On monodromy for a class of surfaces

Ann Lemahieu, Willem Veys

On monodromy for a class of surfaces

Ann Lemahieu [a], Willem Veys [b]

[a] *Fachbereich Mathematik, Universität Kaiserslautern, 67653 Kaiserslautern*

[b] *Departement Wiskunde, Katholieke Universiteit Leuven, Celestijnenlaan 200B, 3001 Leuven*

Abstract

In this note we present a result on the monodromy conjecture for surfaces that are generic with respect to a toric idealistic cluster.

Résumé

Sur la monodromie pour une certaine classe de surfaces. On présente dans cette note un résultat sur la conjecture de monodromie pour les surfaces qui sont génériques pour un amas torique idéalistique.

Version française abrégée

En 1992 Denef et Loeser ont introduit la fonction zêta topologique. Un analogue de la conjecture de monodromie, déjà établie pour la fonction zêta d'Igusa, était né pour cette nouvelle fonction zêta. Loeser a demontré la conjecture de monodromie pour les courbes et pour les polynômes qui sont nondégénérés pour leur polyèdre de Newton et qui satisfont des conditions numériques ([L1,L2]). Artal Bartolo, Cassou-Noguès, Luengo, Melle Hernández ([A-B,C-N,Lu,M-H]), Rodrigues ([Ro]) et Veys ([Ve1]) ont également traité de cette conjecture mystérieuse.
Soient $f : (\mathbb{C}^d, 0) \to (\mathbb{C}, 0)$ le germe d'une fonction holomorphe et $\pi : X \to \mathbb{C}^d$ une résolution plongée des singularités de $f^{-1}\{0\}$. On note $(E_j)_{j\in S}$ les composantes irréductibles de $\pi^{-1}(f^{-1}\{0\})$ et on note respectivement les multiplicités de E_j dans les diviseurs de $f\circ\pi$ et de $\pi^*(dx_1\wedge\ldots\wedge dx_d)$ sur X par N_j et par ν_j-1. Pour $I \subset S$ on pose $E_I := \cap_{i\in I} E_i$ et $E_I^\circ := E_I \setminus (\cup_{j\notin I} E_j)$. La *fonction zêta topologique locale* associée à f est la fonction rationnelle en une variable complexe $Z_{top,f}(s) := \sum_{I\subset S} \chi(E_I^\circ \cap \pi^{-1}\{0\}) \prod_{i\in I} \frac{1}{N_i s+\nu_i}$.
Denef et Loeser ont montré que la fonction zêta topologique ne dépend pas de la résolution plongée choisie ([De,L]). Supposons maintenant que f est une fonction en trois variables et que la surface $f = 0$ a exactement une singularité isolée 0. Alors on peut supposer que π est un isomorphisme sur le complément de l'image inverse de l'origine. Soient $(E_j)_{j\in J:=\{1,\cdots,r\}}$ les composantes irréductibles exceptionnelles de $\pi^{-1}\{0\}$. Dans l'article [A'C] A'Campo montre que la fonction zêta de monodromie ζ_f de f à 0 peut être

Email addresses: lemahieu@mathematik.uni-kl.de (Ann Lemahieu), wim.veys@wis.kuleuven.be (Willem Veys).

Preprint submitted to the Académie des sciences

écrite comme $\zeta_f(t) = \prod_{j=1}^{r}(1-t^{N_j})^{\chi(E_j^\circ)}$. Sous cette description les valeurs propres de la monodromie de f sont les zéros de ζ_f et la conjecture de monodromie devient *'Si s est un pôle de $Z_{top,f}$, alors $e^{2\pi is}$ est un zéro de ζ_f'*. Fixons maintenant un candidat pôle $s := -\nu_j/N_j$, $j \in J$, de $Z_{top,f}$. Nous écrivons ν_j/N_j comme a/b avec a et b premiers entre eux et nous définissons l'ensemble $J_b := \{j \in J \mid \ b|N_j\}$. La formule d'A'Campo implique que $e^{2\pi is}$ est un zéro de ζ_f si et seulement si $\sum_{j\in J_b}\chi(E_j^\circ) \neq 0$.
On considère les surfaces génériques pour lesquelles il existe une résolution plongée des singularités réalisée par l'éclatement d'une constellation torique qui est la constellation des points de base d'un idéal monomial de support fini.

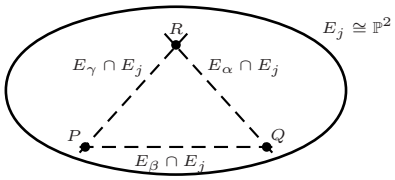


Figure 1

Le contexte des constellations toriques en dimension 3 implique que la configuration dans $E_j \cong \mathbb{P}^2$ est comme dans la Figure 1. Les seules intersections possibles avec d'autres composantes exceptionelles sont les droites de coordonnées $E_\alpha \cap E_j$, $E_\beta \cap E_j$ et $E_\gamma \cap E_j$ et les points P, Q et R sont les seuls points de $E_j \cong \mathbb{P}^2$ dans lesquelles il est permis d'éclater.

En général, il peut y avoir beaucoup d'annulations qui font que $\sum_{j\in J_b}\chi(E_j^\circ) = 0$. On obtient que $\chi(E_j^\circ) < 0$ *si et seulement si la configuration dans $E_j \cong \mathbb{P}^2$ consiste en (au moins trois) droites - éventuellement exceptionelles - qui passent toutes par le même point.* Ce résultat nous permet de démontrer :

Théorème *Si $\chi(E_j^\circ) > 0$, alors $e^{-2\pi i\nu_j/N_j}$ est une valeur propre de la monodromie de f.*

Comme application on peut vérifier la conjecture de monodromie pour 'la plupart' des pôles.

Corollaire *Si $-\nu_j/N_j$ est un candidat pôle de $Z_{top,f}$ d'ordre 1 qui est un pôle, alors $e^{-2\pi i\nu_j/N_j}$ est une valeur propre de la monodromie de f.*

1. Introduction

In 1992 Denef and Loeser introduced the topological zeta function. The monodromy conjecture, first stated for the Igusa zeta function, got an analogue for this new zeta function. Loeser proved the conjecture for curves and for polynomials that are nondegenerate with respect to their Newton polyhedron and that satisfy some numerical conditions ([L1,L2]). Also Artal Bartolo, Cassou-Noguès, Luengo, Melle Hernández ([A-B,C-N,Lu,M-H]), Rodrigues ([Ro]) and Veys ([Ve1]) provided results about this mysterious conjecture. We explain in this note by geometric arguments a phenomenon concerning calculation of monodromy and the monodromy conjecture for surfaces that are generic with respect to a 3-dimensional toric idealistic cluster .
Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function and let $\pi : X \to \mathbb{C}^d$ be an embedded resolution of singularities of $f^{-1}\{0\}$. We write $(E_j)_{j\in S}$ for the irreducible components of $\pi^{-1}(f^{-1}\{0\})$ and we denote by N_j and $\nu_j - 1$ the multiplicities of E_j in the divisor on X of $f\circ\pi$ and $\pi^*(dx_1\wedge\ldots\wedge dx_d)$ respectively. The couples $(N_j,\nu_j)_{j\in S}$ are called the numerical data of the embedded resolution (X,π). For $I \subset S$ we denote also $E_I := \cap_{i\in I}E_i$ and $E_I^\circ := E_I \setminus (\cup_{j\notin I}E_j)$. The *local topological zeta function* associated to f is the rational function in one complex variable $Z_{top,f}(s) := \sum_{I\subset S}\chi(E_I^\circ \cap \pi^{-1}\{0\})\prod_{i\in I}\frac{1}{N_i s+\nu_i}$. Indeed, Denef and Loeser proved that this expression does not depend on the chosen embedded resolution (see [De,L]). From now on suppose that f is a function in three variables and that the surface $f = 0$ has only one isolated singularity 0. Then we may suppose that π is an isomorphism outside the inverse image of the origin. Say that the $(E_j)_{j\in J:=\{1,\cdots,r\}}$ are the irreducible exceptional components of $\pi^{-1}\{0\}$.

In [A'C] A'Campo showed that the zeta function of monodromy ζ_f of f at 0 can then be written as $\zeta_f(t) = \prod_{j=1}^{r}(1-t^{N_j})^{\chi(E_j^\circ)}$. Under this description, the eigenvalues of monodromy of f are the zeroes of ζ_f and the monodromy conjecture becomes *'If s is a pole of $Z_{top,f}$, then $e^{2\pi is}$ is a zero of ζ_f'*. Now fix a candidate pole $s := -\nu_j/N_j$, $j \in J$, of $Z_{top,f}$. We write ν_j/N_j as a/b such that a and b are coprime and we define the set $J_b := \{j \in J \mid b|N_j\}$. It follows from A'Campo's formula that $e^{2\pi is}$ is a zero of ζ_f if and only if $\sum_{j\in J_b}\chi(E_j^\circ) \neq 0$.
To introduce the class of surfaces we work with, we first say something about clusters. For details we refer to [C,G-S,L-J]. Let X be a nonsingular variety of dimension $d \geq 2$ and Z be obtained from X by a finite succession of point blowing-ups. A point $Q \in Z$ is *infinitely near* to a point $P \in X$ if P is in the image of Q. Let $Q_0 \in X =: X_0$. A *constellation with origin* Q_0 is a finite sequence $\mathcal{C} := \{Q_0, Q_1, \cdots, Q_{r-1}\}$ of points such that each Q_j is infinitely near to Q_0 and is a point on the variety X_j obtained by blowing up Q_{j-1} in X_{j-1}, $j \in \{1, \cdots, r-1\}$. For each $Q_i \in \mathcal{C}$, denote the exceptional divisor of the blowing-up in Q_i and its strict transform at any stage by E_i. If $Q_j \in E_i$, then Q_j is *proximate* to Q_i; we write $j \to i$. A d-dimensional *toric constellation* with origin Q_0 is a constellation $\mathcal{C} := \{Q_0, Q_1, \cdots, Q_{r-1}\}$ such that each Q_j is a 0-dimensional orbit in the toric variety X_j obtained by blowing up Q_{j-1} in X_{j-1}, $1 \leq j \leq r-1$. A tree with a root such that each vertex has at most d following adjacent vertices is called a *d-ary tree*. There is a natural bijection between the set of d-dimensional toric constellations with origin and the set of finite d-ary trees with a root, with the edges labeled with positive integers not greater than d, such that two edges with the same source have different labels. The labels indicate in which affine chart the points are created. A *cluster* $\mathcal{A} := (\mathcal{C}, \underline{m})$ consists of a constellation $\mathcal{C} := \{Q_0, \cdots, Q_{r-1}\}$ and a sequence $\underline{m} := (m_0, \cdots, m_{r-1})$ of nonnegative integers. The idea of clusters is to express that a system of hypersurfaces is passing through the points of the constellation with (at least) the given multiplicities. An ideal I in $\mathcal{O}_{X,Q_0}$ is *finitely supported* if I is primary for the maximal ideal of $\mathcal{O}_{X,Q_0}$ and if there exists a constellation $\mathcal{C}$ of X such that $I\mathcal{O}_{X_r}$ is an invertible sheaf. An infinitely near point Q of Q_0 is a *base point* of I if Q belongs to the constellation with the minimal number of points, say r, with the above property. The canonical map $X_r \to X$ associated to the constellation of base points of a finitely supported ideal I to X is an embedded resolution of the subvariety of (X, Q_0) defined by a general enough element in I.
We will consider generic surfaces for which there exists an embedded resolution of singularities realised by the blowing-up of a toric constellation with origin that is the constellation of base points of a finitely supported monomial ideal. In general there can be a lot of cancellations which make that $\sum_{j\in J_b}\chi(E_j^\circ) = 0$. We study when $\chi(E_j^\circ) < 0$ for some $j \in J$. We obtain that $\chi(E_j^\circ) < 0$ *if and only if the configuration in $E_j \cong \mathbb{P}^2$ consists of (at least three) lines - possibly exceptional - that are all going through the same point.* Using this result we prove:

Theorem *If $\chi(E_j^\circ) > 0$, then $e^{-2\pi i\nu_j/N_j}$ is an eigenvalue of monodromy of f.*

As an application we can verify the assertion in the monodromy conjecture for 'most' poles.

Corollary *If $-\nu_j/N_j$ is a candidate pole of $Z_{top,f}$ of order 1 that is a pole, then $e^{-2\pi i\nu_j/N_j}$ is an eigenvalue of monodromy of f.*

The results presented in this note are shown by geometric arguments and cover unknown cases of the monodromy conjecture. In a forthcoming paper we will actually prove the monodromy conjecture in this context. That proof is completely combinatorial.

2. Determination of the cases in which appears a negative $\chi(E_j^\circ)$

Let $I \subset \mathbb{C}[x,y,z]$ be a finitely supported monomial ideal and let S be a surface given by a general element in I. We will write $\hat{S}$ for its strict transform, whatever the stage is. We will denote the curves $E_j \cap \hat{S} \subset E_j \cong \mathbb{P}^2$ by C_j, $j \in J$, and their strict transforms by $\hat{C}_j$.

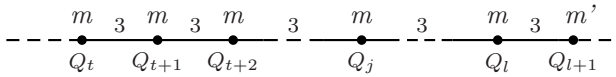
Lemma 2.1 *x, y or z can not be a factor in the equation of C_j.*
Proof. Suppose that x is a factor in the equation of C_j. Then the monomial ideal I would not be finitely supported. Indeed, at the moment that, say x appears as factor in the equation of C_j, the strict transform of C_j in some affine chart takes the form $xg_x + zg_z$, for some polynomials g_x, g_z in $\mathbb{C}[x,y,z]$. It is now obvious that the ideal can not be principalised by blowing up a toric constellation. ■

Lemma 2.2 *The curve $C_j \subset \mathbb{P}^2$ is generically reducible if and only if its equation is of the form $h(m,n) = 0$ with h a homogeneous polynomial in two variables of degree at least 2, and m and n two monomials.*
Proof. The claim follows from [Bo,Deb,Na, Thm. 1.2]. ■

Notice also that the setting of toric constellations in dimension 3 implies that the configuration in $E_j \cong \mathbb{P}^2$ is as in Figure 1. The only possible intersections with other exceptional components are the coordinate lines $E_\alpha \cap E_j$, $E_\beta \cap E_j$ and $E_\gamma \cap E_j$ and the points P, Q and R are the only points in $E_j \cong \mathbb{P}^2$ in which it is allowed to blow up.

Lemma 2.3 *If the equation of C_j contains three variables, then $\chi(E_j^\circ) \geq 0$.*
Proof. Suppose that $\chi(E_j^\circ) < 0$. We split cases according to whether C_j is irreducible or not.
(i) If the curve C_j is irreducible, then $\chi(C_j) \leq 2$ and if $\chi(E_j^\circ) < 0$, the configuration in E_j should consist of C_j and two lines such that these three curves intersect in exactly one point, say P. Moreover $\chi(C_j)$ should then be equal to 2. When the degree d of C_j is at least 2 and if one of the lines would not be a principal tangent line to C_j in P, then this line would intersect C_j in another point. This is a contradiction. Hence, C_j should be analytically reducible in P, and consequently $\chi(C_j) < 2$. So suppose now that $\deg(C_j) = 1$. In an affine chart where one can see P, one can write $E_\alpha \leftrightarrow x = 0$, $E_\beta \leftrightarrow y = 0$ and then C_j should have an equation of the form $c_x x + c_y y = 0$ with c_x and c_y complex numbers. However, the equation of C_j contains the three variables.
(ii) Suppose now that the curve C_j is reducible. Lemma 2.2 implies that the equation of C_j is of the form $h(m,n) = 0$ with h a homogeneous polynomial in two variables of degree at least 2, and m and n two monomials. Then by Lemma 2.1 one easily verifies that (m,n) must be of the form $(x^a y^b, z^{a+b})$. We may suppose moreover that a and b are coprime (otherwise we change h). The decomposition in irreducible components of C_j is then $\prod_{i=1}^{k}(x^a y^b - c_i z^{a+b})$ with the $c_i \in \mathbb{C}^*$. As the curves $x^a y^b - c_i z^{a+b} = 0$ are rational with Euler characteristic 2, we get $\chi(E_j^\circ) = 0$ or $\chi(E_j^\circ) = 1$, depending on the position and the number of exceptional components that are present. ■

We continue to search for cases in which appears an exceptional component E_j for which $\chi(E_j^\circ) < 0$. It follows by Lemma 2.1 that the equation of C_j can not contain just one variable. The remaining case to investigate is when the equation of C_j contains exactly two variables, say x and y. As C_j then has a homogeneous equation in two variables, say of degree m, it follows that the curve C_j consists of m lines having exactly one point in common. From Lemma 2.1 it follows that x^m and y^m certainly appear in the equation of C_j. The point Q_j is then contained in a subcluster of the form



(1)

In this chain Q_t is the point with the lowest level for which an edge with label 3 is leaving and that also has multiplicity m. We suppose that Q_l is the point in the chain with the highest level for which its multiplicity is equal to m. The point Q_j can be equal to Q_t and Q_{l+1} can be absent. As C_j is of degree m, the point Q_j has multiplicity m on the surface. Obviously the multiplicity of Q_{j+1} is also m.
We will now study $\chi(E_j^\circ)$.
- If $Q_j = Q_t$, then the configuration in $E_j \cong \mathbb{P}^2$ is as in Figure 2. If the exceptional line $E_\gamma \cap E_j$ is present, then $\chi(E_j^\circ)$ is always equal to 0. Suppose now that $E_\gamma \cap E_j$ does not appear. This means that Q_t is moreover the point with the lowest level in the chain from which an edge with label 3 is leaving. If Q_j is the origin of the constellation, then $\chi(E_j^\circ) = 2 - m$. If there is exactly one point, say Q_α, for which $j \to \alpha$, then $\chi(E_j^\circ) = 1 - m$. Finally, if there exist two points, say Q_α and Q_β, for which $j \to \alpha$ and $j \to \beta$, then $\chi(E_j^\circ) = -m$.
- If $j \in \{t+1, \cdots, l-1\}$, then the configuration in E_j is as in Figure 3 and then $\chi(E_j^\circ) = 0$.

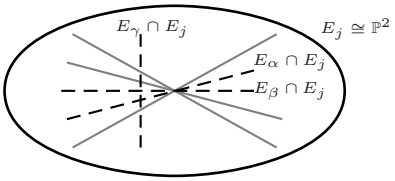


Figure 2

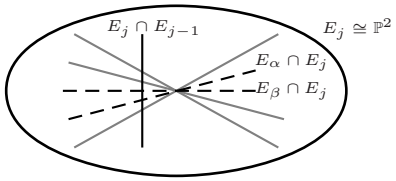


Figure 3

Conclusion: *$\chi(E_j^\circ) < 0$ if and only if the configuration in $E_j \cong \mathbb{P}^2$ consists of (at least three) lines - possibly exceptional - that are all going through the same point.*

3. Application

Lemma 3.1 *Let $\chi(E_t^\circ) < 0$ such that we are in the situation (1).*
(i) If a set J_b contains the index t, then it also contains the indices in $\{t+1, \cdots, l\}$.
(ii) If $\frac{\nu_l}{N_l} = \frac{c}{d}$ with c and d coprime, then $t \notin J_d$.
Proof. If we denote the numerical data of E_t by (N, ν), then, independently of the number of points Q_s for which $t \to s$, one easily computes that the numerical data for $i \in \{t+1, \cdots, l\}$ are $E_i((i-t+1)N, (i-t+1)\nu - (i-t))$. Now the first assertion follows immediately. To see the second claim, suppose that $t \in J_d$. Then $d \mid N$ which implies that $l-t+1 | (l-t+1)\nu - (l-t)$. This contradiction closes the proof. ■

Theorem 3.2 *If $\chi(E_j^\circ) > 0$, then $e^{-2\pi i \nu_j / N_j}$ is an eigenvalue of monodromy of f.*
Proof. Suppose that E_j is an exceptional component such that $\chi(E_j^\circ) > 0$. To prove that $e^{-2\pi i \nu_j / N_j}$ is an eigenvalue of monodromy of f, we show that $e^{-2\pi i \nu_j / N_j}$ is a zero of ζ_f. We write ν_j / N_j as a/b with a and b coprime. If J_b does not contain an index t for which $\chi(E_t^\circ) < 0$, then there is nothing to verify. So suppose now that $\chi(E_t^\circ) < 0$ and that $t \in J_b$. From Lemma 3.1 it follows that $E_j \neq E_l$ and that $l \in J_b$. We will show that $\chi(E_t^\circ) + \chi(E_l^\circ) \geq 0$. Let us therefore study the configuration in $E_l \cong \mathbb{P}^2$.
The equation of C_{l-1} is of the form $c_0 x^m + c_1 x^{m-1} y + \cdots + c_m y^m$, with $c_k \in \mathbb{C}$ $(0 \leq k \leq m)$ and c_0 and c_m different from 0. It follows from Lemma 2.1 that the equation of C_l is then equal to $c_0 x^m + c_1 x^{m-1} y + \cdots + c_m y^m + z g(x, y, z)$ for some polynomial $g \in \mathbb{C}[x, y, z]$, $g \neq 0$. In particular we have that C_l cannot be of the form as in Lemma 2.2 and thus that it is irreducible. Hence $\chi(C_l) \leq 2$.

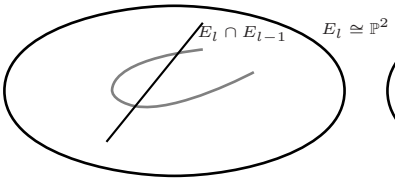


Figure 4

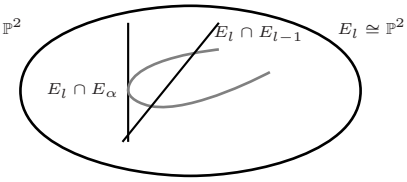


Figure 5

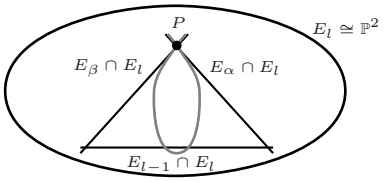


Figure 6

If Q_t is the origin of the constellation, then the configuration in $E_l \cong \mathbb{P}^2$ is like in Figure 4. The curve C_l can be singular. We get $\chi(E_l^\circ) = \chi(E_l) - (\chi(C_l) + (\chi(E_{l-1}) - m)) \geq 3 - (2 + (2 - m)) = m - 1$ and $\chi(E_t^\circ) + \chi(E_l^\circ) \geq (2 - m) + (m - 1) = 1$. Suppose that there exists exactly one point, say Q_α, for which $t \to \alpha$. Then $\chi(E_l^\circ)$ is minimal if C_l and $E_l \cap E_\alpha$ intersect only in one point, as in Figure 5. Again we find $\chi(E_l^\circ) \geq m - 1$. Then $\chi(E_t^\circ) + \chi(E_l^\circ) \geq (1 - m) + (m - 1) = 0$. When there exist two points, say Q_α and Q_β, to which Q_t is proximate, then $\chi(E_l^\circ)$ is minimal when E_α, E_β and C_l intersect in exactly one point, say in P, see Figure 6. Moreover, to have $\chi(E_l^\circ) = m - 1$, one should also require that $\chi(C_l) = 2$. If $m \geq 2$, then $E_\alpha \cap E_l$ and $E_\beta \cap E_l$ are principal tangent lines to C_l in P but then $\chi(C_l) \leq 1$. If $m = 1$, then the equation of C_l must be $x + c_y y + c_z z = 0$ for some $c_y, c_z \in \mathbb{C}^*$. But then C_l intersects the three coordinate lines in three different points and $\chi(E_l^\circ) = 1$. It follows that we always have that $\chi(E_l^\circ) \geq m$ and thus $\chi(E_t^\circ) + \chi(E_l^\circ) \geq -m + m = 0$. This study permits us to conclude that $\sum_{i \in J_b} \chi(E_i^\circ) > 0$. ■

Corollary 3.3 *If $-\nu_j/N_j$ is a candidate pole of $Z_{top,f}$ of order 1 that is a pole, then $e^{-2\pi i \nu_j/N_j}$ is an eigenvalue of monodromy of f.*

Proof. In [Ve2] it is shown that then there exists an exceptional component E_k such that $\nu_k/N_k = \nu_j/N_j$ and such that $\chi(E_k^\circ) > 0$. The result follows now immediately from Theorem 3.2. ■

These results are particular for this context. In [Ve1] are given many examples where appear negative $\chi(E_j^\circ)$ while the configuration in $E_j = \mathbb{P}^2$ does not consist of lines. It also often happens that positive $\chi(E_j^\circ)$ does not imply that $e^{-2\pi i \nu_j/N_j}$ is an eigenvalue of monodromy of f.

References

[A'C] N. A'Campo, *La fonction zêta d'une monodromie*, Comment. Math. Helv. **50** (1975), 233-248.

[A-B,C-N,Lu,M-H] E. Artal-Bartolo, P. Cassou-Noguès, I. Luengo, A. Melle-Hernández, *Monodromy conjecture for some surface singularities*, Ann. Scient. Ec. Norm. Sup. **35** (2002), 605-640.

[Bo,Deb,Na] A. Bodin, P. Dèbes, S. Najib, *Irreducibility of hypersurfaces*, preprint.

[C,G-S,L-J] A. Campillo, G. Gonzalez-Sprinberg, M. Lejeune-Jalabert, *Clusters of infinitely near points*, Math. Ann. **306** (1996), 169-194.

[De,L] J. Denef and F. Loeser, *Caractéristique d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc. **5, 4** (1992), 705-720.

[L1] F. Loeser, *Fonctions d'Igusa p-adiques et polynômes de Bernstein.* Amer. J. Math. **110** (1988), 1-22.

[L2] F. Loeser, *Fonctions d'Igusa p-adiques, polynômes de Bernstein, et polyèdres de Newton*, J. reine angew. Math. **412** (1990), 75-96.

[Ro] B. Rodrigues, *On the monodromy conjecture for curves on normal surfaces*, Math. Proc. of the Cambridge Philosophical Society **136** (2004), 313-324.

[Ve1] W. Veys, *Poles of Igusa's local zeta function and monodromy*, Bull. Soc. Math. France **121** (1993), 545-598.

[Ve2] W. Veys, *Vanishing of principal value integrals on surfaces*, J. Reine Angew. Math. **598** (2006), 139-158.