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# On Mean Values of Dirichlet Polynomials

Michel Weber

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## Abstract

We show the following general lower bound valid for any positive integer  $q$ , and arbitrary reals  $\varphi_1, \dots, \varphi_N$  and non-negative reals  $a_1, \dots, a_N$ ,

$$c_q \left( \sum_{n=1}^N a_n^2 \right)^q \leq \frac{1}{2T} \int_{|t| \leq T} \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|^{2q} dt.$$

## 1 Main Result

The object of this short Note is to prove the following lower bound

**Theorem 1** *For any positive integer  $q$ , there exists a constant  $c_q$ , such that for any reals  $\varphi_1, \dots, \varphi_N$ , any non-negative reals  $a_1, \dots, a_N$ , and any  $T > 0$ ,*

$$c_q \left( \sum_{n=1}^N a_n^2 \right)^q \leq \frac{1}{2T} \int_{|t| \leq T} \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|^{2q} dt.$$

The result is no longer true for arbitrary reals  $a_1, \dots, a_N$  as yields the case  $\varphi_1 = \dots = \varphi_N$ . It also follows that

$$c \left( \sum_{n=1}^N a_n^2 \right)^{1/2} \leq \sup_{t \in \mathbf{R}} \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|. \quad (1)$$

In the case  $\varphi_n = \log n$ , it is known from [5] and [8] that for any  $(a_n)$

$$\sup_{t \in \mathbf{R}} \left| \sum_{n=0}^{N-1} a_n n^{it} \right| \geq \alpha_1 \frac{e^{\beta_1 \sqrt{\log N \log \log N}}}{\sqrt{N}} \left( \sum_{n=0}^{N-1} |a_n| \right) \quad (2)$$

and for some  $(a_n)$

$$\sup_{t \in \mathbf{R}} \left| \sum_{n=0}^{N-1} a_n n^{it} \right| \leq \alpha_2 \frac{e^{\beta_2 \sqrt{\log N \log \log N}}}{\sqrt{N}} \left( \sum_{n=0}^{N-1} |a_n| \right), \quad (3)$$

with some universal constants  $\alpha_1, \alpha_2, \beta_1, \beta_2$ . Then (1) is better than (2) if for instance  $a_n = n^{-\alpha}$ ,  $\alpha > 1/2$ , since

$$\frac{e^{\beta_1 \sqrt{\log N \log \log N}}}{\sqrt{N}} \left( \sum_{n=0}^{N-1} |a_n| \right) \sim e^{\beta_1 \sqrt{\log N \log \log N}} N^{\frac{1}{2} - \alpha} = o(1) \ll \left( \sum_{n=1}^N a_n^2 \right)^{1/2}.$$

The  $L^1$ -case is related to well-known Ingham's inequality [2]. We state the sharper form due to Mordell [7]: let  $0 < \varphi_1 < \dots < \varphi_N$  and let  $\gamma$  be such that  $\min_{1 < n \leq N} \varphi_n - \varphi_{n-1} \geq \gamma > 0$ . Then

$$\sup_{n=1}^N |a_n| \leq \frac{K}{T} \int_{-T}^T \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right| dt \quad \text{with } T = \frac{\pi}{\gamma}, \quad (4)$$

where  $K \leq 1$ .

Further with no restriction, one always have

$$\sup_{n=1}^N |a_n| \leq \limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right| dt \leq \sup_{t \in \mathbf{R}} \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|, \quad (5)$$

a very familiar inequality in the theory of uniformly almost periodic functions. See also [1] where the more complicated inequality is established:

$$|a_n| \leq \frac{1}{\prod_{j=0}^{n-1} \cos\left(\frac{\pi\varphi_j}{2\varphi_n}\right) \cdot \prod_{n+1}^N \cos\left(\frac{\pi\varphi_n}{2\varphi_j}\right)} \cdot \sup_{|t| \leq \frac{\pi}{2} \left( \frac{n}{\varphi_n} + \sum_{j=n+1}^N \frac{1}{\varphi_j} \right)} \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|. \quad (6)$$

In particular, if  $\varphi_1, \dots, \varphi_N$  are linearly independent, and  $T$  is large enough, then

$$b_q \left( \sum_{n=1}^N a_n^2 \right)^q \leq \frac{1}{2T} \int_{|t| \leq T} \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|^{2q} dt \leq B_q \left( \sum_{n=1}^N a_n^2 \right)^q, \quad (7)$$

holds for any nonnegative reals  $a_1, \dots, a_N$  and  $b_q, B_q$  depend on  $q$  only.

The proof of Theorem 1 relies upon the following lemma, which just generalizes a useful majorization argument ([6], p.131) to arbitrary even powers.

**Lemma 2** *Let  $q$  be any positive integer. Let  $c_1, \dots, c_N$  be complex numbers and nonnegative reals  $a_1, \dots, a_N$  such that  $|c_n| \leq a_n$ ,  $n = 1, \dots, N$ . Then for any reals  $T, T_0$  with  $T > 0$*

$$\int_{|t-T_0| \leq T} \left| \sum_{n=1}^N c_n e^{it\varphi_n} \right|^{2q} dt \leq 3 \int_{|t| \leq T} \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|^{2q} dt.$$

*Proof.* Let

$$K_T(t) = K_T(|t|) = (1 - |t|/T) \chi_{\{|t| \leq T\}}$$

Observe that for any reals  $t, H$

- a)  $K_T(t - H) = (1 - |t - H|/T) \chi_{\{|t - H| \leq T\}}$
- b)  $\chi_{\{|t - H| \leq T\}} \leq K_T(t - H) + K_T(t - H + T) + K_T(t - H - T)$
- c)  $\widehat{K}_T(u) = \frac{1}{T} \left( \frac{\sin Tu}{u} \right)^2 \geq 0$ , for all real  $u$ .

Suppose that  $|c_n| \leq a_n$  for  $n = 1, \dots, N$ . From

$$\left( \sum_{n=1}^N c_n e^{it\varphi_n} \right)^q = \sum_{k_1 + \dots + k_N = q} \left( \frac{q!}{k_1! \dots k_N!} \right) \prod_{n=1}^N c_n^{k_n} e^{itk_n\varphi_n}. \quad (8)$$

and

$$\left| \sum_{n=1}^N c_n e^{it\varphi_n} \right|^{2q} = \sum_{\substack{k_1+\dots+k_N=q \\ h_1+\dots+h_N=q}} \left( \frac{(q!)^2}{k_1!h_1! \dots k_N!h_N!} \right) \prod_{n=1}^N c_n^{k_n} \bar{c}_n^{h_n} e^{it(k_n-h_n)\varphi_n}$$

we get

$$\begin{aligned} & \int_{\mathbf{R}} K_T(t-H) \left| \sum_{n=1}^N c_n e^{it\varphi_n} \right|^{2q} dt \\ &= \sum_{\substack{k_1+\dots+k_N=q \\ h_1+\dots+h_N=q}} \frac{(q!)^2}{k_1!h_1! \dots k_N!h_N!} \prod_{n=1}^N c_n^{k_n} \bar{c}_n^{h_n} \int_{\mathbf{R}} K_T(t-H) e^{it \sum_{n=1}^N (k_n-h_n)\varphi_n} dt \\ &= \sum_{\substack{k_1+\dots+k_N=q \\ h_1+\dots+h_N=q}} \frac{(q!)^2}{k_1!h_1! \dots k_N!h_N!} \prod_{n=1}^N c_n^{k_n} \bar{c}_n^{h_n} \int_{\mathbf{R}} K_T(s) e^{i(s+H) \sum_{n=1}^N (k_n-h_n)\varphi_n} ds \\ &= \sum_{\substack{k_1+\dots+k_N=q \\ h_1+\dots+h_N=q}} \frac{(q!)^2}{k_1!h_1! \dots k_N!h_N!} \prod_{n=1}^N (c_n e^{iH\varphi_n})^{k_n} (\bar{c}_n e^{-iH\varphi_n})^{h_n} \widehat{K}_T \left( \sum_{n=1}^N (k_n-h_n)\varphi_n \right) \\ &\leq \sum_{\substack{k_1+\dots+k_N=q \\ h_1+\dots+h_N=q}} \frac{(q!)^2}{k_1!h_1! \dots k_N!h_N!} \prod_{n=1}^N a_n^{k_n+h_n} \widehat{K}_T \left( \sum_{n=1}^N (k_n-h_n)\varphi_n \right) \\ &= \int_{\mathbf{R}} K_T(t) \left[ \sum_{\substack{k_1+\dots+k_N=q \\ h_1+\dots+h_N=q}} \frac{(q!)^2}{k_1!h_1! \dots k_N!h_N!} \prod_{n=1}^N a_n^{k_n+h_n} e^{it \sum_{n=1}^N (k_n-h_n)\varphi_n} \right] dt \\ &= \int_{\mathbf{R}} K_T(t) \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|^{2q} dt. \end{aligned}$$

Hence, if  $|c_n| \leq a_n$  for  $n = 1, \dots, N$

$$\int_{\mathbf{R}} K_T(t-H) \left| \sum_{n=1}^N c_n e^{it\varphi_n} \right|^{2q} dt \leq \int_{\mathbf{R}} K_T(t) \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|^{2q} dt. \quad (9)$$

By applying Lemma 2 with  $H = 0, T_0, -T_0$ , and using b), we get

$$\begin{aligned} & \int_{|t-T_0| \leq T} \left| \sum_{n=1}^N c_n e^{it\varphi_n} \right|^{2q} dt \\ &\leq \int_{\mathbf{R}} \left( K_T(t-T_0) + K_T(t-T_0+T) + K_T(t-T_0-T) \right) \left| \sum_{n=1}^N c_n e^{it\varphi_n} \right|^{2q} dt \\ &\leq 3 \int_{\mathbf{R}} K_T(t) \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|^{2q} dt \leq 3 \int_{|t| \leq T} \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|^{2q} dt. \quad (10) \end{aligned}$$

■

The proof of Theorem 1 is now achieved as follows. First recall the Khintchin-Kahane inequalities [4]. Let  $\{\varepsilon_i, 1 \leq i \leq N\}$  be independent Rademacher random variables, thus satisfying  $\mathbf{P}\{\varepsilon_i = \pm 1\} = 1/2$ , if  $(\Omega, \mathcal{A}, \mathbf{P})$  denotes the underlying basic probability. Then for any  $0 < p < \infty$ , there exist positive finite constants  $c_p, C_p$  depending on  $p$  only, such that for any sequence  $\{a_i, 1 \leq i \leq N\}$  of real numbers

$$c_p \left( \sum_{i=1}^N a_i^2 \right)^{1/2} \leq \left\| \sum_{i=1}^N a_i \varepsilon_i \right\|_p \leq C_p \left( \sum_{i=1}^N a_i^2 \right)^{1/2}. \quad (11)$$

This remains true for complex  $a_n$ . If  $a_n = \alpha_n + i\beta_n$ , then

$$\begin{aligned} \left\| \sum_{j=1}^N a_j \varepsilon_j \right\|_p^p &= \mathbf{E} \left| \sum_{j=1}^N \alpha_j \varepsilon_j + i \sum_{j=1}^N \beta_j \varepsilon_j \right|^p = \mathbf{E} \left( \left| \sum_{j=1}^N \alpha_j \varepsilon_j \right|^2 + \left| \sum_{j=1}^N \beta_j \varepsilon_j \right|^2 \right)^{p/2} \\ &\leq 2^{(p/2)-1} \left( \mathbf{E} \left| \sum_{j=1}^N \alpha_j \varepsilon_j \right|^p + \mathbf{E} \left| \sum_{j=1}^N \beta_j \varepsilon_j \right|^p \right), \end{aligned}$$

where we have denoted by  $\mathbf{E}$  the corresponding expectation symbol. Thus, since  $\sqrt{A} + \sqrt{B} \leq \sqrt{2(A+B)}$ ,  $A, B \geq 0$ ,

$$\begin{aligned} \left\| \sum_{j=1}^N a_j \varepsilon_j \right\|_p &\leq 2^{1/2-1/p} C_p \left[ \left( \sum_{j=1}^N \alpha_j^2 \right)^{1/2} + \left( \sum_{j=1}^N \beta_j^2 \right)^{1/2} \right] \\ &\leq 2^{1-1/p} C_p \left( \sum_{j=1}^N (\alpha_j^2 + \beta_j^2) \right)^{1/2} = C'_p \left( \sum_{j=1}^N |a_j|^2 \right)^{1/2}. \end{aligned}$$

Conversely, from

$$\begin{aligned} \left\| \sum_{j=1}^N a_j \varepsilon_j \right\|_p^p &= \mathbf{E} \left( \left| \sum_{j=1}^N \alpha_j \varepsilon_j \right|^2 + \left| \sum_{j=1}^N \beta_j \varepsilon_j \right|^2 \right)^{p/2} \\ &\geq \max \left( \mathbf{E} \left| \sum_{j=1}^N \alpha_j \varepsilon_j \right|^p, \mathbf{E} \left| \sum_{j=1}^N \beta_j \varepsilon_j \right|^p \right), \end{aligned}$$

we get

$$\begin{aligned} \left\| \sum_{j=1}^N a_j \varepsilon_j \right\|_p &\geq \max \left( \left\| \sum_{j=1}^N \alpha_j \varepsilon_j \right\|_p, \left\| \sum_{j=1}^N \beta_j \varepsilon_j \right\|_p \right) \\ &\geq c_p \max \left( \left( \sum_{j=1}^N |\alpha_j|^2 \right)^{1/2}, \left( \sum_{j=1}^N |\beta_j|^2 \right)^{1/2} \right) \\ &\geq \frac{c_p}{2} \left( \sum_{j=1}^N (|\alpha_j|^2 + |\beta_j|^2) \right)^{1/2} = c'_p \left( \sum_{j=1}^N |a_j|^2 \right)^{1/2}. \end{aligned}$$

Now choose  $c_n = \varepsilon_n a_n$ . Taking expectation in inequality of Lemma 2.1, and using Fubini's Theorem, gives

$$\int_{|t| \leq T} \mathbf{E} \left| \sum_{n=1}^N \varepsilon_n a_n e^{it\varphi_n} \right|^{2q} \leq 3 \int_{|t| \leq T} \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|^{2q} dt. \quad (12)$$

By (11) we have

$$c_q \left( \sum_{n=1}^N a_n^2 \right)^q \leq \mathbf{E} \left| \sum_{n=1}^N \varepsilon_n a_n e^{it\varphi_n} \right|^{2q} \leq C_q \left( \sum_{n=1}^N a_n^2 \right)^q. \quad (13)$$

By reporting

$$2T c_q \left( \sum_{n=1}^N a_n^2 \right)^q \leq 3 \int_{|t| \leq T} \left| \sum_{n=1}^N a_n e^{it\varphi_n} \right|^{2q} dt, \quad (14)$$

which proves our claim. ■

## 2 Application

We shall deduce from Theorem 1 the following lower bound.

**Corollary 3** *For every  $N, T$  and  $\nu$*

$$c_\nu \log^{\nu^2} N \leq \frac{1}{2T} \int_{|t| \leq T} \left| \sum_{n=1}^N \frac{1}{n^{\frac{1}{2}+it}} \right|^{2\nu} dt.$$

In relation with this is Ramachandra's well-known lower bound (see [3] section 9.5, to which we also refer for the estimates used in the proof)

$$c_\nu (\log T)^{\nu^2} \leq \frac{1}{2T} \int_{|t| \leq T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2\nu} dt. \quad (15)$$

*Proof.* Apply Theorem 1 with  $q = 2$  to the sum

$$\left( \sum_{n=1}^N \frac{1}{n^{\frac{1}{2}+it}} \right)^\nu := \sum_{m=1}^{N^\nu} \frac{b_m}{m^{\frac{1}{2}+it}},$$

where

$$b_m = \#\{(n_j)_{j \leq \nu}; n_j \leq N : m = \prod_{j \leq \nu} n_j\}.$$

Thus for all  $N$  and  $T$

$$c_\nu \sum_{m=1}^{N^\nu} \frac{b_m^2}{m} \leq \frac{1}{2T} \int_{|t| \leq T} \left| \sum_{n=1}^N \frac{1}{n^{\frac{1}{2}+it}} \right|^{2\nu} dt.$$

But if  $m \leq N$ ,  $b_m = d_\nu(m)$  where  $d_\nu(m)$  denotes the number of representations of  $m$  as a product of  $\nu$  factors, and we know that

$$\sum_{m \leq x} \frac{d_\nu^2(m)}{m} = (C_\nu + o(1)) \log^{\nu^2} x.$$

Thus

$$\sum_{m=1}^{N^\nu} \frac{b_m^2}{m} \geq \sum_{m=1}^N \frac{b_m^2}{m} \geq c_\nu \log^{\nu^2} N$$

Henceforth

$$c_\nu \log^{\nu^2} N \leq \frac{1}{2T} \int_{|t| \leq T} \left| \sum_{n=1}^N \frac{1}{n^{\frac{1}{2}+it}} \right|^{2\nu} dt. \quad \blacksquare$$

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