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**THE Cos^λ AND Sin^λ TRANSFORMS
AS INTERTWINING OPERATORS BETWEEN
GENERALIZED PRINCIPAL SERIES REPRESENTATIONS
OF $\text{SL}(n + 1, \mathbb{K})$**

GESTUR ÓLAFSSON AND ANGELA PASQUALE

ABSTRACT. In this article we connect topics from convex and integral geometry with well known topics in representation theory of semisimple Lie groups by showing that the Cos^λ and Sin^λ -transforms on the Grassmann manifolds $\text{Gr}_p(\mathbb{K}) = \text{SU}(n + 1, \mathbb{K})/\text{S}(\text{U}(p, \mathbb{K}) \times \text{U}(n + 1 - p, \mathbb{K}))$ are standard intertwining operators between certain generalized principal series representations induced from a maximal parabolic subgroup P_p of $\text{SL}(n + 1, \mathbb{K})$. The index $_p$ indicates the dependence of the parabolic on p . The general results of Knapp and Stein and Vogan and Wallach then show that both transforms have meromorphic extension to \mathbb{C} and are invertible for generic $\lambda \in \mathbb{C}$. Furthermore, known methods from representation theory combined with a Selberg type integral allow us to determine the K -spectrum of those operators.

INTRODUCTION

The Cos^λ and Sin^λ transforms have appeared in several mathematical contexts and have a long history. In the last few years they have been widely studied because of their connection to convex geometry and to some classical integral transforms, like the Funk and Radon transforms on the sphere and their generalizations to Grassmann manifolds.

For the sphere S^n , the Cos^λ transform \mathcal{C}^λ is defined on $L^2(S^n)$ by

$$\mathcal{C}^\lambda(f)(\omega) = \int_{S^n} |(x, \omega)|^{\lambda-\rho} f(x) d\sigma(x), \quad \omega \in S^n, \quad (0.1)$$

where $d\sigma$ is the normalized rotational invariant measure on S^n , (\cdot, \cdot) stands for the usual inner product on \mathbb{R}^n and $\rho = (n + 1)/2$. The ρ -shift is chosen so that \mathcal{C}^λ agrees with a standard intertwining operator between certain principal series representations of $\text{SL}(n + 1, \mathbb{R})$. The integral kernel of (0.1) is a power of the cosine of the angle between x and ω . This is at the origin of the name *cosine transform*, given to (0.1) by E. Lutwak in [27] for the case $\lambda - \rho = 1$.

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It is clear that \mathcal{C}^λ is an integral transform on $L^2(\mathcal{B})$, where $\mathcal{B} := \text{Gr}_1(\mathbb{R})$ is the Grassmann manifold of lines in \mathbb{R}^{n+1} . Denote by $\text{Gr}_n(\mathbb{R})$ the Grassmann manifold of n -dimensional subspaces of \mathbb{R}^{n+1} . The Sin^λ -transform \mathcal{S}^λ maps functions on \mathcal{B} to functions on $\text{Gr}_n(\mathbb{R})$ and is given by

$$\mathcal{S}^\lambda(f)(\omega) = \mathcal{C}^\lambda(f)(\omega^\perp).$$

Note that $\text{Gr}_n(\mathbb{R}) \simeq \mathcal{B}$ so that one can also view \mathcal{S}^λ as an integral transform on $L^2(\mathcal{B})$.

We note that $\mathcal{C}^\lambda(f)$ exists for all $f \in L^2(S^n)$ as long as $\lambda \in \mathbb{C}_\rho := \{z \in \mathbb{C} \mid \text{Re}(z) \geq \rho\}$ and $\lambda \mapsto \mathcal{C}^\lambda(f)$ is holomorphic on \mathbb{C}_ρ° , the interior of \mathbb{C}_ρ . The transform $\mathcal{C}^\lambda : C^\infty(S^n) \rightarrow C^\infty(S^n)$ extends meromorphically to \mathbb{C} . One way to determine the singularities of this extension is to determine the $\text{SO}(n+1)$ -spectrum of \mathcal{C}^λ . The group $K = \text{SO}(n+1)$ acts naturally on $L^2(S^n)$ and, as a representation of K , the space $L^2(S^n)$ decomposes into a direct sum

$$L^2(S^n) \simeq_K \bigoplus_{m \in \mathbb{N}_0} \mathcal{Y}^m$$

where

$$\mathcal{Y}^m = \{p|_{S^n} \mid p \text{ a homogeneous polynomial of degree } m \text{ and } \Delta p = 0\}.$$

Here $\Delta = \partial_1^2 + \cdots + \partial_{n+1}^2$ denotes the Laplace operator on \mathbb{R}^{n+1} . Likewise, by considering Cos^λ as a transform on $L^2(\mathcal{B})$, we have the decomposition

$$L^2(\mathcal{B}) \simeq_K \bigoplus_{m \in 2\mathbb{N}_0} \mathcal{Y}^m.$$

See for instance [43]. As these are multiplicity-one decompositions and \mathcal{C}^λ commutes with rotations, it follows from Schur's Lemma that for each $m \in 2\mathbb{N}_0$ there exists a holomorphic function $\eta_m : \mathbb{C}_\rho \rightarrow \mathbb{C}$ such that

$$\mathcal{C}^\lambda|_{\mathcal{Y}^m} = \eta_m(\lambda) \text{id}_{\mathcal{Y}^m}.$$

Information about η_m gives more detailed information about the meromorphic extension of \mathcal{C}^λ as well as the image and the kernel of the Cos^λ -transform, as clearly both of those spaces are K -invariant.

The functions $\eta_m(\lambda)$ can be for instance determined using a Funk-Hecke type formula [35, 36], and one gets for $m \in 2\mathbb{N}_0$

$$\eta_m(\lambda) = (-1)^{m/2} \frac{\Gamma(\rho)}{\Gamma(1/2)} \frac{\Gamma(\frac{1}{2}(\lambda - \rho + 1)) \Gamma(\frac{1}{2}(-\lambda + \rho + m))}{\Gamma(\frac{1}{2}(\lambda + \rho + m)) \Gamma(\frac{1}{2}(-\lambda + \rho))}.$$

Note that

$$\frac{\Gamma(\frac{1}{2}(-\lambda + \rho + m))}{\Gamma(\frac{1}{2}(-\lambda + \rho))} = \begin{cases} 1 & \text{if } m = 0 \\ 2^{-m/2}(-\lambda + \rho)(-\lambda + \rho + 2) \cdots (-\lambda + \rho + m - 2) & \text{if } m \in 2\mathbb{N} \end{cases}$$

is a polynomial of degree $m/2$.

The first pole of \mathcal{C}^λ is at $\lambda = (n - 1)/2$ and the Funk transform

$$\mathcal{F}(f)(\omega) = \int_{(x,\omega)=0} f(x) dm(x)$$

is then given, up to a constant that can be determined, by $\mathcal{F}(f)(\omega) = c \text{Res}_{\lambda=(n-1)/2} \mathcal{C}^\lambda(f)(\omega)$. A similar discussion relates the Radon transform with the singularities of the Sin^λ -transform.

All of these integral transforms have been generalized to the Grassmann manifolds over \mathbb{R} , \mathbb{C} and \mathbb{H} . We refer to [1, 2, 33, 37, 38, 39, 40, 45] for more details and discussions.

As mentioned earlier, the importance of the Cos^λ and Sin^λ transforms comes from the connection to convex geometry and classical integral transforms like the Funk and Radon transforms. The problem is to characterize its rank and derive an inversion formula. The injectivity for $\lambda - \rho = 1$ was first proved in [42]. The connection to stochastic geometry was established by G. Matheron in [29]. The description of the rank for the same case was first achieved by [2]. In the unpublished manuscript [1], S. Alesker connected the Cos^λ -transform with intertwining operators and showed that the image was an irreducible representation of $\text{GL}(n + 1, \mathbb{R})$ for suitable parameters λ . We would also like to point out [14] where the other Grassmann manifolds were considered, and the work by B. Rubin, A. Koldobsky and coworkers [22, 23, 24, 25, 37, 38, 39, 40] where the Cos^λ and Sin^λ transforms are considered in most generality and the connection to the Busemann-Petty problem plays an important role.

The integral transform in (0.1) appeared independently in representation theory of $\text{SL}(n + 1, \mathbb{K})$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , around the same time as the Cos^λ -transform. The real case was studied in [10], the complex case in [12], and finally the quaternionic case in [34]. In these articles it was shown that the integral (0.1) defines an intertwining operator between generalized principal series representations induced from a maximal parabolic subgroup in $\text{SL}(n + 1, \mathbb{K})$. The K -spectrum was determined. Among the applications was the composition series, some embeddings of the complementary series and the study of the so-called canonical representations on rank-one Riemannian symmetric spaces of the noncompact type, [8, 9, 11]. However the connections to convex geometry and to the Funk and Radon transforms was neither discussed nor mentioned.

As first shown in [45], the Sin^λ -transform can be viewed as a Knapp-Stein intertwining operator. This was used to construct complementary series representations for $\text{GL}(2n, \mathbb{R})$. This transform is then also naturally linked to reflection positivity, which relates complementary series representations of $\text{GL}(2n, \mathbb{R})$ to highest weight representations of $\text{SU}(n, n)$, [17, 18].

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} and let $\text{Gr}_p(\mathbb{K}) = \text{SU}(n + 1, \mathbb{K})/\text{S}(\text{U}(p, \mathbb{K}) \times \text{U}(q, \mathbb{K}))$, with $q = n + 1 - p$, be the Grassmann manifold of p -dimensional subspaces of \mathbb{K}^{n+1} . Our main aim in this article is to show that the Cos^λ and Sin^λ -transforms for $\text{Gr}_p(\mathbb{K})$ are always canonical intertwining operators between generalized principal series representations induced from maximal parabolic subgroups of $\text{SL}(n + 1, \mathbb{K})$. We then use the spectrum generating method developed in [5] to determine the K -spectrum of the Cos^λ and Sin^λ -transforms.

The article is organized as follows. In Section 1 we recall needed notations and facts on simple Lie groups and Lie algebras. In Section 2 we introduce the generalized spherical principal series representations. These are representations induced from a character χ_λ of a parabolic subgroup $P = MAN$. The elements of the representation space can be viewed as L^2 sections of the G -homogeneous line bundle over $\mathcal{B} = G/P$ corresponding to χ_λ . We recall the results by Knapp and Stein and Vogan and Wallach on the meromorphic continuation of standard intertwining operators $J(\lambda)$ between spherical generalized principal series representations, [20, 21, 44]. The material of this section is presented in the general setting of semisimple Lie groups rather than for the special case considered later in this article. Indeed, the analytic properties of the intertwining operators which we discuss here possess a rather unified structure for the different semisimple Lie groups. On the other hand, we will associate a geometric meaning to these operators only for a specific class of semisimple Lie groups. It is therefore a natural question whether one can provide a geometric meaning to the standard intertwining operators for other groups or, more generally, to intertwining operators between generalized principal series representations induced from other parabolic subgroups. We expect that the answer will be positive and that higher rank Cos^λ -transforms are in fact given by such intertwining operators.

In Section 3 we recall the method from [5] on how to find the eigenvalues of the intertwining operators on each of the K -types in case P is a maximal parabolic subgroup and the K -types have multiplicity one. As the situation considered in this article is less general than the one in [5], the results and their proofs are simpler. The principal result is Theorem 3.5, which provides us with an inductive way to compute the spectrum.

We specialize to the case of the Grassmann manifolds $\text{Gr}_p(\mathbb{K})$ in Section 4. We introduce the Cos^λ -transform \mathcal{C}^λ and show that $\mathcal{C}^\lambda = J(\lambda)$. This main result has several applications. Using the well-known fact that the generalized principal series are irreducible for almost all $\lambda \in \mathbb{C}$, we obtain that there exists a meromorphic function $c_P(\lambda) = J(\lambda)1 = \mathcal{C}^\lambda(1)$ such that $\mathcal{C}^{-\lambda} \circ \mathcal{C}^\lambda = \mathcal{C}^\lambda \circ \mathcal{C}^{-\lambda} = c_P(-\lambda)c_P(\lambda)\text{id}$ for almost all λ . Moreover, together with the method from [5] outlined in Section 3, the equality $\mathcal{C}^\lambda = J(\lambda)$ allows us to calculate the K -spectrum of \mathcal{C}^λ . This computation is done in Section 5. The main result is Theorem 5.11. Finally, a similar discussion about the Sin^λ -transform is carried out in Theorem 6.4.

1. SEMISIMPLE LIE GROUPS AND PARABOLIC SUBGROUPS

In this section we introduce the notation and basic facts on semisimple Lie groups, Lie algebras and parabolic subgroups. The material can be found in any standard textbook on semisimple Lie group, e.g. [19].

From now on G will stand for a noncompact connected semisimple Lie group with finite center. The Lie algebra of G is denoted by \mathfrak{g} . In general Lie groups will be denoted by upper case Latin letters and their Lie algebras by the corresponding German letter.

An involution $\theta : G \rightarrow G$ is a Cartan involution if $K = G^\theta = \{a \in G \mid \theta(a) = a\}$ is a maximal compact subgroup of G . In that case K is connected. From now on θ will denote a fixed Cartan involution on G and K the corresponding maximal compact subgroup. Our

standard example will be $G = \text{SL}(n+1, \mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or \mathbb{H} , and $\theta(x) = (\bar{x}^t)^{-1}$ where $\bar{\cdot}$ denotes the canonical conjugation in \mathbb{K} which fixes \mathbb{R} . In this case $K = \{x \in G \mid \bar{x}^t = x\}$. The general structure theory discussed in this section is specialized to this case in Section 4.

The derived involution $\dot{\theta} : \mathfrak{g} \rightarrow \mathfrak{g}$ and its complex linear extension to $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ will for simplicity also be denoted by θ . Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ where $\mathfrak{k} = \mathfrak{g}^\theta$ is the Lie algebra of K and $\mathfrak{s} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. We have $[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{s}] \subseteq \mathfrak{s}$, and $\text{Ad}(K)\mathfrak{s} = \mathfrak{s}$. We denote the representation of K on $\mathfrak{s}_{\mathbb{C}}$ by $\text{Ad}_{\mathfrak{s}}$.

Denote by $\langle \cdot, \cdot \rangle$ the Killing form on \mathfrak{g} , given by $\langle X, Y \rangle = \text{Tr}(\text{ad}(X)\text{ad}(Y))$. Then

$$(X, Y) \mapsto \langle X, Y \rangle_{\theta} := \langle X, \theta(Y) \rangle$$

is an inner product on \mathfrak{g} . If $X \in \mathfrak{g}$ then $\text{ad}(X)^* = -\text{ad}(\theta(X))$. In particular, if $X \in \mathfrak{s}$ then $\text{ad}(X)$ is a symmetric operator and hence diagonalizable over the reals. Let $\mathfrak{a} \subset \mathfrak{s}$ be abelian. For $\alpha \in \mathfrak{a}^*$ let

$$\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g} \mid (\forall H \in \mathfrak{a}) [H, X] = \alpha(H)X\}$$

and let

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\forall H \in \mathfrak{a}) [H, X] = 0 \quad \text{and} \quad X \perp \mathfrak{a}\}.$$

Then $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{g}_0$. We will always assume that the

$$\mathfrak{a} = \mathfrak{z}(\mathfrak{m}) \cap \mathfrak{s} = \{X \in \mathfrak{m} \oplus \mathfrak{a} \mid [X, \mathfrak{m} \oplus \mathfrak{a}] = \{0\}\}.$$

Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{a}) := \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}_{\alpha} \neq \{0\}\}$. The elements in Δ are called roots. We have

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}.$$

As $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$ for $\alpha, \beta \in \Delta \cup \{0\}$ it follows that $[\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{g}_{\alpha}] \subseteq \mathfrak{g}_{\alpha}$ for all $\alpha \in \Delta$.

Let $\mathfrak{a}_r := \{H \in \mathfrak{a} \mid (\forall \alpha \in \Delta) \alpha(H) \neq 0\}$. For a fixed $H \in \mathfrak{a}_r$ let $\Delta^+ := \{\alpha \in \Delta \mid \alpha(H) > 0\}$. Then Δ^+ is system of positive roots, i.e., $\Delta = \Delta^+ \cup -\Delta^+$, $\Delta^+ \cap -\Delta^+ = \emptyset$, and $(\Delta^+ + \Delta^+) \cap \Delta \subset \Delta^+$. In particular,

$$\mathfrak{n} := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\alpha}$$

is a nilpotent subalgebra of \mathfrak{g} normalized by the Lie algebra $\mathfrak{p} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. In fact

$$\mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{n}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{n}] \subseteq \mathfrak{n}\}.$$

\mathfrak{p} is a parabolic subalgebra of \mathfrak{g} . It is maximal, if $\dim \mathfrak{a} = 1$.

Let $P = N_G(\mathfrak{p}) = \{g \in G \mid \text{Ad}(g)\mathfrak{p} \subset \mathfrak{p}\}$. Then P is a closed subgroup of G with Lie algebra \mathfrak{p} . Let $A = \exp \mathfrak{a}$ respectively $N = \exp \mathfrak{n}$ denote the analytic subgroup of G with Lie algebra \mathfrak{a} respectively \mathfrak{n} . Then A and N are closed, A is abelian and N is nilpotent. Let M_o denote the analytic subgroup of G with Lie algebra \mathfrak{m} and $M := Z_K(A)M_o$. Then M is a closed subgroup of G with finitely many connected components, $Z_G(A) = MA$ and

the product map $M \times A \times N \rightarrow P$, $(m, a, n) \mapsto man$, is an analytic diffeomorphism. Let $L = K \cap M$ and $\mathcal{B} := K/L$. Then

$$G = KP \quad \text{and} \quad K \cap P = L. \quad (1.1)$$

Thus $\mathcal{B} = G/P$ and G acts on \mathcal{B} . To describe this action, write $g = k_P(g)m_P(g)a_P(g)n_P(g)$ with $(k_P(g), m_P(g), a_P(g), n_P(g)) \in K \times M \times A \times N$. If the role of P is clear, then we leave out the index P . The map $G \ni g \mapsto (a(g), n(g)) \in A \times N$ is analytic and $g \mapsto a(g)$ is right MN -invariant. We can therefore view $a(\cdot)$ as a map $G/MN \rightarrow A$. The elements $k(g)$ and $m(g)$ are not uniquely defined. However, the maps

$$g \mapsto k(g)L \in \mathcal{B} \quad \text{and} \quad g \mapsto Lm(g) \in L \backslash M$$

are well defined and analytic. Let $b_o = eL$. Then we can describe the G action on \mathcal{B} by $g \cdot (k \cdot b_o) = k(gk) \cdot b_o$. If $b = k \cdot b_o \in \mathcal{B}$ and $g \in G$, then we set $a(gb) := a(gk)$.

Let $\bar{N} := \theta(N)$. The Lie algebra of \bar{N} is $\bar{\mathfrak{n}} = \theta(\mathfrak{n}) = \bigoplus_{\alpha \in -\Delta^+} \mathfrak{g}_\alpha$. Furthermore, $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} = \bar{\mathfrak{n}} \oplus \mathfrak{p}$, and the map $\bar{N} \times M \times A \times N \rightarrow G$, $(\bar{n}, m, a, n) \mapsto \bar{n}man$, is an analytic diffeomorphism onto an open and dense subset $\bar{N}P$ of G of full measure.

Let $\bar{P} = M\bar{N} = \theta(P)$. Then as a K -manifold $\mathcal{B} = G/\bar{P}$ but the G -action in general is different. Denote this action for the moment by $\bar{\cdot}$.

Lemma 1.1. *Let $g \in G$ and $b \in \mathcal{B}$. Then $a_{\bar{P}}(gb) = a_P(\theta(g)b)^{-1}$, $k_{\bar{P}}(gb) = k_P(\theta(g)b)$ and $g\bar{\cdot}b = \theta(g) \cdot b$.*

Proof. Let $k \in K$ be such that $b = k \cdot b_o$. Then $g\bar{\cdot}b = k_{\bar{P}}(gk)L$. We have

$$\begin{aligned} gk &= \theta(\theta(g)k) \\ &= \theta(k_P(\theta(g)k)m_P(\theta(g)k)a_P(\theta(g)k)n_P(\theta(g)k)) \\ &= k_P(\theta(g)k)\theta(m_P(\theta(g)k))a_P(\theta(g)k)^{-1}\theta(n_P(\theta(g)k)). \end{aligned}$$

The Lemma now follows as M is θ -stable and $\theta(n_P(\theta(g)k)) \in \bar{N}$. \square

We note for later use that the the above argument shows that

$$a_{\bar{P}}(gk) = a_P(\theta(g)k)^{-1}. \quad (1.2)$$

We normalize the invariant measure on \mathcal{B} and compact groups so that the total measure is one. If $f \in C(\mathcal{B})$ then

$$\int_{\mathcal{B}} f(b) db = \int_K f(k \cdot b_o) dk.$$

For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ define a homomorphism $\chi_\lambda : P \rightarrow \mathbb{C}$ by

$$\chi_\lambda(m \exp Hn) = e^{\lambda(H)}, \quad m \in M, H \in \mathfrak{a}, \quad \text{and} \quad n \in N.$$

We will also use the shorthand notation p^λ for $\chi_\lambda(p)$. For $\alpha \in \Delta$ let $m_\alpha := \dim \mathfrak{g}_\alpha$ and let $\rho_P := \frac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha \in \mathfrak{a}^*$. As P is fixed most of the time we simply write ρ instead of ρ_P . We have $p^{2\rho} = |\det \text{Ad}(p)|_n$.

Normalize the Haar measure $d\bar{n}$ on \bar{N} such that

$$\int_{\bar{N}} a(\bar{n})^{-2\rho} d\bar{n} = 1$$

and let $dn = \theta(d\bar{n})$. We normalize the invariant measure on A respectively \mathfrak{a} such that the inversion formula for the abelian Fourier transform holds without additional constants, i.e., if $f \in C_c^\infty(A)$ and $\widehat{f}(\lambda) = \int_A f(a)a^{-\lambda} da$, then $f(a) = \int_{\mathfrak{a}^*} \widehat{f}(i\lambda)a^{i\lambda} d\lambda$.

Lemma 1.2. *Let the notation be as above.*

(1) *We can normalize the Haar measure on G and M such that for all $f \in L^1(G)$*

$$\begin{aligned} \int_G f(x) dx &= \int_K \int_M \int_A \int_N f(kman)a^{2\rho} dk dmdadn \\ &= \int_{\bar{N}} \int_M \int_A \int_N f(\bar{n}man)a^{2\rho} d\bar{n} dmdadn. \end{aligned}$$

(2) *The set $\bar{N} \cdot b_o \subset \mathcal{B}$ is open, dense and of full measure. If $f \in L^1(\mathcal{B})$ then*

$$\int_{\mathcal{B}} f(b) db = \int_{\bar{N}} f(\bar{n} \cdot b_o) a(\bar{n})^{-2\rho} d\bar{n}. \quad (1.3)$$

(3) *Let $f \in L^1(\mathcal{B})$ and $x \in G$. Then*

$$\int_{\mathcal{B}} f(x \cdot b) a(xb)^{-2\rho} db = \int_{\mathcal{B}} f(b) db. \quad (1.4)$$

Proof. See [19], pp. 137–140 and p. 170, or [30], §7. □

2. GENERALIZED PRINCIPAL SERIES REPRESENTATIONS

We now introduce the spherical generalized principal series representations and their intertwining operators. We refer to [20, 21, 44] for proofs. Later we will show that these intertwining operators coincide with the Cos^λ-transform.

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ define a continuous representation of G on $L^2(\mathcal{B})$ by

$$[\pi_\lambda(x)f](b) := a(x^{-1}b)^{-\lambda-\rho} f(x^{-1} \cdot b). \quad (2.1)$$

Then $\pi_\lambda = \pi_{P,\lambda} = \text{Ind}_P^G \chi_\lambda$. It follows easily by (1.4) that π_λ is unitary if and only if $\lambda \in i\mathfrak{a}^*$.

Theorem 2.1 (Vogan-Wallach). *There exists a countable collection $\{p_n\}_n$ of non-zero holomorphic polynomials on $\mathfrak{a}_{\mathbb{C}}^*$ such that if $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $p_n(\lambda) \neq 0$, then π_λ is irreducible. In particular, π_λ is irreducible for almost all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.*

Proof. This is Lemma 5.3 in [44]. □

Replace P by \bar{P} , realize \mathcal{B} as G/\bar{P} and then induce the character determined by λ from \bar{P} to G . We denote the corresponding representation by $\pi_{\bar{P},\lambda}$.

Lemma 2.2. *Let $\pi_\lambda^\theta := \pi_{P,\lambda} \circ \theta$. Then $\pi_{\bar{P},\lambda} = \pi_{-\lambda}^\theta$ as a representation of G acting on $L^2(\mathcal{B})$.*

Proof. This follows easily from Lemma 1.1, equation (1.2) and $\rho_{\bar{P}} = -\rho_P$:

$$\begin{aligned} [\pi_{\bar{P},\lambda}(x)f](b) &= a_{\bar{P}}(x^{-1}b)^{-\lambda-\rho_{\bar{P}}}f(x^{-1}\cdot b) \\ &= a_P(\theta(x)^{-1}b)^{\lambda-\rho_P}f(\theta(x)^{-1}\cdot b) \\ &= [\pi_{P,-\lambda}(\theta(x))f](b). \end{aligned} \quad \square$$

For $f \in C^\infty(\mathcal{B})$ and $b = k \cdot b_o$ define the standard intertwining operator $J(\lambda) = J_{\bar{P}|P}(\lambda)$ by

$$J(\lambda)f(b) := \int_{\bar{N}} a(\bar{n})^{-\lambda-\rho}f(k\bar{n} \cdot b_o) d\bar{n} \quad (2.2)$$

whenever the integral exists. We note that this is well defined. Indeed, if $m \in L$ then $a(m\bar{n}m^{-1}) = a(\bar{n})$, and the Haar measure on \bar{N} is invariant under conjugation by m . Let $b = k \cdot b_o = k_1 \cdot b_o$. Then $k = k_1m$ for some $m \in L$. Thus

$$\begin{aligned} \int_{\bar{N}} a(\bar{n})^{-\lambda-\rho}f(k\bar{n} \cdot b_o) d\bar{n} &= \int_{\bar{N}} a(\bar{n})^{-\lambda-\rho}f(k_1m\bar{n} \cdot b_o) d\bar{n} \\ &= \int_{\bar{N}} a(m\bar{n}m^{-1})^{-\lambda-\rho}f(k_1(m\bar{n}m^{-1}) \cdot b_o) d\bar{n} \\ &= \int_{\bar{N}} a(\bar{n})^{-\lambda-\rho}f(k_1\bar{n} \cdot b_o) d\bar{n}. \end{aligned}$$

Recall that the space of smooth vectors for π_λ is $C^\infty(\mathcal{B})$. In particular, it is independent of λ . If $X \in \mathfrak{g}$ then we define as usually $\pi_\lambda^\infty(X) : C^\infty(\mathcal{B}) \rightarrow C^\infty(\mathcal{B})$ by

$$\lim_{t \rightarrow 0} \frac{\pi_\lambda(\exp(tX))f - f}{t} = \left. \frac{d}{dt} \right|_{t=0} \pi_\lambda(\exp(tX))f$$

where the derivative is taken in $L^2(\mathcal{B})$. Then π_λ^∞ is a representation of \mathfrak{g} in $C^\infty(\mathcal{B})$ and extends to a representation of the universal enveloping algebra $U(\mathfrak{g})$. If $u \in U(\mathfrak{g})$ then $\pi_\lambda^\infty(u) : C^\infty(\mathcal{B}) \rightarrow C^\infty(\mathcal{B})$ is a differential operator. If $\lambda \in \mathfrak{a}_\mathbb{C}^*$ write $\lambda = \lambda_R + i\lambda_I$ with $\lambda_R, \lambda_I \in \mathfrak{a}^*$.

Theorem 2.3 (Vogan-Wallach). *The following holds:*

(1) *There exists a constant a_P such that if*

$$\lambda \in \mathfrak{a}_\mathbb{C}^*(a_P) := \{\mu \in \mathfrak{a}_\mathbb{C}^* \mid (\forall \alpha \in \Delta^+) \langle \mu_R, \alpha \rangle \geq a_P\}$$

then the integral (2.2) converges absolutely. Furthermore, there exists a constant $C > 0$ such that for all $\lambda \in \mathfrak{a}_\mathbb{C}^(a_P)$ and $f \in C^\infty(\mathcal{B})$*

$$\|J(\lambda)f\|_\infty \leq C\|f\|_\infty. \quad (2.3)$$

(2) *If $f \in C^\infty(\mathcal{B})$ then*

$$\mathfrak{a}_\mathbb{C}^*(a_P) \ni \lambda \mapsto J(\lambda)f \in C^\infty(\mathcal{B})$$

is continuous and holomorphic on the interior $\{\mu \in \mathfrak{a}_\mathbb{C}^ \mid (\forall \alpha \in \Delta^+) \langle \mu_R, \alpha \rangle > a_P\}$.*

- (3) There exists polynomial functions $b : \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$ and $D : \mathfrak{a}_{\mathbb{C}}^* \rightarrow U(\mathfrak{g})^K$ such that if $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(a_P)$ and $f \in C^\infty(\mathcal{B})$, then

$$b(\lambda)J(\lambda)f = J(\lambda + 4\rho)\pi_{\lambda+4\rho}(D(\lambda))f.$$

In particular $\lambda \mapsto J(\lambda)f$ extends to a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$.

- (4) The operator $J(\lambda)$ intertwines π_λ and $\pi_{-\lambda}^\theta$. Thus, if $x \in G$ and $f \in C^\infty(\mathcal{B})$, then

$$J(\lambda)(\pi_\lambda(x)f) = \pi_{-\lambda}^\theta(x)J(\lambda)f. \quad (2.4)$$

Proof. Referring to [44], we have that (1) is Lemma 1.2, (2) is Lemma 1.3, (3) is Theorem 1.5, and (4) is Lemma 1.2 for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(a_P)$. It extends to $\mathfrak{a}_{\mathbb{C}}^*$ by meromorphic continuation. \square

Our situation is simpler than the general case considered in [44], therefore—and for completeness—we describe the main idea behind the proof. Let

$$a_P := \max_{\alpha \in \Delta^+} \langle \alpha, \rho \rangle > 0. \quad (2.5)$$

According to Lemma A.3.3 in [44] we have $\langle \log a(\bar{n}), \alpha \rangle \geq 0$ for all $\alpha \in \Delta^+$. Thus for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(a_P)$

$$|a(\bar{n})^{-\lambda-\rho}| \leq a(\bar{n})^{-2\rho}$$

and $\bar{n} \mapsto a(\bar{n})^{-2\rho}$ is integrable. Thus, the integral defining $J(\lambda)$ exists and $\lambda \mapsto J(\lambda)f$ is holomorphic and also (2.3) clearly follows. Note that (2.3) says in particular that $J(\lambda)f \in C^\infty(\mathcal{B})$.

Part (2) follows by the above estimates (2.3) that show that the integral defining $J(\lambda)f$ converges uniformly on any domain of the form $\mathfrak{a}_{\mathbb{C}}^*(a)$ with $a > a_P$.

The proof of (3) uses tensoring with finite dimensional representations of G . Another way of showing that $\lambda \mapsto J(\lambda)f$ extends to a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ is to use the last part of (2.2) and ideas from [6, 30, 31] using the Bernstein polynomial for $\bar{n} \mapsto a(\bar{n})^{-\lambda-\rho}$ and equation (2.2).

We will show that $J(\lambda)$ is an intertwining operator after having proved Lemma 2.5 below.

Lemma 2.4. *Let a_P be as in (2.5) and let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(a_P)$. Let $1 \leq p \leq \infty$ and let $f \in L^p(\mathcal{B})$. Then the integral (2.2) exists and $|J(\lambda)f(k)| \leq \|f\|_p$. In particular, $J(\lambda)f \in L^p(\mathcal{B})$ and $\|J(\lambda)f\|_p \leq \|f\|_p$.*

Proof. The case $p = \infty$ is (2.3). We can therefore assume that $p < \infty$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $2\rho = \frac{2}{p}\rho + \frac{2}{q}\rho$. As $|a(\bar{n})^{-\lambda-\rho}| \leq a(\bar{n})^{-2\rho}$ and $\int_{\bar{N}} a(\bar{n})^{-2\rho} d\bar{n} = 1$, it follows

that

$$\begin{aligned}
\left| \int_{\overline{N}} a(\bar{n})^{-\lambda-\rho} f(kk(\bar{n}) \cdot b_o) d\bar{n} \right| &\leq \int_{\overline{N}} a(\bar{n})^{-2\rho} |f(kk(\bar{n}) \cdot b_o)| d\bar{n} \\
&\leq \left(\int_{\overline{N}} \left| a(\bar{n})^{-\frac{2}{p}\rho} f(kk(\bar{n}) \cdot b_o) \right|^p d\bar{n} \right)^{1/p} \\
&= \left(\int_{\overline{N}} |f(kk(\bar{n}) \cdot b_o)|^p a(\bar{n})^{-2\rho} d\bar{n} \right)^{1/p} \\
&= \left(\int_{\mathcal{B}} |f(kb)|^p db \right)^{1/p} \\
&= \|f\|_p.
\end{aligned}$$

Here we used (1.3) to transfer the integral over \overline{N} to an integral over \mathcal{B} . \square

For $x \in \overline{N}P$ write

$$x = \bar{n}_P(x)m_P(x)\alpha_P(x)n_P(x)$$

with $\bar{n}_P(x) \in \overline{N}$, $m_P(x) \in M$, $\alpha_P(x) \in A$, and $n_P(x) \in N$. Note that $x \mapsto \alpha_P(x)$ is right MN -invariant. We leave out the subscript P if it is clear which parabolic subgroup we are using.

Lemma 2.5. *Assume that $\lambda \in \mathfrak{a}_\mathbb{C}^*(a_P)$. Let $f \in C^\infty(\mathcal{B})$. Then for $k \in K$*

$$J(\lambda)f(k) = \int_K \alpha_P(k^{-1}h)^{\lambda-\rho} f(h) dh = \int_{\mathcal{B}} \alpha_P(k^{-1}b)^{\lambda-\rho} f(b) db.$$

In particular, $J(\lambda)$ is a convolution operator on $L^2(\mathcal{B})$ if $\lambda \in \mathfrak{a}_\mathbb{C}^(a_P)$.*

Proof. Let $\bar{n} \in \overline{N}$. Then

$$\begin{aligned}
\bar{n} &= k(\bar{n})m(\bar{n})a(\bar{n})n(\bar{n}) \\
&= \bar{n}(k(\bar{n}))m_{\overline{N}}(k(\bar{n}))\alpha(k(\bar{n}))n_{\overline{N}}(k(\bar{n}))m(\bar{n})a(\bar{n})n(\bar{n}) \\
&= \bar{n}(k(\bar{n}))m'\alpha(k(\bar{n}))a(\bar{n})n'
\end{aligned}$$

for some $m' \in M$ and $n' \in N$. Thus $\alpha(k(\bar{n})) = a(\bar{n})^{-1}$. Assume that $\lambda \in \mathfrak{a}_\mathbb{C}^*(a_P)$. Using (1.3) we get

$$\begin{aligned}
\int_{\overline{N}} a(\bar{n})^{-\lambda-\rho} f(kk(\bar{n}) \cdot b_o) d\bar{n} &= \int_{\overline{N}} \alpha(k(\bar{n}))^{\lambda-\rho} f(kk(\bar{n}) \cdot b_o) a(\bar{n})^{-2\rho} d\bar{n} \\
&= \int_{\mathcal{B}} \alpha(b)^{\lambda-\rho} f(kb) db \\
&= \int_{\mathcal{B}} \alpha(k^{-1}b)^{\lambda-\rho} f(b) db
\end{aligned}$$

as the measure on \mathcal{B} is K -invariant. \square

We now show that $J(\lambda)\pi_\lambda(x) = \pi_{-\lambda}(\theta(x))J(\lambda)$. For this, let $x \in G$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*(a_P)$. Then

$$J(\lambda)(\pi_\lambda(x)f)(k) = \int_K \alpha(k^{-1}h)^{\lambda-\rho} a(x^{-1}h)^{-\lambda-\rho} f(k(x^{-1}h)) dh. \quad (2.6)$$

We have

$$\alpha(k^{-1}h) = \alpha(k^{-1}xx^{-1}h) = \alpha(k^{-1}xk(x^{-1}h))a(x^{-1}h).$$

Hence we can rewrite (2.6) as

$$\begin{aligned} J(\lambda)(\pi_\lambda(x)f)(k) &= \int_K \alpha(k^{-1}xk(x^{-1}h))^{\lambda-\rho} f(k(x^{-1}h))a(x^{-1}h)^{-2\rho} dh \\ &= \int_K \alpha(k^{-1}xh)^{\lambda-\rho} f(h) dh \\ &= \int_K \alpha((x^{-1}k)^{-1}h)^{\lambda-\rho} f(h) dh. \end{aligned} \quad (2.7)$$

Now use Lemma 1.1 to write

$$x^{-1}k = k_{\bar{P}}(x^{-1}k)ma_{\bar{P}}(x^{-1}k)\bar{n} = k_P(\theta(x^{-1})k)ma_P(\theta(x^{-1})k)^{-1}\bar{n}$$

for some $m \in M$ and $n \in N$ and insert this into (2.7) to get

$$J(\lambda)(\pi_\lambda(x)f)(k) = a(\theta(x)^{-1}k)^{\lambda-\rho} \int_K \alpha(k(\theta(x)^{-1}k)^{-1}h)^{\lambda-\rho} f(h) dh = \pi_{-\lambda}(\theta(x))J(\lambda)f(k).$$

As $J(\lambda)$ is K -intertwining and the only K -invariant functions in $L^2(\mathcal{B})$ are multiple of the constant function $1(x) = 1$, it follows that there exists a meromorphic function $c_P : \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$ such that $J(\lambda)1 = c_P(\lambda)$. For $\lambda \in \mathfrak{a}_\mathbb{C}^*(a_P)$ the function $c_P(\lambda)$ is given by the converging integral

$$c_P(\lambda) := J(\lambda)1 = \int_{\bar{N}} a(\bar{n})^{-\lambda-\rho} d\bar{n} = \int_K \alpha(k)^{\lambda-\rho} dk. \quad (2.8)$$

Theorem 2.6. *Let the notation be as above. Then, as an identity of meromorphic functions,*

$$J(-\lambda)J(\lambda) = J(\lambda)J(-\lambda) = c_P(\lambda)c_P(-\lambda)\text{id}.$$

Furthermore, if $\lambda \in i\mathfrak{a}^$, then the normalized operator $c_P(\lambda)^{-1}J(\lambda)$ is unitary.*

Proof. We have

$$J(-\lambda)J(\lambda)\pi_\lambda(x) = J(-\lambda)\pi_{-\lambda}(\theta(x))J(\lambda) = \pi_\lambda(x)J(-\lambda)J(\lambda)$$

as $\theta^2 = \text{id}$. As π_λ is irreducible for λ in an open dense subset of $\mathfrak{a}_\mathbb{C}^*$ it follows that the $J(-\lambda)J(\lambda)$ is a scalar multiple of the identity on that set. As $J(-\lambda)J(\lambda)1 = c_P(-\lambda)c_P(\lambda)$ it follows that $J(-\lambda)J(\lambda) = c(-\lambda)c(\lambda)\text{id}$ as a meromorphic function. The same argument shows that $J(\lambda)J(-\lambda) = c(-\lambda)c(\lambda)\text{id}$ and (1) follows. The unitarity of $c_P(\lambda)^{-1}J(\lambda)$ for λ purely imaginary follows from $[c_P(\lambda)^{-1}J(\lambda)]^* = c_P(-\lambda)^{-1}J(-\lambda)$ as in this case $\bar{\lambda} = -\lambda$. \square

3. THE SPECTRUM GENERATING OPERATOR

We will now specialize the situation to the case where G is simple and P is a maximal parabolic subgroup. Thus $\dim \mathfrak{a} = 1$. Based on our application later on, we will also assume that $\Delta = \{\alpha, -\alpha\}$. Hence \mathfrak{n} is abelian. Finally, we will assume that K is semisimple and that (K, L) is a symmetric pair corresponding to an involution $\tau : K \rightarrow K$. This implies that each irreducible representation of K that occurs in $L^2(\mathcal{B})$ has multiplicity one.

Denote by \widehat{K} the set of equivalence classes of irreducible representations of K . For simplicity we will often write $\delta \in \widehat{K}$ where δ is an irreducible representation of K , i.e., we identify the representation with its equivalence class. Denote by V_δ the finite dimensional Hilbert space on which δ acts and let $d(\delta) := \dim V_\delta$. Let

$$V^L := \{u \in V_\delta \mid (\forall k \in L) \delta(k)u = u\}.$$

Then, as K/L is a symmetric space, either $\dim V_\delta^L = 0$ or $\dim V_\delta^L = 1$. We say that (δ, V_δ) is *spherical* if $V_\delta^L \neq \{0\}$. Let \widehat{K}_L denote the subset of \widehat{K} of spherical representations. To describe \widehat{K}_L write $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{q}$ where $\mathfrak{q} = \{X \in \mathfrak{k} \mid \tau(X) = -X\}$. Let \mathfrak{b} be a maximal abelian subspace of \mathfrak{q} . Denote by $\Delta_\mathfrak{k}$ the set of roots of $\mathfrak{b}_\mathbb{C}$ in $\mathfrak{k}_\mathbb{C}$. Note that $\Delta_\mathfrak{k} \subset i\mathfrak{b}^*$. Let $\Delta_\mathfrak{k}^+$ be a set of positive roots and let $\rho_\mathfrak{k}$ be the corresponding half-sum of roots, counted with multiplicities. Denote by $W_\mathfrak{k}$ the Weyl group of $\Delta_\mathfrak{k}$. Finally we let

$$\Lambda^+ := \{\mu \in i\mathfrak{b}^* \mid (\forall \alpha \in \Delta_\mathfrak{k}^+) \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N}_0\}.$$

Then the map $\pi \mapsto$ (highest weight of π) defines an injective map of \widehat{K}_L into Λ^+ . This map is bijective if and only if \mathcal{B} is simply connected. In general \widehat{K}_L is isomorphic to a sublattice $\Lambda^+(\mathcal{B})$ of Λ^+ . We refer to [15], p. 535, and the discussion in [32], in particular the appendix, for more details. Let $\mu \in \Lambda^+(\mathcal{B})$ and denote by π_μ the corresponding spherical representation. Let V_μ be the space on which π_μ acts and let $(\cdot, \cdot)_{V_\mu}$ the inner product on V_μ making π_μ unitary. We also fix a unit vector $e_\mu \in V_\mu^L$. Define $T_\mu : V_\mu \rightarrow L^2(\mathcal{B})$ by

$$T_\mu(v)(x) := d(\mu)^{1/2}(v, \pi_\mu(x)e_\mu)_{V_\mu}.$$

Then T_μ is an isometric K -intertwining operator between π_μ and the left regular representation L of K on $L^2(\mathcal{B})$. Let $L_\mu^2(\mathcal{B}) := \text{Im}(T_\mu)$. Then L is an irreducible K -representation on $L^2(\mathcal{B})$ and

$$L^2(\mathcal{B}) \simeq_K \bigoplus_{\mu \in \Lambda^+(\mathcal{B})} L_\mu^2(\mathcal{B}).$$

Lemma 3.1. *For $\mu \in \Lambda^+(\mathcal{B})$ let $J_\mu(\lambda) := J(\lambda)|_{L_\mu^2(\mathcal{B})}$. There exists a meromorphic function $\eta_\mu : \mathfrak{a}_\mathbb{C}^* \rightarrow \mathbb{C}$ such that*

- (1) $J_\mu(\lambda) = \eta_\mu(\lambda) \text{id}_{L_\mu^2(\mathcal{B})}$.
- (2) $\eta_0(\lambda) = c_P(\lambda)$,
- (3) $\eta_\mu(\lambda)\eta_\mu(-\lambda) = c_P(\lambda)c_P(-\lambda)$,

Proof. $J(\lambda)$ is a G -intertwining operator between π_λ and $\pi_{-\lambda}^\theta$ by Theorem 2.3 (4). As K -representations, π_λ and $\pi_{-\lambda}^\theta$ agree with L . Hence $J(\lambda)$ is a K -intertwining operator of L with itself. As each $L_\mu^2(\mathcal{B})$ is irreducible and the multiplicity of π_μ in $L^2(\mathcal{B})$ is one, it follows by Schur's Lemma that we can define $\eta_\mu(\lambda)$ by (1). The claim now follows from our normalization $J(\lambda)1 = c_P(\lambda)$ and Theorem 2.6. \square

We call $\{\eta_\mu(\lambda) \mid \mu \in \Lambda^+(\mathcal{B})\}$ the K -spectrum of $J(\lambda)$.

Our aim is to determine the functions η_μ using the results from [5]. Fix $H_o \in \mathfrak{a}$ such that $\alpha(H_o) = 1$. Fix a G -invariant \mathbb{R} -bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} such that $\langle H_o, H_o \rangle = 1$. For $H \in \mathfrak{a}$ we have $\alpha(H) = \langle H, H_o \rangle$. Denote the complex bilinear extension to $\mathfrak{g}_\mathbb{C}$ by the same symbol. Define a map $\omega : \mathfrak{s}_\mathbb{C} \rightarrow C^\infty(\mathcal{B})$ by

$$\omega(Y)(k) := \langle Y, \text{Ad}(k)H_o \rangle$$

and note that

$$\omega(\text{Ad}(h)Y)(k) = \langle \text{Ad}(h)Y, \text{Ad}(k)H_o \rangle = \langle Y, \text{Ad}(h^{-1}k)H_o \rangle = \omega(Y)(h^{-1}k).$$

Thus ω is a K -intertwining operator.

Note that $\langle \cdot, \cdot \rangle$ is negative definite on \mathfrak{k} . Let $X_1, \dots, X_{\dim \mathfrak{q}}$ be an orthonormal basis of \mathfrak{q} such that X_1, \dots, X_p , $p = \dim \mathfrak{b}$, is an orthonormal basis of \mathfrak{b} . Denote by $\Omega = -\sum_j X_j^2$ the corresponding positive definite Laplace operator on \mathcal{B} . The action of Ω on functions on \mathcal{B} is—up to a sign—the same as that of the Casimir element of \mathfrak{k} . Thus $\Omega|_{L_\mu^2(\mathcal{B})} = \langle \mu + 2\rho_\mathfrak{k}, \mu \rangle \text{id}$. For later reference we state this as a Lemma:

Lemma 3.2. *For $\mu \in \Lambda^+(\mathcal{B})$ let*

$$\omega(\mu) = -\sum_{j=1}^p \mu(X_j)(\mu + 2\rho_\mathfrak{k})(X_j) = -\sum_{j=1}^p \left(\mu(X_j)^2 + 2\mu(X_j)\rho_\mathfrak{k}(X_j) \right).$$

Then $\Omega|_{L_\mu^2(\mathcal{B})} = \omega(\mu)\text{id}$.

For $f \in C^\infty(\mathcal{B})$ denote by $M(f) : L^2(\mathcal{B}) \rightarrow L^2(\mathcal{B})$ the multiplication operator $g \mapsto fg$.

Theorem 3.3. *Let $Y \in \mathfrak{s}$. Then $[\Omega, M(\omega(Y))] = 2\pi_0(Y)$.*

Proof. This is Theorem 2.3 in [5]. \square

For $\mu \in \Lambda^+(\mathcal{B})$ define $\Phi_\mu : L_\mu^2(\mathcal{B}) \otimes \mathfrak{s}_\mathbb{C} \rightarrow L^2(\mathcal{B})$ by

$$\Phi_\mu(\varphi \otimes Y) := M(\omega(Y))\varphi.$$

Observe that for $k \in K$, $Y \in \mathfrak{s}_\mathbb{C}$ and $\varphi \in L_\mu^2(\mathcal{B})$ we have

$$L_k(M(\omega(Y))\varphi) = (L_k\omega(Y))(L_k\varphi) = M(\omega(\text{Ad}(k)Y))(L_k\varphi)$$

with $\text{Ad}(k)Y \in \mathfrak{s}_\mathbb{C}$ and $L_k\varphi \in L_\mu^2(\mathcal{B})$. Hence Φ_μ is K -equivariant and $\text{Im } \Phi_\mu$ is K -invariant. Define a finite subset $S(\mu) \subset \Lambda^+(\mathcal{B})$ by

$$\text{Im } \Phi_\mu \simeq_K \bigoplus_{\sigma \in S(\mu)} L_\sigma^2(\mathcal{B}).$$

Denote by pr_σ the orthogonal projection $L^2(\mathcal{B}) \rightarrow L_\sigma^2(\mathcal{B})$.

Lemma 3.4. *Assume that $\mu \in \Lambda^+(\mathcal{B})$. Let $\sigma \in S(\mu)$, $Y \in \mathfrak{s}_\mathbb{C}$ and $r\alpha \in \mathfrak{a}_\mathbb{C}^*$. Let*

$$\omega_{\sigma\mu}(Y) := \text{pr}_\sigma \circ M(\omega(Y))|_{L_\mu^2(\mathcal{B})} : L_\mu^2(\mathcal{B}) \rightarrow L_\sigma^2(\mathcal{B}). \quad (3.1)$$

Then

$$\text{pr}_\sigma \circ \pi_{r\alpha}(Y)|_{L_\mu^2(\mathcal{B})} = \frac{1}{2}(\omega(\sigma) - \omega(\mu) + 2r)\omega_{\sigma\mu}(Y). \quad (3.2)$$

Proof. This is Corollary 2.6 in [5]. □

Theorem 3.5. *Let $\mu \in \Lambda^+(\mathcal{B})$, $\sigma \in S(\mu)$ and $\lambda = r\alpha \in \mathfrak{a}_\mathbb{C}^*$. Then*

$$\frac{\eta_\sigma(\lambda)}{\eta_\mu(\lambda)} = \frac{2r - \omega(\sigma) + \omega(\mu)}{2r + \omega(\sigma) - \omega(\mu)}. \quad (3.3)$$

The K -spectrum $\{\eta_\mu(\lambda)\}_{\mu \in \Lambda^+(\mathcal{B})}$ and hence $J(\lambda)$ is uniquely determined by (3.3) and the normalization $\eta_0(\lambda) = c_P(\lambda)$.

Proof. Applying $J(\lambda)$ to (3.2) from the left, using that $J(\lambda)$ commutes with pr_σ and that $J(\lambda) \circ \pi_\lambda(Y) = \pi_{-\lambda}^\theta(Y) \circ J(\lambda) = -\pi_{-\lambda}(Y) \circ J(\lambda)$, we obtain:

$$(\omega(\sigma) - \omega(\mu) + 2r)\eta_\sigma(\lambda)\omega_{\sigma\mu}(Y) = -(\omega(\sigma) - \omega(\mu) - 2r)\eta_\mu(\lambda)\omega_{\sigma\mu}(Y).$$

As $\omega_{\sigma\delta}(Y)$ is non-zero for generic λ it can be canceled out. The last statement follows from the fact that π_λ is irreducible for generic λ , hence iterated application of (3.1) will in the end reach all K -types starting from the trivial K -type. □

4. THE CASE OF THE GRASSMANN MANIFOLDS

From now on \mathbb{K} denotes the one of the fields \mathbb{R} or \mathbb{C} , or the skew field \mathbb{H} of quaternions. We let $G = \text{SL}(n+1, \mathbb{K})$. For discussion about the determinant function on the space of quaternionic matrices see [4, 13]. We will use that if $x \in \text{M}(n+1, \mathbb{K})$ then $|\det_{\mathbb{R}} x| = |\det_{\mathbb{K}} x|^d$, where $d = \dim_{\mathbb{R}} \mathbb{K}$ is equal to 1, 2, 4 for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. Denote by $G_p(\mathbb{K})$ the Grassmann manifold of k -dimensional subspaces of \mathbb{K}^{n+1} . Note that $G_p(\mathbb{K}) \simeq G_{n+1-p}(\mathbb{K})$. So we can assume without loss of generality that $2p \leq n+1$. We set $q = n+1-p$. Hence $q \geq p$.

In this section we show that $G_p(\mathbb{K}) = G/P = K/L$, where P is a maximal parabolic subgroup of G . Furthermore K/L is a symmetric space. Therefore the results from the previous sections apply to this case. We introduce the Cos^λ -transform for K/L and then show that it agrees with the intertwining operator $J(\lambda)$. In the next section we then apply the spectrum generating operator to determine the K -spectrum.

Define an invariant \mathbb{R} -bilinear form on \mathfrak{g} by

$$\langle X, Y \rangle := \frac{n+1}{pq} \text{Re}(\text{Tr}(XY)).$$

Denote by $z \mapsto \bar{z}$ the conjugation in \mathbb{K} and for $x = (x_{\nu\mu}) \in M(n+1, \mathbb{K})$ let $\bar{x} := (\bar{x}_{\nu\mu})$ and $x^* := (\bar{x}_{\mu\nu}) = \bar{x}^t$. The homomorphism $\theta : G \rightarrow G$, $x \mapsto (x^{-1})^*$ is a Cartan involution on G . The corresponding Cartan involution on \mathfrak{g} is $\theta(X) = -X^*$. We have

$$\begin{aligned} K &= \text{SU}(n+1, \mathbb{K}) := \{x \in G \mid x^* = x^{-1}\} \\ \mathfrak{k} &= \{X \in M(n+1, \mathbb{K}) \mid X^* = -X \text{ and } \text{Tr}(X) = 0\} \\ \mathfrak{s} &= \{X \in M(n+1, \mathbb{K}) \mid X^* = X \text{ and } \text{Tr}(X) = 0\} \end{aligned}$$

In the notation in [16] we have $G = \text{SL}(n+1, \mathbb{R})$ and $K = \text{SO}(n+1)$ for $\mathbb{K} = \mathbb{R}$, $G = \text{SL}(n+1, \mathbb{C})$ and $K = \text{SU}(n+1)$ for $\mathbb{K} = \mathbb{C}$, and $G = \text{SU}^*(2(n+1))$ and $K = \text{Sp}(n+1)$ for $\mathbb{K} = \mathbb{H}$.

For $k \in \mathbb{N}$ denote by I_k the $k \times k$ identity matrix. We write matrices as blocks

$$\begin{pmatrix} X & Y \\ W & Z \end{pmatrix}, \quad X \in M_{p \times p}(\mathbb{K}), Y \in M_{p \times q}(\mathbb{K}), W \in M_{q \times p}(\mathbb{K}) \text{ and } Z \in M_{q \times q}(\mathbb{K}).$$

Let

$$H_o = \begin{pmatrix} \frac{q}{n+1} I_p & 0 \\ 0 & -\frac{p}{n+1} I_q \end{pmatrix} \in \mathfrak{s},$$

and define $\mathfrak{a} := \mathbb{R}H_o$, and $\mathfrak{m} = \{X \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \mid \langle X, H_o \rangle = 0\}$. Then $\mathfrak{z}(\mathfrak{m}) \cap \mathfrak{s} = \mathfrak{a}$. We have $\Delta = \{\alpha, -\alpha\}$ where $\alpha(H_o) = 1$. As $\langle H_o, H_o \rangle = 1$ it follows that $\langle \cdot, \cdot \rangle$ agrees with the invariant form from last section.

Let $\Delta^+ = \{\alpha\}$. Then

$$\begin{aligned} \mathfrak{m} \oplus \mathfrak{a} &= \left\{ M(X, Y) = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \mid \begin{array}{l} X \in M(p, \mathbb{K}) \\ Y \in M(q, \mathbb{K}) \end{array} \text{ and } \text{Tr}X + \text{Tr}Y = 0 \right\} \\ &= \mathfrak{s}(\mathfrak{gl}(p, \mathbb{K}) \times \mathfrak{gl}(q, \mathbb{K})), \\ \mathfrak{n} &= \left\{ N(X) := \begin{pmatrix} 0_{pp} & X \\ 0_{qp} & 0_{qq} \end{pmatrix} \mid X \in M(p \times q, \mathbb{K}) \right\} \simeq M(p \times q, \mathbb{K}) \\ MA &= \left\{ m(a, b) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid \begin{array}{l} a \in \text{GL}(p, \mathbb{K}) \\ b \in \text{GL}(q, \mathbb{K}) \end{array} \text{ and } \det a \det b = 1 \right\} \\ &= \text{S}(\text{GL}(p, \mathbb{K}) \times \text{GL}(q, \mathbb{K})) \\ N &= \left\{ n(X) = \begin{pmatrix} I_p & X \\ 0_{qp} & I_q \end{pmatrix} \mid X \in M(p \times q, \mathbb{K}) \right\} \\ P &= \left\{ p(a, b; X) := \begin{pmatrix} a & X \\ 0 & b \end{pmatrix} \mid \begin{array}{l} a \in \text{GL}(p, \mathbb{K}) \\ b \in \text{GL}(q, \mathbb{K}) \end{array}, \det a \det b = 1, X \in M(p \times q, \mathbb{K}) \right\} \\ L &= \text{S}(\text{U}(p, \mathbb{K}) \times \text{U}(q, \mathbb{K})). \end{aligned}$$

Let e_1, \dots, e_{n+1} be the standard basis for \mathbb{K}^{n+1} and let $b_o = \mathbb{K}e_1 \oplus \dots \oplus \mathbb{K}e_p \in \text{Gr}_p(\mathbb{K})$. Then $\text{Gr}_p(\mathbb{K}) = K \cdot b_o \simeq K/L = G/P$. Thus we can introduce the notation $\mathcal{B} = \text{Gr}_p(\mathbb{K})$, which agrees with the one from the previous sections. Note that $L = \text{S}(\text{O}(p) \times \text{O}(q))$ for

$\mathbb{K} = \mathbb{R}$, $L = \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ for $\mathbb{K} = \mathbb{C}$, and $L = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$ for $\mathbb{K} = \mathbb{H}$. In particular \mathcal{B} is simply connected for $\mathbb{K} = \mathbb{C}$ and \mathbb{H} .

Applying θ to \mathfrak{n} , we see that

$$\bar{\mathfrak{n}} = \left\{ \bar{N}(Y) = \begin{pmatrix} 0_{pp} & 0_{pq} \\ Y & 0_{qq} \end{pmatrix} \mid Y \in \mathrm{M}_{q \times p}(\mathbb{K}) \right\} \simeq \mathrm{M}(q \times p, \mathbb{K})$$

and

$$\bar{N} = \left\{ \bar{n}(Y) = \begin{pmatrix} I_p & 0_{pq} \\ Y & I_q \end{pmatrix} \mid Y \in \mathrm{M}(q \times p, \mathbb{K}) \right\}.$$

The action of MA on \mathfrak{n} , respectively $\bar{\mathfrak{n}}$, is given by

$$m(a, b)N(X)m(a^{-1}, b^{-1}) = N(aXb^{-1}) \quad \text{and} \quad m(a, b)\bar{N}(Y)m(a^{-1}, b^{-1}) = \bar{N}(bYa^{-1}).$$

Lemma 4.1. $\bar{N}P = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \mid A \in \mathrm{GL}(p, \mathbb{K}) \right\}$.

Proof. A simple calculation shows that

$$\bar{n}(Y)m(a, b)n(X) = \begin{pmatrix} a & aX \\ Ya & YaX + b \end{pmatrix}, \quad (4.1)$$

proving one of the inclusions. On the other hand, let $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be such that A is invertible. Define $a = A$, $X = a^{-1}B$, $Y = Ca^{-1}$, and $b = D - Ca^{-1}B$. Then $m(a, b) \in MA$ and $\bar{n}(Y)m(a, b)n(X) = x$. \square

Let $b \in \mathrm{Gr}_p(\mathbb{K})$ and view b as a dp -dimensional real vector space. Fix a convex subset $E \subset b$ containing the zero vector, such that the volume of E is one. For $c \in \mathrm{Gr}_p(\mathbb{K})$ let $P_c : \mathbb{K}^n \rightarrow c$ denote the orthogonal projection onto c . Define

$$|\mathrm{Cos}(b, c)| := \mathrm{Vol}_{\mathbb{R}}(P_c(E))^{1/d} \quad (4.2)$$

The definition is independent of E as we will see in a moment. Recall, if $x : b \rightarrow b$ is linear then

$$\mathrm{Vol}_{\mathbb{R}}(x(E)) = |\det_{\mathbb{R}}(x)| = |\det(x)|^d. \quad (4.3)$$

As the elements in K acts as orthogonal transformations it follows that $|\mathrm{Cos}(k \cdot b, c)| = |\mathrm{Cos}(b, k^{-1}c)|$. In particular, if $b = k \cdot b_o$ and $c = h \cdot b_o$ with $k, h \in K$, then

$$|\mathrm{Cos}(b, c)| = |\mathrm{Cos}(h^{-1}k \cdot b_o, b_o)| = \mathrm{Vol}_{\mathbb{R}}(P_{b_o}(h^{-1}kE))^{1/d}. \quad (4.4)$$

As a motivation for the definition assume that $p = 1$ and $\mathbb{K} = \mathbb{R}$. If $b = \mathbb{R}x, c = \mathbb{R}y \in \mathrm{Gr}_1(\mathbb{R})$, then

$$|\mathrm{Cos}(b, c)| = \frac{|(x, y)|}{\|x\|\|y\|} = |\cos(\sphericalangle(x, y))|$$

where $\sphericalangle(x, y)$ denotes the angle between x and y .

For reasons that will become transparent in a moment, we identify $\mathfrak{a}_{\mathbb{C}}^*$ with \mathbb{C} by

$$\lambda \mapsto \frac{n+1}{pq} \lambda(H_o) \quad \text{with inverse} \quad z \mapsto z \frac{pq}{n+1} \alpha. \quad (4.5)$$

We have $\dim_{\mathbb{R}} \mathfrak{n} = dpq$. Hence

$$\rho = \frac{dpq}{2} \alpha \longleftrightarrow \frac{d(n+1)}{2} \in \mathbb{R}.$$

Then ρ is an integer if and only if $\mathbb{K} = \mathbb{C}, \mathbb{H}$ or $\mathbb{K} = \mathbb{R}$ and n is odd. For a complex number λ , we write from now on $\alpha_P(x)^\lambda$ instead of $\alpha_P(x)^{\lambda \frac{pq}{n+1} \alpha}$.

Theorem 4.2. *Fix a convex subset $E \subset b_o$ containing zero such that $\text{Vol}_{\mathbb{R}}(E) = 1$. For $x \in G$ and $\lambda \in \mathbb{C}$ we have*

$$\alpha_P(x)^\lambda = |\text{Vol}_{\mathbb{R}}(P_{b_o}(x(E)))|^{\lambda/d} = |\text{Cos}(x \cdot b_o, b_o)|^\lambda. \quad (4.6)$$

In particular, if $b = k \cdot b_o$ and $c = h \cdot b_o \in \mathcal{B}$, then

$$\alpha(h^{-1}k)^\lambda = |\text{Cos}(b, c)|^\lambda. \quad (4.7)$$

Proof. For $z = (z_1, \dots, z_p)^t \in \mathbb{K}^p$ we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} Az \\ Cz \end{pmatrix}.$$

Thus

$$P_{b_o} \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} \right) = \begin{pmatrix} Az \\ 0 \end{pmatrix}.$$

Thus, by (4.4),

$$\left| \text{Cos} \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot b_o, b_o \right) \right|^\lambda = |\det_{\mathbb{R}} A|^{\lambda/d}. \quad (4.8)$$

Let $\bar{n}(Y)m(a, b)n(X) \in \bar{N}P$. Write $m(a, b) = m_o \exp(xH_o)$ with $m_o \in M$ and $x \in \mathbb{R}$. Then

$$|\det_{\mathbb{R}} a|^\lambda = \exp \left(\lambda \frac{xdpq}{n+1} \right) = \exp(xH_o)^\lambda \frac{dpq}{n+1} \alpha = \alpha_P(\exp(xH_o))^{d\lambda}.$$

Thus

$$\alpha_P(\bar{n}(Y)m(a, b)n(X))^\lambda = |\det_{\mathbb{R}}(a)|^{\lambda/d} = |\det a|^\lambda \quad (4.9)$$

and (4.6) follows now from (4.8). \square

This shows in particular that the definition of Cos is independent of the convex set E .

Lemma 4.3. *Let $k \in K$. Then $\alpha_P(k) = \alpha_P(k^{-1})$.*

Proof. Write $k = \bar{n}man$. Then, as $\theta(k) = k$ and $\theta(a)^{-1} = a$, we have $k^{-1} = \theta(k)^{-1} = \theta(\bar{n})^{-1} \theta(m)^{-1} a \theta(\bar{n})^{-1}$ and the claim follows. \square

Corollary 4.4. *Let $b, c \in \mathcal{B}$. Then $|\text{Cos}(b, c)|^\lambda = |\text{Cos}(c, b)|^\lambda$.*

Proof. Write $b = k \cdot b_o$ and $c = h \cdot b_o$. By (4.7) and Lemma 4.3

$$|\text{Cos}(b, c)|^\lambda = \alpha(h^{-1}k)^\lambda = \alpha(k^{-1}h)^\lambda = |\text{Cos}(c, b)|^\lambda,$$

which proves the Corollary. \square

The Cos^λ -transform \mathcal{C}^λ is defined for $\text{Re } \lambda > \rho$ by

$$\mathcal{C}^\lambda(f)(c) := \int_{\mathcal{B}} |\text{Cos}(b, c)|^{\lambda-\rho} f(c) dc = \int_K \alpha(k^{-1}h)^{\lambda-\rho} f(h \cdot b_o) dk \quad c = k \cdot b_o \quad (4.10)$$

where we have used Theorem 4.2. Thus, \mathcal{C}^λ is nothing but the standard intertwining operator $J(\lambda)$. We would like to point out, that the proof of the intertwining property of $J(\lambda)$ shows that \mathcal{C}^λ is an intertwining operator without the reference to the general theory. But the general theory described in Section 2 gives us the following theorem for free and allows us to apply the results from [5] described in Section 3.

Theorem 4.5. *The following holds:*

- (1) *The Cos^λ -transform extends to a meromorphic family of intertwining operators $\mathcal{C}^\lambda : C^\infty(\mathcal{B}) \rightarrow C^\infty(\mathcal{B})$.*
- (2) *Let $c_P(\lambda) := \mathcal{C}^\lambda(1)$. Then $c_P : \mathbb{C} \rightarrow \mathbb{C}$ is meromorphic and*

$$\mathcal{C}^{-\lambda} \circ \mathcal{C}^\lambda = \mathcal{C}^\lambda \circ \mathcal{C}^{-\lambda} = c_P(-\lambda)c_P(\lambda)\text{id}.$$

- (3) *If $\lambda \in i\mathbb{R}$, then $\frac{1}{c_P(\lambda)}\mathcal{C}^\lambda$ is unitary.*

5. THE K -SPECTRUM OF THE Cos^λ -TRANSFORM

In this section we determine the K -spectrum of $J(\lambda)$ for the Grassmann manifolds $G_p(\mathbb{K}) = \text{SU}(n+1, \mathbb{K})/\text{S}(\text{U}(p, \mathbb{K}) \times \text{U}(q, \mathbb{K})) \simeq \text{SL}(n+1, \mathbb{K})/P$. This then determines the K -spectrum of the cos^λ transform. As before we always assume that $p \leq q$ and $p + q = n + 1$. We keep the notation from the previous sections.

The following is well known.

Lemma 5.1. *The space \mathcal{B} is a symmetric space and the corresponding involution is*

$$\tau(x) = \begin{pmatrix} \text{I}_p & 0 \\ 0 & -\text{I}_q \end{pmatrix} x \begin{pmatrix} \text{I}_p & 0 \\ 0 & -\text{I}_q \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and $K^\tau = L$.

The derived involution is given by the same formula. Let \mathfrak{l} denote the Lie algebra of L . Then $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{q}$ where

$$\mathfrak{q} = \left\{ Q(X) = \begin{pmatrix} 0_{pp} & X \\ -X^* & 0_{qq} \end{pmatrix} \mid X \in \text{M}(p \times q, \mathbb{K}) \right\}.$$

Let $E_{\nu,\mu}^{(r,s)} = (\delta_{i\nu}\delta_{j\mu})_{i,j}$ denote the matrix in $M(r \times s, \mathbb{K})$ with all entries equal to 0 but the (ν, μ) -th which is equal to 1. For $\mathbf{t} = (t_1, \dots, t_p)^t \in \mathbb{R}^p$ let

$$\begin{aligned} X(\mathbf{t}) &= - \sum_{j=1}^p t_j E_{q-p+j,j}^{(p,q)} \in M(p \times q, \mathbb{K}), \\ Y(\mathbf{t}) &= Q(X(\mathbf{t})) \in \mathfrak{q} \\ \text{diag}(\mathbf{t}) &= \sum_{j=1}^p t_j E_{j,j} \in M(p \times p, \mathbb{K}). \end{aligned}$$

Then $\mathfrak{b} = \{Y(\mathbf{t}) \mid \mathbf{t} \in \mathbb{R}^p\} \simeq \mathbb{R}^p$ is a maximal abelian subspace of \mathfrak{g} .

To describe the set \widehat{K}_L we note first that if $\mathbb{K} = \mathbb{R}$, then \mathcal{B} is not simply connected. In all other cases \mathcal{B} is simply connected. Define $\epsilon_j(Y(\mathbf{t})) := t_j$. We will identify the element $\lambda = \sum_{j=1}^p \lambda_j \epsilon_j$ with the corresponding vector $\lambda = (\lambda_1, \dots, \lambda_p)$.

Lemma 5.2. *We have*

$\Delta_{\mathfrak{k}} = \{\pm \epsilon_i \pm \epsilon_j \ (1 \leq i \neq j \leq p, \pm \text{ independent}), \pm \epsilon_i \ (1 \leq i \leq p), \pm 2\epsilon_i \ (1 \leq i \leq p)\}$
with multiplicities respectively d (and not there in case $p = 1$), $d(q - p)$ (and not there in case $p = q$) and $d - 1$ (and absent if $d = 1$).

Proof. This follows from [16]: the table on page 518, the description of the simple root systems on page 462 ff. and the Satake diagrams on page 532–533. \square

Lemma 5.3. *Let the notation be as above.*

(1) *If $\mathbb{K} = \mathbb{R}$ and $p = q$, then*

$$\Lambda^+(\mathcal{B}) = \left\{ \mu = \sum_{j=1}^p m_j \epsilon_j \mid m_j \in 2\mathbb{N}_0 \text{ and } m_1 \geq \dots \geq m_{p-1} \geq |m_p| \right\}.$$

In all other cases,

$$\Lambda^+(\mathcal{B}) = \left\{ \mu = \sum_{j=1}^p m_j \epsilon_j \mid m_j \in 2\mathbb{N}_0 \text{ and } m_1 \geq \dots \geq m_p \geq 0 \right\}.$$

(2) $\rho_{\mathfrak{k}} = \sum_{j=1}^p [\rho - d(j - 1) - 1] \epsilon_j$.

(3) *Let $\mu \in \Lambda^+(\mathcal{B})$. Then $\omega(\mu) = \frac{pq}{2(n+1)} \sum_{j=1}^p \left(m_j^2 + 2m_j(\rho - d(j - 1) - 1) \right)$.*

Proof. (1) is well known and (2) follows by a simple calculation using the form of $\Delta_{\mathfrak{k}}^+$ and the multiplicities. For (3) let $\widetilde{X}_1, \dots, \widetilde{X}_p \in \mathfrak{b}$ be so that $\epsilon_j(\widetilde{X}_k) = i\delta_{jk}$. Then $-\langle \widetilde{X}_k, \widetilde{X}_k \rangle = 2(n+1)/pq$. Thus the $X_k = \sqrt{pq/[2(n+1)]} \widetilde{X}_k$ form an orthonormal basis for \mathfrak{b} and $-\epsilon_j(X_k)^2 = pq/[2(n+1)]\delta_{jk}$. The claim now follows from Lemma 3.2. \square

Lemma 5.4. *Let $\mu = (m_1, \dots, m_p) \in \Lambda^+(\mathcal{B})$. Then*

$$S(\mu) = \{\mu \pm 2\epsilon_j \mid j = 1, \dots, p\} \cap \Lambda^+(\mathcal{B}).$$

These representations occur with multiplicity one.

Proof. A simple matrix calculation shows that the weights of \mathfrak{h} in $\mathfrak{s}_{\mathbb{C}}$ are $\pm(\epsilon_i + \epsilon_j)$ for $i, j = 1, \dots, p$. Next we note that $\mu \pm (\epsilon_i + \epsilon_j)$, $j \neq i$, is not in $\Lambda^+(\mathcal{B})$ as the i^{th} and j^{th} entries are not even. It follows now from [26] and the fact that each submodule in $V_{\mu} \otimes \mathfrak{s}_{\mathbb{C}}$ that occur in $L^2(\mathcal{B})$ is spherical that $\sigma \in S(\mu)$ if and only if there exists $j = 1, \dots, p$ and $w \in W_{\mathfrak{k}}$ such that $w(\mu \pm 2\epsilon_j)$ is dominant and $\sigma = w(\mu \pm 2\epsilon_j)$. Assume that $\mu + 2\epsilon_j$ is not dominant. Then $m_{j-1} < m_j + 2$. Hence $m_j = m_{j-1}$. Applying the transposition $(j-1, j)$ shows that $\mu + 2\epsilon_j$ is $W_{\mathfrak{k}}$ -conjugate to $\mu + 2\epsilon_{j-1}$. Repeating as long as necessary shows that there exists a k , $1 \leq k < j$ such that $\mu + 2\epsilon_k$ is dominant and $W_{\mathfrak{k}}$ -conjugate to $\mu + 2\epsilon_j$. The case $\mu - 2\epsilon_j$ not dominant is handled in similar way. \square

Lemma 5.5. *Let $\mu = (m_1, \dots, m_p) \in \Lambda^+(\mathcal{B})$ and $\lambda \in \mathbb{C} \simeq \mathfrak{a}_{\mathbb{C}}^*$. Then*

$$\frac{\eta_{\mu+2\epsilon_j}(\lambda)}{\eta_{\mu}(\lambda)} = \frac{\lambda - m_j - \rho + d(j-1)}{\lambda + m_j + \rho - d(j-1)}. \quad (5.1)$$

Proof. By Lemma 5.3 we get

$$\omega(\mu + 2\epsilon_j) - \omega(\mu) = \frac{2pq}{n+1} (m_j + \rho - d(j-1)).$$

The claim now follows from (3.3) and our identification $\mathbb{C} \simeq \mathfrak{a}_{\mathbb{C}}^*$, $z \mapsto z \frac{pq}{n+1} \alpha$. \square

Introduce the Siegel or Gindikin Γ -function $\Gamma_{p,d} : \mathbb{C}^p \rightarrow \mathbb{C}$ by

$$\Gamma_{p,d}(\lambda) = \prod_{j=1}^p \Gamma(\lambda_j - \frac{d}{2}(j-1)). \quad (5.2)$$

Then $\Gamma(p, d; \lambda)$ is meromorphic with singularities on the union of the hyperplanes

$$\mathcal{H}(d, j, n) = \{\lambda \in \mathbb{C}^p \mid \lambda_j = -n + \frac{d}{2}(j-1)\}, \quad j = 1, \dots, p, \quad n \in \mathbb{N}_0.$$

In the following, if $\lambda \in \mathbb{C}$, we will also indicate by λ the vector $(\lambda, \dots, \lambda) \in \mathbb{C}^p$. It will be clear from the context if we are using λ as a complex number or as a vector.

Theorem 5.6. *Let $\lambda \in \mathbb{C}$. Then there exists a meromorphic function $G(\lambda)$ such that for all $\mu \in \Lambda^+(\mathcal{B})$ we have*

$$\eta_{\mu}(\lambda) = (-1)^{|\mu|/2} G(\lambda) \frac{\Gamma_{p,d}(\frac{1}{2}(-\lambda + \rho + \mu))}{\Gamma_{p,d}(\frac{1}{2}(\lambda + \rho + \mu))}, \quad (5.3)$$

where $|\mu| = \sum_{j=1}^p m_j$.

Proof. Denote the right hand side of (5.3) by $\sigma_\mu(\lambda)$. Using $\Gamma(z+1) = z\Gamma(z)$, we get

$$\sigma_{\mu+2\epsilon_j}(\lambda) = \sigma_\mu(\lambda) \frac{\lambda - m_j - \rho + d(j-1)}{\lambda + m_j + \rho - d(j-1)}.$$

Now Lemma 5.5 implies that there exists a meromorphic function $G(\lambda)$ with the requires property. \square

The function $G(\lambda)$ will be explicitly computed in (5.8) below, by comparing the formula for $\eta_0(\lambda) = c_P(\lambda)$. For this, we note the following result which can be found in [3], Theorem 8.1.1.

Lemma 5.7 (Selberg's integral). *Let $\gamma_1, \gamma_2, \alpha \in \mathbb{C}$ with $\text{Re } \gamma_1, \text{Re } \gamma_2, \text{Re } \alpha > 0$. Then*

$$\begin{aligned} & \int_0^1 \cdots \int_0^1 \prod_{j=1}^p t_j^{\gamma_1-1} (1-t_j)^{\gamma_2-1} \prod_{1 \leq i < j \leq p} |t_i - t_j|^{2\alpha} dt_1 \cdots dt_p = \\ & = \prod_{j=1}^p \frac{\Gamma(\alpha j + 1) \Gamma(\alpha(j-1) + \gamma_1) \Gamma(\alpha(j-1) + \gamma_2)}{\Gamma(\alpha + 1) \Gamma(\alpha(p+j-2) + \gamma_1 + \gamma_2)}. \end{aligned}$$

To complete the calculation we need to evaluate $\alpha_P(\exp(Y(\mathbf{t})))$. We have

$$\exp Y(\mathbf{t}) = \begin{pmatrix} \text{diag}(\cos t_1, \dots, \cos t_p) & 0 & -\text{diag}(\sin t_1, \dots, \sin t_p) \\ 0 & I_{q-p} & 0 \\ \text{diag}(\sin t_1, \dots, \sin t_p) & 0 & \text{diag}(\cos t_1, \dots, \cos t_p) \end{pmatrix} \quad (5.4)$$

where the matrix $I_{q-p} = I_{n+1-2p}$ does not occur in case $p = (n+1)/2$.

For the following see also [45], Lemma 3.2:

Lemma 5.8. *Let $\mathbf{t} \in \mathbb{R}^p$ and $\lambda \in \mathbb{C}$. Then $\alpha_P(\exp Y(\mathbf{t}))^\lambda = \prod_{j=1}^p |\cos(t_j)|^\lambda$.*

Proof. This follows from (5.4), (4.9) and (4.1). \square

Theorem 5.9. *Let $c_P(\lambda) = \int_K \alpha_P(k)^{\lambda-\rho} dk = \eta_0(\lambda) = J(\lambda)1$. Then*

$$c_P(\lambda) = \frac{\Gamma_{p,d}(d(n+1)/2) \Gamma_{p,d}(\frac{1}{2}(\lambda - \rho + dp))}{\Gamma_{p,d}(dp/2) \Gamma_{p,d}(\frac{1}{2}(\lambda + \rho))}. \quad (5.5)$$

Proof. In the following c will denote a positive constant that might change from one line to another. Its value will then be determined at the end. Let $\gamma_1 = (\lambda - \rho + d)/2$, $\gamma_2 = d(q-p+1)/2$, $Q = [0, 1]^p$, and $Q_\pi = [0, \pi]^p$. Using that $\sin(2u) = 2\cos(u)\sin(u)$ and

$|\sin(u - v) \sin(u + v)| = |\cos^2(u) - \cos^2(v)|$, we get by the substitution $u_j = \cos^2(t_j)$:

$$\begin{aligned}
c_P(\lambda) &= c \int_{Q_\pi} \prod_{j=1}^p |\cos t_j|^{2\gamma_1-1} \sin(t_j)^{2\gamma_2-1} \prod_{i<j} |\cos^2(t_i) - \cos^2(t_j)|^d dt_1 \dots dt_p \\
&= c \int_{Q_\pi} \prod_{j=1}^p |\cos t_j|^{2\gamma_1-1} (1 - \cos^2 t_j)^{\gamma_2-1} \sin(t_j) \prod_{i<j} |\cos^2(t_i) - \cos^2(t_j)|^d dt_1 \dots dt_p \\
&= c \int_Q \prod_{j=1}^p u_j^{\gamma_1-1} (1 - u_j)^{\gamma_2-1} \prod_{i<j} |u_i - u_j|^d du_1 \dots du_p \\
&= c \prod_{j=1}^p \frac{\Gamma\left(\frac{1}{2}(\lambda - \rho + dj)\right)}{\Gamma\left(\frac{1}{2}(\lambda - \rho + d(q + j))\right)} \\
&= c \prod_{j=1}^p \frac{\Gamma\left(\frac{1}{2}(\lambda - \rho + dp - d(j - 1))\right)}{\Gamma\left(\frac{1}{2}(\lambda + \rho - d(j - 1))\right)}.
\end{aligned}$$

In the last equation we used that $d(q + j) = d(n + 1 - p + j) = 2\rho - dp + dj$ and that for a complex number z

$$\prod_{j=1}^p \Gamma\left(\frac{1}{2}(z + dj)\right) = \prod_{j=1}^p \Gamma\left(\frac{1}{2}(z + dp - d(j - 1))\right). \quad (5.6)$$

Now evaluate c by using that $c_P(\rho) = 1$ and recalling that $\rho = d(n + 1)/2$. \square

Remark 5.10. After circulation of the preprint of the present paper, Ole Warnaar kindly pointed us that the integral in Theorem 5.9 is well known to algebraic combinatorialists and was probably first considered by I. Macdonald in [28], p. 992.

Theorem 5.11. *The K -spectrum the Cos^λ -transform is given by:*

$$\eta_\mu(\lambda) = (-1)^{|\mu|/2} \frac{\Gamma_{p,d}(d(n + 1)/2)}{\Gamma_{p,d}(dp/2)} \frac{\Gamma_{p,d}\left(\frac{1}{2}(\lambda - \rho + dp)\right) \Gamma_{p,d}\left(\frac{1}{2}(-\lambda + \rho + \mu)\right)}{\Gamma_{p,q}\left(\frac{1}{2}(-\lambda + \rho)\right) \Gamma_{p,d}\left(\frac{1}{2}(\lambda + \rho + \mu)\right)}. \quad (5.7)$$

Proof. By taking $\mu = 0$ in (5.3) and using $\eta_0 = c_P$, we get

$$G(\lambda) = \frac{\Gamma_{p,d}\left(\frac{1}{2}(\lambda + \rho)\right)}{\Gamma_{p,d}\left(\frac{1}{2}(-\lambda + \rho)\right)} c_P(\lambda). \quad (5.8)$$

The claim then follows from (5.3) and (5.5). \square

Remark 5.12. In Section 2 we referred to the result of Vogan and Wallach on the meromorphic continuation of the intertwining operator $J(\lambda)$. This result is not needed for the computation of $\eta_\mu(\lambda)$. Indeed, it is enough to know that $J(\lambda)$ is holomorphic on $\mathfrak{a}_\mathbb{C}^*(a_P)$. Then Theorem 5.11 implies the meromorphic continuation of $J(\lambda) = \mathcal{C}^\lambda$ at least at the level of K -finite functions, cf. [21].

Remark 5.13. One can give a more general definition of the Cos^λ -transform. Note first that the definition of $|\text{Cos}(b, c)|^\lambda$ in (4.2) is valid for all $b \in \text{Gr}_{p'}(\mathbb{K})$, $p' \geq p$. For $p' \leq p$ one uses the symmetry relation $|\text{Cos}(b, c)|^\lambda = |\text{Cos}(c, b)|^\lambda$. For $p \leq p'$ define the Radon transform $R_{p', p} : C^\infty(\text{Gr}_{p'}) \rightarrow C^\infty(\text{Gr}_p)$ by

$$R_{p', p}(f)(\omega) := \int_{\omega \subset \eta} f(\eta) d\sigma_{p', \omega}(\eta)$$

where $\sigma_{p'}$ is a well defined probability measure on the set $\{\eta \mid \omega \subset \eta\}$. Then $R_{p', p}$ is a K -intertwining operator whose K -spectrum is independent of λ . Furthermore, defining \mathcal{C}^λ by (4.10), one shows that, with the obvious notation,

$$\mathcal{C}_{p', p}^\lambda = \mathcal{C}^\lambda \circ R_{p', p}.$$

The determination of the K -spectrum of the more general Cos^λ -transform is then reduced to the determination of the λ -independent K -spectrum of $R_{p', p}$ and the determination of the λ -dependent K -spectrum of \mathcal{C}^λ . We note that obviously $R_{p', p}$ is zero on all K -types $\mu \in \Lambda^+(\mathcal{B}_{p'}) \setminus \Lambda^+(\mathcal{B}_p) \cap \Lambda^+(\mathcal{B}_{p'})$, or equivalently, have a nonzero entry m_j with $p < j \leq p'$. As our purpose is to relate the Cos^λ -transform with the standard intertwining operators, we do not go further into this interesting topic, but refer to [45] and the references therein for further discussion.

6. THE Sin^λ -TRANSFORM

In this section we introduce the Sin^λ -transform and determine its K -spectrum. We assume that $p = q$. This in particular implies that $n+1$ is even and $p = \frac{n+1}{2}$. If $\omega \in \text{Gr}_p(\mathbb{K})$ then ω^\perp denotes the orthogonal complement of ω . Define the Sin^λ -transform $\mathcal{S}^\lambda : C^\infty(\mathcal{B}) \rightarrow C^\infty(\mathcal{B})$ by

$$\mathcal{S}^\lambda(f)(\omega) := \mathcal{C}^\lambda(f)(\omega^\perp).$$

This transform has been studied by B. Rubin for more general p , see [39, 40] and references therein, and by G. Zhang [45]. In particular G. Zhang shows, using the \overline{N} realization of the principal series representations, that the Sin^λ -transform is a Knapp-Stein intertwining operator and uses this fact to determine the complementary series for π_λ .

As $p = (n+1)/2$ we have $H_o = \begin{pmatrix} \frac{1}{2}\mathbf{I}_p & 0 \\ 0 & -\frac{1}{2}\mathbf{I}_p \end{pmatrix}$. Let

$$w = \begin{pmatrix} 0 & -\mathbf{I}_p \\ \mathbf{I}_p & 0 \end{pmatrix} = \exp\left(Y\left(\frac{\pi}{2}, \dots, \frac{\pi}{2}\right)\right).$$

Then $wH_o w^{-1} = \text{Ad}(w)H_o = -1$. It follows that $\text{Ad}(w)\overline{N} = N$.

Lemma 6.1. *For $f \in C^\infty(\mathcal{B})$ define $A(\lambda)f(k \cdot b_o) := J(\lambda)(f)(kw \cdot b_o) = R(w)J(\lambda)f(k \cdot b_o)$, where $R(w)$ denotes the right translation by w . Then $A(\lambda) : C^\infty(\mathcal{B}) \rightarrow C^\infty(\mathcal{B})$. Then $\lambda \mapsto A(\lambda)$ is a meromorphic family of intertwining operators: $A(\lambda) \circ \pi_\lambda(x) = \pi_{-\lambda}(x)A(\lambda)$.*

Proof. We have $w^{-1}Lw = L$ as $\text{Ad}(w)$ normalizes \mathfrak{a} . Thus $kmw \cdot b_o = kw \cdot b_o$ if $m \in L$ and $k \in K$. Thus $A(\lambda)$ is well defined. The intertwining property follows from Theorem 2.3 and the fact that $w^{-1}\overline{N}w = N$ and $\text{Ad}(w)|_{\mathfrak{a}} = -1$. \square

Theorem 6.2. *If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*(a_P)$, then $A(\lambda) = \mathcal{S}^\lambda$.*

Proof. We have by definition

$$A(\lambda)f(k \cdot b_o) = \int_K |\text{Cos}(h \cdot b_o, kw \cdot b_o)|^{\lambda-\rho} f(h) dh. \quad (6.1)$$

The claim follows if we can show that $(k \cdot b_o)^\perp = kw \cdot b_o$. As $\text{SU}(n+1, \mathbb{K})$ acts by orthogonal transformations it is enough to show that $b_o^\perp = w \cdot b_o$. But this follows from

$$w(x_1, \dots, x_p, 0, \dots, 0)^t = (0, \dots, 0, x_1, \dots, x_p)^t. \quad \square$$

As for the Cos^λ -transform, it follows that for each $\mu \in \Lambda^+(\mathcal{B})$ there exists a meromorphic function $\nu_\mu(\lambda)$ such that

$$\mathcal{S}^\lambda|_{L_\mu^2(\mathcal{B})} = \nu_\mu(\lambda)I_{L_\mu^2(\mathcal{B})}.$$

As $w^2 \in L$ it follows that $R(w)$ defines a unitary K -intertwining operator of order two on each of the K -types $L_\mu^2(\mathcal{B})$. Thus $R(w)|_{L_\mu^2(\mathcal{B})} = r(\mu)\text{id}_{L_\mu^2(\mathcal{B})}$ where $r(\mu)$ is a constant independent of λ , $r(\mu)^2 = \pm 1$ and $\nu_\mu(\lambda) = r(\mu)\eta_\mu(\lambda)$.

Lemma 6.3. *Let $\mu \in \Lambda^+(\mathcal{B})$. Then $r(\mu) = (-1)^{|\mu|/2}$.*

Proof. Let $e_\mu \in V_\mu^L$ be as in Section 2. Then $\pi_\mu(w)e_\mu = r(\mu)e_\mu$. Let v_μ be a highest weight vector such that $(v_\mu, e_\mu)_{V_\mu} = 1$. Then

$$r(\mu) = (v_\mu, \pi_\mu(w)e_\mu) = (\pi_\mu(w^{-1})v_\mu, e_\mu) = e^{-\frac{\pi i}{2} \sum m_j} (v_\mu, e_\mu) = (-1)^{|\mu|/2}. \quad \square$$

Theorem 6.4. *The K -spectrum of the Sin^λ -transform is given by*

$$\nu_\mu(\lambda) = \frac{\Gamma_{p,d}(\rho)}{\Gamma_{p,d}(\rho/2)} \frac{\Gamma_{p,d}(\frac{\lambda}{2}) \Gamma_{p,d}(\frac{1}{2}(-\lambda + \mu + \rho))}{\Gamma_{p,d}(\frac{1}{2}(-\lambda + \rho)) \Gamma_{p,d}(\frac{1}{2}(\lambda + \mu + \rho))}$$

with $\rho = dp$.

Proof. This follows from Theorem 5.11 and Lemma 6.3. \square

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