, C) \vdash_\pi \phi > \psi \textit{ if and only if } (\mathcal{K}, >) \vdash_c \phi > \psi.$$

More generally, we can show that inference in the sense of system $\mathcal{S}$ is less productive than possibilistic inference.

Proposition 13. *Let (Σ, C) be a consistent symbolic possibilistic base and $(\mathcal{K}, >)$ its translation (according to Definition 12).*

$$(\mathcal{K}, >) \vdash_\mathcal{S} \phi > \psi \Rightarrow (\Sigma, C) \vdash_\pi \phi > \psi$$

Remark 5. Note the difference between the proof of the above proposition and the proof of Proposition 11 when translating $(\mathcal{K}, >)$ into $(\Sigma_\mathcal{K}, C_\mathcal{K})$, in particular for rule RI_1. We are restricted to clauses in $\mathcal{K}$, because the translation $\phi > \psi$ into $\{(\phi, a), (\psi, b)\}$ that attaches independent weights to formulas is problematic if ψ is a conjunction: Example 11 shows that $(\Sigma_\mathcal{K}, C_\mathcal{K})$ does not allow the recovery of this inference rule in possibilistic logic, because the translation does not preserve all the information. But here, we use the symbolic necessity measure N_Σ to compare formulas. It therefore obeys RI_1.

The following example shows that the converse is false in general.

Example 13. Let x, y, z be atoms and $\Sigma = \{(x, a), (y, b), (z, c)\}$ a symbolic possibilistic base with $C = \{a > b, a > c\}$. We have $N_\Sigma(y \vee z) = \max(c, b)$. So we have $(\Sigma, C) \vdash_\pi (x > y \vee z)$. The translation of the possibilistic base is $(\mathcal{K}, >) = \{x > \bot, y > \bot, z > \bot, x > y, x > z\}$. We cannot deduce $x > y \vee z$, using the inference system $\mathcal{S}$.

The converse of Proposition 13 is not true because in possibilistic logic, the principle of least commitment is applied. In the example above, we conclude $N_\Sigma(y \vee z) = \max(c, b)$ while strictly speaking, only the inequality $N_\Sigma(y \vee z) \geq \max(c, b)$ holds for all necessity measures that satisfy the base (Σ, C); equality is reached for the least informative one. However, the following result is a kind of converse valid only for formulas already present in the possibilistic base.

Proposition 14. *Let (Σ, C) be a consistent symbolic possibilistic base and $(\mathcal{K}, >)$ its translation (according to Definition 12). For any formula $\psi \in \Sigma^*$*

$$(\mathcal{K}, >) \vdash_\mathcal{S} \phi > \psi \textit{ if and only if } (\Sigma, C) \vdash_\pi \phi > \psi$$

Proof of Proposition 14. Let $\psi \in \Sigma^*$, so $\psi \in \mathcal{K}$ and $\psi \neq \bot$. We suppose that $(\Sigma, C) \vdash_\pi \phi > \psi$. By Proposition 12 we have $(\mathcal{K}, >) \vdash_c \phi > \psi$.

Due to Proposition 8, we have $(\mathcal{K}, >) \models_N \phi > \psi$, and then due to completeness of system $\mathcal{S}$ (Proposition 6), $(\mathcal{K}, >) \vdash_\mathcal{S} \phi > \psi$. The converse is Proposition 13. □

7. Related works and discussion

In this section, we show that our framework is closely related and in some sense unifies various logics proposed in the past, namely some modal, conditional and non-monotonic formalisms.

[8] Strictly speaking $(\mathcal{K}, >)_\Sigma$.

7.1. Conditional logics

Despite his viewing a partial order as resulting from incomparability, the conditional logic proposed by Halpern [30] is the closest to our work. This logic uses the same atomic sentences (strict comparisons between propositional formulas) but its language is more complete (including negations and disjunctions of atomic comparisons; we excluded the latter as leading to very complex statements). He considers a propositional calculus on relative likelihood atoms $\phi \succ \psi$ expressing partial comparative possibility, while our approach accepts only conjunctions of relative certainty statements. The axiom system is also the same as ours ($\mathcal{S}$). If we translate Halpern axioms in terms of relative certainty, we can match them with ax_1, RI_1, RI_2.

However, there is a major difference pertaining to the semantics. Halpern [30] uses a partial relation $\gg$ on a set of states S that is different from the set of interpretations of $\mathcal{L}$. He assumes formulas in $\mathcal{L}$ can be evaluated on states by assuming a surjective application $f : S \to \Omega$. Then $s \models \phi$ means $f(s) \models \phi$ in the usual sense. The set of interpretations Ω is assimilated to the set of equivalence classes induced by f on S according to the relation $s \approx_f s'$ if and only if $f(s) = f(s')$. Halpern shows that if S is large enough, any relation over 2^Ω can be induced by a single relation on S. He gives an upper bound on the cardinality of S to that effect. The proof of completeness relies on partial order relations $\gg$ on the set of states S, large enough to interpret the syntactic closure of a partially ordered base by means of a semantic inference of the form $(S, \gg) \models \phi > \psi$ if and only if $\forall s$, such that $f(s) \models \psi$, there exists s' such that $f(s') \models \phi$ and $s' \gg s$ (this is the weak optimistic strict dominance). This semantics is akin to the one proposed by Kraus et al. [31] for system P, and is more in the spirit of interpreting partial orders among formulas in terms of incomparability than in terms of incompleteness.

Our semantics is conceptually simpler in the sense that it is based directly on a relation between subsets of interpretations $[\phi], [\psi]$ without resorting to a hypothetical set of states finer than the set of interpretations of the language. The use of such an underlying state space S is a kind of artefact. While one may admit that the language $\mathcal{L}$ cannot describe the states of the world in a very precise way, due to a limited vocabulary, the actual use of S would presuppose a finer language that is not assumed in the model. In this paper we prefer to see the partial order between sets of models as the incomplete specification of a total necessity order on the language $\mathcal{L}$ induced by a plausibility ranking on the set Ω of interpretations. Each such plausibility ranking is easier to interpret than a partial order on a purely abstract set S that our language cannot describe.

Halpern's work actually extends to the partially ordered case pioneering constructions of Lewis [33] devoted to the logic of comparative possibility, interpreting formulas $\phi \geq \psi$ as the inequality $\Pi([\phi]) \geq \Pi([\psi])$ for a possibility distribution π on Ω.[9] This logic is called *VN*. From a semantic point of view, a model of VN can be represented by a relation of weak order (total preorder) on the set of interpretations of the language. A set of formulas equipped with a reflexive relation can be interpreted that way. Fariñas and Herzig [26] have shown that the axiom system of *VN* is equivalent to axioms of comparative possibility in [15]. Bendova and Hajek [3] elaborate another logic of comparative possibility, allowing the nesting of formulas $\phi \geq \psi$ and comparing this formalism to temporal logic (viewing a total plausibility order as a time path).

Finally, we can also show that our logic is closely related the Minimal Epistemic Logic (MEL) of incomplete information [1]. The MEL language is the subjective fragment of S5, without nesting of modalities. This is the minimal language that can express the fact that a propositional formula is known to be true ($\Box\phi$), known to be false ($\Box\neg\phi$) or unknown ($\neg\Box\phi \wedge \neg\Box\neg\phi$). MEL captures possibility theory in its all-or-nothing version, using axioms of the logic KD. It turns out that the inference system $\mathcal{S}$ generalizes a fragment of MEL. Namely, let $\phi > \neg\phi$ be denoted by $\Box\phi$, and restrict the language $\mathcal{L}_>$ to such formulas. Then, a restricted form of the following axioms and inference rules in $\mathcal{S}$ can be expressed in MEL: $ax1$, RI_2, RI_3, RI_7. $ax1$ says that $\Box\phi$ is a tautology when ϕ is so. RI_2 says that $\Box\phi$ implies $\Box\psi$ if $\phi \models \psi$. Rule RI_3 corresponds to

[9] Here, we use our terminology.

the contradiction between $\Box\phi$ and $\Box\neg\phi$. And a special instance of RI_7 is $(\Box\phi) \wedge (\Box\psi) \models \Box(\phi \wedge \psi)$. Likewise, satisfiability in MEL is in terms of non-empty sets E of Ω representing epistemic states: $E \models \Box\phi$ means $E \subseteq [\phi]$. The corresponding model for the MEL-fragment of $\mathcal{S}$ is the partial certainty relation obtained by the closure of $E > \overline{E}$ using properties O and Q^d (see [12] for details). We can easy prove that if we increase the language $\mathcal{L}_>$ with a negation (allowing expressions such as $\neg(\phi > \psi)$), adapting axiom RI_3 accordingly, we recover the MEL logic as a special case using the axioms of $\mathcal{S}$ [12].

7.2. Non-monotonic logics

We can also establish links with the non-monotonic logic of conditional assertions, called system P [31]. This system encapsulates propositional logic in a language equipped with a binary relation that reflects plausible inference between two formulas (rules $\phi \mid\!\sim \psi$ expressing that generally ψ is the case in context ϕ). Intuitively, $\phi \mid\!\sim \psi$ means that $\phi \wedge \psi$ is more plausible than $\phi \wedge \neg\psi$. In terms of relative certainty it can be equivalently expressed as $\phi \to \psi > \phi \to \neg\psi$. If we consider the restriction of our language $\mathcal{L}_>$ to pieces of knowledge of this form, it is easy to prove [12] that the inference system $\mathcal{S}$ captures the inference rules of system P. In other words, a set of conditional assertions can be encoded as a conditional logic base in $\mathcal{L}_>$. Fariñas del Cerro et al. [27] were the first to lay bare this fact and to provide a conditional account of non-monotonic inference in system P in the style of Lewis logics, using a partially ordered set of formulas.

In contrast, Benferhat et al. [7] propose a semantic account of system P, considering the partial order between formulas induced by conditional assertions, as a family of linear plausibility orders on Ω compatible with this partial order. Namely, the partial order between formulas can be viewed as a fragment of possibility ordering 2^Ω induced by a linear order on Ω. Let $>$ be a partial order on 2^Ω, and let $\mathbb{P}_>$ be the set of linear orderings $\rhd^i$ on Ω, and such that $>^i_\Pi$, the comparative possibility on 2^S induced by $\rhd^i$, extends $>$. The following result holds:

Proposition 15. *(See [7,22,11].) Suppose $>$ on 2^Ω satisfies Q and O. Then: $A > B$ if and only if $A >^i_\Pi B$, $\forall \rhd^i \in \mathbb{P}_>$.*

See [34] for a similar result in the area of belief revision. As a consequence, the weak optimistic strict order on subsets is characterized by a family of linear orderings on elements. Given the properties satisfied by $>_{wos}$, this result clearly bridges the gap between weak optimistic dominance at work in the logic system $\mathcal{S}$ and the partially ordered non-monotonic inference setting of Kraus, Lehmann and Magidor [31] encoding the dominance $[\phi] >_{wos} [\psi]$ when $[\phi \wedge \psi] = \varnothing$ as the conditional assertion $\phi \vee \psi \mid\!\sim \neg\psi$. Moreover, the connections between rational closure of Lehmann and Magidor [32] and possibilistic logic are well-known [6].

The inference system $\mathcal{S}$ can also be related to the idea of considering a partially ordered base $(\mathcal{K}, >)$ as the family of possibilistic (totally ordered) bases extending it [39]. A cautious method consists in considering inferences sanctioned by all such possibilistic bases as valid. Then, concluding $\phi > \psi$ comes down to inferring $\phi >^i \psi$ from all totally ordered bases that extend $(\mathcal{K}, >)$. The extension of such totally ordered bases on $\mathcal{L}$ induced by a set of conditional assertions corresponds to what Lehmann et al. [31] call a rational extension. It is clear from the above results that this cautious inference from totally ordered bases extending $(\mathcal{K}, >)$ is equivalent to inference in system $\mathcal{S}$.

8. Conclusion and future works

Inference from a partially ordered knowledge base is often carried out in a conditional logic in the tradition of Lewis [33]. On the other hand, inference from a totally ordered knowledge base is usually encoded in possibilistic logic. This paper explores the links between these two approaches, thus providing a unified view that encompasses one form of non-monotonic reasoning. We have shown important differences between the relative certainty logic and the possibilistic logic points of view:

- A possibilistic logic base can be represented by a single weak order of interpretations, while this is not the case for po-bases. Using a single partial ordering of interpretations induced by $(\mathcal{K}, >)$ loses information on the way. So the usual semantics of possibilistic logic cannot be used in the partially ordered case, which justifies the use of semantics in terms of partial orders between epistemic states, more natural than the one used in Halpern [30].
- Possibilistic inference can be reduced to classical inference on cuts of the totally ordered base, while this approach becomes incomplete in the partially ordered case.
- Possibilistic logic avoids inconsistencies caused by the original ranking of formulas, because the ordering is interpreted as lower certainty bounds on formulas and can be revised, while in the relative certainty approach, the ranking of formulas is enforced and is understood as a fragment of a full-fledged certainty ordering of the language.
- Possibilistic logic relies on the minimal specificity principle that helps selecting a unique ordering of interpretations, in the spirit of rational closure [32] or system Z of Pearl [36], while the relative certainty logic approach does not appeal to it and is more cautious.

The form of possibilistic logic closest to the conditional logic encoding of po-bases is SPL, when weights are symbolic representations of ill-known certainty values with constraints on their ranking. One important contribution of this paper is the comparison between SPL and partially ordered bases, in the form of partial mutual translations. This link is important because procedures for automated reasoning from po-bases could be implemented in possibilistic logic, which is easier to exploit than a full-fledged conditional logic. This is our topic of current investigation [13]. This work also has potential applications for the revision and the fusion of beliefs, as well as preference modeling [23].

In this paper, we focus on strict order relations. However, the study could be extended to partial preorderings, enriching the language system $\mathcal{S}$ in order to take into account weak comparative assertions of the form $\phi \geq \psi$, interpreting such statements as disjunctions $(\phi > \psi) \vee (\phi \sim \psi)$, for some equivalence relation $\sim$.

Besides, a similar analysis could be carried out for partially ordered bases interpreted as fragment of *preadditive* partial orders (that are self-dual like probability relations [22]). In [10], we have studied the case of a preadditive version of the optimistic dominance relation $>_{wos}$, of the form $A >_P B \iff A \cap \overline{B} >_{wos} B \cap \overline{A}$ (known in non-monotonic inference since the 1980s, and particularly studied in [9]); it satisfies natural properties and is more discriminant than $>_{wos}$. A conditional logic system for reasoning with such refined relations is outlined in [12].

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Appendix A. Proofs

Proof of Proposition 4. By definition, we have: $A \succ_N B$ if and only if $\overline{B} \succ_\Pi \overline{A}$.

Due to Proposition 3 we know that the relation $\succ_\Pi$, obtained by weak optimistic dominance, satisfies Qualitativeness (Q), Orderliness (O), Negligibility (N), Conditional Closure by Implication (CCI), Conditional Closure by Conjunction (CCC), Left Disjunction (OR), (CUT), (CM). We only prove the first property for $\succ_N$. The proofs for the other ones are similar (see [12]).

Assume that $C \succ_N A \cap B$ and $B \succ_N A \cap C$. It can be written as $\overline{A \cap B} \succ_\Pi \overline{C}$ and $\overline{A \cap C} \succ_\Pi \overline{B}$. Then we have $\overline{A} \cup \overline{B} \succ_\Pi \overline{C}$ and $\overline{A} \cup \overline{C} \succ_\Pi \overline{B}$. As $\succ_\Pi$ satisfies Qualitativeness (Q), it can be deduced that $\overline{A} \succ_\Pi \overline{B} \cup \overline{C}$, which can be written $B \cap C \succ_N A$. $\square$

Proof of Proposition 6. Let $(\mathcal{K}, >) = \{(\phi_i > \psi_i), i = 1 \cdots n\}$.

- **Soundness:**
  Let $\succ_N$ be a strict partial order on 2^Ω satisfying O and Q^d. We must show that if $\forall i = 1 \cdots n, [\phi_i] \succ_N [\psi_i]$ then $[\phi] \succ_N [\psi]$. Assume that $\phi > \psi$ is an axiom or was obtained from the pairs $(\phi_i > \psi_i)$ by inference rules RI_1 and RI_2. It is enough to prove that the axiom is valid and each of the inference rules is sound.

  ax_1 We must show that $\forall \mathcal{N}, \mathcal{N} \vDash (\top > \bot)$. Or equivalently, $\forall \succ_N$ strict partial relation on 2^Ω satisfying the properties O and Q^d, $[\top] \succ_N [\bot]$. We have $[\top] = \Omega$ and $[\bot] = \varnothing$ and $\Omega \succ_N \varnothing$ for $\succ_N$ to be a partial certainty relation on 2^Ω.

  RI_1 We must show that if $[\phi \wedge \psi] \succ_N [\chi]$ and $[\phi \wedge \chi] \succ_N [\psi]$ then $[\phi] \succ_N [\psi \wedge \chi]$. This is true since $[\phi \wedge \psi] = [\phi] \cap [\psi]$ and the relation $\succ_N$ satisfies Q^d.

  RI_2 We must show that if $[\phi] \succ_N [\psi]$, $\phi \vDash \phi'$ and $\psi' \vDash \psi$ then $[\phi'] \succ_N [\psi']$. This is true since the relation $\succ_N$ satisfies O.

  RI_3 Moreover, we show that the presence of both $\phi > \psi$ and $\psi > \phi$ leads to a semantical contradiction. This is true since the relation $\succ_N$ being asymmetric, we cannot have both $[\psi] \succ_N [\phi]$ and $[\phi] \succ_N [\psi]$.

- **Completeness:**
  We assume that $(\mathcal{K}, >)$ is rc-consistent. Let us suppose that for every order relation $\succ_N$ on 2^Ω satisfying O and Q^d, if $\forall i = 1 \cdots n, \phi_i \succ_N \psi_i$ then $[\phi] \succ_N [\psi]$. We must show that $(\mathcal{K}, >) \vdash_\mathcal{S} \phi > \psi$. If $\phi > \psi$ is in $(\mathcal{K}, >)$, it is proved.
  Otherwise, consider the strict partial order $\succ_N$ defined on 2^Ω as the smallest relation containing the pairs $[\phi_i] \succ_N [\psi_i]$ and closed for the properties Q^d, O. This relation exists since $(\mathcal{K}, >)$ is rc-consistent. According to the hypothesis we have $[\phi] \succ_N [\psi]$. And by definition of $\succ_N$, the pair $([\phi], [\psi])$ is obtained by successive applications of properties Q^d, O. This amounts to obtaining $\phi > \psi$ by successive applications of the inference rules RI_1 and RI_2.
  It remains to prove that if $(\mathcal{K}, >)$ is rc-inconsistent, $(\mathcal{K}, >) \vdash_\mathcal{S} \bot_>$.
  Notice that, as $\mathcal{L}_>$ contains only atomic comparison statements and their conjunction, the only form of inconsistency is by the presence of both $\phi > \psi$ and $\psi > \phi$ deduced from $(\mathcal{K}, >)$. This is the only way to get $(\mathcal{K}, >) \vdash_\mathcal{S} \bot_>$. In this case, we know that $(\mathcal{K}, >)$ has no partial certainty model.
  So, if $(\mathcal{K}, >) \vdash_\mathcal{S} \bot_>$ does not hold, then the relation $>$ obtained on $\mathcal{L}_>$ by syntactic closure will be asymmetric and transitive, and so will be the relation $\succ$ on 2^Ω defined by $[\phi] \succ [\psi]$ if and only if $(\mathcal{K}, >) \vdash_\mathcal{S} \phi > \psi$. Moreover, $\succ$ will be the least relation containing the pairs $([\phi_i], [\psi_i])$ if and only if $\phi_i > \psi_i$ appears in $(\mathcal{K}, >)$, and closed for the properties Q^d, O. It is an $\mathcal{N}$-model of $(\mathcal{K}, >)$, which is thus rc-consistent. $\square$

Proof of Proposition 8. Assume that $(\mathcal{K}, >) \vdash_c \phi > \psi$. By definition, $\mathcal{K}_\psi^> \vdash \phi$. $\mathcal{K}_\psi^> = \{\chi : \chi \in \mathcal{K}$ and $\chi > \psi\}$. Let $\mathcal{K}_\psi^> = \{\chi_1, \ldots, \chi_p\}$, which means $\forall i, \chi_i > \psi \in (\mathcal{K}, >)$. In agreement with Section 5.2, let us consider a strict order $\succ_N$ on 2^Ω satisfying O and Q^d, such that if $\phi_i > \psi_i \in (\mathcal{K}, >), [\phi_i] \succ_N [\psi_i]$. We have $\forall i, [\chi_i] \succ_N [\psi]$. As $\succ_N$ satisfies O and Q^d, it also satisfies Adjunction. So $[\chi_1] \cap [\chi_2] \cap \cdots \cap [\chi_p] \succ_N [\psi]$. That is $[\chi_1 \wedge \chi_2 \cdots \wedge \chi_p] \succ_N [\psi]$. As $\{\chi_1, \ldots \chi_p\} \vdash \phi$, it holds that $[\chi_1 \wedge \chi_2 \cdots \wedge \chi_p] \subseteq [\phi]$. As $\succ_N$ satisfies O, we obtain that $[\phi] \succ_N [\psi]$. So we have proved that $(\mathcal{K}, >) \vDash_N \phi > \psi$. $\square$

Proof of Proposition 9. We assume constraints in C are strict.

1. $\omega \rhd_\Sigma \omega$ if and only if $\forall(\phi_j, p_j) \in \Sigma$ such that $\phi_j \in \overline{\Sigma(\omega)}$, there is some $(\phi_i, p_i) \in \Sigma$ such that $\phi_i \in \overline{\Sigma(\omega')}$ and $p_i > p_j$ for each instantiation of elementary weights appearing in C. Besides,
$$C \vDash \iota_\Sigma(\omega) < \iota_\Sigma(\omega') \text{ if and only if } C \vDash (\max_{j=1,\dots,m} \iota_j(\omega) < \max_{k=1,\dots,\ell} \iota_k(\omega'))$$
if and only if for each instantiation of elementary weights appearing in C, there is an index k such that $\iota_k(\omega') > 0$, and for all h such that $\iota_h(\omega) > 0, \iota_h(\omega) < \iota_k(\omega')$ (the latter expressions are the minima of some p_i or p_j terms)
if and only if for each instantiation of elementary weights appearing in C and all j such that $\omega \nvDash \phi_j$, there is some index i such that $\omega' \nvDash \phi_i :\ p_j < p_i$.
So $\omega \rhd_\Sigma \omega'$ implies $C \vDash \iota_\Sigma(\omega) < \iota_\Sigma(\omega')$, as the former requires that $p_i > p_j$ for each instantiation of elementary weights appearing in C and fixed indices i and j, while in the latter the choice of p_i and p_j depends on each such instantiation.
2. Follows from the previous point and the definition of $N_\Sigma(\phi) = \min_{\omega \notin [\phi]} \iota_\Sigma(\omega)$. $\square$

Proof of Proposition 11. If $(\mathcal{K}, >)$ is rc-consistent, the partial order on $\mathcal{K}$ is a fragment of the resulting partial order on the language. This partial order satisfies Q^d and O, therefore it can be extended by a necessity order. In this case, the order on $(\Sigma_\mathcal{K}, C_\mathcal{K})$ will not be modified by deduction (otherwise we would contradict in $(\Sigma_\mathcal{K}, C_\mathcal{K})$ a formula of $\mathcal{L}_>$ in $(\mathcal{K}, >)$). So we can assume that $\eta(\phi) = N_{\Sigma_\mathcal{K}}(\phi)$ where ϕ is a clause. We will show that the axioms and inference rules of $\mathcal{S}$ are satisfied in possibilistic logic.

- ax_1: the translation of $\top > \bot$ is $(\top, 1)$, an axiom of possibilistic logic.
- We cannot apply RI_1 because $\mathcal{K}$ contains only clauses. We can apply the weaker rule RI_5: $\Sigma = \{(\phi, a), (\psi, b), (\chi, c)\}$ with $a > c$ and $b > c$, implies $N_\Sigma(\phi \wedge \psi) = \min(a, b) > N_\Sigma(\chi) = c$.
- We proceed in the same way for RI_2.
- As we assume rc-consistency, RI_3 is not involved. $\square$

Proof of Proposition 12. $\Rightarrow$) We suppose that $(\Sigma, C) \vdash_\pi \phi > \psi$, this means that $C \vDash (N_\Sigma(\phi) > N_\Sigma(\psi))$. Let $p = N_\Sigma(\phi)$ and $q = N_\Sigma(\psi)$. We have $p > q > 0$. $\mathcal{K}^>_\psi = \{\chi \in \mathcal{K} : \chi > \psi\}$ and $\chi > \psi$ implies $(\chi, r) \in \widehat{\Sigma}$ and $C \vDash r > q$.

As $q > 0$, $\mathcal{K}^>_\psi$ is a classical set of consistent formulas. On the other hand, $\mathcal{K}^>_\psi$ contains $(\Sigma^>_p)^*$ (where $\Sigma^>_p = \{(\phi_j, p_j) \in \widehat{\Sigma} : p_j > p\}$) and $(\Sigma^>_p)^* \vdash \phi$. So $\mathcal{K}^>_\psi \vdash \phi$.

$\Leftarrow$) Conversely, $\mathcal{K}^>_\psi$ is consistent and if $\mathcal{K}^>_\psi \vdash \phi$, we have $\mathcal{K}^>_\psi = \{\varphi_i \in \mathcal{K} : \varphi_i > \psi\} \vdash \phi$. So $(\varphi_i, c_i), (\psi, q) \in \widehat{\Sigma}$ and $C \vDash r_i > q$, and $C \vDash \min(r_i) > q$. Clearly, $\mathit{Inc}(\Sigma \cup (\neg\psi, 1)) = q$ and $N_\Sigma(\phi) \geq \min(r_i) > q = N_\Sigma(\psi))$. So $(\Sigma, C) \vdash_\pi \phi > \psi$. $\square$

Proof of Proposition 13. We assume that $(\mathcal{K}, >) \vdash_\mathcal{S} \phi > \psi$. The proof is by induction on the number of steps using the inference rules of the system $\mathcal{S}$.

Case when $\phi > \psi \in (\mathcal{K}, >)$: it means that $(\phi, p) \in \widehat{\Sigma}, (\psi, q) \in \widehat{\Sigma}$ and $C \vDash p > q$. Or equivalently $C \vDash N_\Sigma(\phi) > N_\Sigma(\psi)$. That is exactly the definition of $(\Sigma, C) \vdash_\pi \phi > \psi$.

If RI_1 is applied: Let $\phi = \varphi \wedge \chi$ with $(\mathcal{K}, >) \vdash_\mathcal{S} \chi > \varphi \wedge \psi$ and $(\mathcal{K}, >) \vdash_\mathcal{S} \varphi > \chi \wedge \psi$. We know that $(\mathcal{K}, >) \vdash_\mathcal{S} \chi \wedge \varphi > \psi$. By induction hypothesis:
$$C \vDash N_\Sigma(\chi) > N_\Sigma(\varphi \wedge \psi) = \min(N_\Sigma(\varphi), N_\Sigma(\psi))$$
$$C \vDash N_\Sigma(\varphi) > N_\Sigma(\chi \wedge \psi) = \min(N_\Sigma(\chi), N_\Sigma(\psi))$$

If $N_\Sigma(\psi) > N_\Sigma(\varphi)$ then $N_\Sigma(\chi) > N_\Sigma(\varphi)$. Thus, $\min(N_\Sigma(\chi), N_\Sigma(\psi)) > N_\Sigma(\varphi)$ (which is impossible). Thus $N_\Sigma(\varphi) \geq N_\Sigma(\psi)$. So $N_\Sigma(\chi) > N_\Sigma(\psi)$ and $N_\Sigma(\varphi) > N_\Sigma(\psi)$. Hence, $N_\Sigma(\varphi \wedge \chi) > N_\Sigma(\psi)$. Thus $N_\Sigma(\phi) > N_\Sigma(\psi)$.

If RI_2 is applied: We have $(\mathcal{K}, >) \vdash_{\mathcal{S}} \phi' > \psi'$ with $\phi' \vDash \phi$ and $\psi \vDash \psi'$.

By hypothesis $C \vDash N_\Sigma(\phi') > N_\Sigma(\psi')$. We also have $N_\Sigma(\phi) \geq N_\Sigma(\phi')$ and $N_\Sigma(\psi') \geq N_\Sigma(\psi)$. Thus $N_\Sigma(\phi) > N_\Sigma(\psi)$.

As we assume that (Σ, C) is consistent, its translation $(\mathcal{K}, >)$ is rc-consistent, so RI_3 is not involved. $\square$

Proof of Proposition 15. Since the strict part of a possibility relation satisfies Q and O, we can consider the set of comparative possibility measures that extend $>$, as it will be not empty. So the if part is obvious. For the only if part, it is also clear by construction, that the intersection of the relations on $2^\Omega \times 2^\Omega$ representing the comparative possibility measures that extend $>$ is the relation that represents $>$. $\square$

Appendix B. Nomenclature of partial order relations

The following table summarizes the nomenclature of relation symbols used in the paper:

Symbol	Description
$>, \succ$	strict partial orders on a set of formulas
$\rhd$	a strict partial order on a set of interpretations
$>_{wos}$	the weak optimistic strict dominance on 2^S induced by $>$ on S
$>_{Sos}$	the strong optimistic strict dominance on 2^S induced by $>$ on S
$\rhd_{wos}$	the weak optimistic strict dominance on 2^Ω induced by $\rhd$ on Ω
$\succ_N$	a partial certainty relation, strict partial order on 2^Ω
$\succ_\Pi$	a partial possibility relation, strict partial order on 2^Ω

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