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# ADMISSIBLE $Q$ -CURVATURES UNDER ISOMETRIES FOR THE CONFORMAL GJMS OPERATORS

FRÉDÉRIC ROBERT

*Dedicated to Jean-Pierre Gossez on the occasion of his 65th birthday*

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let  $M$  be a compact manifold of dimension  $n \geq 3$  and let  $k \geq 1$  be an integer such that  $k \leq \frac{n}{2}$  if  $n$  is even. In their celebrated work, Graham-Jenne-Mason-Sparling [15] provided a systematic construction of conformally invariant operators (GJMS operators for short) based on the ambient metric of Fefferman-Graham [12, 13]. More precisely, letting  $\mathcal{M}$  be the set of Riemannian metrics on  $M$ , then for all  $g \in \mathcal{M}$ , there exists an operator  $P_g : C^\infty(M) \rightarrow C^\infty(M)$  such that

- (i)  $P_g$  is a differential operator and  $P_g = \Delta_g^k + \text{lot}$
- (ii)  $P_g$  is natural, that is  $\varphi^* P_g = P_{\varphi^* g}$  for all smooth diffeomorphism  $\varphi : M \rightarrow M$ .
- (iii)  $P_g$  is self-adjoint with respect to the  $L^2$ -scalar product
- (iv) Given  $\omega \in C^\infty(M)$  and defining  $\hat{g} = e^{2\omega} g$ , we have that

$$(1) \quad P_{\hat{g}}(f) = e^{-\frac{n+2k}{2}\omega} P_g \left( e^{\frac{n-2k}{2}\omega} f \right) \text{ for all } f \in C^\infty(M).$$

Here  $\Delta_g := -\text{div}_g(\nabla)$  is the Laplace-Beltrami operator and  $\text{lot}$  denotes differential terms of lower order. Point (iii) above is due to Graham-Zworski [16]. For instance, on  $\mathbb{R}^n$  endowed with its Euclidean metric  $\xi$ , one has that  $P_\xi = \Delta_\xi^k$ . There is a natural scalar invariant, namely the  $Q$ -curvature, attached to the operator  $P_g$ : this scalar invariant, denoted as  $Q_g$ , was initially introduced by Branson and Ørsted [7] for  $n = 2k = 4$  and generalized by Branson [4, 5]. When  $k = 1$ , the GJMS operator is the conformal Laplacian and the  $Q$ -curvature is the scalar curvature (up to a dimensional constant). When  $k = 2$ , the GJMS operator is the Paneitz operator introduced in [26]. When  $n \neq 2k$ , the  $Q$ -curvature is  $Q_g := \frac{2}{n-2k} P_g(1)$ : when  $n = 2k$ , the definition is much more subtle and involves a continuation in dimension argument (we refer to the survey Branson-Gover [6] and to Juhl [20] for an exposition in book form). In the spirit of classical problems in conformal geometry, our objective here is to prescribe the  $Q$ -curvature in a conformal class; that is, given a conformal Riemannian class  $\mathcal{C}$  on  $M$  and a function  $f \in C^\infty(M)$ , we investigate the existence of a metric  $g \in \mathcal{C}$  such that  $Q_g = f$ . As one checks (see Proposition 3 below), up to multiplication by a constant, this amounts to finding

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critical points of the perturbation of the Hilbert functional

$$\begin{aligned} \mathcal{C} &\rightarrow \mathbb{R} \\ g &\mapsto \frac{\int_M Q_g dv_g}{V_f(M, g)^{\frac{n-2k}{n}}} \end{aligned}$$

where  $V_f(M, g) := \int_M f dv_g$  is the weighted  $f$ -volume of  $(M, g)$ . This structure suggests to apply variational methods to prescribe the  $Q$ -curvature and we define

$$\mu_f(\mathcal{C}) := \inf_{g \in \mathcal{C}} \frac{\int_M Q_g dv_g}{V_f(M, g)^{\frac{n-2k}{n}}}.$$

Given a metric  $g \in \mathcal{C}$ , the conformal class can be described as

$$\mathcal{C} = \{e^{2\omega}g/\omega \in C^\infty(M)\}.$$

We assume that  $n > 2k$ : in this context, it is more convenient to write a metric  $\hat{g} \in \mathcal{C}$  as  $\hat{g} = u^{\frac{4}{n-2k}}g$  with  $u \in C_+^\infty(M)$ , the set of positive smooth functions. With this parametrization, we have that

$$\mathcal{C} = \{u^{\frac{4}{n-2k}}g/u \in C_+^\infty(M)\},$$

and the relation (1) between  $P_g$  and  $P_{\hat{g}}$  rewrites

$$(2) \quad P_{\hat{g}}\varphi = u^{1-2^*} P_g(u\varphi)$$

for all  $\varphi \in C^\infty(M)$ , where  $2^* := \frac{2n}{n-2k}$ . Therefore, taking  $\varphi \equiv 1$ , we have that

$$P_g u = \frac{n-2k}{2} Q_{\hat{g}} u^{2^*-1} \text{ in } M$$

where  $\hat{g} = u^{\frac{4}{n-2k}}g$ , and then finding a metric in  $\mathcal{C}$  with  $f$  as  $Q$ -curvature amounts to solving the variational elliptic equation  $P_g u = \frac{n-2k}{2} f u^{2^*-1}$ . Despite this elegant variational structure, this question gives rise to a crucial intrinsic difficulty due to the essence of the problem, that is the conformal invariance of the operator. More precisely, in the spirit of Bourguignon-Ezin [3], Delanoë and the author proved in [9] that

$$\int_M X(Q_g) dv_g = 0$$

for all conformal Killing field  $X$  on  $(M, \mathcal{C})$ . When  $k = 1$ , this is the celebrated Kazdan-Warner obstruction [21] to the scalar curvature problem. In particular, if  $\varphi \in C^\infty(\mathbb{S}^n) \setminus \{0\}$  is a first eigenfunction of the Laplace-Beltrami operator on the standard sphere  $(\mathbb{S}^n, h)$ , then for any  $\epsilon \neq 0$ ,  $Q_h + \epsilon\varphi$  is not achieved as the  $Q$ -curvature of a metric in the conformal class of the standard sphere. Therefore, a function can be arbitrarily close to a  $Q$ -curvature but not be a  $Q$ -curvature itself: the prescription of the  $Q$ -curvature is then a highly unstable problem, and its underlying analysis is intricate. We refer again to [9] for considerations on the structure of the set of  $Q$ -curvatures. In the case  $k = 1$  and  $n \geq 3$ , the problem of prescribing a constant  $Q$ -curvature is known as the Yamabe problem: it is not the purpose of the present article to make an extensive historical review of the famous resolution of this problem, and we refer to Lee-Parker [22] and the references therein. Concerning fourth order problems, that is for  $k = 2$ , there has been an intensive litterature on the question: here, we refer to the recent surveys of Branson-Gover [6], Chang [8], Malchiodi [24] and the references therein.

In the sequel, we will say that a function is admissible if it can be achieved as the  $Q$ -curvature of a metric in a given conformal class. As seen above, some functions on the sphere are not admissible for the standard conformal class. Moser [25] remarked that functions enjoying some symmetries automatically satisfy the Kazdan-Warner identities: indeed, on the standard sphere, given an isometry  $\sigma$  such that  $\varphi \circ \sigma = -\varphi$  for all first eigenfunction of the Laplace-Beltrami operator (take  $\sigma = -Id$  for instance), then the Kazdan-Warner identity yields 0 for all function invariant by  $\sigma$ . Then, Moser had the idea to impose invariance under a group of isometries to find admissible functions on the sphere for the scalar curvature problem in 2D. This strategy was also used by Escobar-Schoen [11] and Hebey [18] in higher dimensions. In the same spirit, Delanoë and the author [9] proved that a function on the sphere which is close to  $Q_h$  and invariant under a group of isometries acting without fixed point is admissible. In the present article, we relax the condition of being close to  $Q_h$  by imposing cancelation of some derivatives (see Theorem 3 below). In the specific case  $n = 2k + 1$ , very few is required; this is the object our main result:

**Theorem 1.** *Let  $k \geq 1$  and let  $G$  be a subgroup of isometries of  $(\mathbb{S}^{2k+1}, h)$ . Let  $f \in C^\infty(M)$  be a positive  $G$ -invariant function and assume that  $G$  acts without fixed point (that is  $|O_G(x)| \geq 2$  for all  $x \in \mathbb{S}^{2k+1}$ ). Then there exists  $g \in [h]$  such that  $Q_g = f$  and  $G \subset Isom_g(\mathbb{S}^n)$ .*

When  $k = 1, 2$ , this result is due respectively to Hebey [18] and to the author [27]. This theorem is a particular case of more general results proved on arbitrary conformal manifolds (see Proposition 8 and Theorem 3 below). In this article, we make a general analysis of the operator  $P_g$  and of the blow-up phenomenon attached to it on arbitrary conformal manifolds. In the last section, we apply this analysis to the conformal sphere.

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## 2. MISCELLANEOUS ON THE OPERATOR $P_g$

The operator  $P_g$  can be written (partially) as a divergence form (we refer to Branson-Gover [6]): as a preliminary step, we precise this divergence form that will be useful in the sequel:

**Proposition 1.** *Let  $P_g$  be the conformal GJMS operator. Then for any  $l \in \{0, \dots, k - 1\}$ , there exists  $A_{(l)}(g)$  a smooth  $T_{2l}^0$ -tensor field on  $M$  such that*

$$(3) \quad P_g = \Delta_g^k + \sum_{l=0}^{k-1} (-1)^l \nabla^{j_1 \dots j_l} (A_{(l)}(g)_{i_1 \dots i_l j_1 \dots j_l} \nabla^{i_1 \dots i_l}),$$

where the indices are raised via the musical isomorphism. In addition for any  $l \in \{0, \dots, k - 1\}$ ,  $A_{(l)}(g)$  is symmetric in the following sense:  $A_{(l)}(g)(X, Y) =$

$A_{(l)}(g)(Y, X)$  for all  $X, Y$   $T_0^l$ -tensors on  $M$ . In particular, we have that

$$(4) \quad \int_M u P_g(v) dv_g = \int_M \left( \Delta_g^{\frac{k}{2}} u \Delta_g^{\frac{k}{2}} v + \sum_{l=0}^{k-1} A_{(l)}(g)(\nabla^l u, \nabla^l v) \right) dv_g$$

for all  $u, v \in C^\infty(M)$ . Here, we have adopted the convention

$$\Delta_g^{\frac{k}{2}} u \Delta_g^{\frac{k}{2}} v := (\nabla \Delta_g^{\frac{k-1}{2}} u, \nabla \Delta_g^{\frac{k-1}{2}} v)_g$$

when  $k$  is odd.

*Proof.* The proof uses only the self-adjointness of the operator  $P_g$ . In the sequel, we note  $A^*$  the adjoint of any operator  $A$  with respect to the  $L^2$ -product. As a preliminary, we compute the adjoint of some elementary operators. We adopt here Hamilton's convention [17]: the notation  $A \star B$  denotes a linear combination of contraction of the tensors  $A, B, g$  and  $g^{-1}$ . Given  $B$  a smooth  $T_q^0$ -tensor field on  $M$ , we consider the operator  $Bu := B \cdot \nabla^q u = B_{i_1 \dots i_q} \nabla^{i_1 \dots i_q} u$  for all  $u \in C^\infty(M)$ . We claim that

$$B^* = (-1)^q B + \sum_{l=1}^{q-1} \nabla^l u \star \nabla^{q-l} B.$$

We prove the claim. We let  $u, v \in C^\infty(M)$  be two smooth functions on  $M$ . Integrating by parts, we have that

$$\begin{aligned} \int_M u B v dv_g &= \int_M u B_{i_1 \dots i_q} \nabla^{i_1 \dots i_q} v dv_g = (-1)^q \int_M \nabla^{i_q \dots i_1} (u B_{i_1 \dots i_q}) v dv_g \\ &= (-1)^q \int_M \left( B_{i_1 \dots i_q} \nabla^{i_q \dots i_1} u + \sum_{l=0}^{q-1} \nabla^l u \star \nabla^{q-l} B \right) v dv_g. \end{aligned}$$

Therefore,  $B^*$  is defined and

$$B^* u = (-1)^q B_{i_1 \dots i_q} \nabla^{i_q \dots i_1} u + \sum_{l=0}^{q-1} \nabla^l u \star \nabla^{q-l} B.$$

For any smooth tensor field  $T$ , we define  $Asym(T)(X, Y, \dots) := T(X, Y, \dots) - T(Y, X, \dots)$ . It follows from the definition of the curvature tensor that

$$Asym(\nabla^2 T) = T \star R,$$

where  $R$  is the curvature tensor. Therefore, for any permutation  $\sigma$  of  $\{1, \dots, q\}$ , we have that

$$(5) \quad \nabla^q u - \sigma \cdot \nabla^q u = \nabla^{q-2} u \star R,$$

where  $\sigma \cdot T$  permutes the variables of the covariant tensor  $T$  along  $\sigma$ . Therefore, we have that  $\nabla^{i_q \dots i_1} u - \nabla^{i_1 \dots i_q} u$  is a contraction of  $\nabla^{q-2} u$ , and therefore we get that  $B^* = (-1)^q B + lot$ . This proves the claim.

We are now in position to prove Proposition 1. It follows from the definition of  $P_g$  that there exists  $B$ , a smooth  $T_{2k-1}^0$ -tensor field on  $M$ , such that  $P_g u = \Delta_g^k u + Bu + lot$  for all  $u \in C^\infty(M)$ . Since  $P_g$  and  $\Delta_g$  are self-adjoint, we then get that

$$P_g = P_g^* = \Delta_g^k + B^* + lot = \Delta_g^k - B + lot$$

since  $2k - 1$  is odd. In particular,  $Bu = lot$  and therefore,  $Bu = 0$  for all  $u \in C^\infty(M)$ .

We now take  $C$  a smooth  $(2k-2, 0)$ -tensor field such that  $P_g = \Delta_g^k + C \cdot \nabla^{2k-2} + \text{lot}$ . We define  $A$  as the symmetrized tensor of  $C$ , that is via coordinates  $A(X, Y) = (-1)^{k-1} \frac{1}{2} (C(X, Y) + C(Y, X))$  for all  $X, Y$  any  $T_0^{k-1}$ -tensors on  $M$ . As easily checked, since changing the order of differentiation involves only lower order terms via with (5), we have that

$$\begin{aligned} C \cdot \nabla^{2k-2} u &= C_{i_1 \dots i_{k-1} j_1 \dots j_{k-1}} \nabla^{i_1 \dots i_{k-1} j_1 \dots j_{k-1}} u \\ &= (-1)^{k-1} A_{i_1 \dots i_{k-1} j_1 \dots j_{k-1}} \nabla^{i_1 \dots i_{k-1} j_1 \dots j_{k-1}} u + \nabla^{2k-4} u \star R \\ &= (-1)^{k-1} A_{i_1 \dots i_{k-1} j_1 \dots j_{k-1}} \nabla^{j_{k-1} \dots j_1 i_1 \dots i_{k-1}} u + \nabla^{2k-4} u \star R \\ &= (-1)^{k-1} \nabla^{j_{k-1} \dots j_1} (A_{i_1 \dots i_{k-1} j_1 \dots j_{k-1}} \nabla^{i_1 \dots i_{k-1}} u) \\ &\quad + \nabla^{2k-4} u \star R + \sum_{l=1}^{k-1} \nabla^{2k-2-l} u \star \nabla^l A \end{aligned}$$

and then

$$P_g = \Delta_g^k + (-1)^{k-1} \nabla^{j_{k-1} \dots j_1} (A_{i_1 \dots i_{k-1} j_1 \dots j_{k-1}} \nabla^{i_1 \dots i_{k-1}}) + \text{lot}.$$

Iterating these steps yields (3). Integrating by parts then yields (4).  $\square$

Define the norm  $\|u\|_{H_k^2} := \sum_{l=0}^k \|\nabla^l u\|_2$  and the space  $H_k^2(M)$  as the completion of  $C^\infty(M)$  for the norm  $\|\cdot\|_{H_k^2}$ . As a consequence of (4), we get that the bilinear form  $(u, v) \mapsto \int_M u P_g v dv_g$  extends to a continuous symmetrical bilinear form on  $H_k^2(M) \times H_k^2(M)$ . We say that  $P_g$  is coercive if there exists  $c > 0$  such that

$$\int_M u P_g u dv_g \geq c \|u\|_2^2 \text{ for all } u \in H_k^2(M).$$

We then define the norm  $\|u\|_{P_g} := \sqrt{\int_M u P_g u dv_g}$  for all  $u \in H_k^2(M)$ .

**Proposition 2.** *Assume that  $P_g$  is coercive. Then  $\|\cdot\|_{P_g}$  is a norm on  $H_k^2$  equivalent to  $\|\cdot\|_{H_k^2}$ .*

*Proof.* Clearly  $\|\cdot\|_{P_g}$  is a norm and there exists  $C > 0$  such that  $\|\cdot\|_{P_g} \leq C \|\cdot\|_{H_k^2}$ . We now argue by contradiction and we assume that the two norms are not equivalent: then there exists  $(u_i)_{i \in \mathbb{N}} \in H_k^2(M)$  such that

$$(6) \quad \|u_i\|_{H_k^2} = 1 \text{ and } \|u_i\|_{P_g} = o(1)$$

when  $i \rightarrow +\infty$ . Up to a subsequence, still denoted as  $(u_i)$ , there exists  $u \in H_k^2(M)$  such that  $u_i \rightharpoonup u$  weakly in  $H_k^2(M)$  and  $u_i \rightarrow u$  strongly in  $H_{k-1}^2(M)$  when  $i \rightarrow +\infty$ . The coercivity of  $P_g$  yields  $\|u_i\|_2 = o(1)$  when  $i \rightarrow +\infty$ , and then  $u \equiv 0$ . Therefore, we have that

$$(7) \quad u_i \rightharpoonup 0 \text{ weakly in } H_k^2(M) \text{ and } u_i \rightarrow 0 \text{ strongly in } H_{k-1}^2(M)$$

when  $i \rightarrow +\infty$ . Consequently, (6) rewrites

$$(8) \quad \lim_{i \rightarrow +\infty} \int_M |\nabla^k u_i|_g^2 dv_g = 1 \text{ and } \lim_{i \rightarrow +\infty} \int_M (\Delta_g^{\frac{k}{2}} u_i)^2 dv_g = 0.$$

The contradiction comes from a Bochner-Lichnerowicz-Weitzenbock type formula. Here again, we use (5). We fix  $u, v \in C^\infty(M)$ : we have that (the notation  $a \equiv b$

means that the terms are equal up to a divergence)

$$\begin{aligned}
(\nabla^k u, \nabla^k v)_g &\equiv g^{\alpha_1 \beta_1} \dots g^{\alpha_k \beta_k} \nabla_{\alpha_1 \dots \alpha_k} u \nabla_{\beta_1 \dots \beta_k} v \\
&\equiv -g^{\alpha_1 \beta_1} \dots g^{\alpha_k \beta_k} \nabla_{\beta_1 \alpha_1 \dots \alpha_k} u \nabla_{\beta_2 \dots \beta_k} v \\
&\equiv -g^{\alpha_1 \beta_1} \dots g^{\alpha_k \beta_k} \nabla_{\alpha_2 \dots \alpha_k \beta_1 \alpha_1} u \nabla_{\beta_2 \dots \beta_k} v + \nabla^{k-1} u \star \nabla^{k-1} v \star R \\
&\equiv -g^{\alpha_2 \beta_2} \dots g^{\alpha_k \beta_k} \nabla_{\alpha_2 \dots \alpha_k} g^{\alpha_1 \beta_1} \nabla_{\beta_1 \alpha_1} u \nabla_{\beta_2 \dots \beta_k} v + \nabla^{k-1} u \star \nabla^{k-1} v \\
&\equiv g^{\alpha_2 \beta_2} \dots g^{\alpha_k \beta_k} \nabla_{\alpha_2 \dots \alpha_k} \Delta_g u \nabla_{\beta_2 \dots \beta_k} v + \nabla^{k-1} u \star \nabla^{k-1} v \star R \\
&\equiv (\nabla^{k-1} \Delta_g u, \nabla^{k-1} v)_g + \nabla^{k-1} u \star \nabla^{k-1} v \star R.
\end{aligned}$$

the same procedure applied to  $(\nabla^{k-1} v, \nabla^{k-1} \Delta_g u)_g$  yields

$$\begin{aligned}
(\nabla^k u, \nabla^k v)_g &\equiv (\nabla^{k-2} \Delta_g u, \nabla^{k-2} \Delta_g v)_g \\
&\quad + \nabla^{k-1} u \star \nabla^{k-1} v \star R + \nabla^{k-2} \Delta_g u \star \nabla^{k-2} v \star R.
\end{aligned}$$

Taking  $u = v = u_i$ , integrating over  $M$  and using (7) yields

$$\int_M |\nabla^k u_i|_g^2 dv_g = \int_M |\nabla^{k-2} \Delta_g u_i|_g^2 dv_g + o(1)$$

when  $i \rightarrow +\infty$ . Iterating this process and considering separately the cases  $k$  odd and  $k$  even, we get that

$$\int_M |\nabla^k u_i|_g^2 dv_g = \int_M (\Delta_g^{\frac{k}{2}} u_i)^2 dv_g + o(1)$$

when  $i \rightarrow +\infty$ . This is a contradiction with (8) and Proposition 2 is proved.  $\square$

### 3. GENERAL CONSIDERATIONS ON THE EQUIVARIANT YAMABE INVARIANT

We let  $(M, \mathcal{C})$  be a conformal Riemannian manifold. We let  $G \subset \text{Diff}(M)$  be a subgroup of diffeomorphisms of  $M$ . We define

$$\mathcal{C}_G := \{g \in \mathcal{C} / G \subset \text{Isom}_g(M)\},$$

and we assume that  $\mathcal{C}_G \neq \emptyset$ . In particular,  $G$  is contained in a compact group. Therefore, without loss of generality, we assume that  $G$  is a compact group. As easily checked, for any  $g \in \mathcal{C}_G$ , we have that

$$\mathcal{C}_G = \{e^{2\omega} g / \omega \in C_G^\infty(M)\}$$

where  $C_G^\infty(\Omega) = \{\omega \in C^\infty(M) / \omega \circ \sigma = \omega \text{ for all } \sigma \in G\}$  is the set of  $G$ -invariant smooth functions on  $M$ . We assume that  $n > 2k$ : in this context, it is more convenient to write a metric  $\hat{g} \in \mathcal{C}$  as  $\hat{g} = u^{\frac{4}{n-2k}} g$  with  $u \in C_+^\infty(M)$ . The relation between  $P_g$  and  $P_{\hat{g}}$  is given by (2). With the new parametrization, we have that

$$\mathcal{C}_G = \{u^{\frac{4}{n-2k}} g / u \in C_{G,+}^\infty(M)\},$$

where  $C_{G,+}^\infty(M) := \{u \in C_G^\infty(M) / u > 0\}$ . Let  $f \in C_{G,+}^\infty(M)$  be a smooth positive  $G$ -invariant function. By analogy with the Yamabe invariant, we define

$$\mu_f(\mathcal{C}_G) := \inf_{g \in \mathcal{C}_G} \frac{\int_M Q_g dv_g}{V_f(M, g)^{\frac{2}{2^*}}}$$

where  $V_f(M, g)$  is the  $f$ -volume defined in the introduction and  $2^* := \frac{2n}{n-2k}$ . We fix  $g \in \mathcal{C}_G$ : as easily checked, we have that

$$\mu_f(\mathcal{C}_G) = \frac{2}{n-2k} \inf_{u \in C_{G,+}^\infty(M)} I_g(u)$$

where

$$I_g(u) := \frac{\int_M u P_g u \, dv_g}{\left(\int_M f |u|^{2^*} \, dv_g\right)^{\frac{2}{2^*}}}$$

for all  $u \in H_k^2(M) \setminus \{0\}$ .

**Proposition 3.** *A metric  $g \in \mathcal{C}_G$  is a critical point of the functional  $g \mapsto \frac{\int_M Q_g \, dv_g}{V_f(M, g)^{\frac{2}{2^*}}}$  if and only if there exists  $\lambda \in \mathbb{R}$  such that  $Q_g = \lambda f$ .*

*Proof.* We fix  $g \in \mathcal{C}_G$  and  $t \mapsto g(t) \in \mathcal{C}_G$  a differentiable family of metrics conformal to  $g$  such that  $g(0) = g$ . In particular, there exists a differentiable family  $t \mapsto u(t) \in C_{G,+}^\infty(M)$  such that  $g(t) = u(t)^{\frac{4}{n-2k}} g$  and  $u(0) = 1$ . We define  $\dot{u} := u'(0)$ . Using the self-adjointness of  $P_g$ , straightforward computations yield

$$\frac{d}{dt} \left( \frac{\int_M Q_{g(t)} \, dv_{g(t)}}{V_f(M, g(t))^{\frac{2}{2^*}}} \right)_{t=0} = 2 \frac{\int_M \dot{u} (Q_g - f \bar{Q}_g^f) \, dv_g}{V_f(M, g(t))^{\frac{2}{2^*}}}$$

where

$$\bar{Q}_g^f = \frac{\int_M Q_g \, dv_g}{V_f(M, g)}.$$

Since  $u$  is  $G$ -invariant, the function  $\dot{u}$  ranges  $C_G^\infty(M)$ . Fix  $v \in C^\infty(M)$  and let  $v_G$  be its symmetrization via the Haar measure (which is well-defined since  $G$  is compact). We then define  $u(t) := 1 + tv_G$  for all  $t \in \mathbb{R}$ : since  $f$  and  $Q_g$  are  $G$ -invariant (this is a consequence of point (ii) of the characterization of  $P_g$  and of the definition of  $Q_g$ ), we get that

$$\int_M \dot{u} (Q_g - f \bar{Q}_g^f) \, dv_g = \int_M v_G (Q_g - f \bar{Q}_g^f) \, dv_g = \int_M v (Q_g - f \bar{Q}_g^f) \, dv_g.$$

Therefore,  $g$  is a critical point if and only if  $Q_g = f \bar{Q}_g^f$ . This proves Proposition 3.  $\square$

To carry out the analysis, coercivity and positivity preserving property are required. More precisely, we assume that there exists  $g \in \mathcal{C}$  such that

$$\left\{ \begin{array}{l} (C) \quad \text{the operator } P_g \text{ is coercive} \\ (PPP) \quad \text{for any } u \in C^\infty(M) \text{ such that } P_g \geq 0 \text{ then either } u > 0 \text{ or } u \equiv 0 \end{array} \right\}.$$

Note that (C) and (PPP) are conformally invariant: they hold for some  $g \in \mathcal{C}$  iff they hold for all  $g \in \mathcal{C}$ .

**Proposition 4.** *Assume that the metric  $g$  is Einstein with positive scalar curvature and  $n > 2k$ , then  $P_g$  satisfies (C) and (PPP).*

*Proof.* This relies essentially on the the explicit expression of the GJMS operator in the Einstein case: see Proposition 7.9 of Fefferman-Graham [13] and also Gover [14] for a proof via tractors. Indeed, for an Einstein metric  $g$ ,  $P_g$  expresses as an explicit product of second-order operators with constant coefficients depending only on the scalar curvature. For positive curvature, a direct consequence is that  $P_g$  satisfies (PPP) by  $k$  applications of the second-order comparison principle. Moreover, still in this case, since  $P_g = S(\Delta_g)$  with  $S$  a polynomial with positive constant coefficients, it follows from Hebey-Robert [19] that the first eigenvalue of  $P_g$  is  $S(0) > 0$  (0 is the first eigenvalue of  $\Delta_g$ ), and then  $P_g$  satisfies (C).  $\square$

Due to the lack of compactness of the embedding  $H_k^2(M) \hookrightarrow L^{2^*}(M)$ , it is standard to use the subcritical method. Given  $q \in (2, 2^*]$ , we define

$$I_{g,q}(u) := \frac{\int_M u P_g u \, dv_g}{\left(\int_M f |u|^q \, dv_g\right)^{\frac{2}{q}}}$$

for all  $u \in H_k^2(M) \setminus \{0\}$ , and

$$\mu_q := \inf_{u \in H_{k,G}^2(M) \setminus \{0\}} I_{g,q}(u),$$

where  $H_{k,G}^2(M) := \{u \in H_k^2(M) / u \circ \sigma = u \text{ a.e. for all } \sigma \in G\}$ . The first result is that  $\mu_q$  is achieved at a smooth positive minimizer when  $q < 2^*$ :

**Proposition 5.** *We fix  $q \in (2, 2^*)$ , we assume that (C) and (PPP) hold and that  $\mathcal{C}_G \neq \emptyset$ . Then  $\mu_q > 0$  is achieved. Moreover, there exists  $u_q \in C_{G,+}^\infty(M)$  a smooth positive function such that  $\mu_q = I_{g,q}(u_q)$  and*

$$(9) \quad P_g u_q = \mu_q f u_q^{q-1} \text{ in } M \text{ with } \int_M f u_q^q \, dv_g = 1.$$

*Proof.* Since  $P_g$  is coercive, the norms  $\|\cdot\|_{H_k^2}$  and  $\|\cdot\|_{P_g}$  are equivalent, and then, it follows from Hölder's and Sobolev's inequality that

$$(10) \quad \begin{aligned} \left(\int_M f |u|^q \, dv_g\right)^{\frac{2}{q}} &\leq \left(\int_M f \, dv_g\right)^{\frac{2}{q} - \frac{2}{2^*}} \left(\int_M f |u|^{2^*} \, dv_g\right)^{\frac{2}{2^*}} \\ &\leq C \left(\int_M f \, dv_g\right)^{\frac{2}{q} - \frac{2}{2^*}} \|u\|_{H_k^2}^2 \leq C' \left(\int_M f \, dv_g\right)^{\frac{2}{q} - \frac{2}{2^*}} \|u\|_{P_g}^2, \end{aligned}$$

and then  $I_{g,q}(u) \geq (C')^{-1} \left(\int_M f \, dv_g\right)^{-\frac{2}{q} + \frac{2}{2^*}}$  for all  $u \in H_k^2(M) \setminus \{0\}$ , and therefore  $\mu_q > 0$ . The existence of a minimizer is standard and we omit it. Let us take then  $u \in H_{k,G}^2(M) \setminus \{0\}$  be a minimizer. Without loss of generality, we can assume that  $\int_M f |u|^q \, dv_g = 1$ .

The Euler-Lagrange equation for  $I_{g,q}$  yields  $I'_{g,q}(u)\varphi = 0$  for all  $\varphi \in H_{k,G}^2(M)$ . Using the Haar measure and arguing as in the proof of Proposition 3 (see also [18]), we get that this equality holds for all  $\varphi \in H_k^2(M)$ . Since the exponent  $q$  is subcritical, we get with standard bootstrap arguments that  $u \in C_G^{2k}(M)$  and  $P_g u = \mu_q f |u|^{q-2} u$ . We are left with proving that  $u > 0$  or  $u < 0$ . We let  $v \in C_G^{2k}(M)$  be such that  $P_g v = |P_g u|$  in  $M$ . Since  $u \not\equiv 0$ , it follows from (PPP) that  $v \geq |u|$  and  $v > 0$ . Using again the definition of  $\mu_q$ , we have that

$$\begin{aligned} \mu_q &\leq \frac{\int_M v P_g v \, dv_g}{\left(\int_M f v^q \, dv_g\right)^{\frac{2}{q}}} = \mu_q \frac{\int_M f v |u|^{q-1} \, dv_g}{\left(\int_M f v^q \, dv_g\right)^{\frac{2}{q}}} \\ &\leq \mu_q \frac{\left(\int_M f v^q \, dv_g\right)^{\frac{1}{q}} \left(\int_M f |u|^q \, dv_g\right)^{\frac{q-1}{q}}}{\left(\int_M f v^q \, dv_g\right)^{\frac{2}{q}}} \\ &\leq \mu_q \left(\int_M f |u|^q \, dv_g\right)^{\frac{q-2}{q}} = \mu_q \text{ since } v \geq |u| \end{aligned}$$

Therefore equality holds everywhere and  $|u| = v > 0$ . In particular  $u$  does not change sign, and we can assume that it is positive. Bootstrap and regularity theory (see [1]) then yield  $u \in C_{G,+}^\infty(M)$ , and Proposition 5 is proved with  $u_q := u$ .  $\square$

**Proposition 6.** *We claim that  $\lim_{q \rightarrow 2^*} \mu_q = \mu_{2^*} = \frac{n-2k}{2} \mu_f(\mathcal{C}_G)$ .*

*Proof.* Using the Hölder's inequality (10), we get that

$$I_{g,2^*}(u) \leq I_{g,q}(u) V_f(M, g)^{\frac{2}{q} - \frac{2}{2^*}}$$

for all  $u \in H_k^2(M) \setminus \{0\}$ , and then  $\mu_{2^*} \leq \mu_q V_f(M, g)^{\frac{2}{q} - \frac{2}{2^*}}$ , which yields  $\mu_{2^*} \leq \liminf_{q \rightarrow 2^*} \mu_q$ . Conversely, fix  $\epsilon > 0$  and let  $u \in H_{k,G}^2(M) \setminus \{0\}$  be such that  $I_{g,2^*}(u) < \mu_{2^*} + \epsilon$ . Since  $\lim_{q \rightarrow 2^*} I_{g,q}(u) = I_{g,2^*}(u)$ , we then get that there exists  $q_0 < 2^*$  such that  $\mu_q < \mu_{2^*} + \epsilon$  for  $q \in (q_0, 2^*)$ , and then  $\limsup_{q \rightarrow 2^*} \mu_q \leq \mu_{2^*}$ . Therefore,  $\lim_{q \rightarrow 2^*} \mu_q = \mu_{2^*}$ .

For  $q \in (2, 2^*)$ , we define  $\mu_{q,+} := \inf\{I_{g,q}(u)/u \in H_{k,G}^2(M) \setminus \{0\} \text{ and } u \geq 0 \text{ a.e.}\}$ . Arguing as above, we get that  $\lim_{q \rightarrow 2^*} \mu_{q,+} = \mu_{2^*,+}$ . Since  $\mu_{q,+} = \mu_q$  for all  $q < 2^*$  with Proposition 5, we then get that  $\mu_{2^*} = \mu_{2^*,+}$ .

We claim that  $\mu_{2^*,+} = \frac{n-2k}{2} \mu_f(\mathcal{C}_G)$ . Indeed, via local convolutions with a positive kernel, we get that  $C_+^\infty(M)$  is dense in  $H_{k,+}^2(M)$  for the  $H_k^2$ -norm. A symmetrization via the Haar measure then yields that  $C_{G,+}^\infty(M)$  is dense in  $H_{k,G,+}^2(M)$ : clearly this yields  $\mu_{2^*,+} = \frac{n-2k}{2} \mu_f(\mathcal{C}_G)$ , and the claim is proved.  $\square$

We define  $D_k^2(\mathbb{R}^n)$  as the completion of  $C_c^\infty(\mathbb{R}^n)$  for the norm  $u \mapsto \|\Delta_\xi^{\frac{k}{2}} u\|_2$  and we define

$$(11) \quad \frac{1}{K(n, k)} := \inf_{u \in D_k^2(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} u)^2 dv_\xi}{\left(\int_{\mathbb{R}^n} |u|^{2^*} dv_\xi\right)^{\frac{2}{2^*}}}.$$

It follows from Sobolev's embedding theorem that  $K(n, k) > 0$ . Moreover, it follows from Lions [23] that the infimum is achieved by  $U : x \mapsto (1 + |x|^2)^{k - \frac{n}{2}}$ , and that all minimizers are compositions of  $U$  by translations and homotheties.

**Proposition 7.** *We have that*

$$(12) \quad \mu_f(\mathcal{C}_G) \leq \frac{2}{n-2k} \cdot \frac{|O_G(x)|^{\frac{2k}{n}}}{f(x)^{\frac{2}{2^*}} K(n, k)}$$

for all  $x \in M$ , where  $|O_G(x)|$  denotes the cardinal (possibly  $\infty$ ) of the orbit  $O_G(x)$ .

*Proof.* We fix  $x \in M$ . Without loss of generality, we assume that  $m := |O_G(x)| < +\infty$  (otherwise (12) is clear). We let  $\sigma_1 = Id_M, \dots, \sigma_m \in G$  be such that  $O_G(x) = \{x_1, \dots, x_m\}$  where  $x_i = \sigma_i(x)$  for all  $i \in \{1, \dots, m\}$  are distinct. We let  $u \in C_c^\infty(\mathbb{R}^n)$  be a radially symmetrical smooth function and we define for  $\epsilon > 0$  small the function

$$u_{\epsilon,i}(z) := u\left(\frac{1}{\epsilon} \exp_{x_i}^{-1}(z)\right) \text{ if } d_g(z, x_i) < i_g(M) \text{ and } 0 \text{ otherwise.}$$

Clearly,  $u_{\epsilon,i} \in C^\infty(M)$  for  $\epsilon > 0$  small enough. We now define

$$u_\epsilon := \sum_{i=1}^m u_{\epsilon,i}.$$

As one checks, since  $u$  is radially symmetrical, we have that  $u_\epsilon \in C_G^\infty(M)$  is  $G$ -invariant for  $\epsilon > 0$  small enough.

Let us compute  $I_{g,2^*}(u_\epsilon)$ . We fix  $\delta \in (0, i_g(M))$  and we define the metric  $g_\epsilon := (\exp_g^*)(\epsilon)$ : since the elements of  $G$  are isometries (and then  $P_g = P_{\sigma^*g} = \sigma^*P_g$  for all  $\sigma \in G$ ) and the  $u_{\epsilon,i}$ 's have disjoint supports, we get that

$$\begin{aligned} \int_M u_\epsilon P_g u_\epsilon dv_g &= \sum_{i,j=1}^m \int_M u_{\epsilon,i} P_g u_{\epsilon,j} dv_g = \sum_{i=1}^m \int_M u_{\epsilon,i} P_g u_{\epsilon,i} dv_g \\ &= \sum_{i=1}^m \int_{B_\delta(x_i)} u_{\epsilon,1} \circ \sigma_i^{-1} P_g (u_{\epsilon,1} \circ \sigma_i^{-1}) dv_g \\ &= m \int_{B_\delta(x)} u_{\epsilon,1} P_g u_{\epsilon,1} dv_g = m \epsilon^{n-2k} \int_{B_{\epsilon^{-1}\delta}(0)} u P_{g_\epsilon} u dv_{g_\epsilon} \end{aligned}$$

since  $\lim_{\epsilon \rightarrow 0} g_\epsilon = \xi$ , the Euclidean metric, we get that

$$\int_M u_\epsilon P_g u_\epsilon dv_g = \epsilon^{n-2k} \left( m \int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} u)^2 dv_\xi + o(1) \right)$$

when  $\epsilon \rightarrow 0$ . Similarly, using the  $G$ -invariance of  $f$ , we get that

$$\int_M f |u_\epsilon|^{2^*} dv_g = \epsilon^n \left( m f(x) \int_{\mathbb{R}^n} |u|^{2^*} dv_\xi + o(1) \right)$$

when  $\epsilon \rightarrow 0$ , and then

$$I_{g,2^*}(u_\epsilon) = \frac{m^{\frac{2k}{n}}}{f(x)^{\frac{2^*}{2^*}}} \cdot \frac{\int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} u)^2 dv_\xi}{\left( \int_{\mathbb{R}^n} |u|^{2^*} dv_\xi \right)^{\frac{2^*}{2^*}}} + o(1)$$

when  $\epsilon \rightarrow 0$ . Therefore, since  $\mu_f(\mathcal{C}_G) = \mu_{2^*}$ , taking the limit  $\epsilon \rightarrow 0$  and taking the infimum on the  $u$ 's, we get that

$$\mu_{2^*} \leq \frac{|O_G(x)|^{\frac{2k}{n}}}{f(x)^{\frac{2^*}{2^*}}} \cdot \inf_{u \in C_c^\infty(\mathbb{R}^n) \setminus \{0\}} \text{radial} \frac{\int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} u)^2 dv_\xi}{\left( \int_{\mathbb{R}^n} |u|^{2^*} dv_\xi \right)^{\frac{2^*}{2^*}}}$$

It follows from Lions [23] that the infimum  $K(n, k)^{-1}$  is achieved at smooth radially symmetrical functions, therefore we obtain (12).  $\square$

#### 4. THE QUANTIZATION OF THE FORMATION OF SINGULARITIES

The objective of this section is to prove the following result:

**Theorem 2.** *Let  $(M, \mathcal{C})$  be a conformal Riemannian manifold of dimension  $n \geq 3$  and let  $k \in \mathbb{N}^*$  be such that  $2k < n$ . Let  $G$  be a group of diffeomorphisms such that  $\mathcal{C}_G \neq \emptyset$  and let  $f \in C_{G,+}^\infty(M)$  be a positive  $G$ -invariant function. Assume that there exists  $g \in \mathcal{C}$  such that  $P_g$  satisfies (C) and (PPP). For any  $q \in (2, 2^*)$ , we let  $u_q \in C_{G,+}^\infty(M)$  as in Proposition 5. Then:*

(i) *either  $\limsup_{q \rightarrow +\infty} \|u_q\|_\infty = +\infty$ , and there exists  $x \in M$  such that  $\nabla f(x) = 0$  and*

$$\mu_f(\mathcal{C}_G) = \frac{2}{n-2k} \cdot \frac{|O_G(x)|^{\frac{2k}{n}}}{f(x)^{\frac{2^*}{2^*}} K(n, k)},$$

(ii) *or  $\|u_q\|_\infty \leq C$  for all  $q < 2^*$ , and there exists  $u \in C_{G,+}^\infty(M)$  such that  $\lim_{q \rightarrow 2^*} u_q = u$  in  $C^{2k}(M)$  and  $P_g u = \frac{n-2k}{2} \mu_f(\mathcal{C}_G) f u^{2^*-1}$  in  $M$ . In particular, there exists  $\hat{g} \in \mathcal{C}_G$  such that  $Q_{\hat{g}} = f$  and the infimum  $\mu_f(\mathcal{C}_G)$  is achieved.*

This type of result is classical. The proof of Theorem 2 goes through nine steps. For  $q \in (2, 2^*)$ , we let  $u_q \in C_{G,+}^\infty(M)$  be as in Proposition 5 (this is relevant since (C) and (PPP) hold).

**Step 1:** We assume that there exists  $C > 0$  such that  $\|u_q\|_\infty \leq C$  for all  $q < 2^*$ . We claim that (ii) of Theorem 2 holds.

We prove the claim. Indeed, it follows from (9), Proposition 6, the uniform bound of  $(u_q)_q$  in  $L^\infty$  and standard elliptic (see for instance [1]), that, up to a subsequence, there exists  $u \in C^{2k}(M)$  nonnegative such that  $\lim_{q \rightarrow 2^*} u_q = u$  in  $C^{2k}(M)$ : therefore,  $P_g u = \mu_{2^*} f u^{2^*-1}$  in  $M$  and  $\int_M f u^{2^*} dv_g = 1$ . In particular,  $P_g u \geq 0$  and  $u \not\equiv 0$ , and then it follows from (PPP) that  $u > 0$ . Since  $u_q$  is  $G$ -invariant for all  $q \in (2, 2^*)$ , we get that  $u \in C_{G,+}^\infty(M)$ . Moreover,  $I_g(u) = \mu_{2^*}$ , and then the metric  $u^{\frac{4}{n-2k}} g$  is extremal for  $\mu_f(\mathcal{C}_G)$ : it then follows from Proposition 3 that  $\hat{g} := (\mu_f(\mathcal{C}_G))^{1/k} u^{\frac{4}{n-2k}} g$  is also an extremal for  $\mu_f(\mathcal{C}_G)$  and  $Q_{\hat{g}} = f$ . This ends Step 1.

From now on, we assume that  $\limsup_{q \rightarrow 2^*} \|u_q\|_\infty = +\infty$ . For the sake of clearness, we will write  $(u_q)$  even for a subsequence of  $(u_q)$ . For any  $q \in (2, 2^*)$ , we let  $x_q \in M$  be such that

$$(13) \quad u_q(x_q) = \max_M u_q \text{ and } \lim_{q \rightarrow 2^*} u_q(x_q) = +\infty.$$

We define

$$\alpha_q := u_q(x_q)^{-\frac{2}{n-2k}} \text{ and } \beta_q := \alpha_q^{\frac{q-2}{2^*-2}}$$

for all  $q \in (2, 2^*)$ . It follows from (13) that

$$(14) \quad \lim_{q \rightarrow 2^*} \alpha_q = 0 \text{ and } \beta_q \geq \alpha_q \text{ for } q \rightarrow 2^*.$$

We define

$$(15) \quad \tilde{u}_q(x) := \alpha_q^{\frac{n-2k}{2}} u_q(\exp_{x_q}(\beta_q x))$$

for all  $x \in B_{\beta_q^{-1}\delta}(0)$ , where  $\delta \in (0, i_g(M))$ .

**Step 2:** We claim that there exists  $\tilde{u} \in C^{2k}(\mathbb{R}^n)$  such that  $\lim_{q \rightarrow 2^*} \tilde{u}_q = \tilde{u}$  in  $C_{loc}^{2k}(\mathbb{R}^n)$  where

$$(16) \quad 0 \leq \tilde{u} \leq \tilde{u}(0) = 1 \text{ and } \Delta_\xi^k \tilde{u} = \mu_{2^*} f(x_\infty) \tilde{u}^{2^*-1} \text{ in } \mathbb{R}^n,$$

and  $x_\infty := \lim_{q \rightarrow 2^*} x_q$ .

We prove the claim. It follows of the naturality of the geometric operator  $P_g$  and of (9) that

$$(17) \quad P_{g_q} \tilde{u}_q = \mu_q f(\exp_{x_q}(\beta_q \cdot)) \tilde{u}_q^q \text{ in } B_{\beta_q^{-1}\delta}(0)$$

for all  $q \in (2, 2^*)$ , where  $g_q := (\exp_{x_q}^* g)(\beta_q \cdot)$ . In particular, since the exponential is a normal chart at  $x_q$ , we have that  $\lim_{q \rightarrow 2^*} g_q = \xi$  in  $C_{loc}^{2k}(\mathbb{R}^n)$ . Since  $0 \leq \tilde{u}_q \leq \tilde{u}_q(0) = 1$ , it follows from standard elliptic theory (see for instance [1]) that there exists  $\tilde{u} \in C^{2k}(\mathbb{R}^n)$  such that  $\lim_{q \rightarrow 2^*} \tilde{u}_q = \tilde{u}$  in  $C_{loc}^{2k}(\mathbb{R}^n)$ . In addition, using that  $P_\xi = \Delta_\xi^k$ , passing to the limit in (17) yields (16). This proves the claim.

**Step 3:** We claim that there exists  $C > 0$  such that

$$(18) \quad \alpha_q \leq \beta_q \leq C \alpha_q$$

when  $q \rightarrow 2^*$ .

We prove the claim. We fix  $R > 0$  and we let  $q$  be in  $(2, 2^*)$ : a change of variable and Sobolev's embedding yields

$$\int_{B_R(0)} \tilde{u}_q^{2^*} dv_{g_q} = \left(\frac{\alpha_q}{\beta_q}\right)^n \int_{B_{R\beta_q}(x_q)} u_q^{2^*} dv_g \leq C \left(\frac{\alpha_q}{\beta_q}\right)^n \|u_q\|_{P_g}^{2^*}$$

for all  $q \in (2, 2^*)$ . Using (9) and Proposition 6, letting  $q \rightarrow 2^*$ , we get that

$$\left(\frac{\beta_q}{\alpha_q}\right)^n \leq \frac{C'}{\int_{B_R(0)} \tilde{u}^{2^*} dv_\xi} + o(1)$$

when  $q \rightarrow 2^*$ . Since  $\tilde{u}(0) > 0$ , we get that  $\beta_q = O(\alpha_q)$  when  $q \rightarrow 2^*$ . This inequality combined with (14) yields (18). This proves the claim.

**Step 4:** We claim that  $\tilde{u} \in D_k^2(\mathbb{R}^n)$ .

We prove the claim. Indeed, for all  $i \in \{0, \dots, k\}$ , it follows from (18) and a change of variable that  $\|\nabla^i \tilde{u}_q\|_{L^{p_i}(B_R(0))} \leq C \|\nabla^i u_q\|_{L^{p_i}(B_{R\beta_q}(x_q))} \leq \|\nabla^i \tilde{u}_q\|_{L^{p_i}(M)}$  for all  $q \in (2, 2^*)$ , all  $R > 0$  and where  $p_i := \frac{2n}{n-2k+2i}$ . It follows from Sobolev's inequalities that the right-hand-side is dominated by  $\|u_q\|_{H_k^2}$ , and therefore, letting  $q \rightarrow 2^*$  and  $R \rightarrow +\infty$  yields  $\nabla^i \tilde{u} \in L^{p_i}(\mathbb{R}^n)$  for all  $i \in \{0, \dots, k\}$ . We let  $\eta \in C_c^\infty(\mathbb{R}^n)$  be such that  $\eta|_{B_1(0)} \equiv 1$ : as easily checked,  $(\eta(m^{-1}\cdot)\tilde{u})_m \in C_c^\infty(\mathbb{R}^n)$  is a Cauchy sequence for the  $D_k^2$ -norm, and therefore  $\tilde{u} \in D_k^2(\mathbb{R}^n)$ . This proves the claim.

**Step 5:** We claim that

$$(19) \quad \mu_{2^*} = \frac{|O_G(x_\infty)|^{\frac{2k}{n}}}{f(x_\infty)^{\frac{2}{2^*}} K(n, k)} \text{ and } \lim_{\alpha \rightarrow +\infty} \frac{\beta_q}{\alpha_q} = 1$$

We prove the claim. Since  $\tilde{u} \in D_k^2(\mathbb{R}^n)$ , we multiply (16) by  $\tilde{u}$  and integrate to get

$\int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} \tilde{u})^2 dv_\xi = \mu_{2^*} f(x_\infty) \int_{\mathbb{R}^n} \tilde{u}^{2^*} dv_\xi$ . Since  $\tilde{u} \not\equiv 0$ , plugging this identity in the Sobolev inequality (11) yields

$$(20) \quad \int_{\mathbb{R}^n} \tilde{u}^{2^*} dv_\xi \geq \left( \frac{1}{\mu_{2^*} f(x_\infty) K(n, k)} \right)^{\frac{2^*}{2^*-2}}$$

We let  $m := |O_G(x_\infty)|$  if  $|O_G(x_\infty)| < \infty$ , and any  $m \in \mathbb{N} \setminus \{0\}$  otherwise. We let  $\sigma_1, \dots, \sigma_m \in G$  be such that  $\sigma_i(x_\infty) \neq \sigma_j(x_\infty)$  for all  $i, j \in \{1, \dots, m\}$ ,  $i \neq j$ . We fix  $\delta < \min_{i \neq j} \{d_g(z, z') / z \neq z' \in O_G(x_\infty)\}$ . The  $G$ -invariance yields

$$(21) \quad \begin{aligned} 1 &= \int_M f u_q^q dv_g \geq \sum_{i=1}^m \int_{B_\delta(\sigma_i(x_\infty))} f u_q^q dv_g = m \int_{B_\delta(x_\infty)} f u_q^q dv_g \\ &\geq m \int_{B_{R\beta_q}(x_q)} f u_q^q dv_g = m \left(\frac{\beta_q}{\alpha_q}\right)^{n-2k} \int_{B_R(0)} f(\exp_{x_q}(\beta_q \cdot)) \tilde{u}_q^q dv_{g_q} \end{aligned}$$

for all  $q \in (2, 2^*)$  and all  $R > 0$ . Letting  $q \rightarrow +\infty$ , and then  $R \rightarrow +\infty$  and using (20), we get that

$$1 \geq \left( \lim_{q \rightarrow 2^*} \frac{\beta_q}{\alpha_q} \right)^{n-2k} \frac{m f(x_\infty)}{(\mu_{2^*} f(x_\infty) K(n, k))^{\frac{2^*}{2^*-2}}}.$$

In particular, since  $\beta_q \geq \alpha_q$  with (18), we get an upper-bound for  $m$ , and therefore  $|O_G(x)| < \infty$ , and we take  $m = |O_G(x)|$ . The inequality rewrites

$$\mu_f(\mathcal{C}_G) \geq \frac{2}{n-2k} \cdot \frac{|O_G(x_\infty)|^{\frac{2k}{n}}}{f(x_\infty)^{\frac{2}{2^*}} K(n,k)} \cdot \left( \lim_{q \rightarrow 2^*} \frac{\beta_q}{\alpha_q} \right)^{\frac{2k(n-2k)}{n}}.$$

It then follows from (12) and (18) that (19) holds. Moreover, we also get that equality holds in (20) and that  $\tilde{u}$  is an extremal for the Sobolev inequality (11). This proves the claim.

**Step 6:** We claim that

$$(22) \quad fu_q^q dv_g \rightharpoonup \frac{1}{|O_G(x)|} \delta_{O_G(x)} \text{ in the sense of measure when } q \rightarrow 2^*.$$

We prove the claim. Since equality holds in (20), that  $\lim_{q \rightarrow 2^*} \frac{\alpha_q}{\beta_q} = 1$  and that (19) holds, we get with a change of variables that

$$(23) \quad \lim_{R \rightarrow +\infty} \lim_{q \rightarrow 2^*} \int_{B_{R\beta_q}(x_q)} fu_q^q dv_g = f(x_\infty) \int_{\mathbb{R}^n} \tilde{u}^{2^*} dv_\xi = \frac{1}{m}.$$

For  $\delta > 0$ , we let  $B_\delta(O_G(x_\infty))$  be the union of balls of radius  $\delta$  centered at the orbit. Therefore, since  $\int_M fu_q^q dv_g = 1$ , (21), (23) and the  $G$ -invariance yield

$$(24) \quad \lim_{q \rightarrow 2^*} \int_{M \setminus B_\delta(O_G(x_\infty))} fu_q^q dv_g = 0$$

for all  $\delta > 0$ . Consequently,  $\lim_{q \rightarrow 2^*} \int_{B_\delta(z)} fu_q^q dv_g = \frac{1}{m}$  for all  $\delta > 0$  small enough and all  $z \in O_G(x)$ . Assertion (22) then follows. This proves the claim.

**Step 7:** We claim that there exists  $C > 0$  such that

$$(25) \quad d(x, O_G(x_q))^{\frac{n-2k}{2}} u_q(x) \leq C$$

for all  $x \in M$  and all  $q \in (2, 2^*)$ .

We prove the claim. This pointwise inequality has its origins in Druet [10]. We define  $w_q(x) := d(x, O_G(x_q))^{\frac{n-2k}{2}} u_q(x)$  for all  $q \in (2, 2^*)$  and all  $x \in M$ . We argue by contradiction and assume that  $\lim_{q \rightarrow 2^*} \|w_q\|_\infty = +\infty$ . We define  $(y_q)_{q \in (2, 2^*)} \in M$  such that

$$(26) \quad \max_{y \in M} w_q(y) = w_q(y_q) \rightarrow +\infty$$

when  $q \rightarrow 2^*$ . We define  $\gamma_q := u_q(y_q)^{-\frac{2}{n-2k}}$  for all  $q \in (2, 2^*)$ . It follows from (26) that

$$(27) \quad \lim_{q \rightarrow 2^*} u_q(y_q) = +\infty \text{ and } \lim_{q \rightarrow 2^*} \gamma_q = 0.$$

As easily checked, coming back to the definitions of  $\alpha_q$  and  $\beta_q$ , it follows from (19) that  $\lim_{q \rightarrow 2^*} u_q(x_q)^{2^*-q} = 1$ . Therefore, since  $u_q(y_q) \leq u_q(x_q)$  for all  $q$  and (27) holds, we get that  $\lim_{q \rightarrow 2^*} \gamma_q^{2^*-q} = 1$ . We define

$$\bar{u}_q(x) := \gamma_q^{\frac{n-2k}{2}} u_q(\exp_{y_q}(\gamma_q x))$$

for all  $q \in (2, 2^*)$  and all  $x \in B_{\delta\gamma_q^{-1}}(0)$  where  $\delta \in (0, i_g(M))$ . Arguing as in Step 2 and using that  $\lim_{q \rightarrow 2^*} \gamma_q^{2^*-q} = 1$ , we get that

$$(28) \quad P_{\bar{g}_q} \bar{u}_q = \mu_q(1 + o(1)) f(\exp_{y_q}(\gamma_q \cdot)) \bar{u}_q^q \text{ in } B_{\delta\gamma_q^{-1}}(0)$$

for all  $q \in (2, 2^*)$ , where  $\lim_{q \rightarrow 2^*} o(1) = 0$  uniformly. We fix  $R > 0$ . It follows from the definition (26) of  $w_q$  and  $y_q$  that

$$(29) \quad d(\exp_{y_q}(\gamma_q x), O_G(x_q))^{\frac{n-2k}{2}} \bar{u}_q(x) \leq d(y_q, O_G(x_q))^{\frac{n-2k}{2}}$$

for all  $x \in B_R(0)$  and  $q \in (2, 2^*)$ . The limit  $w_q(y_q) \rightarrow +\infty$  when  $q \rightarrow 2^*$  rewrites  $\lim_{q \rightarrow 2^*} \gamma_q^{-1} d_g(y_q, O_G(x_q)) = +\infty$ : therefore, there exists  $q_0 \in (2, 2^*)$  such that  $d(\exp_{y_q}(\gamma_q x), O_G(x_q)) \geq d(y_q, O_G(x_q))/2$  for all  $x \in B_R(0)$  and all  $q \in (q_0, 2^*)$ , and it follows from (29) that  $0 \leq \bar{u}_q(x) \leq 2^{\frac{n-2k}{2}}$  for all  $x \in B_R(0)$  and all  $q \in (q_0, 2^*)$ . It then follows from (28) and standard elliptic theory (see for instance [1]) that there exists  $\bar{u} \in C_{loc}^{2k}(\mathbb{R}^n)$  such that  $\lim_{q \rightarrow 2^*} \bar{u}_q = \bar{u}$  in  $C_{loc}^{2k}(\mathbb{R}^n)$ . Moreover,  $\bar{u} \geq 0$  and  $\bar{u}(0) = \lim_{q \rightarrow 2^*} \bar{u}_q(0) = 1$ , and then  $\bar{u} \not\equiv 0$ . In particular,

$$(30) \quad \lim_{R \rightarrow +\infty} \lim_{q \rightarrow 2^*} \int_{B_{R\gamma_q}(y_q)} f u_q^q dv_g = f(y_\infty) \int_{\mathbb{R}^n} \bar{u}^{2^*} dv_\xi$$

where  $y_\infty := \lim_{q \rightarrow 2^*} y_q$ . Since  $\lim_{q \rightarrow 2^*} \gamma_q^{-1} d_g(y_q, O_G(x_q)) = +\infty$  and  $\gamma_q \geq \alpha_q = (1 + o(1))\beta_q$  when  $q \rightarrow 2^*$ , we get that for any  $R, R' > 0$

$$B_{R\gamma_q}(y_q) \cap B_{R'\beta_q}(O_G(x_q)) = \emptyset$$

where  $q \rightarrow 2^*$ . We let  $\sigma_1, \dots, \sigma_m \in G$  be such that  $O_G(x_\infty) = \{\sigma_1(x_\infty), \dots, \sigma_m(x_\infty)\}$  and these points are distinct: as easily checked, we have that  $\cup_{i=1}^m B_{R'\beta_q}(\sigma_i(x_q)) \subset B_{R'\beta_q}(O_G(x_q))$  and the balls are distinct. Therefore

$$1 = \int_M f u_q^q dv_g \geq \int_{B_{R\gamma_q}(y_q)} f u_q^q dv_g + \sum_{i=1}^m \int_{B_{R'\beta_q}(\sigma_i(x_q))} f u_q^q dv_g$$

for all  $q \in (2, 2^*)$  and  $R, R' > 0$ . Letting  $q \rightarrow 2^*$ , then  $R, R' \rightarrow +\infty$  and using (23) and (30), we get that

$$1 \geq f(y_\infty) \int_{\mathbb{R}^n} \bar{u}^{2^*} dv_\xi + 1,$$

a contradiction since  $\bar{u} \not\equiv 0$ . Then (26) does not hold and therefore (25) holds. This proves the claim.

**Step 8:** We claim that

$$(31) \quad \lim_{q \rightarrow 2^*} u_q = 0 \text{ in } C_{loc}^{2k}(M \setminus O_G(x_\infty)).$$

We prove the claim. We fix  $\Omega \subset\subset M \setminus O_G(x_\infty)$  a relatively compact subset. It follows from Step 7 that there exists  $C(\Omega) > 0$  such that  $u_q(x) \leq C(\Omega)$  for all  $x \in \Omega$  and all  $q \in (2, 2^*)$ . It then follows from (9) and standard elliptic theory (see for instance [1]) that there exists  $u_\infty \in C^\infty(M \setminus O_G(x_\infty))$  such that  $\lim_{q \rightarrow 2^*} u_q = u_\infty$  in  $C_{loc}^{2k}(\Omega)$ . It then follows from (24) that  $u_\infty \equiv 0$ , and then (31) holds. This proves the claim.

The following remark will be useful in the sequel: since  $\|u_q\|_{P_g}^2 = \mu_q \rightarrow \mu_{2^*}$  when  $q \rightarrow 2^*$  and  $u_q \rightarrow 0$  in  $C^{2k}$  outside the orbit, we get from the compact embedding  $H_k^2 \hookrightarrow H_{k-1}^2$  that

$$(32) \quad \lim_{q \rightarrow 2^*} u_q = 0 \text{ strongly in } H_{k-1}^2(M)$$

**Step 9:** We claim that  $\nabla f(x_\infty) = 0$ .

We prove the claim. Indeed, this is equivalent to proving that  $X(f)(x_\infty) = 0$  for all vector field  $X$  on  $M$ . With no loss of generality, we assume that  $\nabla X(x_\infty) = 0$  (this is always possible by modifying  $X$  in a normal chart at  $x_\infty$ ) and that  $X$  has its support in  $B_\delta(x_\infty)$ , where  $\delta < \min\{d_g(z, z')/z \neq z' \in O_G(x_\infty)\}$ . We are going to estimate  $\int_M X(u_q)\Delta_g^k u_q dv_g$  with two different methods. We detail here the case  $k = 2l$  even and we leave the odd case to the reader.

Integrating by parts, we have that

$$\begin{aligned} \int_M X(u_q)\Delta_g^{2l} u_q dv_g &= \int_M \Delta_g^l (X(u_q))\Delta_g^l u_q dv_g = \int_M X(\Delta_g^l u_q)\Delta_g^l u_q dv_g \\ &+ \sum_{i=1}^l \int_M \Delta_g^l u_q \Delta_g^{l-i} (\Delta_g(X(\Delta_g^{i-1} u_q)) - X(\Delta_g^i u_q)) dv_g. \end{aligned}$$

Using the explicit contraction in (5), we get that

$$\Delta_g(X(v)) - X(\Delta_g v) = (\Delta_g X)(\nabla v) - 2(\nabla X, \nabla^2 v) - Ric_g(X, \nabla v),$$

where  $v \in C^\infty(M)$  and  $\Delta_g X$  is the rough Laplacian, that is  $(\Delta_g X)^\alpha = -g^{ij}\nabla_{ij}X^\alpha$ . Therefore, we have that (for convenience, we omit the curvature tensor  $R$ )

$$\Delta_g(X(\Delta_g^{i-1} u_q)) - X(\Delta_g^i u_q) = \nabla^2 X \star \nabla^{2i-1} u_q + \nabla X \star \nabla^{2i} u_q + X \star \nabla^{2i-1} u_q$$

for all  $i \in \{1, \dots, l\}$ , and then, denoting as  $\nabla^{\{m\}}T$  any linear combination of covariant derivatives of  $T$  up to order  $m$ , we get that

$$\begin{aligned} &\Delta_g^{l-i} (\Delta_g(X(\Delta_g^{i-1} u_q)) - X(\Delta_g^i u_q)) \\ &= \Delta_g^{l-i} (\nabla^2 X \star \nabla^{2i-1} u_q + \nabla X \star \nabla^{2i} u_q + X \star \nabla^{2i-1} u_q) \\ &= \nabla^{\{2l-2i+2\}} X \star \nabla^{\{2l-1\}} u_q + \nabla X \star \nabla^{2l} u_q, \end{aligned}$$

and then

$$\begin{aligned} \int_M X(u_q)\Delta_g^{2l} u_q dv_g &= \int_M X(\Delta_g^l u_q)\Delta_g^l u_q dv_g \\ &+ \int_M \Delta_g^l u_q (\nabla^{\{2+2l\}} X \star \nabla^{\{2l-1\}} u_q + \nabla X \star \nabla^{2l} u_q) dv_g \\ &= \int_M X(\Delta_g^l u_q)\Delta_g^l u_q dv_g + \int_M \Delta_g^l u_q \star \nabla^{\{2l+2\}} X \star \nabla^{\{2l-1\}} u_q dv_g \\ &+ \int_M \Delta_g^l u_q \nabla X \star \nabla^{2l} u_q dv_g \end{aligned}$$

Since  $k = 2l$ , it follows from (32) and the Cauchy-Schwarz inequality that

$$\int_M \Delta_g^l u_q \star \nabla^{\{2l+2\}} X \star \nabla^{\{2l-1\}} u_q dv_g = O\left(\|u_q\|_{H_k^2} \|u_q\|_{H_{k-1}^2}\right) = o(1)$$

when  $q \rightarrow 2^*$ . Moreover, since  $\nabla X(x_\infty) = 0$  and (31) holds, we get that

$$\int_M \Delta_g^l u_q \nabla X \star \nabla^{2l} u_q dv_g = o(\|u_q\|_{H_k^2}) = o(1)$$

when  $q \rightarrow 2^*$ . Therefore, integrating by parts, we get that

$$\begin{aligned} \int_M X(u_q) \Delta_g^{2l} u_q dv_g &= \int_M X(\Delta_g^l u_q) \Delta_g^l u_q dv_g + o(1) \\ &= \int_M X \left( \frac{(\Delta_g^l u_q)^2}{2} \right) dv_g + o(1) = - \int_M \frac{\operatorname{div}_g(X)}{2} (\Delta_g^l u_q)^2 + o(1) \end{aligned}$$

when  $q \rightarrow 2^*$  and where  $\operatorname{div}_g(X) = \nabla_i X^i$ . Since  $\nabla X(x_\infty) = 0$ , (31) holds and  $\|u_q\|_{H_k^2} \leq C$  for all  $q \rightarrow 2^*$ , we get that the right-hand-side above goes to zero, and then

$$(33) \quad \lim_{q \rightarrow 2^*} \int_M X(u_q) \Delta_g^{2l} u_q dv_g = 0.$$

We now estimate  $\int_M X(u_q) \Delta_g^{2l} u_q dv_g$  using equation (9). It follows from (4) that

$$\int_M X(u_q) P_g u_q dv_g = \int_M \Delta_g^l X(u_q) \Delta_g^l u_q dv_g + \sum_{l=0}^{k-1} \int_M A_{(l)}(\nabla^l X(u_q), \nabla^l u_q) dv_g$$

It then follows from (32) and an integration by parts that

$$\int_M X(u_q) \Delta_g^{2l} u_q dv_g = \int_M X(u_q) P_g u_q dv_g + o(1)$$

when  $q \rightarrow 2^*$ . We now use equation (9) to get that

$$\begin{aligned} \int_M X(u_q) \Delta_g^{2l} u_q dv_g &= \mu_q \int_M f X(u_q) u_q^{q-1} dv_g + o(1) \\ &= \mu_q \int_M f X \left( \frac{u_q^q}{q} \right) dv_g = - \frac{\mu_q}{q} \int_M (X(f) + f \operatorname{div}_g(X)) u_q^q dv_g + o(1) \end{aligned}$$

when  $q \rightarrow 2^*$ . It now follows from Proposition 6, (22) and  $\nabla X(x_\infty) = 0$  that

$$\lim_{q \rightarrow 2^*} \int_M X(u_q) \Delta_g^{2l} u_q dv_g = - \frac{\mu_{2^*} X(f)(x_\infty)}{2^* |O_G(x_\infty)| f(x_\infty)}.$$

This limit combined with (33) yields  $X(f)(x_\infty) = 0$ , which, as already mentioned, proves that  $\nabla f(x_\infty) = 0$ . This ends Step 9.

Theorem 2 is a direct consequence of Steps 1 to 9.

As a direct byproduct of Theorem 2, we have the following proposition:

**Proposition 8.** *Let  $(M, \mathcal{C})$  be a conformal Riemannian manifold of dimension  $n \geq 3$  and let  $k \in \mathbb{N}^*$  be such that  $2k < n$ . Let  $G$  be a group of diffeomorphisms such that  $\mathcal{C}_G \neq \emptyset$  and let  $f \in C_{G,+}^\infty(M)$  be a positive  $G$ -invariant function. Assume that there exists  $g \in \mathcal{C}_G$  such that  $P_g$  satisfies (C) and (PPP). We assume that*

$$\mu_f(\mathcal{C}_G) < \frac{2}{n-2k} \cdot \frac{|O_G(x)|^{\frac{2k}{n}}}{f(x)^{\frac{2}{2^*}} K(n, k)},$$

for all  $x \in M$ . Then there exists  $\hat{g} \in \mathcal{C}_G$  such that  $Q_{\hat{g}} = f$  and the infimum  $\mu_f(\mathcal{C}_G)$  is achieved.

A similar result was proved in [18] for  $k = 1$  and in [2] when  $n = 2k$ .

## 5. THE CASE OF THE SPHERE

We consider here the standard unit  $n$ -sphere  $\mathbb{S}^n$  endowed with its standard round metric  $h$  and the associated conformal class  $\mathcal{C} := [h]$ .

**Proposition 9.** *Let  $G$  be a subgroup of  $\text{Isom}_h(\mathbb{S}^n)$  and let  $f \in C_{G,+}^\infty(\mathbb{S}^n)$  be a smooth positive function. Let  $p \in \mathbb{S}^n$  be such that  $\nabla^i f(p) = 0$  for all  $i \in \{1, \dots, n-2k\}$  and  $|O_G(p)| \geq 2$ . Then*

$$\mu_f(\mathcal{C}_G) < \frac{2}{n-2k} \cdot \frac{|O_G(p)|^{\frac{2k}{n}}}{K(n,k)f(p)^{\frac{2}{2^*}}}.$$

*Proof.* Given  $\lambda > 1$  and  $x_0 \in \mathbb{S}^n$ , we let  $\phi_\lambda : \mathbb{S}^n \rightarrow \mathbb{S}^n$  be such that  $\phi_\lambda(x) = \pi_{x_0}^{-1}(\lambda^{-1}\pi_{x_0}(x))$  if  $x \neq x_0$  and  $\phi_\lambda(x_0) = x_0$ , where  $\pi_{x_0}$  is the stereographic projection of pole  $x_0$ . Up to a rotation, we can assume that  $x_0 := (0, \dots, 0, 1)$  is the north pole: then we have that  $(\pi_N^{-1})^*h = U_1^{\frac{4}{n-2k}}\xi$ , where  $U_1(x) := ((1+|x|^2)/2)^{k-n/2}$ . As easily checked,  $\phi_\lambda$  is a conformal diffeomorphism and standard computations yield  $\phi_\lambda^*h = u_{x_0,\beta}^{\frac{4}{n-2k}}h$  where  $\beta := (\lambda^2+1)(\lambda^2-1)^{-1}$  and

$$u_{x_0,\beta}(x) := \left( \frac{\sqrt{\beta^2-1}}{\beta - \cos d_h(x, x_0)} \right)^{\frac{n-2k}{2}}$$

for all  $x \in \mathbb{S}^n$  and  $\beta > 1$ . In particular, we have that

$$(34) \quad \int_{\mathbb{S}^n} u_{p,\beta}^{2^*} dv_h = \omega_n$$

where  $\omega_n > 0$  is the volume of  $(\mathbb{S}^n, h)$ . It follows from the conformal law (2) that

$$(35) \quad P_h u_{x_0,\beta} = c_{n,k} Q_h u_{x_0,\beta}^{2^*-1} \text{ in } \mathbb{S}^n \text{ with } c_{n,k} := \frac{n-2k}{2}.$$

We now fix  $p \in \mathbb{S}^n$  as in the statement of Proposition 9 and we let  $\sigma_1, \dots, \sigma_m \in G$  be such that  $O_G(p) = \{\sigma_1(p), \dots, \sigma_m(p)\}$  and  $|O_G(p)| = m \geq 2$ . We define

$$u_\beta := \sum_{i=1}^m u_{\sigma_i(p),\beta}$$

for all  $\beta > 1$ . One checks that  $u_\beta$  is positive and  $G$ -invariant. Let us estimate

$$I_h(u_\beta) := \frac{\int_{\mathbb{S}^n} u_\beta P_h u_\beta dv_h}{\left( \int_{\mathbb{S}^n} f u_\beta^{2^*} dv_h \right)^{\frac{2}{2^*}}}.$$

The  $G$ -invariance and (35) yield

$$\int_{\mathbb{S}^n} u_\beta P_h u_\beta dv_h = c_{n,k} Q_h \sum_{i,j=1}^m \int_{\mathbb{S}^n} u_{\sigma_i(p),\beta} u_{\sigma_j(p),\beta}^{2^*-1} dv_h = m c_{n,k} Q_h (\omega_n + d_\beta)$$

where we have used (34) and where

$$d_\beta := \sum_{i=2}^m \int_{\mathbb{S}^n} u_{\beta,p} u_{\beta,\sigma_i(p)}^{2^*-1} dv_h$$

for all  $\beta > 1$ . Standard computations yield

$$d_\beta = (1 + o(1)) \Lambda_{p,G} (\beta^2 - 1)^{\frac{n-2k}{2}}$$

when  $\beta \rightarrow 1$ , where

$$\Lambda_{p,G} := \left( \int_{\mathbb{S}^n} (1 - \cos d_h(x,p))^{k-n/2} dv_h \right) \cdot \sum_{i=2}^m (1 - \cos d_h(p, \sigma_i(p)))^{k-n/2} dv_h > 0.$$

Concerning the denominator, it follows from the cancelation hypothesis on the derivatives of  $f$  that  $|f(x) - f(p)| \leq C d_h(x, O_G(p))^{n-2k+1}$  for all  $x \in \mathbb{S}^n$ . Therefore, rough estimates yield

$$\left| \int_{\mathbb{S}^n} (f - f(p)) u_\beta^{2^*} dv_h \right| \leq C (\beta^2 - 1)^{\frac{n-2k+1}{2}}$$

for all  $\beta > 1$ . A convexity inequality yields

$$\begin{aligned} \int_{\mathbb{S}^n} u_\beta^{2^*} dv_h &\geq \sum_{i=1}^m \int_{\mathbb{S}^n} u_{\beta, \sigma_i(p)}^{2^*} dv_h + 2^* \sum_{i \neq j} \int_{\mathbb{S}^n} u_{\sigma_i(p), \beta} u_{\sigma_j(p), \beta}^{2^*-1} dv_h \\ &\geq m (\omega_n + 2^* d_\beta) \end{aligned}$$

Noting  $\Lambda_{p,G} > 0$  and that  $c_{n,k} Q_h \omega_n^{\frac{2^*-2}{2^*}} = K(n,k)^{-1}$  (since pulling back  $u_{\beta,p}$  by the stereographic projections gives  $U_1$ , an extremal for (11)), these estimates yield

$$\begin{aligned} I_h(u_\beta) &\leq \frac{|O_G(p)|^{\frac{2k}{n}}}{f(p)^{\frac{2}{2^*}} K(n,k)} \cdot \left( 1 - \frac{\Lambda_{p,G}}{\omega_n} (\beta^2 - 1)^{\frac{n-2k}{2}} + o((\beta^2 - 1)^{\frac{n-2k}{2}}) \right) \\ &< \frac{|O_G(p)|^{\frac{2k}{n}}}{f(p)^{\frac{2}{2^*}} K(n,k)}. \end{aligned}$$

Coming back to the definition of  $\mu_f(\mathcal{C}_G)$ , this proves Proposition 9.  $\square$

**Proof of Theorem 1:** In the case  $n = 2k + 1$ , it follows from Proposition 4 and 9 that Case (i) of Theorem 2 cannot hold. Therefore Case (ii) holds, and Theorem 1 is proved.

More generally, Propositions 4 and 7, Theorem 2 and Proposition 9, yield:

**Theorem 3.** *Let  $k \geq 1$  and let  $G$  be a subgroup of isometries of  $(\mathbb{S}^n, h)$ ,  $n > 2k$ . Let  $f \in C^\infty(M)$  be a positive  $G$ -invariant function and assume that  $G$  acts without fixed point (that is  $|O_G(x)| \geq 2$  for all  $x \in \mathbb{S}^n$ ). Assume that there exists  $p \in \mathbb{S}^n$  such that*

$$\frac{|O_G(p)|^{\frac{2k}{n}}}{f(p)^{\frac{2}{2^*}}} \leq \frac{|O_G(x)|^{\frac{2k}{n}}}{f(x)^{\frac{2}{2^*}}}$$

*for all  $x \in \mathbb{S}^n$  and that  $\nabla^i f(p) = 0$  for all  $i \in \{1, \dots, n - 2k\}$ . Then there exists  $g \in [h]$  such that  $Q_g = f$  and  $G \subset \text{Isom}_g(\mathbb{S}^n)$ .*

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