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Abstract

Recently the second author and Van Proeyen proved the monodromy conjecture on topological zeta functions for all non-degenerate surface singularities. In this paper, we obtain some higher-dimensional analogues of their results. First we study configurations of B_1-pyramid facets which produce fake poles. Secondly, we introduce fully supermodular functions which are useful to find eigenvalues of monodromies. Finally, we obtain a result which would be useful to treat the monodromy conjecture for non-degenerate hypersurfaces whose Newton polyhedron is not convenient. In particular, we prove the conjecture for non-degenerate hypersurface singularities in $\mathbb{C}^4$ under some additional assumptions.

1 Introduction

Over the fields $\mathbb{R}$ and $\mathbb{C}$ it is well-known that the poles of the local zeta function associated to a polynomial f are contained in the set of roots of the Bernstein-Sato polynomial and integer shifts of them. By a celebrated theorem of Kashiwara and Malgrange, this implies that for any such pole $s_0 \in \mathbb{Q}$ the complex number $\exp(2\pi i s_0) \in \mathbb{C}$ is an eigenvalue of the monodromies of the complex hypersurface defined by f. Igusa predicted a similar beautiful relationship between the poles of p-adic integrals and the complex monodromies. This is now called the monodromy conjecture (see Denef [4] and Nicaise [25] for excellent reviews on this subject). Later in [6] Denef and Loeser introduced the local topological zeta function $Z_{\mathrm{top},f}(s)$ associated to f and proposed a weaker version of the monodromy

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conjecture. For important contributions to this Denef-Loeser conjecture, see Loeser [19], [20], Artal Bartolo-Cassou-Noguès-Luengo-Melle Hernández [2], Veys [30] etc. Recently the second author and Van Proeyen [17] proved it for all non-degenerate surface singularities. The aim of this paper is to exploit to what extent the results of [17] hold true in higher dimensions. In Section 3, as a straightforward generalization of the notion of B_1-facets in [17] we introduce B_1-pyramid facets of the Newton polyhedron $\Gamma_+(f)$ of f and show that for some configurations of them the candidate poles of $Z_{\mathrm{top},f}(s)$ contributed only by them are fake i.e. not actual poles. Moreover in Sections 4, 5 and 6, following the strategy of [17] we prove some key results to show that the candidate poles of $Z_{\mathrm{top},f}(s)$ contributed by some non-B_1-pyramid facets of $\Gamma_+(f)$ yield monodromy eigenvalues at some points in a neighborhood of the origin $0 \in \mathbb{C}^n$. In particular we obtain Theorem 4.3. Its proof partially relies upon the new notion of a fully supermodular function, which is inspired by supermodular functions in game theory and may be of independent interest. See Section 5 for the details. In Section 6 we prove Proposition 6.1 which enables us to reduce the problem of non-convenient polynomials to that of convenient ones. In this way, we can confirm the monodromy conjecture of Denef-Loeser [6] for most of non-degenerate hypersurfaces in higher dimensions. Indeed in Section 7 we prove the conjecture for non-degenerate polynomials of four variables under some additional assumptions (see Theorem 7.10 for the details).

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2 The monodromy conjecture for topological zeta functions

In this section, we briefly recall the monodromy conjecture for local topological zeta functions and related results. Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a germ of a non-trivial analytic function. We assume that f is defined on an open neighborhood X of the origin $0 \in \mathbb{C}^n$. Let $\pi : Y \longrightarrow X$ be an embedded resolution of the complex hypersurface $f^{-1}(0) \subset X$ and E_j $(j \in J)$ the irreducible components of the normal crossing divisor $\pi^{-1}(f^{-1}(0)) \subset Y$. For $j \in J$ we denote by N_j (resp. $\nu_j - 1$) the multiplicity of the divisor associated to $f \circ \pi$ (resp. $\pi^*(dx_1 \wedge \cdots \wedge dx_n)$) along $E_j \subset Y$. For a non-empty subset $I \subset J$ we set

$$E_I = \bigcap_{i \in I} E_i, \qquad E_I^\circ = E_I \setminus \Big(\bigcup_{j \notin I} E_j \Big). \tag{2.1}$$

In [6] Denef and Loeser defined the local topological zeta function $Z_{\mathrm{top},f}(s) \in \mathbb{C}(s)$ associated to f (at the origin) by

$$Z_{\mathrm{top},f}(s) = \sum_{I \neq \emptyset} \chi(E_I^\circ \cap \pi^{-1}(0)) \times \prod_{i \in I} \frac{1}{N_i s + \nu_i}, \tag{2.2}$$

where $\chi(\cdot)$ denotes the topological Euler characteristic. More precisely, they introduced $Z_{\mathrm{top},f}(s)$ by p-adic integrals and showed that it does not depend on the choice of the embedded resolution $\pi : Y \longrightarrow X$ by algebraic methods. Later in [7] and [8], they redefined $Z_{\mathrm{top},f}(s)$ by using the motivic zeta function of f and reproved this independence

of π more elegantly. For a point $x \in f^{-1}(0)$ let $F_x \subset X \setminus f^{-1}(0)$ be the Milnor fiber of f at x and $\Phi_{j,x} : H^j(F_x;\mathbb{C}) \xrightarrow{\sim} H^j(F_x;\mathbb{C})$ $(j \in \mathbb{Z})$ the Milnor monodromies associated to it. Then the monodromy conjecture of Denef-Loeser for the local topological zeta function $Z_{\mathrm{top},f}(s)$ is stated as follows.

Conjecture: (Denef-Loeser [6, Conjecture 3.3.2]) Assume that $s_0 \in \mathbb{C}$ is a pole of $Z_{\mathrm{top},f}(s)$. Then $\exp(2\pi i s_0) \in \mathbb{C}$ is an eigenvalue of the monodromy $\Phi_{j,x} : H^j(F_x;\mathbb{C}) \xrightarrow{\sim} H^j(F_x;\mathbb{C})$ for some point $x \in f^{-1}(0)$ in a neighborhood of the origin $0 \in \mathbb{C}^n$ and $j \in \mathbb{Z}$.

In [6] the authors also fomulated an even stronger conjecture conerning the Bernstein-Sato polynomial $b_f(s)$ of f. Namely they conjectured that the poles of $Z_{\mathrm{top},f}(s)$ are roots of $b_f(s)$. From now on, we assume that f is a non-trivial polynomial on $\mathbb{C}^n$ such that $f(0) = 0$ and recall the results of Denef-Loeser [6, Section 5] and Varchenko [29]. For $f(x) = \sum_{v \in \mathbb{Z}_+^n} c_v x^v \in \mathbb{C}[x_1, \ldots, x_n]$ we define its support $\mathrm{supp} f \subset \mathbb{Z}_+^n$ by

$$\mathrm{supp} f = \{v \in \mathbb{Z}_+^n \mid c_v \neq 0\} \subset \mathbb{Z}_+^n. \tag{2.3}$$

We denote by $\Gamma_+(f) \subset \mathbb{R}_+^n$ the convex hull of $\cup_{v \in \mathrm{supp} f}(v + \mathbb{R}_+^n)$ in $\mathbb{R}_+^n$ and call it the Newton polyhedron of f at the origin $0 \in \mathbb{C}^n$. The polynomial f such that $f(0) = 0$ is called convenient if $\Gamma_+(f)$ intersects the positive part of any coordinate axis of $\mathbb{R}^n$.

Definition 2.1. *(Kouchnirenko [15]) We say that f is non-degenerate (at the origin $0 \in \mathbb{C}^n$) if for any compact face $\tau \prec \Gamma_+(f)$ the complex hypersurface*

$$\{x \in (\mathbb{C}^*)^n \mid f_\tau(x) = 0\} \subset (\mathbb{C}^*)^n \tag{2.4}$$

in $(\mathbb{C}^*)^n$ is smooth and reduced, where we set

$$f_\tau(x) = \sum_{v \in \tau \cap \mathbb{Z}_+^n} c_v x^v \in \mathbb{C}[x_1, \ldots, x_n]. \tag{2.5}$$

It is well-known that generic polynomials having a fixed Newton polyhedron are non-degenerate (see for example [27, Chapter V, paragraph 2]). For $a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ we set

$$N(a) = \min_{v \in \Gamma_+(f)} \langle a, v \rangle, \quad \nu(a) = |a| = \sum_{i=1}^n a_i \tag{2.6}$$

and

$$F(a) = \{v \in \Gamma_+(f) \mid \langle a, v \rangle = N(a)\} \prec \Gamma_+(f). \tag{2.7}$$

We call $F(a)$ the supporting face of the vector $a \in \mathbb{R}_+^n$ on $\Gamma_+(f)$. For a face $\tau \prec \Gamma_+(f)$ we set

$$\tau^\circ = \overline{\{a \in \mathbb{R}_+^n \mid F(a) = \tau\}} \subset \mathbb{R}_+^n. \tag{2.8}$$

Note that τ° is an $(n - \dim \tau)$-dimensional rational polyhedral convex cone in $\mathbb{R}_+^n$. We call it the dual cone of τ. Let $\Delta = \mathbb{R}_+ a(1) + \cdots + \mathbb{R}_+ a(l)$ $(a(i) \in \mathbb{Z}_+^n)$ be a rational simplicial cone in $\mathbb{R}_+^n$, where the $a(i)$ are linearly independent over $\mathbb{R}$ and primitive. Let $\mathrm{aff}(\Delta) \simeq \mathbb{R}^l$ be the affine span of Δ in $\mathbb{R}^n$ and $s(\Delta) \subset \Delta$ the l-dimensional lattice simplex whose vertices are $a(1), \ldots, a(l)$ and the origin $0 \in \Delta \subset \mathbb{R}_+^n$. We denote by $\mathrm{mult}(\Delta) \in \mathbb{Z}_{>0}$ the l-dimensional normalized volume $\mathrm{Vol}_{\mathbb{Z}}(s(\Delta))$ of $s(\Delta)$ i.e. $l!$ times the usual volume of

$s(\Delta)$ with respect to the affine lattice $\mathrm{aff}(\Delta) \cap \mathbb{Z}^n \simeq \mathbb{Z}^l$ in $\mathrm{aff}(\Delta)$. By using this integer $\mathrm{mult}(\Delta)$ we set

$$J_\Delta(s) = \frac{\mathrm{mult}(\Delta)}{\prod_{i=1}^{l}\{N(a(i))s + \nu(a(i))\}} \in \mathbb{C}(s). \tag{2.9}$$

For a face $\tau \prec \Gamma_+(f)$ we choose a decomposition $\tau^\circ = \cup_{1\leq i\leq r}\Delta_i$ of its dual cone τ° into rational simplicial cones Δ_i of dimension $l := \dim\tau^\circ$ such that $\dim(\Delta_i \cap \Delta_j) < l$ $(i \neq j)$ and set

$$J_\tau(s) = \sum_{i=1}^{r} J_{\Delta_i}(s) \in \mathbb{C}(s). \tag{2.10}$$

According to [6, Lemme 5.1.1], this rational function $J_\tau(s)$ does not depend on the choice of the decomposition of τ°. It is also well-known that we can decompose τ° into rational simplicial cones without adding new edges. Then we have the following formula for $Z_{\mathrm{top},f}(s)$.

Theorem 2.2. *(Denef-Loeser [6, Théorème 5.3 (ii)]) Assume that $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ is non-degenerate. Then we have*

$$Z_{\mathrm{top},f}(s) = \sum_{\gamma} J_\gamma(s) + \frac{s}{s+1}\sum_{\tau}(-1)^{\dim\tau}\mathrm{Vol}_{\mathbb{Z}}(\tau)\cdot J_\tau(s), \tag{2.11}$$

where in the sum $\sum_\gamma$ (resp. $\sum_\tau$) the face $\gamma \prec \Gamma_+(f)$ (resp. $\tau \prec \Gamma_+(f)$) ranges through the vertices of $\Gamma_+(f)$ (resp. the compact ones such that $\dim\tau \geq 1$) and $\mathrm{Vol}_{\mathbb{Z}}(\tau) \in \mathbb{Z}_{>0}$ is the $(\dim\tau)$-dimensional normalized volume of τ with respect to the affine lattice $\mathrm{aff}(\tau)\cap\mathbb{Z}^n \simeq \mathbb{Z}^{\dim\tau}$ in $\mathrm{aff}(\tau) \simeq \mathbb{R}^{\dim\tau}$.

Recall that a face τ of $\Gamma_+(f)$ is called a facet if $\dim\tau = n-1$. For a facet $\tau \prec \Gamma_+(f)$ let $a(\tau) = (a(\tau)_1, \ldots, a(\tau)_n) \in \tau^\circ \cap \mathbb{Z}^n_+$ be its primitive conormal vector and set

$$N(\tau) = \min_{v\in\Gamma_+(f)}\langle a(\tau), v\rangle, \tag{2.12}$$

$$\nu(\tau) = |a(\tau)| = \sum_{i=1}^{n} a(\tau)_i = \left\langle a(\tau), \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right\rangle. \tag{2.13}$$

We call $N(\tau)$ the lattice distance of τ from the origin $0 \in \mathbb{R}^n$. It follows from Theorem 2.2 that any pole $s_0 \neq -1$ of $Z_{\mathrm{top},f}(s)$ is contained in the finite set

$$\left\{-\frac{\nu(\tau)}{N(\tau)} \mid \tau \prec \Gamma_+(f) \quad \text{is a facet not lying in a coordinate hyperplane}\right\} \subset \mathbb{Q}. \tag{2.14}$$

Its elements are called candidate poles of $Z_{\mathrm{top},f}(s)$. We say that a candidate pole $s_0 \in \mathbb{C}$ of $Z_{\mathrm{top},f}(s)$ is contributed by a facet $\tau \prec \Gamma_+(f)$ if we have $s_0 = -\frac{\nu(\tau)}{N(\tau)}$.

Finally we recall the result of Varchenko [29]. For a polynomial $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f(0) = 0$, we define its monodromy zeta function $\zeta_{f,0}(t) \in \mathbb{C}(t)$ at the origin $0 \in \mathbb{C}^n$ by

$$\zeta_{f,0}(t) = \prod_{j\in\mathbb{Z}}\left\{\det\Big(\mathrm{id} - t\Phi_{j,0}\Big)\right\}^{(-1)^j} \in \mathbb{C}(t). \tag{2.15}$$

Similarly one can define also $\zeta_{f,x}(t) \in \mathbb{C}(t)$ for any point $x \in f^{-1}(0)$. Then by considering the decomposition of the nearby cycle perverse sheaf $\psi_f(\mathbb{C}_{\mathbb{C}^n})[n-1]$ with respect to the monodromy eigenvalues of f and the concentrations of its components at generic points $x \in f^{-1}(0)$ (see e.g. [9] and [14]), in order to prove the monodromy conjecture, it suffices to show that for any pole $s_0 \in \mathbb{C}$ of $Z_{\mathrm{top},f}(s)$ the complex number $\exp(2\pi i s_0) \in \mathbb{C}$ is a root or a pole of $\zeta_{f,x}(t)$ for some point $x \in f^{-1}(0)$ in a neighborhood of the origin $0 \in \mathbb{C}^n$ (see Denef [5, Lemma 4.6]). For a subset $S \subset \{1, 2, \ldots, n\}$ we define a coordinate subspace $\mathbb{R}^S \simeq \mathbb{R}^{|S|}$ of $\mathbb{R}^n$ by

$$\mathbb{R}^S = \{v = (v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_i = 0 \quad \text{for any} \quad i \notin S\} \tag{2.16}$$

and set

$$\mathbb{R}^S_+ = \mathbb{R}^S \cap \mathbb{R}^n_+ \simeq \mathbb{R}^{|S|}_+. \tag{2.17}$$

For a compact face $\tau \prec \Gamma_+(f)$ we take the minimal coordinate subspace $\mathbb{R}^S$ of $\mathbb{R}^n$ containing τ and set $s_\tau = |S|$. If τ satisfies the condition $\dim\tau = s_\tau - 1$ we set

$$\zeta_\tau(t) = \Big(1 - t^{N(\tau)}\Big)^{\mathrm{Vol}_{\mathbb{Z}}(\tau)} \in \mathbb{C}[t], \tag{2.18}$$

where $N(\tau) \in \mathbb{Z}_{>0}$ is the lattice distance of the affine hyperplane $\mathrm{aff}(\tau) \simeq \mathbb{R}^{\dim\tau}$ in $\mathbb{R}^S$ from the origin $0 \in \mathbb{R}^S$.

Theorem 2.3. *(Varchenko [29]) Assume that $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ is non-degenerate. Then we have*

$$\zeta_{f,0}(t) = \prod_\tau \Big\{\zeta_\tau(t)\Big\}^{(-1)^{\dim\tau}}, \tag{2.19}$$

where in the product $\prod_\tau$ the face $\tau \prec \Gamma_+(f)$ ranges through the compact ones satisfying the condition $\dim\tau = s_\tau - 1$.

Definition 2.4. *We say that a face τ of $\Gamma_+(f)$ is a V-face (or a Varchenko face) if it is compact and satisfies the condition $\dim\tau = s_\tau - 1$.*

To end this section, we note the following simple fact. Also for a general face $\tau \prec \Gamma_+(f)$ we define its lattice distance $N(\tau) \in \mathbb{Z}_{>0}$ to be that of the affine hyperplane $\mathrm{aff}(\tau) \simeq \mathbb{R}^{\dim\tau}$ in $\mathrm{aff}(\tau \cup \{0\}) \simeq \mathbb{R}^{\dim\tau+1}$ from the origin $0 \in \mathrm{aff}(\tau \cup \{0\}) \subset \mathbb{R}^n$.

Lemma 2.5. *For two faces $\tau, \gamma \prec \Gamma_+(f)$ such that $\gamma \prec \tau$, we have $N(\gamma)|N(\tau)$.*

Proof. We may assume that γ is a facet of τ. Let Σ_0 be the dual fan of $\Gamma_+(f)$ and Σ a smooth subdivision of Σ_0. Then there exists an $(n - \dim\gamma)$-dimensional smooth cone $\Delta \in \Sigma$ contained in γ° such that $\dim(\Delta \cap \tau^\circ) = \dim\tau^\circ = n - \dim\tau = (n - \dim\gamma) - 1$. Set $l = n - \dim\gamma$ and let $a(1), \ldots, a(l) \in \mathbb{Z}^n_+$ be the primitive vectors on the edges of Δ. Then it is easy to see that for any $v \in \gamma$ we have

$$N(\tau) = \gcd\Big(\langle a(1), v\rangle, \ldots, \langle a(l), v\rangle\Big). \tag{2.20}$$

We have also a similar description of $N(\tau)$ in terms of the primitive vectors on the edges of $\Delta \cap \tau^\circ \prec \Delta$. If we use a point $v \in \gamma \prec \tau$ to express $N(\gamma)$ and $N(\tau)$ simultaneously, we find $N(\gamma)|N(\tau)$. This completes the proof. □

Remark 2.6. *For two V-faces $\tau, \gamma \prec \Gamma_+(f)$ such that $\gamma \prec \tau$ we can prove Lemma 2.5 more easily as follows. Let $\mathbb{R}^{S_\tau}$ (resp. $\mathbb{R}^{S_\gamma}$) be the minimal coordinate subspace of $\mathbb{R}^n$ containing τ (resp. γ) and*

$$\mathrm{aff}(\tau): \ \langle a(\tau), v\rangle = N(\tau) \tag{2.21}$$

the equation of the affine hyperplane $\mathrm{aff}(\tau) \subset \mathbb{R}^{S_\tau}$. Then by restricting it to $\mathbb{R}^{S_\gamma} \subset \mathbb{R}^{S_\tau}$ to find that of $\mathrm{aff}(\gamma)$ we obtain the equality

$$\gcd\Big\{a(\tau)_i \ (i \in S_\tau)\Big\} \cdot N(\gamma) = N(\tau). \tag{2.22}$$

3 Fake poles of topological zeta functions

In this section, we define B_1-pyramid facets of the Newton polyhedron $\Gamma_+(f)$ and show that for some configurations of them the candidate poles of $Z_{\mathrm{top},f}(s)$ contributed only by them are fake i.e. not actual poles. Our definition is a straightforward generalization of that of Lemahieu-Van Proeyen [17].

For a subset $S \subset \{1, 2, \ldots, n\}$ let $\pi_S : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^{S^c}_+ \simeq \mathbb{R}^{n-|S|}_+$ be the natural projection. We say that an $(n-1)$-dimensional polyhedron τ in $\mathbb{R}^n_+$ is non-compact for $S \subset \{1, 2, \ldots, n\}$ if the Minkowski sum $\tau + \mathbb{R}^S_+$ is contained in τ.

Definition 3.1. *(cf. Lemahieu-Van Proeyen [17]) Let τ be an $(n-1)$-dimensional polyhedron in $\mathbb{R}^n_+$.*

1. *We say that τ is a B_1-pyramid of compact type for the variable v_i if τ is a compact pyramid over the base $\gamma = \tau \cap \{v_i = 0\}$ and its unique vertex $P \prec \tau$ such that $P \notin \gamma$ has height one from the hyperplane $\{v_i = 0\} \subset \mathbb{R}^n_+$.*

2. *We say that τ is a B_1-pyramid of non-compact type if there exists a non-empty subset $S \subset \{1, 2, \ldots, n\}$ such that τ is non-compact for S and $\pi_S(\tau) \subset \mathbb{R}^{S^c}_+ \simeq \mathbb{R}^{n-|S|}_+$ is a B_1-pyramid of compact type for some variable v_i $(i \notin S)$.*

3. *We say that τ is a B_1-pyramid if it is a B_1-pyramid of compact or non-compact type.*

Let us recall the condition in Loeser [20]. For two distinct facets τ and τ' of $\Gamma_+(f)$ let $\beta(\tau, \tau') \in \mathbb{Z}$ be the greatest common divisor of the 2×2 minors of the matrix $(a(\tau), a(\tau')) \in M(n, 2; \mathbb{Z})$. If $N(\tau) \neq 0$ (e.g. if τ is compact) we set

$$\lambda(\tau, \tau') = \nu(\tau') - \frac{\nu(\tau)}{N(\tau)} N(\tau'), \quad \mu(\tau, \tau') = \frac{\lambda(\tau, \tau')}{\beta(\tau, \tau')} \quad \in \mathbb{Q}. \tag{3.1}$$

In [20] the author considered only compact facets τ of $\Gamma_+(f)$ which satisfy the following technical condition:

"For any facet $\tau' \prec \Gamma_+(f)$ such that $\tau' \neq \tau$ and $\tau' \cap \tau \neq \emptyset$ we have $\mu(\tau, \tau') \notin \mathbb{Z}$."

He showed that the candidate pole of $Z_{\mathrm{top},f}(s)$ associated to such a compact facet τ is a root of the local Bernstein-Sato polynomial of f. Now let $\tau \prec \Gamma_+(f)$ be a facet containing a B_1-pyramid of compact type for the variable v_i and set $\gamma = \tau \cap \{v_i = 0\}$. Let $\tau_0 \prec \Gamma_+(f)$ be the unique (non-compact) facet such that $\gamma \prec \tau_0$, $\tau_0 \neq \tau$ and $\tau_0 \subset \{v_i = 0\}$.

Then we can easily show that $\beta(\tau,\tau_0)=\mu(\tau,\tau_0)=1$. Namely such a facet τ does not satisfy the above-mentioned condition of [20]. In this paper, we treat also candidate poles associated to non-compact facets of $\Gamma_+(f)$. Roughly speaking, non-isolated singularities of the hypersurface $f^{-1}(0)\subset\mathbb{C}^n$ correspond to non-convenient Newton polyhedra (see [16]).

Proposition 3.2. *Assume that f is non-degenerate and let the facet $\tau\prec\Gamma_+(f)$ be a B_1-pyramid. Assume also that the candidate pole*

$$s_0=-\frac{\nu(\tau)}{N(\tau)}\neq -1 \tag{3.2}$$

of $Z_{\mathrm{top},f}(s)$ is contributed only by τ. Then s_0 is fake i.e. not an actual pole of $Z_{\mathrm{top},f}(s)$.

Proof. Since the proof for B_1-pyramids of non-compact type is similar, we prove the assertion only for B_1-pyramids of compact type. Without loss of generality we may assume that τ is a compact pyramid over the base $\gamma=\tau\cap\{v_n=0\}$ and its unique vertex $P\prec\tau$ such that $P\notin\gamma$ has height one from the hyperplane $\{v_n=0\}\subset\mathbb{R}^n_+$. First let us consider the simplest case where γ and hence τ are simplices. Let $A_1,A_2,\ldots,A_{n-1}$ be the vertices of the $(n-2)$-dimensional simplex γ. For $1\leq i\leq n-1$ denote by σ_i the facet of τ whose vertices are P and the A_j $(j\neq i)$. As in the proof of [17, Proposition 14] let $\tau_i\prec\Gamma_+(f)$ $(1\leq i\leq n-1)$ (resp. $\tau_0\prec\Gamma_+(f)$) be the unique facet such that $\sigma_i\prec\tau_i$ and $\tau_i\neq\tau$ (resp. $\gamma\prec\tau_0$, $\tau_0\neq\tau$ and $\tau_0\subset\{v_n=0\}$). Then it is easy to see that the primitive conormal vector $a(\tau)\in\mathbb{Z}^n_+$ of τ is in the interior of the cone generated by $a(\tau_1),\ldots,a(\tau_{n-1})$ and $a(\tau_0)=(0,0,\ldots,0,1)$. For $1\leq i\leq n-1$ denote by V_{A_i} the multiplicity of the (n-dimensional) cone generated by $a(\tau)$, $a(\tau_0)$ and $a(\tau_j)$ $(j\neq i)$. Also we denote by V_P the multiplicity of the (n-dimensional) cone generated by $a(\tau)$ and $a(\tau_1),\ldots,a(\tau_{n-1})$. Finally we set $l(\tau)(s)=N(\tau)s+\nu(\tau)$ and $l(\tau_i)(s)=N(\tau_i)s+\nu(\tau_i)$ $(1\leq i\leq n-1)$. Then by the argument in Case 1 of the proof of [17, Proposition 14] we obtain an equality

$$Z_{\mathrm{top},f}(s)\equiv\frac{V_P}{l(\tau)(s)\prod_{j=1}^{n-1}l(\tau_j)(s)}+\frac{1}{s+1}\sum_{i=1}^{n-1}\frac{V_{A_i}}{l(\tau)(s)\prod_{j\neq i}l(\tau_j)(s)} \tag{3.3}$$

modulo holomorphic functions at $s=s_0\in\mathbb{C}$. In order to prove the assertion, it suffices to show that the polynomial in the right hand side of the equality

$$(s+1)l(\tau)(s)\left(\prod_{j=1}^{n-1}l(\tau_j)(s)\right)Z_{\mathrm{top},f}(s)\equiv V_P(s+1)+\sum_{i=1}^{n-1}V_{A_i}l(\tau_i)(s) \tag{3.4}$$

is divisible by the factor $l(\tau)(s)=N(\tau)s+\nu(\tau)$. Let $w=(*,*,\ldots,*,1)\in\mathbb{Z}^n_+$ be the coordinate of the point $P\in\tau\subset\mathbb{R}^n_+$ for which we have $\langle a(\tau_0),w\rangle=1$. Let V be the multiplicity of the (n-dimensional) cone generated by $a(\tau_0),a(\tau_1),\ldots,a(\tau_{n-1})$. Now observe that we have the geometric equality

$$Va(\tau)=V_Pa(\tau_0)+\sum_{i=1}^{n-1}V_{A_i}a(\tau_i). \tag{3.5}$$

Then we obtain

$$V_P(s+1) + \sum_{i=1}^{n-1} V_{A_i} l(\tau_i)(s) \tag{3.6}$$

$$= \langle V_P a(\tau_0) + \sum_{i=1}^{n-1} V_{A_i} a(\tau_i), w\rangle s + \left\langle V_P a(\tau_0) + \sum_{i=1}^{n-1} V_{A_i} a(\tau_i), \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle \tag{3.7}$$

$$= \langle V a(\tau), w\rangle s + \left\langle V a(\tau), \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle = V l(\tau)(s) \tag{3.8}$$

and the proof for the case where γ and τ are simplices is now complete.

Now let us consider the general case where τ is a B_1-pyramid of compact type over the base $\gamma \prec \tau$ such that $\gamma \subset \{v_n = 0\}$. Let $\tau_0 \prec \Gamma_+(f)$ be the unique facet such that $\gamma \prec \tau_0$, $\tau_0 \neq \tau$ and $\tau_0 \subset \{v_n = 0\}$. Let $A_1, A_2, \ldots, A_m$ $(m \geq n-1)$ be the vertices of γ. For $1 \leq i \leq m$ we denote the dual cone A_i° of $A_i \prec \Gamma_+(f)$ by C_{A_i}. Similarly we set $C_P = P^\circ$. Then we have

$$a(\tau) \in \mathrm{Int}(C_P \cup C_{A_1} \cup \cdots \cup C_{A_m}). \tag{3.9}$$

In order to construct nice decompositions of $C_P, C_{A_1}, \ldots, C_{A_m}$ into n-dimensional rational simplicial cones, we shall introduce a new dummy vector $b \in \mathrm{Int}C_P \cap \mathbb{Z}_+^n$ satisfying the condition

$$-\frac{\nu(b)}{N(b)} \neq s_0 \tag{3.10}$$

as follows. First, by our assumption $s_0 \neq -1$ the coordinate vector $w = (*, *, \ldots, *, 1) \in \mathbb{Z}_+^n$ of the summit P of the pyramid τ is not parallel to the vector

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{Z}_+^n. \tag{3.11}$$

Then for a sufficiently large primitive vector $b \in \mathrm{Int}C_P \cap \mathbb{Z}_+^n$ we can achieve the desired condition $N(b)s_0 + \nu(b) \neq 0$. For $1 \leq i \leq m$ let $e_i \prec \tau$ be the edge of τ connecting the two points P and A_i and $F_i \prec C_P$ the corresponding facet of the cone C_P containing its edge $\tau^\circ = \mathbb{R}_+ a(\tau) \prec C_P$. All the facets of C_P containing τ° are obtained in this way. Since the point $A_i \prec e_i$ is a vertex of τ_0, its dual cone $C_{A_i} = A_i^\circ$ contains not only F_i but also the 1-dimensional cone $\tau_0^\circ = \mathbb{R}_+ a(\tau_0)$. For $1 \leq i \leq m$ set

$$F_i^\sharp = \mathbb{R}_+ a(\tau_0) + F_i, \qquad F_i^\flat = \mathbb{R}_+ b + F_i. \tag{3.12}$$

In Figure 1 below we presented the transversal hyperplane sections of the cones F_i, $F_i^\sharp$ and $F_i^\flat$.

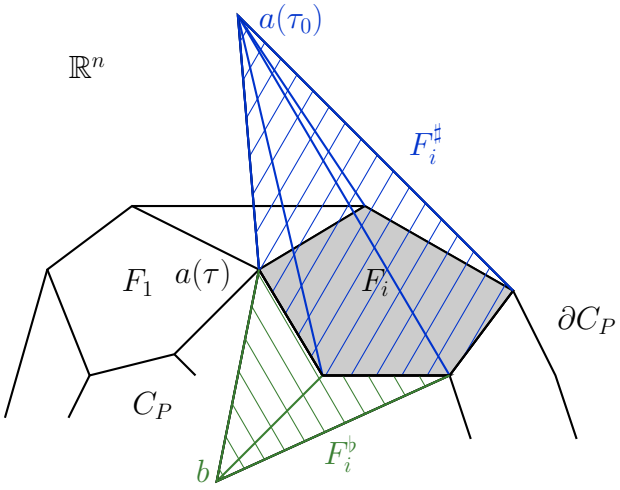

Figure 1

Then by our construction $\cup_{i=1}^m (F_i^\sharp \cup F_i^\flat)$ is an n-dimensional cone in $\mathbb{R}^n$ such that

$$a(\tau) \in \mathrm{Int}\Big\{\bigcup_{i=1}^m (F_i^\sharp \cup F_i^\flat)\Big\}. \tag{3.13}$$

Now we decompose $F_1, F_2, \ldots, F_m$ into rational simplicial cones. By using secondary polytopes (see [12, Chapter 7]), we can obtain such a subdivision of $F_1 \cup \cdots \cup F_m \subset \partial C_P$ without adding new edges. We may assume also that it satisfies the axiom of fans. Let $\Delta_1, \ldots, \Delta_r$ be the $(n-1)$-dimensional simplicial cones thus obtained in $F_1 \cup \cdots \cup F_m \subset \partial C_P$ and containing the edge $\tau^\circ = \mathbb{R}_+ a(\tau)$. For $1 \leq i \leq r$ set

$$\Delta_i^\sharp = \mathbb{R}_+ a(\tau_0) + \Delta_i, \qquad \Delta_i^\flat = \mathbb{R}_+ b + \Delta_i. \tag{3.14}$$

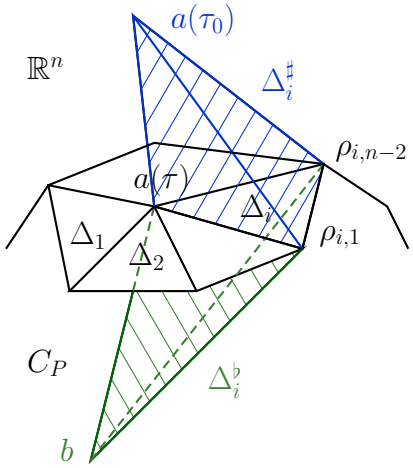

Figure 2

Then $\cup_{i=1}^r (\Delta_i^\sharp \cup \Delta_i^\flat)$ is an n-dimensional cone in $\mathbb{R}^n$ such that

$$a(\tau) \in \mathrm{Int}\Big\{\bigcup_{i=1}^r (\Delta_i^\sharp \cup \Delta_i^\flat)\Big\}. \tag{3.15}$$

By the argument in Case 1 of the proof of [17, Proposition 14] the contributions to $Z_{\mathrm{top},f}(s)$ from the faces $\sigma \prec \tau$ of τ such that $\dim \sigma \geq 1$ cancel each other and it suffices to calculate

only the ones from the vertices $P, A_1, \ldots, A_m \in \tau$ inside the cone $\cup_{i=1}^r (\Delta_i^\sharp \cup \Delta_i^\flat)$. For $1 \leq i \leq r$ let $\rho_{i,1}, \ldots, \rho_{i,n-2} \in \partial C_P$ be the primitive vectors on the edges of Δ_i such that $\rho_{i,j} \notin \tau^\circ = \mathbb{R}_+ a(\tau)$ and set

$$l_{i,j}(s) = N(\rho_{i,j})s + \nu(\rho_{i,j}) \in \mathbb{C}[s] \quad (1 \leq j \leq n-2). \tag{3.16}$$

Moreover we set $l(b)(s) = N(b)s + \nu(b)$. For $1 \leq i \leq r$ we denote by V_i (resp. U_i) the multiplicity of the (n-dimensional) cone generated by $a(\tau)$, $\rho_{i,1}, \ldots, \rho_{i,n-2}$ and $a(\tau_0)$ (resp. b) in $\mathbb{R}^n$. Namely we set $\mathrm{mult}(\Delta_i^\sharp) = V_i$ and $\mathrm{mult}(\Delta_i^\flat) = U_i$. Then we obtain an equality

$$Z_{\mathrm{top},f}(s) \equiv \sum_{i=1}^r \Big\{ \frac{V_i}{(s+1)l(\tau)(s)\prod_{j=1}^{n-2} l_{i,j}(s)} + \frac{U_i}{l(b)(s)l(\tau)(s)\prod_{j=1}^{n-2} l_{i,j}(s)} \Big\} \tag{3.17}$$

modulo holomorphic functions at $s = s_0 \in \mathbb{C}$. Let k be the cardinality of the set $\cup_{i=1}^r \{\rho_{i,1}, \rho_{i,2}, \ldots, \rho_{i,n-2}\} \subset \mathbb{R}^n$ and set

$$\{\rho_1, \rho_2, \ldots, \rho_k\} := \bigcup_{i=1}^r \{\rho_{i,1}, \rho_{i,2}, \ldots, \rho_{i,n-2}\} \tag{3.18}$$

and

$$l_j(s) = N(\rho_j)s + \nu(\rho_j) \in \mathbb{C}[s] \quad (1 \leq j \leq k). \tag{3.19}$$

In order to prove the assertion, it suffices to show that the function

$$(s+1)l(\tau)(s)l(b)(s)\left(\prod_{j=1}^k l_j(s)\right) Z_{\mathrm{top},f}(s) \tag{3.20}$$

$$\equiv \sum_{i=1}^r \left(\prod_{j:\rho_j \notin \Delta_i} l_j(s)\right) \cdot \Big\{ V_i l(b)(s) + U_i(s+1) \Big\} \tag{3.21}$$

is divisible by the factor $l(\tau)(s) = N(\tau)s + \nu(\tau)$. For $1 \leq i \leq r$ we define an n-dimensional simplicial cone $\square_i$ in $\mathbb{R}^n$ by

$$\square_i = \mathbb{R}_+ a(\tau_0) + \mathbb{R}_+ \rho_{i,1} + \cdots + \mathbb{R}_+ \rho_{i,n-2} + \mathbb{R}_+ b. \tag{3.22}$$

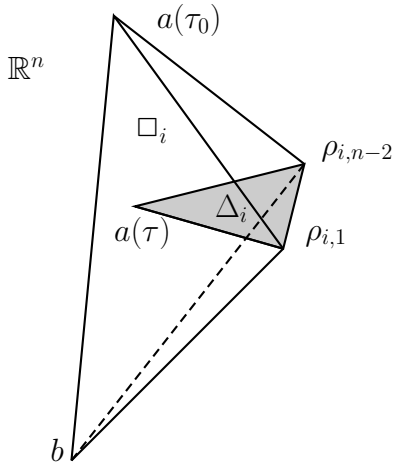

Figure 3

Then by applying the argument in the case where τ is a simplex to $\square_1, \ldots, \square_r$, we can prove that (3.20) is divisible by $l(\tau)(s)$. Note that between two simplicial cones $\square_i$ and $\square_j$ $(i \neq j)$ having a common facet there is a nice cancelling in our calculation. This completes the proof. $\square$

We can show also a similar result even in some cases where a candidate pole of $Z_{\mathrm{top},f}(s)$ is contributed by several (adjacent) B_1-pyramid facets of $\Gamma_+(f)$. Indeed we have the following higher-dimensional analogue of the result in the proof of [17, Proposition 14].

Proposition 3.3. *Assume that f is non-degenerate and let the facets $\tau_1, \ldots, \tau_k \prec \Gamma_+(f)$ be B_1-pyramids such that*

$$s_0 := -\frac{\nu(\tau_1)}{N(\tau_1)} = \cdots\cdots = -\frac{\nu(\tau_k)}{N(\tau_k)} \neq -1 \tag{3.23}$$

and their common candidate pole $s_0 \in \mathbb{Q}$ of $Z_{\mathrm{top},f}(s)$ is contributed only by $\tau_1, \ldots, \tau_k$. Assume also that if τ_i and τ_j $(i \neq j)$ have a common facet they are B_1-pyramids for the same variable. Then s_0 is fake i.e. not an actual pole of $Z_{\mathrm{top},f}(s)$.

Proof. If τ_i and τ_j $(i \neq j)$ do not have a common facet, then by the proof of Proposition 3.2 after a suitable subdivision of the dual fan of $\Gamma_+(f)$ into rational simplicial cones we can calculate their contributions to $Z_{\mathrm{top},f}(s)$ separately. So we may assume that the B_1-pyramid facets $\tau_1, \ldots, \tau_k$ have the common summit $P \in \tau_1 \cap \cdots \cap \tau_k$. For the sake of simplicity, here we shall treat only the case where $k = 2$, τ_1 (resp. τ_2) is a B_1-pyramid over the base $\gamma_1 = \tau_1 \cap \{v_n = 0\}$ (resp. $\gamma_2 = \tau_2 \cap \{v_n = 0\}$) and $\tau_1 \cap \tau_2$ is the (unique) common facet of τ_1 and τ_2. The proofs for the other cases are similar. Let $\tau_0 \prec \Gamma_+(f)$ be the unique facet of $\Gamma_+(f)$ such that $\gamma_1, \gamma_2 \prec \tau_0$, $\tau_0 \neq \tau_i$ $(i = 1, 2)$ and $\tau_0 \subset \{v_n = 0\}$. We denote by P the common summit of τ_1 and τ_2. As in the proof of Proposition 3.2, let $F_1, F_2, \ldots, F_m$ be the facets of the dual cone C_P of $P \prec \Gamma_+(f)$ containing $\tau_1^\circ = \mathbb{R}_+ a(\tau_1)$ or $\tau_2^\circ = \mathbb{R}_+ a(\tau_2)$ and subdivide $F_1 \cup \cdots \cup F_m \subset \partial C_P$ into rational simplicial cones without adding new edges. We may assume that this subdivision satisfies the axiom of fans. Let $\Delta_1, \ldots, \Delta_r$ be the $(n-1)$-dimensional simplicial cones thus obtained in $F_1 \cup \cdots \cup F_m \subset \partial C_P$ and containing τ_1° or τ_2°. As in the proof of Proposition 3.2, we take a new primitive vector $b \in \mathrm{Int} C_P \cap \mathbb{Z}_+^n$ such that

$$-\frac{\nu(b)}{N(b)} \neq s_0. \tag{3.24}$$

For $1 \leq i \leq r$ set

$$\Delta_i^\sharp = \mathbb{R}_+ a(\tau_0) + \Delta_i, \qquad \Delta_i^\flat = \mathbb{R}_+ b + \Delta_i. \tag{3.25}$$

Then $\square := \cup_{i=1}^r (\Delta_i^\sharp \cup \Delta_i^\flat)$ is an n-dimensional cone in $\mathbb{R}^n$ such that

$$a(\tau_1), a(\tau_2) \in \mathrm{Int}\square. \tag{3.26}$$

After a suitable change of the numbering of $\Delta_1, \ldots, \Delta_r$, for some $0 < l < r$ we have also

$$\mathbb{R}_+ a(\tau_1) + \mathbb{R}_+ a(\tau_2) \prec \Delta_i \iff 1 \leq i \leq l. \tag{3.27}$$

Then we obtain an n-dimensional cone $\square' := \cup_{i=1}^l (\Delta_i^\sharp \cup \Delta_i^\flat)$ in $\square$ such that $\mathbb{R}_+ a(\tau_1) + \mathbb{R}_+ a(\tau_2) \subset \square'$. Let us introduce a new primitive vector $b' \in \mathrm{Int}\square' \cap \mathrm{Int} C_P \cap \mathbb{Z}_+^n$ such that

$$-\frac{\nu(b')}{N(b')} \neq s_0 \tag{3.28}$$

and decompose $\square' \subset \square$ into rational simplicial cones by taking the convex hulls of $\mathbb{R}_+ b'$ with the facets of $\square'$.

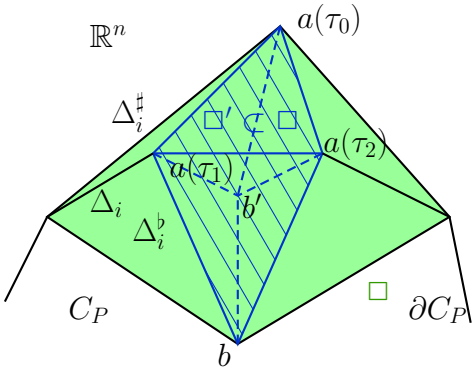

Figure 4

Note that the resulting new decomposition of $\square$ is different from the original one $\square = \cup_{i=1}^r (\Delta_i^\sharp \cup \Delta_i^\flat)$ inside $\square'$ but still satisfies the axiom of fans. By (a slight modification of) [6, Lemme 5.1.1] and the condition $b' \in \mathrm{Int} C_P$ we can use it for the calculation of $Z_{\mathrm{top},f}(s)$. Then the contributions to $Z_{\mathrm{top},f}(s)$ from τ_1 and τ_2 can be calculated separately, because the stars of the cones $\mathbb{R}_+ a(\tau_1)$ and $\mathbb{R}_+ a(\tau_2)$ are disjoint. Now the assertion follows immediately from the proof of Proposition 3.2. $\square$

As we see in the next proposition, in the case $n = 4$ there are also some other facets $\tau \prec \Gamma_+(f)$ whose candidate poles of $Z_{\mathrm{top},f}(s)$ are fake.

Proposition 3.4. *In the case $n = 4$ assume that f is non-degenerate and let $\tau \prec \Gamma_+(f)$ be a facet with six vertices A, B, P, Q, X, Y of the form*

$$\begin{cases} A = & (1, 0, *, *) \\ B = & (1, 0, *, *) \\ P = & (0, 1, *, *) \\ Q = & (0, 1, *, *) \\ X = & (0, 0, *, *) \\ Y = & (0, 0, *, *) \end{cases} \tag{3.29}$$

as in Figure 5 below. It can be degenerated so that $P = Q$ etc. Assume also that the candidate pole

$$s_0 = -\frac{\nu(\tau)}{N(\tau)} \neq -1 \tag{3.30}$$

of $Z_{\mathrm{top},f}(s)$ is contributed only by τ. Then s_0 is fake i.e. not an actual pole of $Z_{\mathrm{top},f}(s)$.

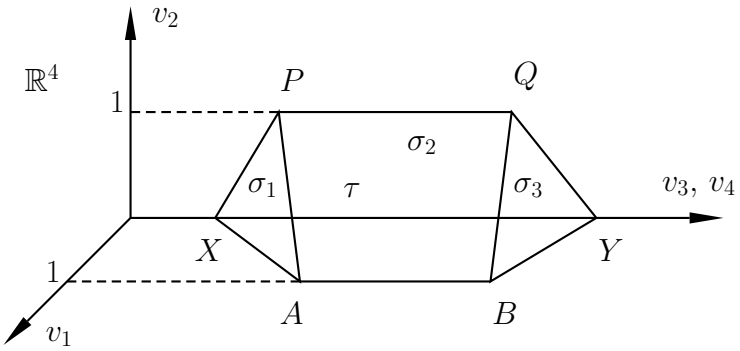

Figure 5

Proof. Note that τ itself is not a B_1-pyramid but it is a union of two B_1-pyramids e.g. $AXPQ$ and $QXABY$. We define facets $\sigma_1, \sigma_2, \sigma_3$ of τ by $\sigma_1 = XAP$, $\sigma_2 = PABQ$, $\sigma_3 = YQB$ respectively. As in the proof of [17, Proposition 14] let $\tau_i \prec \Gamma_+(f)$ $(1 \leq i \leq 3)$ be the unique facet such that $\sigma_i \prec \tau_i$ and $\tau_i \neq \tau$. Moreover for $i = 1, 2$ let $\rho_i \prec \Gamma_+(f)$ be the unique facet such that $\rho_i \subset \{v_i = 0\} \simeq \mathbb{R}^3$, $\tau \cap \{v_i = 0\} \prec \rho_i$ and $\rho_i \neq \tau$. As in the proof of Proposition 3.2 we define the multiplicities $V_A, V_B, V_P, V_Q, V_X, V_Y \in \mathbb{Z}$ of the (n-dimensional) cones associated to the points A, B, P, Q, X, Y and the polynomials $l(\tau)(s), l(\tau_1)(s), l(\tau_2)(s), l(\tau_3)(s)$ of degree one. Then by the argument in Case 1 of the proof of [17, Proposition 14] we obtain an equality

$$Z_{\mathrm{top},f}(s) \equiv \frac{1}{(s+1)l(\tau)}\Big\{\frac{V_X}{l(\tau_1)} + \frac{V_A}{l(\tau_1)l(\tau_2)} + \frac{V_B}{l(\tau_2)l(\tau_3)} + \frac{V_Y}{l(\tau_3)}\Big\} \tag{3.31}$$

$$+\frac{V_P}{l(\tau)l(\tau_1)l(\tau_2)} + \frac{V_Q}{l(\tau)l(\tau_2)l(\tau_3)}. \tag{3.32}$$

modulo holomorphic functions at $s_0 \in \mathbb{C}$. In order to prove the assertion, it suffices to show that the polynomial in the right hand side of the equality

$$(s+1)l(\tau)\left(\prod_{i=1}^{3} l(\tau_i)\right) Z_{\mathrm{top},f}(s) \equiv l(\tau_3)\Big\{l(\tau_2)V_X + V_A + (s+1)V_P\Big\} \tag{3.33}$$

$$+l(\tau_1)\Big\{l(\tau_2)V_Y + V_B + (s+1)V_Q\Big\} \tag{3.34}$$

is divisible by the factor $l(\tau)(s) = N(\tau)s + \nu(\tau)$. Let K_1 (resp. K_1') be the multiplicity of the (4-dimensional) cone generated by $a(\rho_1), a(\rho_2), a(\tau_1), a(\tau_2)$ (resp. $a(\rho_1), a(\rho_2), a(\tau), a(\tau_2)$). Then we have the geometric equality

$$K_1 a(\tau) = V_P a(\rho_2) + V_X a(\tau_2) + V_A a(\rho_1) + K_1' a(\tau_1). \tag{3.35}$$

From this we obtain

$$l(\tau_2)V_X + V_A + (s+1)V_P = l(\tau)K_1 - l(\tau_1)K_1'. \tag{3.36}$$

Moreover let K_2 be the multiplicity of the (4-dimensional) cone generated by $a(\rho_1), a(\rho_2), a(\tau_2), a(\tau_3)$. Then similarly we obtain

$$l(\tau_2)V_Y + V_B + (s+1)V_Q = l(\tau)K_2 + l(\tau_3)K_1'. \tag{3.37}$$

Now the assertion immediately follows. □

Note that the facet in Proposition 3.4 splits into two B_1-pyramids for different variables whose intersection does not contain any 1-dimensional V-face. Motivated by Proposition 3.4 we introduce the following definition.

Definition 3.5. *We call the face τ of $\Gamma_+(f)$ in Proposition 3.4 for $n = 4$ a B_2-facet.*

Lemma 3.6. *For $n = 4$ if a compact facet $\tau \prec \Gamma_+(f)$ is not a B_1-pyramid nor a B_2-facet then it splits into lattice simplices so that one of them is not of type B_1.*

Proof. The facet τ contains a face F not contained in a coordinate hyperplane. Note the following facts about F:

(1): F has at most 4 vertices. Otherwise it contains a triangle whose sides are not in coordinate hyperplanes, and the union of this triangle and any vertex of $\tau \setminus F$ gives a non-B_1-simplex in τ.

(2): If F is a quadrilateral, then some pair of its opposite edges are contained in coordinate hyperplanes, say, $\{v_1 = 0\}$ and $\{v_2 = 0\}$, otherwise we get the same contradiction as in **(1)**. In this case, if a vertex of F at the hyperplane $\{v_1 = 0\}$ has $v_2 > 1$, then this vertex, the two vertices of $F \cap \{v_2 = 0\}$ and any other vertex of $\tau \setminus F$ form a non-B_1-simplex in τ. Thus both vertices of F in the hyperplane $\{v_1 = 0\}$ have $v_2 = 1$ and vice versa. Thus τ is a B_2-facet.

(3): If F is a triangle, then at least one of its edges is contained in a coordinate hyperplane, otherwise we get the same contradiction as in **(1)**.

(3.1): If the triangle F has exactly one edge in a coordinate hyperplane, say, $\{v_1 = 0\}$, then the coordinate v_1 of the other vertex of F equals 1, otherwise F together with any vertex from $\tau \setminus F$ form a non-B_1-simplex in τ. Also in this case, all other vertices of τ should be in the hyperplane $\{v_1 = 0\}$, because otherwise such vertex together with F form a non-B_1-simplex in τ. Thus τ is a B_1-pyramid for v_1.

(3.2): If the triangle F has all three edges in coordinate hyperplanes, then τ is a B_1-tetrahedron.

(3.3): The only remaining case is that F is a triangle, exactly two of whose faces are in coordinate hyperplanes. Since its third edge is not in a coordinate hyperplane, then it should be an edge of another 2-dimensional face G of τ not contained in a coordinate hyperplane. If G is also a triangle, exactly two of whose faces are in coordinate hyperplanes, then τ is a B_1-tetrahedron. If not, then G is of one of the types **(2)** or **(3.1)**, and thus τ is B_1 or B_2 as shown in the corresponding paragraphs. □

4 Candidate poles and eigenvalues of monodromies

In this section, we introduce some higher-dimensional analogues of the results in Lemahieu-Van Proeyen [17, Section 3]. Let $f(x)$ be a polynomial on $\mathbb{C}^n$ such that $f(0) = 0$. Also for lattice simplices τ contained in compact facets of $\Gamma_+(f)$ we define a polynomial $\zeta_\tau(t) = (1 - t^{N(\tau)})^{\mathrm{Vol}_{\mathbb{Z}}(\tau)} \in \mathbb{C}[t]$ similarly and use them freely in this section.

Proposition 4.1. *Let $\tau \prec \Gamma_+(f)$ be a compact facet such that $\gamma = \tau \cap \{v_i = 0\}$ is its facet. Then $\frac{\zeta_\tau}{\zeta_\gamma} \in \mathbb{C}(t)$ is a polynomial of t. If we assume moreover that τ is not a B_1-pyramid for the variable v_i, then the complex number*

$$\lambda = \exp\Big(-2\pi i \frac{\nu(\tau)}{N(\tau)}\Big) \in \mathbb{C} \tag{4.1}$$

is a root of the polynomial.

Proof. By Lemma 2.5 we can easily prove that $\frac{\zeta_\tau}{\zeta_\gamma} \in \mathbb{C}(t)$ is a polynomial. Let us prove the remaining assertion. If τ is not a pyramid over $\gamma = \tau \cap \{v_i = 0\}$, then we have $\mathrm{Vol}_{\mathbb{Z}}(\tau) > \mathrm{Vol}_{\mathbb{Z}}(\gamma)$ and the assertion is obvious. So it suffices to consider the case where τ is a pyramid over $\gamma = \tau \cap \{v_i = 0\}$ but its unique vertex $P \prec \tau$ such that $P \notin \gamma$ has height $h \geq 2$ from the hyperplane $\{v_i = 0\} \subset \mathbb{R}^n_+$. In this case, we define two hyperplanes H_τ and L_τ in $\mathbb{R}^n$ by

$$H_\tau = \{v \in \mathbb{R}^n \mid \langle a(\tau), v\rangle = N(\tau)\}, \tag{4.2}$$

$$L_\tau = \{v \in \mathbb{R}^n \mid \langle a(\tau), v\rangle = \Big\langle a(\tau), \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \Big\rangle = \nu(\tau)\}. \tag{4.3}$$

Note that $P \in \tau \subset H_\tau$ and L_τ is the hyperplane passing through the point $(1, 1, \ldots, 1) \in \mathbb{R}^n_+$ and parallel to H_τ. Namely H_τ is the affine span $\mathrm{aff}(\tau) \simeq \mathbb{R}^{n-1}$ of τ. Moreover the affine subspace $L_\tau \cap \{v_i = 0\} \subset \mathbb{R}^n$ is parallel to the affine span $H_\tau \cap \{v_i = 0\} \subset \mathbb{R}^n$ of $\gamma = \tau \cap \{v_i = 0\}$. This implies that $\lambda = \exp(-2\pi i \frac{\nu(\tau)}{N(\tau)}) \in \mathbb{C}$ is a root of $\zeta_\gamma(t)$ if and only if $L_\tau \cap \{v_i = 0\}$ is rational i.e. $L_\tau \cap \{v_i = 0\} \cap \mathbb{Z}^n \neq \emptyset$. On the other hand, it is easy to see that the affine subspace $H_\tau \cap \{v_i = h - 1\} \subset \mathbb{R}^n$ is a parallel translation of $L_\tau \cap \{v_i = 0\}$ by a lattice vector. Hence if $L_\tau \cap \{v_i = 0\}$ is rational, then $H_\tau \cap \{v_i = h - 1\} \cap \mathbb{Z}^n \neq \emptyset$ and the lattice height of the pyramid τ from its base $\gamma = \tau \cap \{v_i = 0\}$ is $h \geq 2$ i.e. $\mathrm{Vol}_{\mathbb{Z}}(\tau) \geq 2\mathrm{Vol}_{\mathbb{Z}}(\gamma)$. It follows that the polynomial $\frac{\zeta_\tau}{\zeta_\gamma}$ is divided by the factor $t - \lambda$. This completes the proof. □

Motivated by this proposition, we introduce the following definitions.

Definition 4.2. *Let τ be an $(n-1)$-dimensional lattice simplex contained in a compact facet of $\Gamma_+(f)$.*

1. *We say that τ has a (non-empty) corner of codimension r if there exist $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that $\tau \cap \{v_i = 0\}$ is a facet of τ if and only if $i \in \{i_1, i_2, \ldots, i_r\}$ and $\tau \cap (\cap_{j=1}^r \{v_{i_j} = 0\}) \neq \emptyset$.*

2. *If τ has a corner of codimension r for $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, then we set*

$$F_\tau(t) = \prod_{I \subset \{i_1, i_2, \ldots, i_r\}} \Big\{\zeta_{\tau \cap \{v_i = 0 \ (i \in I)\}}(t)\Big\}^{(-1)^{|I|}} \in \mathbb{C}(t). \tag{4.4}$$

In the next section we will prove the following result.

Theorem 4.3. *Let τ be an $(n-1)$-dimensional lattice simplex contained in a compact facet of $\Gamma_+(f)$. Assume that for some $r \geq 1$ it has a (non-empty) corner of codimension r. Then $F_\tau(t) \in \mathbb{C}(t)$ is a polynomial of t. If we assume moreover that τ is not a B_1-pyramid, then the complex number*

$$\lambda = \exp\Big(-2\pi i \frac{\nu(\tau)}{N(\tau)}\Big) \in \mathbb{C} \tag{4.5}$$

is a root of the polynomial.

We can generalize Theorem 4.3 slightly to allow also simplices with empty corners as follows.

Proposition 4.4. *Let $\tau = A_1 A_2 \cdots A_n$ be an $(n-1)$-dimensional lattice simplex in $\mathbb{R}^n_+$. Assume that for any $1 \leq i \leq n$ its vertex A_i is on the positive part of the i-th coordinate axis of $\mathbb{R}^n$ and set*

$$F_\tau(t) = \prod_{\sigma \prec \tau} \Big\{ \zeta_\sigma(t) \Big\}^{(-1)^{n-1-\dim\sigma}} \in \mathbb{C}(t), \tag{4.6}$$

where in the product $\prod_{\sigma \prec \tau}$ the face $\sigma \prec \tau$ of τ ranges through the non-empty ones. Then

$$F_\tau(t) \cdot \Big(1-t\Big)^{(-1)^n} \in \mathbb{C}(t) \tag{4.7}$$

is a polynomial. If we assume moreover that τ is not a B_1-simplex, then the complex number

$$\lambda = \exp\Big(-2\pi i \frac{\nu(\tau)}{N(\tau)}\Big) \in \mathbb{C} \tag{4.8}$$

is a root of the polynomial.

Proof. By the embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}$, $v \longmapsto (v, 0)$ we regard τ as a lattice simplex in $\mathbb{R}^n \times \mathbb{R}$ and set $Q(0,1) \in \mathbb{R}^n \times \mathbb{R}$. Let τ' be the convex hull of $\{Q\} \cup \tau$ in $\mathbb{R}^n \times \mathbb{R}$. Then by considering the vertex $Q \prec \tau'$ as the corner of the simplex τ' we define a polynomial $F_{\tau'}(t) \in \mathbb{C}[t]$ by Theorem 4.3. In this situation, it is easy to see that we have an equality

$$F_{\tau'}(t) = F_\tau(t) \cdot \Big(1-t\Big)^{(-1)^n} \tag{4.9}$$

from which the first assertion immediately follows. Since we have $N(\tau') = N(\tau)$ and $\nu(\tau') = \nu(\tau) + N(\tau)$, the second assertion also follows from Theorem 4.3. □

Together with Proposition 3.2, following the strategy of Lemahieu-Van Proeyen [17] we can now confirm the monodromy conjecture for non-degenerate hypersurfaces in many cases also for $n \geq 4$. In the case $n = 4$ there are some B_2-facets of $\Gamma_+(f)$ (along the intersections of two coordinate hyperplanes in $\mathbb{R}^4_+$) which are not B_1-pyramids but are divided into two B_1-pyramids. See Proposition 3.4 for the details. Their contributions to the monodromy zeta function $\zeta_{f,0}(t)$ are trivial.

5 The proof of Theorem 4.3

5.1 Preliminaries

For the proof of Theorem 4.3 we shall introduce some new notions and their basic properties. Let S be a finite set and denote its power set by 2^S. Namely elements of 2^S are subsets $I \subset S$ of S. Then for a function $\phi : 2^S \longrightarrow \mathbb{Z}$ we define new ones $\phi^{\downarrow}, \phi^{\uparrow} : 2^S \longrightarrow \mathbb{Z}$ by

$$\phi^{\downarrow}(I) = \sum_{J \subset I} \phi(J), \qquad \phi^{\uparrow}(I) = \sum_{J \subset I} (-1)^{|I|-|J|} \phi(J). \tag{5.1}$$

We call $\phi^{\downarrow}$ (resp. $\phi^{\uparrow}$) the antiderivative (resp. derivative) of ϕ. Then we can easily check that $\phi^{\uparrow\downarrow} = \phi^{\downarrow\uparrow} = \phi$.

Definition 5.1. *(i) We say that the function ϕ is fully supermodular if $\phi^{\uparrow}(I) \geq 0$ for any subset $I \subset S$.*

(ii) The function ϕ is called strictly fully supermodular if it is fully supermodular and $\phi^{\uparrow}(S) > 0$.

Lemma 5.2. *The product of two fully supermodular functions $\phi, \psi : 2^S \longrightarrow \mathbb{Z}$ is fully supermodular. Moreover it is strictly fully supermodular if and only if there exist subsets $I, J \subset S$ of S such that $I \cup J = S$ and $\phi^{\uparrow}(I), \psi^{\uparrow}(J) > 0$.*

Proof. For any subset $R \subset S$ of S we have

$$(\phi\psi)^{\uparrow}(R) = (\phi^{\uparrow\downarrow}\psi^{\uparrow\downarrow})^{\uparrow}(R) = \sum_{I \cup J \subset U \subset R} (-1)^{|R|-|U|} \phi^{\uparrow}(I)\psi^{\uparrow}(J) \tag{5.2}$$

$$= \sum_{I \cup J = R} \phi^{\uparrow}(I)\psi^{\uparrow}(J). \tag{5.3}$$

Then the assertion immediately follows. □

5.2 Reduction to the case $r = n - 1$

First we shall reduce the proof of Theorem 4.3 to the case $r = n - 1$. For simplicity assume that the corner $\gamma \prec \tau$ of the simplex $\tau \subset \partial\Gamma_+(f)$ is defined by $\gamma = \tau \cap \{v_1 = v_2 = \cdots = v_r = 0\}$. We set

$$\lambda = \exp\Big(-2\pi i \frac{\nu(\tau)}{N(\tau)}\Big) \in \mathbb{C}. \tag{5.4}$$

As in the proof of Proposition 4.1 we define two parallel affine hyperplanes H_τ and L_τ in $\mathbb{R}^n$ by

$$H_\tau = \{v \in \mathbb{R}^n \mid \langle a(\tau), v\rangle = N(\tau)\}, \tag{5.5}$$

$$L_\tau = \{v \in \mathbb{R}^n \mid \langle a(\tau), v\rangle = \Big\langle a(\tau), \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \Big\rangle = \nu(\tau)\}. \tag{5.6}$$

Let $W = \{v_1 = v_2 = \cdots = v_r = 0\} \simeq \mathbb{R}^{n-r} \subset \mathbb{R}^n$ be the linear subspace of $\mathbb{R}^n$ spanned by γ. Similarly, for a face σ of τ containing γ let $W_\sigma \simeq \mathbb{R}^{\dim\sigma+1} \subset \mathbb{R}^n$ be the linear subspace

of $\mathbb{R}^n$ spanned by σ. Then $\zeta_\sigma(\lambda) = 0$ if and only if the affine hyperplane $L_\tau \cap W_\sigma \subset W_\sigma$ of W_σ is rational i.e. $L_\tau \cap W_\sigma \cap \mathbb{Z}^n \neq \emptyset$. Let $\Phi_0 : W \xrightarrow{\sim} W$ be a unimodular transformation of W such that $\Phi_0(\gamma) \subset W \cap \{v_n = k\}$ for some $k \in \mathbb{Z}_{>0}$. Then we can easily extend it to a unimodular transformation $\Phi : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ of $\mathbb{R}^n$ which preserves W_σ for any $\sigma \prec \tau$ containing γ and the point

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n. \tag{5.7}$$

We can choose such Φ so that the heights of τ and $\Phi(\tau)$ from each coordinate hyperplane in $\mathbb{R}^n$ containing W are the same. Indeed, for the invertible matrix $A_0 \in \mathrm{GL}_{n-r}(\mathbb{Z})$ representing $\Phi_0 : W \xrightarrow{\sim} W$ it suffices to define $\Phi : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ by taking an invertible matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ of the form

$$A = \left(\begin{array}{c|c} I_r & 0 \\ \hline * & A_0 \end{array}\right) \in \mathrm{GL}_n(\mathbb{Z}) \tag{5.8}$$

such that

$$A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{5.9}$$

where $I_r \in \mathrm{GL}_r(\mathbb{Z})$ stands for the identity matrix of size r. By this construction of Φ, τ is a B_1-pyramid if and only if $\Phi(\tau)$ is so. Set $\tau' = \Phi(\tau)$ and define two parallel affine hyperplanes $H_{\tau'}$ and $L_{\tau'}$ in $\mathbb{R}^n$ similarly to the case of τ so that we have $\Phi(H_\tau) = H_{\tau'}$. Since $\Phi(L_\tau)$ is parallel to $\Phi(H_\tau) = H_{\tau'}$ and passes through the point

$$\Phi\Big(\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\Big) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n, \tag{5.10}$$

we have also $\Phi(L_\tau) = L_{\tau'}$. Now let us set

$$\lambda' = \exp\Big(-2\pi i \frac{\nu(\tau')}{N(\tau')}\Big) \in \mathbb{C}. \tag{5.11}$$

Then for any $\sigma \prec \tau$ containing γ we have $\zeta_\sigma(\lambda) = 0 \Longleftrightarrow \zeta_{\Phi(\sigma)}(\lambda') = 0$. Since the lattice distance $N(\sigma) > 0$ of σ (resp. $N(\Phi(\sigma)) > 0$ of $\Phi(\sigma)$) from the origin $0 \in W_\sigma \simeq \mathbb{R}^{\dim\sigma+1}$ is equal to the number of rational hyperplanes in $W_\sigma \simeq \mathbb{R}^{\dim\sigma+1}$ parallel to σ (resp. $\Phi(\sigma)$) between $\mathrm{aff}(\sigma)$ (resp. $\mathrm{aff}(\Phi(\sigma))$) and the origin, we have $N(\sigma) = N(\Phi(\sigma))$ and hence $\zeta_\sigma(t) \equiv \zeta_{\Phi(\sigma)}(t)$. Then we obtain an equality $F_\tau(t) = F_{\tau'}(t)$, where we slightly generalized Definition 4.2 in an obvious way to define $F_{\tau'}(t)$. Hence, to prove Theorem 4.3 we may assume that the corner γ of τ is contained in $W \cap \{v_n = k\}$ for some $k \in \mathbb{Z}_{>0}$. Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{r+1}$, $v \longmapsto (v_1, \ldots, v_r, v_n)$ be the projection. Then by the definition of normalized volumes, for any face σ of τ containing its corner $\gamma \subset W \cap \{v_n = k\}$ we have $\mathrm{Vol}_{\mathbb{Z}}(\sigma) = \mathrm{Vol}_{\mathbb{Z}}(\pi(\sigma)) \cdot \mathrm{Vol}_{\mathbb{Z}}(\gamma)$ and hence $\zeta_\sigma(t) = \{\zeta_{\pi(\sigma)}(t)\}^{\mathrm{Vol}_{\mathbb{Z}}(\gamma)}$. We thus obtain an equality $F_\tau(t) = \{F_{\pi(\tau)}(t)\}^{\mathrm{Vol}_{\mathbb{Z}}(\gamma)}$. Moreover we have $N(\tau) = N(\pi(\tau))$ and $\nu(\tau) = \nu(\pi(\tau))$. This implies that we have only to consider the case $r = n - 1$.

5.3 The proof of the case $r = n - 1$

We have reduced our proof to the case where $r = n - 1$, a vertex Q of our simplex $\tau = QA_1A_2 \cdots A_{n-1}$ has the form $Q = (0, 0, \ldots, 0, k)$ for some $k \in \mathbb{Z}_{>0}$ and its edges are given by

$$\overrightarrow{QA_1} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \\ b_1 \end{pmatrix}, \qquad \overrightarrow{QA_2} = \begin{pmatrix} 0 \\ a_2 \\ \vdots \\ 0 \\ b_2 \end{pmatrix}, \ \ldots\ldots, \qquad \overrightarrow{QA_{n-1}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_{n-1} \\ b_{n-1} \end{pmatrix}, \tag{5.12}$$

where $a_1, a_2, \ldots, a_{n-1} \in \mathbb{Z}_{>0}$ and $b_1, b_2, \ldots, b_{n-1} \in \mathbb{Z}$. We set

$$D = \prod_{i=1}^{n-1} a_i, \qquad K_i = \frac{b_i}{a_i} \cdot D \ (1 \leq i \leq n-1) \tag{5.13}$$

and $K = \sum_{i=1}^{n-1} K_i$. Moreover for a subset $I \subset S = \{1, 2, \ldots, n-1\}$ we denote by $\tau_I \prec \tau$ the face of τ whose vertices are Q and A_i $(i \in I)$ and set

$$D_I = \prod_{i \in I} a_i, \qquad \mathrm{gcd}_I = \mathrm{GCD}\Big(D, K_i \ (i \in I)\Big). \tag{5.14}$$

Then it is easy to see that the $|I|$-dimensional normalized volume $\mathrm{Vol}_{\mathbb{Z}}(\tau_I)$ of τ_I is given by the formula

$$\mathrm{Vol}_{\mathbb{Z}}(\tau_I) = \mathrm{gcd}_I \cdot \frac{D_I}{D} \tag{5.15}$$

and hence we have

$$N(\tau_I) = \frac{D_I}{\mathrm{Vol}_{\mathbb{Z}}(\tau_I)} \cdot k = \frac{D}{\mathrm{gcd}_I} \cdot k. \tag{5.16}$$

In particular, we obtain

$$N(\tau) = \frac{D}{\mathrm{gcd}_S} \cdot k. \tag{5.17}$$

For a subset $I \subset S = \{1, 2, \ldots, n-1\}$ we set

$$\zeta_I(t) = \Big\{1 - t^{N(\tau_I)}\Big\}^{\mathrm{Vol}_{\mathbb{Z}}(\tau_I)} \in \mathbb{C}[t] \tag{5.18}$$

so that we have the equality

$$F_\tau(t) = \prod_{I \subset S} \Big\{\zeta_I(t)\Big\}^{(-1)^{n-1-|I|}}. \tag{5.19}$$

Lemma 5.3. *The complex number*

$$\lambda = \exp\Big(-2\pi i \frac{\nu(\tau)}{N(\tau)}\Big) \in \mathbb{C} \tag{5.20}$$

is a root of the polynomial $\zeta_I(t)$ if and only if $\mathrm{gcd}_I | K$.

Proof. Since the primitive conormal vector of the $(n-1)$-dimensional simplex τ is equal to

$$a(\tau) = \frac{1}{\gcd_S} \begin{pmatrix} -K_1 \\ -K_2 \\ \vdots \\ -K_{n-1} \\ D \end{pmatrix}, \tag{5.21}$$

we have

$$\nu(\tau) = \frac{D-K}{\gcd_S} \tag{5.22}$$

and

$$\frac{\nu(\tau)N(\tau_I)}{N(\tau)} = \frac{D-K}{\gcd_I}. \tag{5.23}$$

Note that λ is a root of $\zeta_I(t)$ if and only if $\frac{\nu(\tau)N(\tau_I)}{N(\tau)}$ is an integer. Then the assertion follows immediately from (5.23) and the fact $\gcd_I | D$. This completes the proof. □

By this lemma the multiplicity of $t-\lambda$ in the rational function $Z_\tau(t)$ is equal to

$$\sum_{I:\ \gcd_I | K} (-1)^{n-1-|I|} \gcd_I \cdot \frac{D_I}{D}. \tag{5.24}$$

Similarly we obtain the following result.

Lemma 5.4. *For any $m \in \mathbb{Z}$ the complex number $\exp(2\pi i \frac{m}{N(\tau)}) \in \mathbb{C}$ is a root of the polynomial $\zeta_I(t)$ if and only if $\gcd_I | (m \cdot \gcd_S)$.*

Proposition 5.5. *The function $F_\tau(t)$ is a polynomial in t.*

Proof. By Lemma 5.4 it suffices to show that for any $m \in \mathbb{Z}$ the alternating sum

$$G_m = \sum_{I:\ \gcd_I | (m \cdot \gcd_S)} (-1)^{n-1-|I|} \gcd_I \cdot \frac{D_I}{D} \tag{5.25}$$

is non-negative. Fix $m \in \mathbb{Z}$ and for a prime number p denote its multiplicities in the prime decompositions of a_i, b_i and m by $\alpha(p)_i, \beta(p)_i$ and $\delta(p)$ respectively. We set

$$\gamma(p) = \delta(p) + \min_{1 \le i \le n-1} \{\beta(p)_i - \alpha(p)_i, 0\} \tag{5.26}$$

and define a function $\phi_p : 2^S \longrightarrow \mathbb{Z}$ by

$$\phi_p(I) = \begin{cases} p^{\min_{i \in I}\{\beta(p)_i - \alpha(p)_i, 0\} + \sum_{i \in I} \alpha(p)_i} & \Big(\min_{i \in I}\{\beta(p)_i - \alpha(p)_i, 0\} \le \gamma(p)\Big), \\ 0 & (\text{otherwise}). \end{cases} \tag{5.27}$$

Then it is easy to see that for the function $\phi = \prod_{p:\ \text{prime}} \phi_p : 2^S \longrightarrow \mathbb{Z}$ we have

$$\phi^\uparrow(S) = G_m. \tag{5.28}$$

By Lemma 5.2 we have only to prove that for any prime number p the function $\phi_p : 2^S \longrightarrow \mathbb{Z}$ is fully supermodular. For this purpose, we reorder the pairs (a_i, b_i) $(1 \leq i \leq n-1)$ so that we have

$$\beta(p)_1 - \alpha(p)_1 \leq \beta(p)_2 - \alpha(p)_2 \leq \cdots\cdots \leq \beta(p)_{n-1} - \alpha(p)_{n-1}. \tag{5.29}$$

Fix a subset $I = \{i_1, i_2, i_3, \ldots\} \subset S = \{1, 2, \ldots, n-1\}$ $(i_1 < i_2 < i_3 < \cdots)$ of S. We will show the non-negativity of the alternating sum

$$\phi_p^{\uparrow}(I) = \sum_{J \subset I} (-1)^{|I|-|J|} \phi_p(J). \tag{5.30}$$

We define $q \geq 0$ to be the maximal number such that $\beta(p)_{i_q} - \alpha(p)_{i_q} < 0$ (resp. $\beta(p)_{i_q} - \alpha(p)_{i_q} \leq \gamma(p)$) in the case $\gamma(p) \geq 0$ (resp. $\gamma(p) < 0$). First let us consider the case $\gamma(p) \geq 0$. Then for $1 \leq l \leq q$ the part of the alternating sum (5.30) over the subsets $J \subset I$ such that $\min J = i_l$ is equal to

$$(-1)^{l-1} p^{\beta(p)_{i_l}} \prod_{j>l} (p^{\alpha(p)_{i_j}} - 1). \tag{5.31}$$

Moreover the remaining part of (5.30) is equal to

$$(-1)^q \prod_{j>q} (p^{\alpha(p)_{i_j}} - 1). \tag{5.32}$$

We thus obtain the equality

$$\phi_p^{\uparrow}(I) = p^{\beta(p)_{i_1}} \prod_{j>1} (p^{\alpha(p)_{i_j}} - 1) - p^{\beta(p)_{i_2}} \prod_{j>2} (p^{\alpha(p)_{i_j}} - 1) + \cdots\cdots \tag{5.33}$$

$$\cdots + (-1)^{q-1} p^{\beta(p)_{i_q}} \prod_{j>q} (p^{\alpha(p)_{i_j}} - 1) + (-1)^q \prod_{j>q} (p^{\alpha(p)_{i_j}} - 1). \tag{5.34}$$

Note that for any $1 \leq j \leq q$ we have $\beta(p)_{i_j} - \alpha(p)_{i_j} < 0$ and obtain an inequality

$$p^{\beta(p)_{i_{j-1}}} (p^{\alpha(p)_{i_j}} - 1) \geq p^{\alpha(p)_{i_j}} - 1 \geq p^{\beta(p)_{i_j}}. \tag{5.35}$$

Thus, subdividing the terms in the above expression of $\phi_p^{\uparrow}(I)$ into pairs, we get the desired non-negativity $\phi_p^{\uparrow}(I) \geq 0$. Finally let us consider the case $\gamma(p) < 0$. In this case, we have the following expression of $\phi_p^{\uparrow}(I)$:

$$\phi_p^{\uparrow}(I) = p^{\beta(p)_{i_1}} \prod_{j>1} (p^{\alpha(p)_{i_j}} - 1) - p^{\beta(p)_{i_2}} \prod_{j>2} (p^{\alpha(p)_{i_j}} - 1) + \cdots + (-1)^{q-1} p^{\beta(p)_{i_q}} \prod_{j>q} (p^{\alpha(p)_{i_j}} - 1). \tag{5.36}$$

Then by using the inequality (5.35) we can prove the non-negativity $\phi_p^{\uparrow}(I) \geq 0$ as in the previous case $\gamma(p) \geq 0$. This completes the proof. □

Proposition 5.6. *Assume that τ is not a B_1-simplex. Then the complex number*

$$\lambda = \exp\Big(-2\pi i \frac{\nu(\tau)}{N(\tau)}\Big) \in \mathbb{C} \tag{5.37}$$

is a root of the polynomial $F_\tau(t)$.

Proof. By Lemma 5.3 it suffices to show that the alternating sum

$$G = \sum_{I:\ \gcd_I | K} (-1)^{n-1-|I|}\ \gcd_I \cdot \frac{D_I}{D} \tag{5.38}$$

is positive. For a prime number p denote its multiplicities in the prime decompositions of a_i, b_i and K by $\alpha(p)_i, \beta(p)_i$ and $\kappa(p)$ respectively. We set

$$\mu(p) = \kappa(p) - \sum_{i=1}^{n-1} \alpha(p)_i \tag{5.39}$$

and define a function $\psi_p : 2^S \longrightarrow \mathbb{Z}$ by

$$\psi_p(I) = \begin{cases} p^{\min_{i\in I}\{\beta(p)_i - \alpha(p)_i, 0\} + \sum_{i\in I}\alpha(p)_i} & \Big(\min_{i\in I}\{\beta(p)_i - \alpha(p)_i, 0\} \leq \mu(p)\Big), \\ 0 & (\text{otherwise}). \end{cases} \tag{5.40}$$

Then it is easy to see that for the function $\psi = \prod_{p:\ \text{prime}} \psi_p : 2^S \longrightarrow \mathbb{Z}$ we have

$$\psi^{\uparrow}(S) = G. \tag{5.41}$$

Now let us set $S_p = \{1 \leq i \leq n-1 \mid \alpha(p)_i = 0\}$ and $I_p = S \setminus S_p = \{1 \leq i \leq n-1 \mid \alpha(p)_i > 0\}$. By our assumption we have $a_i > 1$ for any $1 \leq i \leq n-1$ and hence $\cup_{p:\ \text{prime}} I_p = S$. By Lemma 5.2, in order to show the positivity $\psi^{\uparrow}(S) > 0$ it suffices to prove that for any prime number p we have $\psi_p^{\uparrow}(I_p) > 0$. As in the proof of Proposition 5.5 we reorder the pairs (a_i, b_i) $(1 \leq i \leq n-1)$ so that we have

$$\beta(p)_1 - \alpha(p)_1 \leq \beta(p)_2 - \alpha(p)_2 \leq \cdots\cdots \leq \beta(p)_{n-1} - \alpha(p)_{n-1} \tag{5.42}$$

and $\alpha(p)_i \geq \alpha(p)_{i+1}$ whenever $\beta(p)_i - \alpha(p)_i = \beta(p)_{i+1} - \alpha(p)_{i+1}$. Moreover we set $I_p = \{i_1, i_2, i_3, \ldots\}$ $(i_1 < i_2 < i_3 < \cdots)$. We define $q \geq 0$ to be the maximal number such that $\beta(p)_{i_q} - \alpha(p)_{i_q} < 0$ (resp. $\beta(p)_{i_q} - \alpha(p)_{i_q} \leq \mu(p)$) in the case $\mu(p) \geq 0$ (resp. $\mu(p) < 0$). Then we have the same expressions of $\psi_p^{\uparrow}(I_p) > 0$ as in the proof of Proposition 5.5. In the case $\mu(p) \geq 0$ we have

$$\psi_p^{\uparrow}(I_p) = p^{\beta(p)_{i_1}} \prod_{j>1}(p^{\alpha(p)_{i_j}} - 1) - p^{\beta(p)_{i_2}} \prod_{j>2}(p^{\alpha(p)_{i_j}} - 1) + \cdots\cdots \tag{5.43}$$

$$\cdots + (-1)^{q-1} p^{\beta(p)_{i_q}} \prod_{j>q}(p^{\alpha(p)_{i_j}} - 1) + (-1)^q \prod_{j>q}(p^{\alpha(p)_{i_j}} - 1). \tag{5.44}$$

In the case $\mu(p) < 0$ we have

$$\psi_p^{\uparrow}(I_p) = p^{\beta(p)_{i_1}} \prod_{j>1}(p^{\alpha(p)_{i_j}} - 1) - p^{\beta(p)_{i_2}} \prod_{j>2}(p^{\alpha(p)_{i_j}} - 1) + \cdots + (-1)^{q-1} p^{\beta(p)_{i_q}} \prod_{j>q}(p^{\alpha(p)_{i_j}} - 1). \tag{5.45}$$

By the definitions of $I_p = \{i_1, i_2, i_3, \ldots\} \subset S$ and $q \geq 0$ we have $i \in I_p$ for any $i \leq i_q$. Eventually we find that $i_j = j$ for any $j \leq q$. First let us consider the case $I_p = \emptyset$. Then we have

$$\psi_p(I_p) = \begin{cases} p^0 = 1 & (\mu(p) \geq 0), \\ \\ 0 & (\mu(p) < 0). \end{cases} \tag{5.46}$$

In the case $\mu(p) \geq 0$ we thus obtain the positivity $\psi_p^{\uparrow}(I_p) > 0$. But in the case $\mu(p) < 0$ the condition $I_p = \emptyset$ implies $q = 0$ and such a case cannot occur by the following lemma.

Lemma 5.7. *The case $q = 0$ and $\mu(p) < 0$ cannot occur.*

Proof. Assume that $q = 0$ and $\mu(p) < 0$. By the definition of $\mu(p)$ we have

$$\mathrm{mult}_p(K) = \mathrm{mult}_p(D \cdot p^{\mu(p)}). \tag{5.47}$$

On the other hand, it follows from the condition $\mu(p) < 0$ and $q = 0$ that

$$\mu(p) < \beta(p)_{i_1} - \alpha(p)_{i_1} \leq \beta(p)_{i_2} - \alpha(p)_{i_2} \leq \cdots\cdots . \tag{5.48}$$

Namely for any $i \in I_p$ we have $\mu(p) < \beta(p)_i - \alpha(p)_i$ and hence

$$\mathrm{mult}_p(K_i) > \mathrm{mult}_p(D \cdot p^{\mu(p)}). \tag{5.49}$$

Moreover for any $i \in S_p = S \setminus I_p$ we have

$$\mathrm{mult}_p(K_i) \geq \mathrm{mult}_p(D) > \mathrm{mult}_p(D \cdot p^{\mu(p)}), \tag{5.50}$$

where we used the condition $\mu(p) < 0$ in the second inequality. We thus obtain the inequality

$$\mathrm{mult}_p(K) = \mathrm{mult}_p(\sum_{i \in S} K) > \mathrm{mult}_p(D \cdot p^{\mu(p)}) \tag{5.51}$$

which contradicts (5.47). □

By this lemma, it remains for us to treat the case $I_p \neq \emptyset$. From now on, we assume that $I_p \neq \emptyset$. Note that the inequality (5.35) becomes an equality only in the case $p = 2$, $\beta(p)_{i_{j-1}} = \beta(p)_{i_j} = 0$ and $\alpha(p)_{i_j} = 1$. By Lemma 5.7 this means that the sums (5.43) and (5.45) may be zero only in the following two cases:

Case 1: $p = 2$, $\mu(p) \geq 0$, $q = 2m + 1$ for $m \geq 0$ and $(\alpha(p)_1, \beta(p)_1) = (a, 0)$ for $a > 0$, $(\alpha(p)_2, \beta(p)_2) = \cdots\cdots = (\alpha(p)_q, \beta(p)_q) = (1, 0)$.

Case 2: $p = 2$, $\mu(p) < 0$, $q = 2m$ for $m \geq 1$ and $(\alpha(p)_1, \beta(p)_1) = (a, 0)$ for $a > 0$, $(\alpha(p)_2, \beta(p)_2) = \cdots\cdots = (\alpha(p)_q, \beta(p)_q) = (1, 0)$.

Indeed, in the case $\mu(p) \geq 0$ and $q = 2m$ for $m \geq 0$, if $q < |I_p|$ the last term $(-1)^q \prod_{j>q}(p^{\alpha(p)_{i_j}} - 1)$ of the alternating sum (5.43) is positive. Even if $q = |I_p|$ we still have the positivity

$$(-1)^q \prod_{j>q}(p^{\alpha(p)_{i_j}} - 1) = \psi_p(\emptyset) = 1 > 0. \tag{5.52}$$

Let us show that none of the above two cases can occur.

Case 1: Set $\alpha(p) = \sum_{i \in S} \alpha(p)_i$. Then $2^{\alpha(p)} | D$ and for any $i \in S_2$ we have $2^{\alpha(p)} | K_i$. We thus obtain the equality

$$K \equiv 2^{\alpha(p)-a} \cdot \text{odd} + (q-1) \cdot 2^{\alpha(p)-1} \cdot \text{odd} + \sum_{j>q} 2^{\alpha(p)+\beta(p)_{i_j}-\alpha(p)_{i_j}} \tag{5.53}$$

$$\equiv 2^{\alpha(p)-a} \cdot \text{odd} + (q-1) \cdot 2^{\alpha(p)-1} \cdot \text{odd} \equiv 2^{\alpha(p)-a} \cdot \text{odd} \tag{5.54}$$

of mod $2^{\alpha(p)}$, where we used also the fact that $\beta(p)_{i_j} - \alpha(p)_{i_j} \geq 0$ for any $j > q$. We conclude that $2^{\alpha(p)}$ does not divide K, which contradicts our assumption $\mu(p) \geq 0$.

Case 2: By the condition $q \geq 2$ we have $-1 = \beta(p)_{i_2} - \alpha(p)_{i_2} \leq \mu(p)$. Then by $\mu(p) < 0$ we obtain $\mu(p) = -1$. As in **Case 1**, by using the fact that $q-1$ is odd and $\mu(p) = -1$, if $a = 1$ we obtain the equality

$$K \equiv \sum_{j>q} 2^{\alpha(p)+\beta(p)_{i_j}-\alpha(p)_{i_j}} \equiv 0 \tag{5.55}$$

of mod $2^{\alpha(p)}$. But this result $2^{\alpha(p)} | K$ contradicts our assumption $\mu(p) < 0$. If $a > 1$ we obtain the equality

$$K \equiv 2^{\alpha(p)-a} \cdot \text{odd} \tag{5.56}$$

of mod $2^{\alpha(p)-1}$. But it also contradicts $\mu(p) = -1$.

This completes the proof. □

6 Some auxiliary results

In this section, to reduce the problem of non-convenient polynomials to that of convenient ones we prepare some technical results. Especially we have the following generalization of [17, Lemma 9].

Proposition 6.1. *Assume that f is non-degenerate at the origin $0 \in \mathbb{C}^n$. Then except for finitely many $c \in \mathbb{C}$ the polynomial $f(x_1, x_2, \ldots, x_{n-1}, x_n + c)$ is non-degenerate at the origin $0 \in \mathbb{C}^n$.*

Proof. Let $\tau' \prec \Gamma_+(f)$ be a face of $\Gamma_+(f)$ which is non-compact for the variable v_n and denote its image by the projection $\mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ by $\sigma \subset \mathbb{R}^{n-1}$. Assume that σ is compact. Here we shall treat only the case where τ' is a facet and hence $\dim \sigma = n-2$. The other cases can be treated similarly. By a unimodular transformation of $\mathbb{R}^n$ we regard τ' as a lattice polytope in its affine span $\text{aff}(\tau') \simeq \mathbb{R}^{n-1}$ and the τ'-part $f_{\tau'}$ of f as a Laurent polynomial on $T' = (\mathbb{C}^*)^{n-1}$. Then by our assumption for any compact face τ of τ' the hypersurface $\{f_\tau = 0\} \subset T'$ is smooth and reduced. Moreover the σ-part of the polynomial $f(x_1, x_2, \ldots, x_{n-1}, x_n + c)$ is naturally identified with $f_{\tau'}(x_1, \ldots, x_{n-2}, c)$. Therefore, in order to prove the assertion, it suffices to show that except for finitely many $c \in \mathbb{C}$ the hypersurface

$$W_c = \{(x_1, \ldots, x_{n-2}) \mid f_{\tau'}(x_1, \ldots, x_{n-2}, c) = 0\} \subset (\mathbb{C}^*)^{n-2} \tag{6.1}$$

in $T' \cap \{x_{n-1} = c\} \simeq (\mathbb{C}^*)^{n-2}$ is smooth and reduced. Let $h : T' = (\mathbb{C}^*)^{n-1} \longrightarrow \mathbb{C}$ be the function defined by $h(x_1, \ldots, x_{n-1}) = x_{n-1}$. Then the set of $c \in \mathbb{C}$ for which $W_c \subset (\mathbb{C}^*)^{n-2}$ is not smooth or not reduced is contained in the discriminant variety of the map $h|_{\{f_{\tau'}=0\}} : \{f_{\tau'} = 0\} \longrightarrow \mathbb{C}$. For $\varepsilon > 0$ let $B(0;\varepsilon)^* = \{c \in \mathbb{C} \mid 0 < |c| < \varepsilon\}$ be the punctured disk centered at the origin $0 \in \mathbb{C}$. Then there exists a sufficiently small $0 < \varepsilon \ll 1$ such that the hypersurface $W_c \subset (\mathbb{C}^*)^{n-2}$ is smooth and reduced for any $c \in B(0;\varepsilon)^*$. Indeed, let $\Delta = \tau' \cap \{v_n \leq l\} \subset \tau'$ $(l \gg 0)$ be the truncation of τ'. Let Σ_0 be the dual fan of the $(n-1)$-dimensional polytope Δ in $\mathbb{R}^{n-1}$ and Σ its smooth subdivision. We denote by X_Σ the toric variety associated to Σ (see [11] and [26] etc.). Then X_Σ is a smooth compactification of $T' = (\mathbb{C}^*)^{n-1}$. Recall that $T' = (\mathbb{C}^*)^{n-1}$ acts naturally on X_Σ and the T'-orbits in it are parametrized by the cones in the smooth fan Σ. For a cone $C \in \Sigma$ denote by $T_C \simeq (\mathbb{C}^*)^{n-1-\dim C} \subset X_\Sigma$ the T'-orbit associated to C. By our assumption above, if $C \in \Sigma$ corresponds to a compact face τ of τ' then the hypersurface $W = \overline{\{f_{\tau'} = 0\}} \subset X_\Sigma$ intersects $T_C \subset X_\Sigma$ transversally. We denote the meromorphic extension of $h : T' = (\mathbb{C}^*)^{n-1} \longrightarrow \mathbb{C}$ to X_Σ by the same letter h. Note that h has no point of indeterminacy on the whole X_Σ (because it is a monomial). Then as $|c| \longrightarrow 0$ the level set $h^{-1}(c) \subset X_\Sigma$ of h tends to the union of the T'-orbits which correspond to the compact faces of τ'. More precisely, if a cone $C \in \Sigma$ corresponds to a compact face of τ' then there exists an affine chart $\mathbb{C}_y^{n-1}$ of X_Σ on which

$$T_C = \Big\{ y = (y_1, \ldots, y_{n-1}) \mid y_i = 0 \ (1 \leq i \leq \dim C),\ y_i \neq 0 \ (\dim C + 1 \leq i \leq n-1) \Big\} \quad (6.2)$$

and $h(y) = y_1^{m_1} y_2^{m_2} \cdots y_k^{m_k}$ $(m_i \in \mathbb{Z}_{>0})$ for some $k \geq 1$. By this explicit description of h we see that for $0 < |c| \ll 1$ the hypersurface $h^{-1}(c)$ intersects W transversally. It follows that

$$W_c = W \cap h^{-1}(c) \cap T' \subset h^{-1}(c) \cap T' \simeq (\mathbb{C}^*)^{n-2} \quad (6.3)$$

is smooth and reduced for $0 < |c| \ll 1$. This completes the proof. □

7 The monodromy conjecture for $n = 4$

In this section, we will prove the monodromy conjecture for non-degenerate polynomials of four variables under an additional assumption. From now on, let us consider the case $n = 4$. Assume that $f(x) \in \mathbb{C}[x_1, \ldots, x_4]$ is non-degenerate at the origin $0 \in \mathbb{C}^4$. After subdividing the compact facets of $\Gamma_+(f)$ into 3-dimensional lattice simplices, we have the following lemmas. Also for such simplices τ we define their V-faces (see Definition 2.4) and the integers $N(\tau)$, $\nu(\tau)$ etc. in an obvious way.

Definition 7.1. *Let τ be a 3-dimensional lattice simplex in a compact facet of $\Gamma_+(f)$ and σ a V-face in $\Gamma_+(f)$. Then we say that σ contributes with respect to τ if the complex number*

$$\lambda = \exp\Big(-2\pi i \frac{\nu(\tau)}{N(\tau)}\Big) \in \mathbb{C} \quad (7.1)$$

is a root of the polynomial $\zeta_\sigma(t)$.

Lemma 7.2. *Let $\tau = APQR$ be a 3-dimensional lattice simplex in a compact facet of $\Gamma_+(f)$ such that the vertex $v = A \prec \tau$ and τ itself are its only contributing V-faces with*

respect to τ. Assume that $\mathrm{Vol}_{\mathbb{Z}}(\tau) = 1$. Then there exists another 3-dimensional lattice simplex $\sigma \neq \tau$ in a compact facet of $\Gamma_+(f)$ containing $v = A$ which has no V-face γ containing $v = A$ such that $\gamma \neq \sigma$ and $\gamma \neq v = A$.

Proof. Let us denote $A(\alpha, 0, 0, 0), P(p_0, p_1, p_2, p_3), Q(q_0, q_1, q_2, q_3)$ and $R(r_0, r_1, r_2, r_3)$. We will show that $p_1 = q_2 = r_3 = 0$ contradicts our assumption $\mathrm{Vol}_{\mathbb{Z}}(\tau) = 1$. This implies the existence of at least one more 3-dimensional lattice simplex $\sigma \neq \tau$ in the star of $v = A$ which has no V-face γ containing $v = A$ such that $\gamma \neq \sigma$ and $\gamma \neq v = A$. So suppose now that $p_1 = q_2 = r_3 = 0$. We define a matrix M by

$$M = \begin{pmatrix} 0 & p_2 & p_3 \\ q_1 & 0 & q_3 \\ r_1 & r_2 & 0 \end{pmatrix}. \tag{7.2}$$

Since the vertex $v = A \prec \tau$ and τ itself are the only V-faces of τ by our assumption, $p_2, p_3, q_1, q_3, r_1, r_2$ are positive integers. Let $\widehat{\tau}$ be the convex hull of $\{0\} \cup \tau$ in $\mathbb{R}^4$. Then it follows from $\mathrm{Vol}_{\mathbb{Z}}(\tau) = 1$ that we have

$$\alpha \cdot \det(M) = \mathrm{Vol}_{\mathbb{Z}}(\widehat{\tau}) = \mathrm{Vol}_{\mathbb{Z}}(\tau) \cdot N(\tau) = N(\tau). \tag{7.3}$$

Let

$$\mathrm{aff}(\tau): \ ax + by + cz + dw = N(\tau) = a \cdot \alpha \tag{7.4}$$

be the equation of $\mathrm{aff}(\tau)$ with $\gcd(a, b, c, d) = 1$. Then we get $a = \det(M)$. Also, as $\mathrm{Vol}_{\mathbb{Z}}(\tau) = 1$, the integers a, b, c and d are the 3×3 minors of the matrix

$$\begin{pmatrix} p_0 - \alpha & 0 & p_2 & p_3 \\ q_0 - \alpha & q_1 & 0 & q_3 \\ r_0 - \alpha & r_1 & r_2 & 0 \end{pmatrix}. \tag{7.5}$$

As we supposed that $v = A$ contributes with respect to τ, we have $a|(b + c + d)$. For the integer $k = (b + c + d)/a$ we thus obtain

$$\begin{vmatrix} p_0 - \alpha & 0 & p_2 & p_3 \\ q_0 - \alpha & q_1 & 0 & q_3 \\ r_0 - \alpha & r_1 & r_2 & 0 \\ -k & 1 & 1 & 1 \end{vmatrix} = 0. \tag{7.6}$$

Hence $(-k, 1, 1, 1)$ is a rational linear combination of the vectors $\overrightarrow{AP}, \overrightarrow{AQ}$ and $\overrightarrow{AR}$. As $\mathrm{Vol}_{\mathbb{Z}}(\tau) = 1$, the cone generated by $\overrightarrow{AP}, \overrightarrow{AQ}$ and $\overrightarrow{AR}$ is smooth and hence $(-k, 1, 1, 1)$ is an integer linear combination of these vectors. In particular there should be an integer solution $(x, y, z) \in \mathbb{Z}^3$ for the equation

$$\begin{pmatrix} 0 & q_1 & r_1 \\ p_2 & 0 & r_2 \\ p_3 & q_3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{7.7}$$

By Cramer's rule we then find in particular that

$$x = \frac{q_1 r_2 + q_3(r_1 - r_2)}{\det(M)}. \tag{7.8}$$

Without loss of generality we may assume that $r_1 \geq r_2$. Then we have

$$\det(M) = p_3 q_1 r_2 + p_2 q_3 r_1 \leq q_1 r_2 + q_3(r_1 - r_2), \tag{7.9}$$

which contradicts the positivity of $p_2, p_3, q_1, q_3, r_1, r_2$. This completes the proof. □

Lemma 7.3. *Let $\tau = APQR$ be a 3-dimensional lattice simplex in a compact facet of $\Gamma_+(f)$ such that $v = A(\alpha, 0, 0, 0)$ and $\sigma = PQR$ are its V-faces. Then*

$$\frac{\zeta_\tau(t) \cdot (1-t)}{\zeta_v(t) \cdot \zeta_\sigma(t)} \tag{7.10}$$

is a polynomial.

Proof. If $\mathrm{Vol}_{\mathbb{Z}}(\tau) > \mathrm{Vol}_{\mathbb{Z}}(\sigma)$ then the assertion is obvious. So suppose that $\mathrm{Vol}_{\mathbb{Z}}(\tau) = \mathrm{Vol}_{\mathbb{Z}}(\sigma)$. Let

$$\mathrm{aff}(\tau): \ ax + by + cz + dw = N(\tau) \tag{7.11}$$

be the equation of $\mathrm{aff}(\tau)$ with $\gcd(a, b, c, d) = 1$. Since we have

$$N(v) = \alpha = \frac{N(\tau)}{a}, \quad N(\sigma) = \frac{N(\tau)}{\gcd(b, c, d)} \tag{7.12}$$

and $\gcd(a, b, c, d) = 1$, the condition

$$\mathrm{Vol}_{\mathbb{Z}}(\tau) = \mathrm{Vol}_{\mathbb{Z}}(\sigma) \iff \gcd(N(v), N(\sigma)) = 1 \tag{7.13}$$

implies that $N(\tau) = a \cdot \gcd(b, c, d)$, $N(\sigma) = a$ and $N(v) = \alpha = \gcd(b, c, d)$. Hence we get

$$\frac{\zeta_\tau(t)}{\zeta_v(t) \cdot \zeta_\sigma(t)} = \frac{(1 - t^{a \cdot \alpha})^{\mathrm{Vol}_{\mathbb{Z}}(\sigma)}}{(1 - t^\alpha) \cdot (1 - t^a)^{\mathrm{Vol}_{\mathbb{Z}}(\sigma)}}. \tag{7.14}$$

However, the only common zero of $\zeta_v(t)$ and $\zeta_\sigma(t)$ is equal to 1. □

Lemma 7.4. *Let $\tau = APQR$ for $A(\alpha, 0, 0, 0), P(0, 0, p_2, p_3), Q(0, q_1, 0, q_3), R(0, r_1, r_2, 0)$ be a 3-dimensional lattice simplex in a compact facet of $\Gamma_+(f)$ that is not of type B_1 ($\iff \alpha \geq 2$). Assume that $v = A \prec \tau$ and $\sigma = PQR \prec \tau$ are its contributing V-faces with respect to τ and*

$$\lambda = \exp\Big(-2\pi i \frac{\nu(\tau)}{N(\tau)}\Big) \neq 1. \tag{7.15}$$

Then for the polynomial

$$F(t) = \frac{\zeta_\tau(t) \cdot (1-t)}{\zeta_v(t) \cdot \zeta_\sigma(t)} \in \mathbb{C}[t] \tag{7.16}$$

(see Lemma 7.3) we have $F(\lambda) = 0$.

Proof. Since the V-face $\sigma = PQR \prec \tau$ contributes with respect to τ, by the proof of Proposition 4.1 we have $\mathrm{Vol}_{\mathbb{Z}}(\tau) = \alpha \cdot \mathrm{Vol}_{\mathbb{Z}}(\sigma)$. Let

$$\mathrm{aff}(\tau): \ ax + by + cz + dw = N(\tau) = a \cdot \alpha \tag{7.17}$$

be the equation of $\mathrm{aff}(\tau)$ with $\gcd(a,b,c,d)=1$. Then by the proof of Lemma 7.3 the condition

$$\mathrm{Vol}_{\mathbb{Z}}(\tau)=\alpha\cdot\mathrm{Vol}_{\mathbb{Z}}(\sigma)\iff\gcd(N(v),N(\sigma))=\alpha \tag{7.18}$$

implies that $N(\tau)=a\cdot\alpha\cdot\gcd(b,c,d)$, $N(\sigma)=a\cdot\alpha$ and $N(v)=\alpha=\alpha\cdot\gcd(b,c,d)$. Hence we get $\gcd(b,c,d)=1$, $N(\tau)=N(\sigma)=a\cdot\alpha$ and

$$F(t)=\frac{(1-t^{a\cdot\alpha})^{\alpha\cdot\mathrm{Vol}_{\mathbb{Z}}(\sigma)}\cdot(1-t)}{(1-t^{\alpha})\cdot(1-t^{a\cdot\alpha})^{\mathrm{Vol}_{\mathbb{Z}}(\sigma)}}. \tag{7.19}$$

It follows from our assumption $\alpha\geq 2$ that we have $F(\lambda)=0$ unless $\alpha=2$ and $\mathrm{Vol}_{\mathbb{Z}}(\sigma)=1$. Let us show that the case where $\alpha=2$, $\mathrm{Vol}_{\mathbb{Z}}(\sigma)=1$ and $\mathrm{Vol}_{\mathbb{Z}}(\tau)=2$ cannot occur. Assume that $\mathrm{Vol}_{\mathbb{Z}}(\tau)=2$ and $\mathrm{Vol}_{\mathbb{Z}}(\sigma)=1$. Since another V-face $v=A\prec\tau$ contributes with respect to τ, we have also $a|(b+c+d)$. Then by $N(\tau)=2a$ we obtain $e^{-2\pi i\nu(\tau)/N(\tau)}=1$ or $e^{-2\pi i\nu(\tau)/N(\tau)}=-1$. It suffices to study what happens when $\lambda=e^{-2\pi i\nu(\tau)/N(\tau)}=-1$ $\iff 2a|(b+c+d)$. As $\mathrm{Vol}_{\mathbb{Z}}(\tau)=2$ and $\alpha=2$, the even integers $2a,2b,2c$ and $2d$ are the 3×3 minors of the matrix

$$\begin{pmatrix}-2&0&p_2&p_3\\-2&q_1&0&q_3\\-2&r_1&r_2&0\end{pmatrix},$$

and hence the expressions for a,b,c and d become

$$\begin{aligned}a&=\frac{q_1r_2p_3+r_1p_2q_3}{2},\qquad b=r_2p_3+p_2q_3-q_3r_2,\\ c&=p_3q_1+q_3r_1-r_1p_3,\qquad d=q_1r_2+r_1p_2-p_2q_1.\end{aligned} \tag{7.20}$$

As in the proof of Lemma 7.2 for the integer $k=(b+c+d)/a$ we can show that the vector $(-k,1,1,1)$ is a rational linear combination of $\overrightarrow{AP},\overrightarrow{AQ}$ and $\overrightarrow{AR}$. As $\mathrm{Vol}_{\mathbb{Z}}(\tau)=2$, the cone generated by $\overrightarrow{AP},\overrightarrow{AQ}$ and $\overrightarrow{AR}$ has multiplicity two and so in particular there should be an integer solution $(x,y,z)\in\mathbb{Z}^3$ for the equation

$$\begin{pmatrix}0&q_1&r_1\\p_2&0&r_2\\p_3&q_3&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\2\\2\end{pmatrix}. \tag{7.21}$$

We define a matrix M by

$$M=\begin{pmatrix}0&q_1&r_1\\p_2&0&r_2\\p_3&q_3&0\end{pmatrix}. \tag{7.22}$$

Then by Cramer's rule we then find that

$$x=\frac{2(r_2q_1+r_1q_3-r_2q_3)}{\det(M)}\quad,\quad y=\frac{2(r_2p_3+r_1p_2-r_1p_3)}{\det(M)}, \tag{7.23}$$

$$\text{and}\quad z=\frac{2(q_1p_3+p_2q_3-p_2q_1)}{\det(M)}. \tag{7.24}$$

Now we study the possible signs of x,y and z. If $p_2\geq p_3$ and $q_1\geq q_3$, then $y>0$ and $x>0$. If $p_2\geq p_3$ and $q_1\leq q_3$, then $y>0$ and $z>0$. If $p_2\leq p_3$ and $r_1\leq r_2$, then

$z > 0$ and $y > 0$ and so on. Thus we find that at least two of the integers x, y and z are always positive. By permuting them, we may assume that $x > 0$ and $y > 0$. As none of $p_2, p_3, q_1, q_3, r_1, r_2$ is equal to 0, the equation $p_3x + q_3y = 2$ obtained by (7.21) implies that $p_3 = q_3 = 1$ and $x = y = 1$. Consequently we get

$$a = \frac{q_1r_2 + r_1p_2}{2},\ b = p_2,\ c = q_1,\ d = q_1r_2 + r_1p_2 - p_2q_1 \tag{7.25}$$

and $\det(M) = q_1r_2 + r_1p_2$. As we supposed that $2a|(b + c + d)$, we have

$$(q_1r_2 + r_1p_2)|(p_2 + q_1 + q_1r_2 + r_1p_2 - p_2q_1) \iff \det(M)|(p_2 + q_1 - p_2q_1) \tag{7.26}$$

and z is an even integer. Hence, again by (7.21) and by using that $x = y = 1$, we find that p_2 and q_1 should be even. Then we have

$$\gcd(b, c, d) = \gcd(p_2, q_1, q_1r_2 + r_1p_2 - p_2q_1) \geq 2. \tag{7.27}$$

However, it contradicts our previous result $\gcd(b, c, d) = 1$. This completes the proof. □

Definition 7.5. *For a (not necessarily compact) facet τ of $\Gamma_+(f)$ we define its essential dimension* $\mathrm{ess.dim}\tau$ *by*

$$\mathrm{ess.dim}\tau = \max_{\sigma \prec \tau : \text{compact}} \dim\sigma. \tag{7.28}$$

Note that a facet τ of $\Gamma_+(f)$ is compact if and only if $\mathrm{ess.dim}\tau = 3$. Let $\mathcal{F}$ be the set of V-faces of $\Gamma_+(f)$. First define a subset $\mathcal{F}_3 \subset \mathcal{F}$ of $\mathcal{F}$ by $\sigma \in \mathcal{F}_3 \iff \sigma$ is contained in a compact facet τ of $\Gamma_+(f)$. Next define a subset $\mathcal{F}_2 \subset \mathcal{F} \setminus \mathcal{F}_3$ by $\sigma \in \mathcal{F}_2 \iff \sigma \notin \mathcal{F}_3$ and σ is contained in a facet τ of $\Gamma_+(f)$ such that $\mathrm{ess.dim}\tau = 2$. Similarly we define a subset $\mathcal{F}_1 \subset \mathcal{F} \setminus (\mathcal{F}_2 \cup \mathcal{F}_3)$ by $\sigma \in \mathcal{F}_1 \iff \sigma \notin \mathcal{F}_2 \cup \mathcal{F}_3$ and σ is contained in a facet τ of $\Gamma_+(f)$ such that $\mathrm{ess.dim}\tau = 1$. Finally we set $\mathcal{F}_0 = \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$. Namely the V-faces in $\mathcal{F}_0$ are those contained in facets τ of $\Gamma_+(f)$ such that $\mathrm{ess.dim}\tau = 0$. Note that for any $0 \leq i \leq 3$ and $\sigma \in \mathcal{F}_i$ we have $\dim\sigma \leq i$. Thus $\mathcal{F}$ is a disjoint union of $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$:

$$\mathcal{F} = \mathcal{F}_0 \sqcup \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \mathcal{F}_3. \tag{7.29}$$

For $0 \leq i \leq 3$ we define a rational function $R_i(t) \in \mathbb{C}(t)$ by

$$R_i(t) = \prod_{\sigma \in \mathcal{F}_i} \Big\{\zeta_\sigma(t)\Big\}^{(-1)^{i-\dim\sigma}} \in \mathbb{C}(t) \tag{7.30}$$

so that we have the product decomposition

$$\zeta_{f,0}^{-1}(t) = R_3(t) \cdot R_2^{-1}(t) \cdot R_1(t) \cdot R_0^{-1}(t) \tag{7.31}$$

of $\zeta_{f,0}^{-1}(t)$ by Theorem 2.3.

Definition 7.6. *We say that $\Gamma_+(f)$ is 0-convenient if for any 0-dimensional V-face $v \prec \Gamma_+(f)$ the set $\mathrm{Int}\mathbb{R}^n_+ \cap \partial\Gamma_+(f)$ is contained in the union of compact facets of $\Gamma_+(f)$ in a neighborhood of v.*

Now we have the following result.

Theorem 7.7. *Assume that $f(x) \in \mathbb{C}[x_1, \ldots, x_4]$ is non-degenerate at the origin $0 \in \mathbb{C}^4$ and $\Gamma_+(f)$ is 0-convenient. Let $\tau \prec \Gamma_+(f)$ be a (not necessarily compact) facet that is not a B_1-pyramid nor a B_2-facet. Then the complex number*

$$\lambda = \exp\Big(-2\pi i \frac{\nu(\tau)}{N(\tau)}\Big) \in \mathbb{C} \tag{7.32}$$

associated to it is an eigenvalue of the monodromy of f at some point in a neighborhood of the origin $0 \in \mathbb{C}^4$.

Proof. We follow the strategy in the proof of [17, Theorem 10]. Here we shall treat only the case where τ is compact. The other cases can be treated similarly as in the proof of [17, Theorem 10]. We subdivide compact facets of $\Gamma_+(f)$ into 3-dimensional lattice simplices $\tau_1, \ldots, \tau_k$ and for $1 \leq i \leq k$ define a polynomial $F_{\tau_i}(t) \in \mathbb{C}[t]$ by Theorem 4.3 and Proposition 4.4. Then we have

$$R_3(t) = \prod_{i=1}^{k} F_{\tau_i}(t) \cdot \prod_{\rho} \zeta_\rho(t) \cdot \prod_{\gamma} \zeta_\gamma^{-1}(t), \tag{7.33}$$

where in the product $\prod_\rho$ (resp. $\prod_\gamma$) the V-face ρ (resp. γ) ranges through the 1-dimensional (resp. 0-dimensional) ones in $\mathcal{F}_3$ which were not used in the construction of $F_{\tau_1}(t), \ldots, F_{\tau_k}(t)$. Since τ contains a non-B_1-simplex by Lemma 3.6, as in the proof of [17, Theorem 10] by using Theorem 4.3, Proposition 4.4 and Lemmas 7.2, 7.3 and 7.4 we can show that $R_3(\lambda) = 0$. Indeed, if there exists a 0-dimensional V-face γ in (7.33) not contained in any 1-dimensional V-face ρ in (7.33) then by the 0-convenience of $\Gamma_+(f)$ we are in the situation of Lemmas 7.2, 7.3 and 7.4 and can use them to deduce $R_3(\lambda) = 0$. First assume that $R_2(\lambda) = 0$. In this case, we subdivide 2-dimensional V-faces in $\mathcal{F}_2$ into 2-dimensional lattice simplices $\sigma_1, \ldots, \sigma_l$ and for $1 \leq i \leq l$ define a polynomial $F_{\sigma_i}(t) \in \mathbb{C}[t]$ by Theorem 4.3. Then we have

$$R_2(t) = \prod_{i=1}^{l} F_{\sigma_i}(t) \cdot \prod_{\gamma} \zeta_\gamma(t), \tag{7.34}$$

where in the product $\prod_\gamma$ the V-face $\gamma \in \mathcal{F}_2$ ranges through the 0-dimensional ones which were not used in the construction of $F_{\sigma_1}(t), \ldots, F_{\sigma_l}(t)$. By our assumption $R_2(\lambda) = 0$ there exists $1 \leq i \leq l$ such that λ is a root of the polynomial

$$Q(t) := F_{\sigma_i}(t) \cdot \prod_{\gamma \prec \sigma_i} \zeta_\gamma(t). \tag{7.35}$$

Let $\widetilde{\sigma} \prec \Gamma_+(f)$ be the (unique) non-compact facet containing σ_i. For simplicity, assume that $\widetilde{\sigma}$ is non-compact for the variable v_4. For a sufficiently generic small complex number $c \in \mathbb{C}$ let us set

$$g(x_1, \ldots, x_4) = f(x_1, x_2, x_3, x_4 + c) \in \mathbb{C}[x_1, \ldots, x_4]. \tag{7.36}$$

Then by Proposition 6.1 the new polynomial $g(x)$ is also non-degenerate at the origin $0 \in \mathbb{C}^4$ and its Newton polyhedron $\Gamma_+(g)$ is the projection of $\Gamma_+(f)$ with respect to the

variable v_4. By this construction, we can easily see that the polynomial $Q(t)$ shows up in the product decomposition of $\zeta_{g,0}(t)$ by Theorem 2.3. If $\zeta_{g,0}(\lambda) \neq 0$ then there exists a 1-dimensional V-face $\rho \prec \Gamma_+(g) \cap \{v_4 = 0\}$ of $\Gamma_+(g) \cap \{v_4 = 0\} \subset \{v_4 = 0\} \simeq \mathbb{R}^3$ satisfying the condition $F_\rho(\lambda) = 0$ and contained in a non-compact facet $\widetilde{\rho}$ of $\Gamma_+(g) \cap \{v_4 = 0\}$ such that $\mathrm{ess.dim}\widetilde{\rho} = 1$. For simplicity assume that $\widetilde{\rho}$ is non-compact for the variable v_3 and for a sufficiently generic small $c' \in \mathbb{C}$ set

$$h(x_1, \ldots, x_4) = g(x_1, x_2, x_3 + c', x_4) \in \mathbb{C}[x_1, \ldots, x_4]. \tag{7.37}$$

Then by Proposition 6.1 the polynomial $h(x)$ is also non-degenerate at the origin $0 \in \mathbb{C}^4$ and its Newton polyhedron $\Gamma_+(h)$ is the projection of $\Gamma_+(g)$ with respect to the variable v_3. Now it follows from the condition $F_\rho(\lambda) = 0$ we have $\zeta_{h,0}^{-1}(\lambda) = 0$. Finally let us consider the remaining case $R_2(\lambda) \neq 0$. Since $R_1(t)$ is a polynomial, it suffices to treat only the case $R_0(\lambda) = 0$. However by the 0-convenience of $\Gamma_+(f)$ we have $\mathcal{F}_0 = \emptyset$ and hence $R_0(t) \equiv 1$. This completes the proof. □

Definition 7.8. *A B-wall is a triangle in $\mathbb{R}^4_+$ with vertices of the form $(0, 0, a, b)$, $(1, 0, c, d)$, $(0, 1, e, f)$ up to reordering the coordinates.*

Definition 7.9. *We say that $\Gamma_+(f)$ is good if it is 0-convenient and no two of its B_1-pyramid facets for different variables contributing to the same candidate pole of $Z_{\mathrm{top},f}(s)$ have a common 1-dimensional V-face containing a 0-dimensional one.*

Finally we obtain the following result.

Theorem 7.10. *Assume that $f(x) \in \mathbb{C}[x_1, \ldots, x_4]$ is non-degenerate at the origin $0 \in \mathbb{C}^4$, $\Gamma_+(f)$ is good and no two B_i-facets ($i = 1, 2$) contributing to the same candidate pole of $Z_{\mathrm{top},f}(s)$ intersect by a B-wall. Let $s_0 \in \mathbb{C}$ be a pole of $Z_{\mathrm{top},f}(s)$. Then the complex number $\exp(2\pi i s_0) \in \mathbb{C}$ is an eigenvalue of the monodromy of f at some point in a neighborhood of the origin $0 \in \mathbb{C}^4$.*

Proof. By the results in Section 3, Theorems 4.3, 7.7 and Proposition 4.4, we can prove the assertion following the strategy in proof of [17, Theorem 15] as follows. By Propositions 3.3, 4.4 and Theorems 4.3, 7.7 it remains to consider only the case where the pole s_0 of $Z_{\mathrm{top},f}(s)$ is contributed by several B_1-pyramid facets of $\Gamma_+(f)$ for different variables. Suppose that there exist two B_1-pyramid facets $\tau, \sigma \prec \Gamma_+(f)$ for different variables such that

$$s_0 = -\frac{\nu(\tau)}{N(\tau)} = -\frac{\nu(\sigma)}{N(\sigma)}. \tag{7.38}$$

If $\tau \cap \sigma$ is their common facet, then by reducing the problem to the case where $\tau \cap \sigma$ is compact by Proposition 6.1 and using a V-face in it we can get the corresponding monodromy eigenvalue $\exp(2\pi i s_0) \in \mathbb{C}$ at some point in a neighborhood of the origin $0 \in \mathbb{C}^4$ by the proof of [17, Theorem 15]. Otherwise, their contributions to $Z_{\mathrm{top},f}(s)$ can be calculated separately and by Proposition 3.2 they produce only a fake pole. □

By our results obtained in this paper, we can prove also a similar assertion in higher dimensions $n \geq 5$ under some weak combinatorial assumption on $\Gamma_+(f)$.

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