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Ann Lemahieu, Willem Veys. Zeta Functions and Monodromy for Surfaces that are General for a Toric Idealistic Cluster. *International Mathematics Research Notices*, 2009, 9 (1), pp.11-62. 10.1093/imrn/rnn122 . hal-01279765

HAL Id: hal-01279765

<https://hal.science/hal-01279765>

Submitted on 26 May 2016

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Zeta functions and monodromy for surfaces that are general for a toric idealistic cluster

Ann Lemahieu and Willem Veys*

Abstract.— *In this article we consider surfaces that are general with respect to a 3-dimensional toric idealistic cluster. In particular, this means that blowing up a toric constellation provides an embedded resolution of singularities for these surfaces. First we give a formula for the topological zeta function directly in terms of the cluster. Then we study the eigenvalues of monodromy. In particular, we derive a useful criterion to be an eigenvalue. In a third part we prove the monodromy and the holomorphy conjecture for these surfaces.*

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1. INTRODUCTION

In [We65], Weil introduced some zeta functions $\mathcal{Z}(K, f)$ that are integrals over a p -adic field K and that are associated to a polynomial $f(\underline{x}) \in K[\underline{x}]$. Using embedded resolution of singularities, Igusa showed that these zeta functions are rational and he studied their poles (see [Ig75] and [Ig78]). One can define the analogous integrals over $K = \mathbb{R}$ or \mathbb{C} . Also these zeta functions are rational (see for example [At70] and [BeGe69]) and it is known that their poles are contained in the set of roots - and roots shifted by a negative integer - of the Bernstein polynomial b_f . By Malgrange ([Ma83]), if α is a root of b_f , then $e^{2\pi i\alpha}$

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is an eigenvalue of the local monodromy of f at some point of $f^{-1}(0)$. So when $K = \mathbb{R}$ or \mathbb{C} , then the poles of the zeta function induce eigenvalues of the local monodromy. This result was a motivation to study this relation at the p -adic side. The study of concrete examples made it natural to propose the following conjecture.

Monodromy conjecture. ([Ig88]) Let $F \subset \mathbb{C}$ be a number field and $f \in F[x]$. For almost all p -adic completions K of F , if s_0 is a pole of $\mathcal{Z}(K, f)$, then $e^{2\pi i \operatorname{Re}(s_0)}$ is an eigenvalue of the local monodromy of f at some point of the hypersurface $f = 0$.

Loeser verified this conjecture for plane curves (see [L88]). He also gave a proof for a class of polynomials in higher dimensions; the polynomial should be nondegenerate with respect to its Newton polyhedron and should satisfy some numerical conditions ([L90] and Section 3).

When Denef and Loeser introduced the topological zeta function in 1992 in [De,L92], an analogous version of the monodromy conjecture arose. This monodromy conjecture relates the poles of the topological zeta function $Z_{top,f}$ associated to a polynomial function or a germ of a holomorphic function f with the eigenvalues of monodromy of the hypersurface $f = 0$.

Monodromy conjecture. If s_0 is a pole of $Z_{top,f}$, then $e^{2\pi i s_0}$ is an eigenvalue of the local monodromy of f at some point of the hypersurface $f = 0$.

By the original definition of the topological zeta function, it follows that the monodromy conjecture for the Igusa zeta function implies the monodromy conjecture for the topological zeta function. Artal Bartolo, Cassou-Noguès, Luengo and Melle Hernández proved the monodromy conjecture for some surface singularities, such as the superisolated ones (see [A-B,C-N,Lu,M-H02]), and for quasi-ordinary polynomials in [A-B,C-N,Lu,M-H06]. The second author provided results in [Ve93B], [Ve95] and [Ve06], and together with Rodrigues in [Rve03]. In [LeVe07], the authors consider the same context as in this paper but they had to impose a restricting condition on the surfaces. Via geometrical arguments they showed that the monodromy conjecture holds for candidate poles of the topological zeta function of order 1 that are poles.

There are more conjectures relating the poles of the topological zeta function (Igusa zeta function) and the eigenvalues of monodromy. There exist the rational functions $Z_{top,f}^{(r)}$ ($r \in \mathbb{Z}_{>0}$) that are variants of the topological zeta function and that play a role in the holomorphy conjecture, which was stated by Denef.

Holomorphy conjecture. ([De91]) If $r \in \mathbb{Z}_{>0}$ does not divide the order of any eigenvalue of the local monodromy of f at any point of $f^{-1}\{0\}$, then $Z_{top,f}^{(r)}$ is holomorphic on \mathbb{C} .

Originally the holomorphy conjecture was formulated for the Igusa zeta func-

tion. We refer to [De91] for the inspiration. Denef showed that the conjecture is true for the relative invariants of a few prehomogeneous vector spaces. The second author proved the conjecture for plane curves (see [Ve93A]) and together with Rodrigues for homogeneous polynomials (see [RVe01]).

Although the monodromy conjecture and/or holomorphy conjecture has been proven for these kinds of singularities, one did not get a better understanding of the deep reason why the conjectures hold for them. Until now, the attempts are thus restricted to prove the conjecture for classes of singularities.

This article deals with the class of surfaces that are general with respect to a 3-dimensional toric idealistic cluster. This implies that we work with surfaces for which there exists an embedded resolution of singularities by blowing up in points that are orbits for the action of the torus, i.e. in a toric constellation. We refer to Section 2 for a recap about clusters and in Section 3 we explain the objects that play the main role in the conjecture. In Section 4 we show how the topological zeta function can be computed directly in terms of the toric cluster for the surfaces that we consider. We use the embedded resolution provided by the blowing up of the constellation. Let $\pi : Z \rightarrow \mathbb{C}^3$ be that resolution of such a surface $f = 0$ and let $E_j, j \in S$, be the irreducible components obtained by this resolution of which E_1, \dots, E_r are the exceptional ones. We will denote $E_j^\circ := E_j \setminus (\cup_{i \in S \setminus \{j\}} E_i)$, for $j \in S$. We write N_j and $\nu_j - 1$ for the multiplicities of E_j in the divisor on Z of $f \circ \pi$ and $\pi^*(dx \wedge dy \wedge dz)$, respectively. The numbers $-\nu_j/N_j, j \in S$, form a complete list of candidate poles of $Z_{top,f}$.

We compute in particular the Euler characteristic of the spaces $E_j^\circ, 1 \leq j \leq r$, in terms of the cluster. They show up in A'Campo's formula for the eigenvalues of monodromy and they are very relevant for the monodromy conjecture. In a fifth section we analyse these Euler characteristics. Our goal is to determine when these numbers are less than or equal to 0. A geometric argument will show that we can reduce this job to the investigation of a finite number of families of constellations. We complete Section 5 with combinatorial preparations. These make it possible to determine the sign of the Euler characteristics that we are looking for. We carry this out in Section 6. We then prove the following result.

Theorem. If $\chi(E_j^\circ) > 0$, then $e^{-2\pi i \frac{\nu_j}{N_j}}$ is an eigenvalue of monodromy of f .

Using this result, we prove in Section 7 the monodromy conjecture for candidate poles of order 1 that are poles and in Section 8 the monodromy conjecture for candidate poles of order 2 or 3 that are poles. Hence, we obtain:

Theorem. Let f be a germ of a polynomial map that is general with respect to a 3-dimensional toric idealistic cluster. If s_0 is a pole of $Z_{top,f}$, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy of f at some point of the hypersurface $f = 0$.

In Section 9 we prove the holomorphy conjecture for these surfaces.

Theorem. Let f be a germ of a polynomial map that is general with respect to a 3-dimensional toric idealistic cluster. If $r \in \mathbb{Z}_{>0}$ does not divide the order of any eigenvalue of the local monodromy of f at any point of $f = 0$, then $Z_{top,f}^{(r)}$ is holomorphic on \mathbb{C} .

2. TORIC CLUSTERS

In this section we introduce the terminology of infinitely near points, (toric) clusters etc. according to [C,G-S,L-J96]. We would like to refer to [C,G-S,L-J96] for some historical notes on clusters. See also [Ca90], [Ca00], [En,Ch15], [Li69], [Li78], [Li94] and [Z38] for more details on the theory of clusters.

2.1. Clusters.— Let X be a nonsingular variety of dimension $d \geq 2$ and let Z be a variety obtained from X by a finite succession of point blowing-ups. A point $Q \in Z$ is said to be *infinitely near* to a point $P \in X$ if P is in the image of Q ; we write $Q \geq P$. A *constellation* is a finite sequence $\mathcal{C} := \{Q_1, Q_2, \dots, Q_r\}$ of infinitely near points of X with $Q_1 \in X =: X_0$ and each Q_{j+1} is a point on the variety X_j obtained by blowing up Q_j in X_{j-1} , $j \in \{1, \dots, r-1\}$. The variety $X(\mathcal{C}) := X_r$ obtained by blowing up Q_r in X_{r-1} is called the *sky*.

The relation ‘ \geq ’ gives rise to a partial ordering on the points of a constellation. In the case that they are totally ordered, so $Q_r \geq \dots \geq Q_1$, the constellation \mathcal{C} is called a *chain*. For every Q_j in \mathcal{C} , the subsequence $\mathcal{C}^j := \{Q_i \mid Q_j \geq Q_i\}$ of \mathcal{C} is a chain. The integer $l(Q_j) := \#\mathcal{C}^j - 1$ is called the *level* of Q_j . In particular Q_1 has level 0. If no other point of \mathcal{C} has level 0 then Q_1 is called the *origin* of \mathcal{C} . We will always work with constellations that have an origin and we will also denote the origin of the constellation by o . If $Q_j \geq Q_i$ and $l(Q_j) = l(Q_i) + 1$, we will write $Q_j \succ Q_i$ or $j \succ i$.

For each $Q_i \in \mathcal{C}$, denote the exceptional divisor of the blowing-up in Q_i by E_i , as well as its strict transform at some intermediate stage (including the final stage) X_j , $i \leq j \leq r$. The total transform at some intermediate stage (including the final stage) will be denoted by E_i^* . If $Q_j \in E_i$, then one says that Q_j is *proximate* to Q_i . This will be denoted as $Q_j \rightarrow Q_i$ or $j \rightarrow i$. As $E_i = E_i^* - \sum_{j \rightarrow i} E_j^*$, it follows that also $\{E_1^*, \dots, E_r^*\}$ is a basis of the group of divisors with exceptional support $\bigoplus_{j=1}^r \mathbb{Z}E_j$.

A pair $\mathcal{A} := (\mathcal{C}, \underline{m})$ consisting of a constellation $\mathcal{C} := \{Q_1, \dots, Q_r\}$ and a sequence $\underline{m} := (m_1, \dots, m_r)$ of nonnegative integers is called a *cluster*. One calls m_j the *weight* or *multiplicity* of Q_j in the cluster and we write $D(\mathcal{A}) := \sum_{j=1}^r m_j E_j^*$. Introducing the numbers v_j , $1 \leq j \leq r$, by setting $m_j := v_j - \sum_{j \rightarrow i} v_i$, allows us to write also $D(\mathcal{A}) = \sum_{j=1}^r v_j E_j$.

The idea of clusters is to express that a system of hypersurfaces is passing through the points of the constellation with (at least) the given multiplicities. Blowing up a point $Q_i \in \mathcal{C}$ induces a discrete valuation ν_i on $\mathbb{C}(X) \setminus \{0\}$: for

$g \in \mathbb{C}(X) \setminus \{0\}$, the value $\nu_i(g)$ is the order of the pullback of g (at the stage X_i) along E_i . To a cluster we can then associate the (complete) ideal

$$I(v_1, \dots, v_r) = \{g \in \mathcal{O}_{X,o} \mid \nu_j(g) \geq v_j, 1 \leq j \leq r\} \cup \{0\}.$$

If we want that these ideals principalise by blowing up the points of the constellation, we require the ideals to be finitely supported. Formally, an ideal I in $\mathcal{O}_{X,o}$ is called *finitely supported* if I is primary for the maximal ideal \mathfrak{m} of $\mathcal{O}_{X,o}$ - so supported at the closed point - and if there exists a constellation \mathcal{C} of infinitely near points of X such that $I\mathcal{O}_{X(\mathcal{C})}$ is an invertible sheaf.

On the other hand, given a finitely supported ideal I , one can associate a cluster to it. Let $\mathcal{C}_I =: \{Q_1, \dots, Q_r\}$ be the constellation of base points of I , i.e. the minimal constellation \mathcal{C} such that $I\mathcal{O}_{X(\mathcal{C})}$ is an invertible sheaf. Let m_j be the order of the point Q_j , $1 \leq j \leq r$ in the strict transform of the ideal I in \mathcal{O}_{X_j, Q_j} . Then the ideal sheaf $I\mathcal{O}_{X(\mathcal{C}_I)}$ is associated to $-D(\mathcal{A}) := -\sum_{j=1}^r m_j E_j^*$.

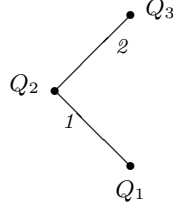
If \mathcal{C} is a constellation with origin at Q_1 , the cluster $\mathcal{A} := (\mathcal{C}, \underline{m})$ is called *idealistic* if there exists a finitely supported ideal I in \mathcal{O}_{X, Q_1} such that $I\mathcal{O}_{X(\mathcal{C})}$ is the ideal sheaf associated to $-D(\mathcal{A})$. For an idealistic cluster \mathcal{A} , Lipman proved that there exists a unique finitely supported complete ideal $I_{\mathcal{A}}$ such that $I_{\mathcal{A}}\mathcal{O}_{X(\mathcal{C})} = \mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{A}))$, namely that given by the direct image of $\mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{A}))$ in X , see [Li78].

In the next subsection we will illustrate these notions in the context of our results.

2.2. Toric clusters in \mathbb{C}^3 .— From now on suppose that X is the affine toric variety \mathbb{C}^3 . Let Q_1 be the origin of $\mathbb{C}^3 = X_0$. A 3-dimensional *toric constellation* of infinitely near points with origin Q_1 is a constellation $\mathcal{C} := \{Q_1, Q_2, \dots, Q_r\}$ such that each Q_{j+1} is a 0-dimensional orbit in the toric variety X_j obtained by blowing up Q_j in X_{j-1} , $1 \leq j \leq r-1$. Blowing up in orbits of smooth varieties corresponds to making star subdivisions of the fan corresponding to the variety (see for example [O78]). In this way each blowing-up in a 0-dimensional orbit induces the creation of three cones of dimension 3 and thus of three new 0-dimensional orbits. Hence, the choice of a point Q_i in a toric chain is equivalent to the choice of an integer $a_i \in \{1, 2, 3\}$, which determines a 3-dimensional cone in the fan. A tree with a root such that each vertex has at most three following adjacent vertices is called a *3-nary tree*. The above observation shows that there is a natural bijection between the set of 3-dimensional toric constellations with origin and the set of finite 3-nary trees with a root, with the edges labeled with positive integers not greater than 3, such that two edges with the same source have different labels.

A cluster $\mathcal{A} := (\mathcal{C}, \underline{m})$ is called *toric* if the constellation \mathcal{C} is toric. The blowing-ups now induce monomial valuations (i.e. valuations determined by their values on monomials) and the ideal $I(v_1, \dots, v_r)$ associated to a toric cluster is thus monomial.

Example 1



Suppose $d = 3$ and \mathcal{C} is the constellation pictured at the left. It represents the following resolution process: by blowing up in the origin Q_1 we get an exceptional variety $E_1 \cong \mathbb{P}^2$. In E_1 there is one point in which we blow up, namely Q_2 . The labels indicate in which affine chart the points of the constellation are created.

We call the affine chart with label 1 that one in which the equation of E_1 is $x = 0$. In the chart with label 2 one has $E_1 \leftrightarrow y = 0$ and in the chart with label 3 one has $E_1 \leftrightarrow z = 0$. The point Q_2 is the origin of the affine chart with label 1. After blowing up in Q_2 we get an exceptional variety $E_2 \cong \mathbb{P}^2$, where we blow up in the point Q_3 that is the origin of chart 1.2. There one has $E_2 \leftrightarrow y = 0$ and (the transform of) $E_1 \leftrightarrow x = 0$.

We now point out how the induced valuations ν_1, ν_2 and ν_3 act. For $a, b, c \in \mathbb{Z}_{\geq 0}$, $\nu_1(x^a y^b z^c) = a + b + c$ because the pullback of $x^a y^b z^c$ in chart 1 is $x^{a+b+c} y^b z^c$. The pullback in chart 1.2 becomes $x^{a+b+c} y^{a+2b+2c} z^c$ and thus $\nu_2(x^a y^b z^c) = a + 2b + 2c$. Analogously we find $\nu_3(x^a y^b z^c) = 2a + 3b + 4c$. We can represent these valuations by the following vectors in the lattice \mathbb{N}^3 :

$$\nu_1 \leftrightarrow (1, 1, 1) \quad \nu_2 \leftrightarrow (1, 2, 2) \quad \nu_3 \leftrightarrow (2, 3, 4).$$

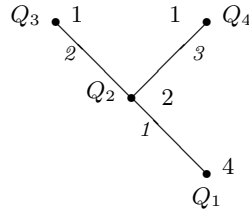
We have $I(\nu_1, \nu_2, \nu_3) = (x^a y^b z^c \mid a + b + c \geq \nu_1, a + 2b + 2c \geq \nu_2, 2a + 3b + 4c \geq \nu_3)$. To compute such an ideal, one can picture the hyperplanes induced by the valuations. We compute this ideal for the cluster $\mathcal{A} := (\mathcal{C}, \underline{m})$ with $(m_1, m_2, m_3) = (2, 1, 1)$. Then $(\nu_1, \nu_2, \nu_3) = (2, 3, 6)$ and one can see from

that $I(2, 3, 6) = (x^3, y^2, z^2, xz, x^2y, yz)$. One can verify that this ideal is finitely supported and that the cluster associated to this ideal is exactly the cluster \mathcal{A} . Hence this cluster is idealistic.

Let us now consider the cluster \mathcal{B} consisting of the above constellation and for which $(m_1, m_2, m_3) = (4, 1, 2)$ or $(\nu_1, \nu_2, \nu_3) = (4, 5, 11)$. Analogously one finds that $I := I(4, 5, 11)$ is equal to

$$(x^6, y^4, z^4, xy^3, x^3y^2, x^4y, x^3z^2, x^4z, xz^3, yz^3, y^2z^2, y^3z, xyz^2, xy^2z, x^2yz).$$

One finds that this ideal is finitely supported but the cluster associated to this ideal is



From this we can deduce that \mathcal{B} is not an idealistic cluster. Indeed, if \mathcal{B} was idealistic, then there would exist a finitely supported complete ideal J such that $J\mathcal{O}_{X(\mathcal{C})} = \mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{B}))$ and we also know that J would be included in I . As I and J are both complete ideals, they should be equal but we mentioned already that $I\mathcal{O}_{X(\mathcal{C})} \neq \mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{B}))$.

Finally let us consider the cluster consisting of the constellation \mathcal{C} with $(m_1, m_2, m_3) = (2, 2, 2)$ or $(v_1, v_2, v_3) = (2, 4, 8)$. One can check that

$$I(2, 4, 8) = (x^4, y^3, z^2, xy^2, x^3y, x^2z, y^2z, xyz)$$

and that this ideal is even not finitely supported. In the same way we can conclude that this cluster is not idealistic. \square

2.3. Properties.— In this subsection we recall some properties about clusters, in particular about toric clusters.

(1) In the case of toric clusters, there exists a combinatorial characterisation for the idealistic clusters. Fix a point Q_i in a toric 3-dimensional constellation \mathcal{C} and some integers a, b such that $a, b \in \{1, 2, 3\}$ and $a \neq b$. For $s, t \in \mathbb{Z}_{\geq 0}$, let $Q_i(a^s, b^t)$ be the terminal point of the chain with origin Q_i coded by $(a, \dots, a, b, \dots, b)$ where a appears s times and b appears t times. If $t = 0$, it is denoted by $Q_i(a^s)$. The point $Q_i(a^s, b^t)$ may not belong to \mathcal{C} . A point $Q_j \in \mathcal{C}$ that is infinitely near to Q_i is said to be *linearly proximate* to Q_i , if $Q_j = Q_i(a, b^t)$, with a, b and t as above. We denote this relation by $Q_j \rightsquigarrow Q_i$ or $j \rightsquigarrow i$. Then we have that Q_j is linearly proximate to Q_i if and only if there exists a 1-dimensional orbit l in B_i such that Q_j belongs to the strict transform of the closure of l in E_i . This explains the terminology. Denote $M_{Q_i}(a, b) := \sum_{t \geq 0} m_{Q_i(a, b^t)}$. Campillo, Gonzalez-Sprinberg and Lejeune-Jalabert show the following.

1. A toric cluster $\mathcal{A} = (\mathcal{C}, \underline{m})$ is idealistic if and only if for each point Q_i of the constellation \mathcal{C} and for each pair of integers a and b such that $a, b \in \{1, 2, 3\}$ and $a \neq b$, the following inequality is satisfied:

$$M_{Q_i}(a, b) + M_{Q_i}(b, a) \leq m_{Q_i}.$$

These inequalities are called the *linear proximity inequalities*.

2. Let $\mathcal{A} = (\mathcal{C}, \underline{m})$ be a 3-dimensional toric idealistic cluster with associated divisor $D(\mathcal{A}) = \sum_{j=1}^r m_j E_j^* = \sum_{j=1}^r v_j E_j$ and let ν_1, \dots, ν_r be the induced discrete valuations. Such a valuation is called Rees for the ideal $I(\underline{v}) := I(v_1, \dots, v_r)$ if it is a valuation induced by an irreducible component of the exceptional divisor of the normalised blowing-up $\overline{Bl}_{I(\underline{v})} X$ of $I(\underline{v})$. Then

$$\forall Q_i \in \mathcal{C} : m_i^2 \geq \sum_{j \rightarrow i} m_j^2 \quad \text{and} \quad (1)$$

$$\nu_i \text{ is Rees for } I(\underline{v}) \text{ if and only if } m_i^2 > \sum_{j \rightarrow i} m_j^2. \quad (2)$$

(2) To a monomial ideal I one can associate a Newton polyhedron \mathcal{N}_I . It is the convex hull of $m + \mathbb{R}_{\geq 0}^3$ as m runs through the set of exponents of monomials in I . We refer to [Ke,Kn,M,S73] for the proofs of the following properties.

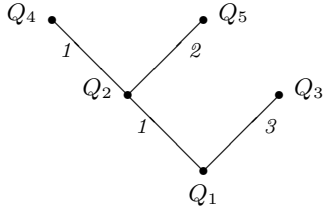
1. The compact facets of \mathcal{N}_I correspond with the Rees valuations of I .
2. A monomial ideal is complete if and only if it contains every monomial whose exponent is a point of $\mathcal{N}_I \cap \mathbb{Z}_{\geq 0}^3$.

(3) Campillo, Gonzalez-Sprinberg and Lejeune-Jalabert generate a very interesting set of ‘general’ hypersurfaces in [C,G-S,L-J96]:

Theorem 1 *The canonical map from the sky of the constellation of base points of a finitely supported ideal I to X is an embedded resolution of the subvariety of (X, o) defined by a general enough element in I .*

We will call these ‘general enough’ elements *general for I* or *general for \mathcal{C}_I* . We will prove the monodromy and holomorphy conjectures for the class of surfaces that are general for a finitely supported monomial ideal. In particular this means that our results apply to all surfaces for which there exists an embedded resolution consisting of toric point blowing ups and for which the corresponding toric cluster is idealistic. According to Theorem 1 such surfaces in a finitely supported ideal form an open dense set.

Example 2



Suppose $d = 3$ and \mathcal{C} is the constellation pictured at the left. It represents the following resolution process: by blowing up in the origin Q_1 we get an exceptional variety $E_1 \cong \mathbb{P}^2$. In E_1 there are two points in which we blow up, namely Q_2 and Q_3 . After blowing up in Q_2 we get an exceptional variety $E_2 \cong \mathbb{P}^2$, where again we blow up in two points.

The induced valuations are represented by the following vectors in the lattice \mathbb{N}^3 :

$$\nu_1 \leftrightarrow (1, 1, 1) \quad \nu_2 \leftrightarrow (1, 2, 2) \quad \nu_3 \leftrightarrow (2, 2, 1) \quad \nu_4 \leftrightarrow (1, 3, 3) \quad \nu_5 \leftrightarrow (2, 3, 4).$$

Consider the multiplicities $(m_1, m_2, m_3, m_4, m_5) = (3, 2, 1, 1, 1)$ for the points of this constellation, equivalently $(v_1, v_2, v_3, v_4, v_5) = (3, 5, 4, 6, 9)$. By the linear proximity inequalities it follows that the cluster $\mathcal{A} := (\mathcal{C}, \underline{m})$ is idealistic. Now let $I_{\mathcal{A}}$ be the ideal generated by the monomials whose exponents are in the associated Newton polyhedron. We find

$$I_{\mathcal{A}} = (x^6, y^3, z^4, x^3y, x^2y^2, yz^2, y^2z, x^3z, xz^2, xyz).$$

The blowing-up of the constellation gives an embedded resolution for a general element of $I_{\mathcal{A}}$, such as for example $h(x, y, z) := x^6 + y^3 + z^4 + x^3y + x^2y^2 + yz^2 + y^2z + x^3z + xz^2 - xyz$. \square

Example 3 Let us consider the non-complete ideal $I = (x^3, y^2, z^2, xz, x^2y)$. This ideal is finitely supported and the associated cluster is the idealistic cluster \mathcal{A} from Example 1. Theorem 1 says then that the blowing up of that constellation gives an embedded resolution for a general element of I , such as for example $x^3 + y^2 + z^2 + xz + x^2y$. \square

(4) We now first recall the notion for a polynomial to be *nondegenerate with respect to its Newton polyhedron*. Let $f \in \mathbb{C}[x_1, \dots, x_d]$ be a non-constant polynomial vanishing in the origin. Write $\underline{x}^{\underline{k}} := x_1^{k_1} \dots x_d^{k_d}$ and $f := \sum_{\underline{k} \in \mathbb{N}^d} c_{\underline{k}} \underline{x}^{\underline{k}}$. The *support* of f is $\text{supp}(f) := \{\underline{k} \in \mathbb{N}^d \mid c_{\underline{k}} \neq 0\}$. The *Newton polyhedron* Γ of f is the convex hull of $\text{supp}(f) + \mathbb{R}_{\geq 0}^d$. For a face τ of Γ we write $f_{\tau} := \sum_{\underline{k} \in \tau} c_{\underline{k}} \underline{x}^{\underline{k}}$. A polynomial f is called *nondegenerate with respect to Γ* if for every compact face τ of Γ , the polynomials f_{τ} and $\partial f_{\tau} / \partial x_i$ have no common zeroes in $(\mathbb{C}^*)^d$, $1 \leq i \leq d$.

Proposition 2 *Every hypersurface that is general with respect to some 3-dimensional toric idealistic cluster is nondegenerate with respect to its Newton polyhedron.*

Proof. Let $\mathcal{A} = (\mathcal{C}, \underline{m})$ be a toric idealistic cluster such that f is general with respect to \mathcal{A} . Suppose that f is degenerate with respect to $\mathcal{N}(f)$.

Let τ be a compact face of $\mathcal{N}(f)$ for which there exists a point $p \in (\mathbb{C}^*)^3$ such that $f_{\tau}(p) = \partial f_{\tau} / \partial x(p) = \partial f_{\tau} / \partial y(p) = \partial f_{\tau} / \partial z(p) = 0$.

If τ is a facet, then τ corresponds to some exceptional irreducible component created by the blowing up of the constellation, say to E_i . More specifically, the strict transform of f_{τ} is equal to $E_0 \cap E_i$. As p is not an orbit, it follows that there exists a point in which $E_0 \cap E_i$ does not have normal crossings and that is not an orbit. If the dimension of τ is one and if τ is the intersection of two compact facets, then analogously we have that there exist two irreducible exceptional components E_i and E_j such that $E_0 \cap E_i \cap E_j$ does not have normal crossings in a point that is not an orbit. Remains the case that τ is the intersection of a compact facet and a coordinate plane. Suppose that that compact facet corresponds to E_i and that the coordinate plane is given by $\{x = 0\}$. Again we get that then $E_0 \cap E_i$ does not have normal crossings in a point that is not an orbit. Indeed, if E_i has equation $y = 0$ in some affine chart, then there is a point $(0, 0, p_z)$ with $p_z \neq 0$ in which there are no normal crossings. \blacksquare

3. CONJECTURES

Let f be a complex polynomial in d variables and let $\pi : Z \rightarrow \mathbb{C}^d$ be an embedded resolution of singularities of $f^{-1}\{0\}$. We write $E_j, j \in S$, for the irreducible components of $\pi^{-1}(f^{-1}\{0\})$ and we denote by N_j and by $\nu_j - 1$ the multiplicities of E_j in the divisor on Z of $f \circ \pi$ and $\pi^*(dx_1 \wedge \dots \wedge dx_d)$, respectively. The couples $(\nu_j, N_j), j \in S$, are called the numerical data of the embedded resolution (Z, π) . We denote also $E_j^{\circ} := E_j \setminus (\cup_{i \in S \setminus \{j\}} E_i)$, for $j \in S$. Let the E_j ,

$j \in J := \{1, \dots, r\} \subset S$, be the exceptional irreducible components of $\pi^{-1}(\{0\})$.

3.1. Monodromy.— We assume that $f(b) = 0$. Take $\epsilon > 0$ small enough such that the open ball B_ϵ with radius ϵ around b in \mathbb{C}^d intersects the fibre $f^{-1}(0)$ transversally. Then choose $\epsilon \gg \eta > 0$ such that for t in the disc $D_\eta \subset \mathbb{C}$ around the origin, the fibre $f^{-1}(t)$ intersects B_ϵ transversally. Write $X := f^{-1}(D_\eta) \cap B_\epsilon$, $X_t := f^{-1}(t) \cap B_\epsilon$ for $t \in D_\eta$ and $D_\eta^* := D_\eta \setminus \{0\}$ for the pointed disc. Milnor showed that $f|_{X \setminus X_0} : X \setminus X_0 \rightarrow D_\eta^*$ is a locally trivial fibration, see [Mil68]. A fibre X_t of this bundle is called *Milnor fibre of f at b* . We will denote it by F_b . Consider the loop γ encircling the origin once counterclockwise. Since $f|_{X \setminus X_0}$ is a locally trivial fibration, the loop γ lifts to a diffeomorphism h of the Milnor fibre F_b , which is well determined up to homotopy. In this way γ induces an automorphism $h^* : H^i(F_b, \mathbb{C}) \rightarrow H^i(F_b, \mathbb{C})$, $i \geq 0$, that is called the *monodromy transformation*.

The surfaces for which we will prove the monodromy conjecture have exactly one isolated singularity in the origin. A result of Milnor (see [Mil68]) then says that $H^i(F_0, \mathbb{C}) = 0$, for $i \neq 0$ and $i \neq d - 1$, and $H^0(F_0, \mathbb{C}) = \mathbb{C}$ with trivial monodromy action. The formula of A'Campo ([A'C75]) describes the characteristic polynomial of the monodromy action on $H^{d-1}(F_0, \mathbb{C})$ in terms of an embedded resolution of the hypersurface $f^{-1}(0)$.

We may suppose that π is an isomorphism outside the inverse image of the origin.

Theorem 3 (*A'Campo*) *The characteristic polynomial of the monodromy action on $H^{d-1}(F_0, \mathbb{C})$ is equal to*

$$\left[\frac{\prod_{j=1}^r (1 - t^{N_j})^{\chi(E_j^\circ)}}{1 - t} \right]^{(-1)^{d-1}}.$$

3.2. Topological zeta function.— In 1992 Denef and Loeser created a new zeta function which they called the topological zeta function because of the topological Euler–Poincaré characteristic $\chi(\cdot)$ turning up in it. It is associated to a complex polynomial f with $f(0) = 0$. If $E_I := \cap_{i \in I} E_i$ and $E_I^\circ := E_I \setminus (\cup_{j \notin I} E_j)$, then they introduced it in [De,L92] in the following way.

Definition 4 *The local topological zeta function associated to f is the rational function in one complex variable*

$$Z_{top,f}(s) := \sum_{I \subset S} \chi(E_I^\circ \cap \pi^{-1}\{0\}) \prod_{i \in I} \frac{1}{N_i s + \nu_i}.$$

Denef and Loeser proved that every embedded resolution gives rise to the same function, so the topological zeta function is a well-defined singularity invariant (see [De,L92]). Once the motivic Igusa zeta function was introduced, they proved this result alternatively in [De,L98] by showing that this more general zeta function specialises to the topological one. There exists a global version,

replacing $E_I^\circ \cap \pi^{-1}\{0\}$ by E_I° .

3.3. Monodromy conjecture.— One calls α an *eigenvalue of monodromy of f* at $b \in f^{-1}\{0\}$ if α is an eigenvalue for some $h^* : H^i(F_b, \mathbb{C}) \rightarrow H^i(F_b, \mathbb{C})$.

Conjecture 5 (Monodromy Conjecture) *If s_0 is a pole of $Z_{top,f}$, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy of f at some point of the germ at 0 of the hypersurface $f = 0$.*

Let f be a polynomial that is general with respect to a 3-dimensional toric idealistic cluster. Consider the embedded resolution $\pi : Z \rightarrow \mathbb{C}^3$ of $f^{-1}\{0\}$ that corresponds to the blowing up of the constellation. We fix a candidate pole $s_0 = -\nu_j/N_j$ of $Z_{top,f}$. If E_j is not an exceptional component, then $\nu_1 = 1$ and $N_1 = 1$. As 1 is always an eigenvalue of the local monodromy of f , this candidate pole does not pose any difficulty. If $s_0 = -\nu_j/N_j$ is a candidate pole of $Z_{top,f}$ induced by an exceptional component E_j , then we write ν_j/N_j as a/b such that a and b are coprime. We define the set $J_b := \{j \in J \mid b \text{ divides } N_j\}$. It follows from A'Campo's formula that

$$e^{2\pi i s_0} \text{ is an eigenvalue of monodromy of } f \text{ at the origin } 0$$

$$\Updownarrow$$

$$\sum_{j \in J_b} \chi(E_j^\circ) \neq 0.$$

In general there can be a lot of cancelations which make that $\sum_{j \in J_b} \chi(E_j^\circ) = 0$. To control this, we will determine when $\chi(E_j^\circ)$ is positive, negative or zero. We will see that the cases where $\chi(E_j^\circ) \leq 0$ are very rare in this context.

3.4. Holomorphy conjecture.— For every $r \in \mathbb{Z}_{>0}$, one can define a variant $Z_{top,f}^{(r)}$ of the topological zeta function that is also a rational function in one complex variable.

Definition 6

$$Z_{top,f}^{(r)} := \sum_{\substack{I \subset S \\ \forall i \in I: r | N_i}} \chi(E_I^\circ \cap \pi^{-1}\{0\}) \prod_{i \in I} \frac{1}{N_i s + \nu_i}.$$

The functions $Z_{top,f}^{(r)}$ are limits of more general Igusa zeta function associated to a polynomial and a character (see [De91]). In particular $Z_{top,f}^{(1)} = Z_{top,f}$. Clearly they are holomorphic on \mathbb{C} if and only if they do not have a pole. The holomorphy conjecture stated by Denef predicts the following relation.

Conjecture 7 (Holomorphy conjecture) *If $r \in \mathbb{Z}_{>0}$ does not divide the order of any eigenvalue of monodromy of f , then $Z_{top,f}^{(r)}$ is holomorphic on \mathbb{C} .*

In Section 9 we provide a proof of the holomorphy conjecture for the surfaces we are studying. Again, the classification of $\chi(E_j^\circ)$ according to the sign will be the key to solve the conjecture.