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Dynamic Erdős-Rényi random graph with forbidden degree

Lucas MERCIER

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As suggested by Itai Benjamini, we introduced a variant of the Erdős-Rényi random graph process with a forbidden degree k , in which every edge adjacent to a vertex v is removed when the degree of v reaches k (but the removed edges may very well be added again later). We study the existence of a giant component, depending on the forbidden degree k and the time parameter t . We prove that for $k > 4$ a giant component appears at some

point, while for $k < 4$, a giant component never occurs. The main tool of our study is the local limit of the random graph process: it provides useful information about the cases $k > 4$, but also the threshold case $k=4$.

1 Introduction

1.1 The model

We consider a sequence of multigraph-valued stochastic processes $G_n^k = (G_{n,t}^k)_{t \geq 0}$, in which n and k , two positive integers, stand for the number of vertices and for the forbidden degree, respectively. The vertices are labelled from 1 to n , and E denotes the set of their $\binom{n}{2}$ eventual edges.

The multiplicity of the edges depends on a Poisson point process Π with intensity $\frac{1}{n} \mathbf{1}_{t \geq 0} dt \otimes \mu$ on $\mathbb{R}_+ \times E$, where μ is the counting measure on E . Π will be seen as a marked point process, with elements in \mathbb{R}_+ , marked by an edge in E . For any $e \in E$, Π_e denotes the point process on \mathbb{R}_+ of the elements of Π marked by e , and is the set of times when the multiplicity of the edge e is increased by 1. More precisely, initially, $G_{n,0}^k$ does not contain any edge. At each time $t \in \Pi_e$, the multiplicity of the edge e is increased by 1, and, for any endpoint w of e that reaches degree k at this step, every edge incident to w , including e , is removed (when both endpoints reach degree k simultaneously, the destruction of edges takes place on both sides)¹. A given edge can be added several times to the process, therefore an edge can be removed, and added again at a later time; multiple edges can also occur. A vertex of $G_{n,t}^k$ is said *saturated* if its degree is $k-1$, the maximum possible degree.

At some point, a discrete time version of the continuous time process G_n^k will be needed: the unmarked version of Π is a Poisson point process with intensity $\frac{n-1}{2}$ on \mathbb{R}_+ , so it is a.s. possible to order the points of Π in increasing order. For any integer i , let $\tau_i = \inf\{t \geq 0 : \Pi([0, t] \times E) \geq i\}$. By the memoryless property of the Poisson point process, the times τ_i are independent of the discrete process $(G_{n,\tau_i}^k)_i$. This discrete process can be described as follows:

- At step $i = 0$, there is no edge.
- At each step $i \geq 1$, choose an edge uniformly at random among the $\binom{n}{2}$ elements of E and increase its multiplicity by one.
- For each endpoint reaching degree k , remove every edge incident to this endpoint.

¹Technically, the endpoints will never reach degree k , going directly from degree $k-1$ to 0, but we will describe this situation as reaching degree k and immediately going to degree 0.

The multigraph-valued stochastic process in which the edges appear according to $(\Pi_e)_{e \in E}$, but are never erased, is denoted $G_n^\infty = (G_{n,t}^\infty)_{t \geq 0}$. We shall call it the *Erdős-Rényi multigraph process*, or the *Erdős-Rényi multigraph*. For any graph G , $C_{\max}(G)$ will denote the size (the number of vertices) of the largest connected component of G .

Possible generalisations Let G be a locally finite simple graph. For every edge e , let Π_e be a locally finite subset of \mathbb{R}_+ . Define the forbidden degree version of G as the multigraph process with the same set of vertices as G evolving in the following way:

- Initially no edge is present.
- For every point t of Π_e , increase the multiplicity of e by one at time t .
- If a vertex reaches the degree k , remove every adjacent edges.

This model is not always well-defined when the graph is infinite. A sufficient condition will be described in Section 4.2.

It is possible to allow the forbidden degree to depend on the vertex by using the following generalisation:

- Let $(k_v)_{v \in V}$ be a sequence of integers, indexed by the set of vertices. The integer k_v will be called *the forbidden degree of v* .
- The edges are added as previously. Whenever a vertex v reaches degree k_v , remove every edge adjacent to v .

This generalization allows to interpolate between two forbidden degrees, by setting the proportion of vertices of each forbidden degree, *e.g.* by having a proportion λ of the vertices with forbidden degree $k + 1$ and a proportion $1 - \lambda$ with forbidden degree k .

Heuristically, we expect that the resulting random graph process is stochastically increasing with the forbidden degree, even if it is not deterministically increasing, as explained in part 1.3.

1.2 Context

In [ER60], Erdős and Rényi obtained a striking result: the largest component of the Erdős-Rényi graph with tn edges and n vertices has two radically different behaviors depending on t . If $t \leq \frac{1}{2}$, $C_{\max}(G_{n,m}^\infty) = o(n)$ a.a.s whereas as soon as $t > \frac{1}{2}$ $C_{\max}(G) = \Theta(n)$. For this reason we will study the evolution of $C_{\max}(G_{n,t}^k)$, depending on (k, t, n) .

Another model with a degree constraint is the graph process with degree restriction [RW92]. There also exists several models with local constraints, *e.g.* the triangle-free process [Boh09] and H -free processes [ESW95, BR00,

OT01] where H is a given subgraph. These models differ with the forbidden degree on many aspects, but in our eye the essential difference is that the graph processes are increasing in the former models, while in the forbidden degree model the graph process is not, as edges are routinely removed. Another model with edge removal is the Drossel-Schwab forest-fire model[DS92], where full components are removed with a rate proportional to their size.

1.3 Issues we have to deal with

Several points need to be considered when studying this process:

1. *Chronology.* For a fixed t , the Erdős-Rényi multigraph $G_{n,t}^\infty$ does not depend on the chronology of the apparition of edges, but only on the set of added edges, while due the forbidden degree constraint, chronology suddenly matters for $G_{n,t}^k$.
2. *Monotony.* G_n^k is not an increasing process, and the existence of a giant component at a given time does not imply the existence of a giant component at a later time. Actually, having more edges at a given time can lead to a smaller graph at a later time.
3. *Locality.* The degree of a given vertex depends on which adjacent edges have been removed. In turn, this depends on the degrees of the other endpoints of these edges, and these degrees depend on the neighbors of these endpoints, and so on... As a consequence, the existence of a local limit is questionable, much more than for the Erdős-Rényi random multigraph.

1.4 Results

The main results obtained in this article are the following:

Theorem 1.1. *For every $k \leq 3$ and for every sequence of non negative real numbers $(t_n)_{n \geq 0}$,*

$$C_{\max}(G_{n,t_n}^k) = o(n).$$

Theorem 1.1 means that there is no giant component for $k \leq 3$.

Theorem 1.2. *If $k \geq 5$, there exists an interval I such that for any time $t \in I$, $C_{\max}(G_{n,t}^k) = \Theta(n)$.*

According to Theorem 1.2, there exists a giant component at some point if $k \geq 5$, but this does not entail that a giant component still exists at a later time.

Theorem 1.3. *Let v be a random vertex of $G_{n,t}$, chosen uniformly among the vertices of $G_{n,t}$ independently of $(G_{n,t}^\infty)_{t \geq 0}$. Then for any $k \in \mathbb{N} \cup \{\infty\}$, $(G_{n,t}^k, v)$ converges in distribution, for the local limit topology, to T_t^k , when n tends to ∞ .*

T_t^∞ is a Galton-Watson tree with Poissonian offspring distribution. T_t^k is the forbidden degree version of T_t^∞ , and turns out to be a two-stages multitype branching process. The convergence is with respect to the local topology, as introduced by Benjamini and Schramm [BS01].

Theorem 1.4. *For any $t \geq 0$ and $k \in \mathbb{N}$,*

$$\frac{C_{\max}(G_{n,t}^k)}{n} \xrightarrow[n \rightarrow \infty]{p} \mathbb{P}(|T_t^k| = \infty).$$

Expectedly, the behavior of the local limit predicts somehow the existence of the giant component, as the supercriticality of the local limit T_t^k is equivalent to the existence of a giant component.

Section 2 will be about Theorem 1.1, Theorem 1.2 will be discussed in Section 3, the existence and properties of the local limit in Section 4 and the Theorem 1.4 will be discussed in Section 5.

2 There is no giant component if $k \leq 3$

If $k \leq 2$, no component has more than 2 vertices. Thus, this section dealing only with the case $k = 3$, $G_{n,t}^3$ is denoted $G_{n,t}$. The largest allowed degree being 2, the connected components are either paths or cycles, which limits their growth. Theorem 1.1 follows from the next proposition.

Proposition 2.1. *There exists a constant A such that for any $\ell \in \mathbb{N}$, and for $n \geq 10$,*

$$\mathbb{E} [C_{\max}(G_{n,\tau_\ell})^2] \leq An.$$

Proof of Theorem 1.1. Let $|a|$ denote the cardinality of a finite set a , and set $N(t) = |\Pi \cap [0, t] \times E|$, resp. $N_e(t) = |\Pi_e \cap [0, t]|$. Since the two families $(\tau_i)_{i \in \mathbb{N}}$ and $(G_{n,\tau_i})_{i \in \mathbb{N}}$ are independent, we obtain:

$$\begin{aligned} \mathbb{E} [C_{\max}(G_{n,t_n})^2] &\leq \sum_i \mathbb{E} [C_{\max}(G_{n,t_n})^2 | N(t_n) = i] \mathbb{P}(N(t_n) = i) \\ &\leq \sum_i \mathbb{E} [C_{\max}(G_{n,\tau_i})^2] \mathbb{P}(N(t_n) = i) \\ &\leq An. \end{aligned}$$

□

Proof of Proposition 2.1. For any n and ℓ , let B_ℓ (resp. A_ℓ, C_ℓ) denote the set of connected components of G_{n,τ_ℓ} (resp. the set of acyclic connected components of G_{n,τ_ℓ} , the set of cycles of G_{n,τ_ℓ}). Consider Z_ℓ defined by:

$$Z_\ell = \sum_{a \in A_\ell} |a|^2 + 2 \sum_{b \in C_\ell} |b|^2. \quad (1)$$

The reason for counting cycles twice will be clear later. Proposition 2.1 follows at once from the next Proposition.

Proposition 2.2. *There exists a positive constant A such that for all integers n and ℓ , $\mathbb{E}(Z_\ell) \leq An$.*

Proof. Set

$$u_\ell = \mathbb{E} \left[\frac{Z_\ell}{n} \right].$$

Let \mathcal{F}_ℓ be the σ -algebra generated by $(G_{n,t})_{t \leq \tau_\ell}$, and let Δ denote the variation of Z_ℓ :

$$\Delta_{\ell+1} = Z_{\ell+1} - Z_\ell.$$

We shall prove that, for suitable constants α and β , both larger than 1,

$$\mathbb{E}(\Delta_{\ell+1} | \mathcal{F}_\ell) \leq \alpha + \beta \frac{Z_\ell}{n} - \frac{1}{4} \frac{Z_\ell^2}{n^2}. \quad (2)$$

As a consequence,

$$n(u_{\ell+1} - u_\ell) \leq \alpha + \beta u_\ell - \frac{1}{4} u_\ell^2,$$

Note that $u_0 = 1$ and let r denote the positive root of $\alpha + \beta X - \frac{1}{4} X^2$. Now,

- if $r \leq u_\ell \leq r + \alpha + \beta^2$, then (2) entails that $u_{\ell+1} \leq u_\ell$;
- if $0 \leq u_\ell \leq r$, then (2) entails that $u_{\ell+1} - u_\ell \leq \frac{1}{n}(\alpha + \beta^2)$.

Thus $A = r + \alpha + \beta^2$ is a suitable choice since $u_0 = 1 \leq A$. \square

Now, due to (1),

$$Z_\ell \geq C_{\max}(G_{n,\tau_\ell})^2,$$

which concludes the proof of Proposition 2.1, assuming relation (2). \square

Proof of (2). Consider a graph process $\Gamma = (\Gamma_\ell)_{\ell \geq 0}$ that starts from n vertices and no edges. At each step $\ell \geq 1$ a first random vertex v_ℓ is picked uniformly and a second random vertex w_ℓ different from v_ℓ is picked uniformly too. Then the multiplicity of the edge $e_\ell = \{v_\ell, w_\ell\}$ is increased by 1, provided that the forbidden degree rule allows it. If either v_ℓ or w_ℓ has degree 2 in $\Gamma_{\ell-1}$ then its edges are erased and e_ℓ is not added. Since Γ has the same distribution as $(G_{n,\tau_\ell})_{\ell \geq 0}$, we shall consider, in what follows, that the process (Z_ℓ) is a functional of Γ , rather than a functional of $(G_{n,\tau_\ell})_{\ell \geq 0}$.

Let us decompose the variation Δ_ℓ according to the connected components of v_ℓ and of w_ℓ . For $c, d \in B_\ell$, set

$$X_{c,d} = \Delta_\ell \mathbf{1}_{v_\ell \in c} \mathbf{1}_{w_\ell \in d},$$

so that

$$\Delta_\ell = \sum_{(c,d) \in B_\ell^2} X_{c,d}.$$

Since A_ℓ, B_ℓ, C_ℓ are measurable with respect to \mathcal{F}_ℓ ,

$$\mathbb{E}[\Delta_\ell | \mathcal{F}_\ell] = \sum_{(c,d) \in B_\ell^2} \mathbb{E}[X_{c,d} | \mathcal{F}_\ell].$$

Let us list the cases in which Z_ℓ increases:

1. **The edge e_ℓ makes c a cycle.** This entails that $c = d \in A_\ell$ and that v_ℓ and w_ℓ are the two endpoints of c , which happens with probability at most $\frac{2}{n(n-1)}$. In this case

$$\Delta_\ell = |c|^2.$$

2. **The edge e_ℓ merges c and d .** The components c and d merge only if $c, d \in A_\ell$, $c \neq d$ and v_ℓ and w_ℓ are endpoints of c and d . There are at most 2 endpoints per component, so, given c and d , this happens with probability at most $\frac{4}{n(n-1)}$. In this case

$$\Delta_\ell = (|c| + |d|)^2 - |c|^2 - |d|^2 = 2|c||d|.$$

Now let us list two of the cases in which Z_ℓ decreases:

3. **c is a cycle.** If $c \neq d$, the cycle c is split into a path of length $|c| - 1$ on one hand and an isolated vertex on the other hand. If $c = d$ the cycle can even be split in smaller parts, but in both cases $X_{c,d} \leq -|c|^2$ with a probability $\frac{|c|}{n}$.
4. **c has endpoints but v_ℓ is not one of them.** Let us say that v_ℓ is the m th vertex of c . If $c \neq d$, the path c is split into three paths, of length $m - 1$, $|c| - m$ and 1. If $c = d$, one of the three previous paths can be split again due to the effect of w_ℓ . Thus, for any m with $1 < m < |c| - 1$, with probability $\frac{1}{n}$,

$$X_{c,d} \leq (m - 1)^2 + (|c| - m)^2 + 1 - |c|^2 \leq 2m(m - |c|).$$

These 4 cases cover the possible contributions of (c, v_ℓ) to Δ_ℓ . The two first cases, in which the two sides of e_ℓ play symmetric rôles, also cover the

positive contributions of the other side of e_ℓ , (d, w_ℓ) , to Δ_ℓ . As we aim to provide an upper bound for Δ_ℓ , we do not need to discuss the negative contributions of the other side, and considering the 4 cases, we obtain:

$$\sum_{(c,d) \in B_\ell^2} \mathbb{E}[X_{c,d} | \mathcal{F}_\ell] \leq \sum_{1 \leq i \leq 4} D_i,$$

in which

$$\begin{aligned} D_1 &= \frac{2}{n(n-1)} \sum_{c \in A_\ell} |c|^2 \leq \frac{2Z_\ell}{n(n-1)}, \\ D_2 &= \frac{8}{n(n-1)} \sum_{c,d \in A_\ell} |c||d| \leq \frac{8n}{n-1}, \\ D_3 &= -\frac{1}{n} \sum_{c \in C_\ell} |c|^3 \end{aligned}$$

and

$$\begin{aligned} D_4 &= \frac{1}{n} \sum_{c \in A_\ell} \sum_{m=2}^{|c|-1} 2m(m-|c|) \\ &= \frac{1}{3n} \sum_{c \in A_\ell} (-|c|^3 + 7|c| - 6) \mathbf{1}_{|c| \geq 2} \\ &\leq \frac{7}{3} - \frac{1}{3n} \sum_{c \in A_\ell} |c|^3. \end{aligned}$$

Thus, for $n \geq 9$,

$$\mathbb{E}[\Delta_\ell | \mathcal{F}_\ell] \leq 12 + \frac{3Z_\ell}{n^2} - \frac{1}{4n} \left(\sum_{c \in A_\ell} |c|^3 + \sum_{c \in C_\ell} 4|c|^3 \right).$$

Let $(a_c)_{c \in B_\ell}$ and $(b_c)_{c \in B_\ell}$ denote two sequences defined as follows:

- if $c \in A_\ell$, $a_c = \sqrt{|c|}$ and $b_c = |c|^{\frac{3}{2}}$;
- if $c \in C_\ell$, $a_c = \sqrt{|c|}$ and $b_c = 2|c|^{\frac{3}{2}}$.

Then, by the Cauchy-Schwarz inequality, we have:

$$(Z_\ell)^2 = \left(\sum_{c \in B_\ell} a_c b_c \right)^2 \leq \sum_{c \in B_\ell} a_c^2 \sum_{c \in B_\ell} b_c^2 = n \left(\sum_{c \in A_\ell} |c|^3 + \sum_{c \in C_\ell} 4|c|^3 \right).$$

As a consequence, (2) holds true for $(\alpha, \beta) = (12, 3)$. \square

3 At some point, there is a giant component if $k \geq 5$

In this part, rather than considering the process with forbidden degree, we shall consider a lower bound, i.e. a new process in which all the edges incident to a vertex to which at least k edges have been added are removed. For $k \geq 5$, this new process is supercritical, allowing us to prove Theorem 1.2.

3.1 Approximation by a simpler model

Consider the graph $g_{n,t}^k$ obtained when one erases all the edges of $G_{n,t}^\infty$ that are incident to a vertex with degree k or more. As opposed to $G_{n,t}^k$, $g_{n,t}^k$ depends on Π only through $(N_e(t))_{e \in E}$, thus it does not depend on the chronology. Let T_k denote the transformation that maps $(N_e(t))_{e \in E}$ to $g_{n,t}^k$:

$$g_{n,t}^k = T_k(N.(t)).$$

Lemma 3.1. *For any n and t ,*

$$g_{n,t}^k \leq G_{n,t}^k \leq G_{n,t}^\infty.$$

If $G(e)$ denotes the multiplicity of the edge e in G , the order we consider in this section is the product order for $(G(e))_{e \in E}$.

Proof. The second inequality is clear. For the first one, note that if $0 \leq G_{n,t}^k(e) < G_{n,t}^\infty(e)$, then one of the endpoints of e reaches the threshold k in G_n^∞ at some point s before t , and, as a consequence, $g_{n,t}^k(e) = 0$. Else none of the endpoints' degrees of e reach the threshold k in $G_{n,t}^\infty$, and as a consequence $g_{n,t}^k(e) = G_{n,t}^k(e) = G_{n,t}^\infty(e)$. \square

The graph $g_{n,t}^k$, conditionally on its degree sequence, is distributed as a configuration model conditioned on being loopless, as will be proven in Corollary 3.3. The configuration model, introduced by Bender and Canfield [BC78, page 297] is defined as follow:

Definition 3.1. Let $(c_v)_{1 \leq v \leq n}$ be a finite sequence of non-negative integers, such that $\sum_v c_v$ is even. The configuration model associated to $(c_v)_{1 \leq v \leq n}$ is the random graph constructed as follow:

- The set of vertices V is $\{1, \dots, n\}$.
- Let L be the multiset containing c_v copies of each vertex v of V .
- A random uniform pairing E of the elements of L is chosen.
- The set of edges is defined by the pairing, with each pair (x, y) corresponding to an edge between x and y .

By construction, the vertex v has degree c_v in the resulting graph.

Remark. The configuration model is sometimes defined from the sequence $(d_j)_{j \geq 0}$ where $d_j = \#\{v : c_v = j\}$ denotes the number of vertices of degree j , instead of the degree sequence c .

Let $c = (c_i)_{1 \leq i \leq n}$ be a sequence of integers smaller than k such that $\sum c_i$ is even and let \mathcal{G}_c denote the set of loopless multigraphs with degree sequence c . An element of \mathcal{G}_c is described by the sequence $(g_e)_{e \in E}$ of its edges' multiplicity.

Lemma 3.2. *For $g_1, g_2 \in \mathcal{G}_c$, let m_i^j denote the number of edges with multiplicity i in g_j . Then for any $t \geq 0$,*

$$\mathbb{P}(g_{n,t}^k = g_1) \prod_{i \geq 2} (i!)^{m_i^1} = \mathbb{P}(g_{n,t}^k = g_2) \prod_{i \geq 2} (i!)^{m_i^2}.$$

Corollary 3.3. *The graph $g_{n,t}^k$, conditionally given its degree sequence, is a configuration model conditioned to be loopless.*

In the configuration model, the number of configurations corresponding to a given multigraph is $\frac{\prod_v c_v!}{\prod_i (i!)^{m_i}}$, thus Lemma 3.2 implies Corollary 3.3.

Proof of Lemma 3.2. Set $B = \{v \in V : c_v > 0\}$. Let $g \in \mathcal{G}_c$. Let $s(e)_{e \in E}$ be a multigraph on the set of vertices $\{1, \dots, n\}$.

Lemma 3.4. *For all $e \in E$, let $r(e) = s(e) - g(e)$. $T_k(s) = g$ if and only if the following three conditions hold:*

1. *For all e in E , $r(e) \geq 0$.*
2. *For all $v \in B$, the degree of v in the graph r is strictly smaller than $k - c_v$.*
3. *For all $e \in E$ such that $r(e) > 0$, one of the endpoints of e is not in B and has degree at least k in r .*

Proof. Let us first assume that $T_k(s) = g$. The graph g is obtained from s by removing every edge adjacent to a vertex of degree at least k in s . Therefore g is a subgraph of s , r denotes the removed edges, and Condition 1 holds. If the degree of v is at least k in s , then every edge adjacent to v is removed when building the graph $T_k(s)$. Therefore, if v is in B (i.e. if the degree of v is positive in g), the degree of v in s is strictly smaller than k . As v has degree c_v in g , v has degree strictly smaller than $k - c_v$ in r , proving Condition 2. If $r(e) > 0$, this means that the edge e is removed when building $T_k(s)$, i.e. that one of its endpoint v has degree at least k in s . As by Condition 2, v cannot be in B , this means that $c_v = 0$ and therefore that v has degree at least k in r , proving Condition 3.

Conversely, let us assume that r satisfies conditions 1-3. Then by Condition 1, g is a subgraph of $s = g + r$. The endpoints of the edges e of g have positive degree in g and therefore they belong to B . By Condition 2, their degree in s is strictly smaller than k , and therefore the edges of g are not removed when building $T_k(s)$. The edges of r are removed, as by Condition 3, one of their endpoints has degree at least k in r and therefore degree at least k in $s = g + r$. □

Let \mathcal{E}_c denote the set of graphs satisfying Conditions 1-3 of Lemma 3.4. It should be noted that \mathcal{E}_c only depends on c , not on g . It should also be noted that for any $r \in \mathcal{E}_c$ and $g \in \mathcal{G}_c$, the set of edges of r and g are disjoint, as the endpoints of edges of g are in B , whereas at least one endpoint of the edges of r is not in B , by Condition 3.

Since $N_e(t)$ is a Poisson random variable with parameter $\lambda = \frac{t}{n}$, for any graph s :

$$\begin{aligned} \mathbb{P}(G_{n,t}^\infty = s) &= \prod_{e \in E} \mathbb{P}(N_e(t) = s(e)) \\ &= e^{-(n-1)t/2} \lambda^{\sum_e s(e)} \prod_{e \in E} \frac{1}{s(e)!} \end{aligned}$$

Therefore, for $g \in \mathcal{G}_c$,

$$\begin{aligned} \mathbb{P}(g_{n,t}^k = g) &= \sum_{s: T_k(s)=g} \mathbb{P}(G_{n,t}^\infty = s) \\ &= \sum_{r \in \mathcal{E}_c} \mathbb{P}(G_{n,t}^\infty = g + r) \\ &= e^{-(n-1)t/2} \sum_{r \in \mathcal{E}_c} \lambda^{\sum_e g(e)+r(e)} \prod_{e \in E} \frac{1}{(g(e) + r(e))!} \\ &= e^{-(n-1)t/2} \frac{\lambda^{\sum_e g(e)}}{\prod_{e \in E} g(e)!} \sum_{r \in \mathcal{E}_c} \lambda^{\sum_e r(e)} \prod_{e \in E} \frac{1}{r(e)!}. \end{aligned}$$

Note that for any edge e , $r(e)g(e) = 0$, hence $(g(e) + r(e))! = g(e)!r(e)!$ holds. Thus

$$\begin{aligned} \mathbb{P}(g_{n,t}^k = g) \prod_{i \geq 1} (i!)^{m_i} &= \mathbb{P}(g_{n,t}^k = g) \prod_{e \in E} g(e)! \\ &= e^{-(n-1)t/2} \lambda^{\sum_e g(e)} \Psi \\ &= e^{-(n-1)t/2} \lambda^{\sum_{i=1}^n c_i/2} \Psi, \end{aligned}$$

in which Ψ , defined by

$$\Psi = \sum_{r \in \mathcal{E}_c} \lambda^{\sum_e r(e)} \prod_{e \in E} \frac{1}{r(e)!}$$

depends only on c , as the set \mathcal{E}_c does. \square

In the rest of the section, t is fixed, so $G_{n,t}^\infty$ and $g_{n,t}^k$ are abridged in G_n^∞ and g_n^k for readability. In [MR95], Molloy and Reed first study the configuration model associated to a degree sequence, and then extend the results to the configuration model conditioned on being simple (this conditional model is called random graph with given degree sequence). In our case, we will first study the configuration model associated to the degree sequence of g_n^k , using [MR95, Theorem 1] and then extend the result to g_n^k by mirroring the proof used for [MR95, Lemma 2]. For a vertex v , the degree of v in G_n^∞ (resp. in g_n^k) is denoted $C_n(v)$ (resp. $c_n(v)$). Set

$$d_n(i) = \#\{v : c_n(v) = i\}.$$

Let G_n^{conf} be the configuration model associated to the degree sequence c_n . We shall see that the sequence d_n satisfies the assumptions of [MR95, Theorem 1], i.e. for each i , there exists a constant $p_{t,i}$ such that

$$\lim_n \frac{d_n}{n}(i) = p_{t,i}.$$

As the degree in the graph is bounded by $k - 1$, the sequence d_n is sparse and well-behaved (with the vocabulary of [MR95]).

Let $\mathfrak{P}\mathfrak{G}\mathfrak{W}(t)$ denote a Galton-Watson tree whose offspring is Poisson distributed with parameter t . By [DM10, Proposition 2.6], for any given vertex v , the rooted graph (G_n^∞, v) converges in distribution to $\mathfrak{P}\mathfrak{G}\mathfrak{W}(t)$ when n tends to infinity. Since $c_n(v)$ depends only on the vertices of the ball with radius 2 in $(G_{n,t}^\infty, v)$ and on the edges between them, the asymptotic distribution of $c_n(v)$ can be read on the first two levels of $\mathfrak{P}\mathfrak{G}\mathfrak{W}(t)$. Two cases arise:

- if $C_n(v) \geq k$, the degree of v in G_n^∞ exceeds the threshold k , and $c_n(v) = 0$;
- if $C_n(v) < k$, an edge $\{v, w\}$ incident to v in G_n^∞ is erased in g_n^k if and only if $C_n(w)$ exceeds the threshold k .

Owing to [DM10, Proposition 2.6], $(C_n(v), C_n(w))$ weakly converges to $(C(v), C(w))$ such that $(C(v), C(w) - 1)$ are i.i.d. and Poisson distributed with parameter t , for both its offspring and its father v contribute to the degree of w in $\mathfrak{P}\mathfrak{G}\mathfrak{W}(t)$. Thus the degree of a neighbor of v is less than $k - 1$ with an asymptotic probability:

$$\pi_k(t) = e^{-t} \sum_{i=0}^{k-2} \frac{t^i}{i!},$$

and the asymptotic distribution $c(v)$ of $c_n(v)$ is obtained by the following algorithm:

1. draw a Poisson random variable $C(v)$ with parameter t ;
2. build a Poisson random variable Y with parameter $t\pi_k(t)$ by a thinning of $C(v)$ with parameter $\pi_k(t)$, so that the conditional distribution of Y given $C(v)$ is binomial with parameters $C(v)$ and $\pi_k(t)$;
3. set $c(v) = Y\mathbf{1}_{C(v)\leq k-1}$.

Set

$$p_{t,i} = \mathbb{P}(c(v) = i) = \lim_n \mathbb{P}(c_n(v) = i).$$

Lemma 3.5. For $0 \leq i < k$, $d_n(i)/n \xrightarrow[n \rightarrow \infty]{(P)} p_{t,i}$.

Proof. Since $\mathbb{P}(c_n(1) = i) \xrightarrow[n \rightarrow \infty]{} p_{t,i}$, and the vertices play symmetric rôles,

$$\mathbb{E}[d_n(i)] = \sum_{i=1}^n \mathbb{P}(c_n(v) = i) = n\mathbb{P}(c_n(1) = i),$$

and the limit holds for expectations. Let us show that it holds in probability.

$$\begin{aligned} \mathbb{E}[d_n(i)^2] &= \sum_{v,w \in [n]} \mathbb{P}(c_n(v) = i = c_n(w)) \\ &= \sum_v \mathbb{P}(c_n(v) = i) + \sum_{v \neq w} \mathbb{P}(c_n(v) = i = c_n(w)) \\ &= n\mathbb{P}(c_n(1) = i) + n(n-1)\mathbb{P}(c_n(1) = i = c_n(2)) \end{aligned}$$

As the birooted local limit of $G_{n,t}^\infty$ is a couple of independent $\mathfrak{B}\mathfrak{G}\mathfrak{W}(t)$, $\mathbb{P}(c_n(1) = c_n(2) = i)$ converges to $p_{t,i}^2$, and:

$$\mathbb{E}[d_n(i)^2] = n^2 p_{t,i}^2 + o(n^2).$$

Thus $\text{Var}(d_n(i)) = o(n^2)$, leading to the desired result. \square

Let

$$Q_t = \sum_i i(i-2)p_{t,i}.$$

According to [MR95, Theorem 1], if $Q_t > 0$, there exists a constant α

$$\mathbb{P}(C_{\max}(G_n^{\text{conf}}) \geq \alpha n | c_n) \xrightarrow[n \rightarrow \infty]{(P)} 1.$$

Moreover, the law of G_n^{conf} conditionally on c_n and on G_n^{conf} being loopless is equal to the law of g_t^k conditionally on c_n . By the main result of [McK85], there exists a constant $\beta > 0$ such that $\mathbb{P}(G_n^{\text{conf}} \text{ is loopless} | c_n) > \beta$ a.a.s, and therefore, if $Q_t > 0$:

$$\mathbb{P}(C_{\max}(g_n^k) \geq \alpha n | c_n) \xrightarrow[n \rightarrow \infty]{(P)} 1.$$

To conclude, we need to compute the sign of

$$Q_t = \mathbb{E}[c(v)(c(v) - 2)] = \mathbb{E}[c(v)(c(v) - 1)] - \mathbb{E}[c(v)].$$

According to the points 2 and 3 of the description of the law of $c(v)$ above, $\mathbb{E}[c(v)|C(v)] = \pi_k(t)C(v)\mathbf{1}_{\mathbf{C}(v) < \mathbf{k}}$ and $\mathbb{E}[c(v)] = \pi_k(t)\mathbb{E}[C(v)\mathbf{1}_{\mathbf{C}(v) < \mathbf{k}}]$. Similarly, $\mathbb{E}[c(v)(c(v) - 1)] = \pi_k(t)^2\mathbb{E}[C(v)(C(v) - 1)\mathbf{1}_{\mathbf{C}(v) < \mathbf{k}}]$. Moreover, we have:

$$\begin{aligned} \mathbb{E}(C(v)\mathbf{1}_{\mathbf{C}(v) < \mathbf{k}}) &= e^{-t} \sum_{i=1}^{k-1} i \frac{t^i}{i!} \\ &= e^{-t} \sum_{i=1}^{k-1} \frac{t^i}{(i-1)!} \\ &= e^{-t} \sum_{i=0}^{k-2} \frac{t^{i+1}}{i!} \\ &= t\pi_k(t) \end{aligned}$$

Thus the equation for the existence of a giant component become successively:

$$\begin{aligned} \pi_k(t)^2\mathbb{E}(C(v)(C(v) - 1)\mathbf{1}_{\mathbf{C}(v) < \mathbf{k}}) &> \pi_k(t)\mathbb{E}(C(v)\mathbf{1}_{\mathbf{C}(v) < \mathbf{k}}) \\ \pi_k(t)^2\mathbb{E}(C(v)(C(v) - 1)\mathbf{1}_{\mathbf{C}(v) < \mathbf{k}}) &> t\pi_k(t)^2 \\ e^{-t} \sum_{i=2}^{k-1} i(i-1) \frac{t^i}{i!} &> t \\ te^{-t} \sum_{i=0}^{k-3} \frac{t^i}{i!} &> 1 \end{aligned}$$

- If $k \geq 5$, there exists a t such that the condition is satisfied. (e.g. $t = 2$).
- If $k \leq 4$, the condition does not hold for any t .

This ends the proof of Theorem 1.2.

Remark. This part does not prove the existence (nor the non-existence) of a giant component for $k = 4$, and only proves that there is a giant component for some bounded interval of t (and not, as it could be expected, for every t larger than some t_0) for $k \geq 5$.

4 The local limit

The aim of this part is to prove the existence of a local limit, and to study the link between this local limit and the existence of a giant component.

4.1 Local topology

For the purpose of this study, the objects used are multigraphs with labelled edges, where the edges are labelled by the time of addition. A root of a graph G is either a vertex or an edge of G . We consider the local topology, as introduced by Benjamini and Schramm[BS01].

Definition 4.1. Given a graph G , an edge $e = (v_1, v_2)$ of G , a vertex v of G , and a non-negative integer l , $B_l(G, v)$ denotes the set of vertices of the ball of radius l centered at v in G , and $B_l(G, e)$ denotes $B_l(G, v_1) \cup B_l(G, v_2)$. The notation $B_l(G, v)$ (resp. $B_l(G, e)$) will also denote the subgraph of G induced by the set of vertices $B_l(G, v)$ (resp. $B_l(G, e)$), and $S_l(G, v)$ (resp. $S_l(G, e)$) will denote the sphere of radius l centered at v (resp. e) *i.e.* the set of vertices $B_l(G, v) \setminus B_{l-1}(G, v)$ (resp. $B_l(G, e) \setminus B_{l-1}(G, e)$), where by convention $B_{-1} = \emptyset$. The ball (resp. sphere) of radius l will be called l -ball (resp. l -sphere).

Definition 4.2. For an integer l , a l -rooted multigraph (G, r) with labelled edges is the data of a multigraph G with labelled edges equipped with a l -tuple $r = (r_1, \dots, r_l)$ of roots of G : the set of roots is ordered, and a given root can appear several times.

For an integer j and an l -rooted graph (G, r) , $B_j(G, (r))$ denotes the l -rooted graph $\cup_{1 \leq i \leq l} B_j(G, r_i)$.

Definition 4.3 (Isomorphism and distance).

- Two l -rooted multigraphs (G, r) and (\tilde{G}, \tilde{r}) are said to be isomorphic if there is a graph isomorphism Φ from G to \tilde{G} preserving the edges' labels such that for all i , $\Phi(r_i) = \tilde{r}_i$.
- For any integer j , (G, r) and (\tilde{G}, \tilde{r}) are said to be j -isomorphic if their j -balls are isomorphic.
- The pseudo-distance between two l -rooted graphs G and \tilde{G} is defined as 2^{-j} , with j the largest integer such that G and \tilde{G} are j -isomorphic (as l -rooted labelled multigraphs). The convergence according to this distance is called the local convergence (this is a straightforward extension of the definition of the local convergence, according to Benjamini *et al*).

If (G, r) and (\tilde{G}, \tilde{r}) are isomorphic, then they are j -isomorphic for any j . The converse holds if there is at least one root in each connected component of each graph, but can be false otherwise.

Definition 4.4. Let T_t^∞ be the labelled random graph defined by:

- The shape of T_t^∞ is a Galton-Watson tree with Poisson offspring with mean t .

- Conditionally on the shape of T_t^∞ , the edges' labels are uniform on $[0, t]$ and independent.

Lemma 4.1. *For any integer l , the unlabelled graph $G_{n,t}^\infty$, rooted at l vertices chosen independently uniformly at random, converges toward l independent copies of a Galton-Watson tree with Poisson offspring with mean t , while the labelled graph $G_{n,t}^\infty$, rooted at l vertices chosen independently uniformly at random, converges toward l independent copies of T_t^∞ .*

The one-rooted version of the first part of Lemma 4.1 is a well-known result, for various versions of the Erdős-Rényi graph. One instance can be found in [DM10, Proposition 2.6], for $G_{n,t}^\infty$ with no multiple edges. For completeness, a proof of this multi-roots, multigraph version of the result in [DM10] can be found below:

Proof. For l and j some positive integers, consider (G, r) , $r = (r_i)_{1 \leq i \leq l}$, a l -rooted finite graph with radius not larger than j (such that $\cup_{1 \leq i \leq l} B_j(G, r_i) = G$). Let us compute $p_n(G)$, the probability that $B_j(G_{n,t}^\infty, v)$ is isomorphic to (G, r) , in which $(v_i)_{1 \leq i \leq l}$ is a sequence of l vertices chosen independently at random. We need a few notations:

- V_G is the vertex set of G , and $N_V = |V_G|$;
- E_G is the edge set of G and $N_E = |E_G|$;
- for any edge $e \in E_G$, let m_e denote the multiplicity of e in G .
- N_V^b is the number of vertices of G that are in $B_{j-1}(G, r)$;
- $(w_i)_{1 \leq i \leq N_V}$ is a specific ordering of V_G , such that $(w_i)_{1 \leq i \leq N_V^b}$ are the elements of $B_{j-1}(G, r)$, and $(w_i)_{N_V^b+1 \leq i \leq N_V}$ are the remaining vertices of G .

There are at most l components of $G = \cup_{1 \leq i \leq l} B_j(G, r_i)$, therefore $l + N_E \geq N_V$. If $l + N_E = N_V$, then G is a forest of l simple trees.

Lemma 4.2. *If $l + N_E > N_V$, then $p_n(G) \rightarrow 0$.*

Proof of Lemma 4.2. Let Φ be an injection from V_G to $\{1, \dots, n\}$. If Φ induces an isomorphism between (G, r) and $B_j(G_{n,t}^\infty, v)$, then the following conditions holds:

1. For every $1 \leq i \leq l$, $v_i = \Phi(r_i)$.
2. For every edge (w_i, w_j) of G , there is an edge, with the same multiplicity, between $\Phi(w_i)$ et $\Phi(w_j)$, in $G_{n,t}^\infty$.

These conditions are only necessary, not sufficient. For any given Φ , the probability of the first condition is $\frac{1}{n!}$. The probability of the second condition, being of the following form:

$$\prod_{e \in E_G} \frac{t^{m_e}}{n^{m_e} m_e!}$$

is smaller than $(\frac{t}{n})^{N_E}$. Therefore, by independence between $G_{n,t}^\infty$ and v , the probability that Φ induces an isomorphism between (G, r) and $G_{n,t}^\infty(v)$ is smaller than $\frac{t^{N_E}}{n^{l+N_E}}$. By union bounds, as there are less than n^{N_V} injections between $(w_i)_{1 \leq i \leq N_V}$ to $\{1, \dots, n\}$, we obtain:

$$p_n(G) \leq t^{N_E} n^{N_V - l - N_E}$$

proving Lemma 4.2. □

A Galton-Watson is usually described as a genealogy tree, *i.e.* the planer embedding of a rooted tree, or "plane tree": a plane tree is a rooted tree with an order relation between the children of the same node (see [FS09][Annex A.9]). An isomorphism of plane trees (resp. of a sequence of plane trees) is an isomorphism preserving the root (resp. the sequence of roots, with order) and the children's order. Assuming that $B_j(G_{n,t}^\infty, v)$ is a forest with l components, let $B_j^*(G_{n,t}^\infty, v)$ be the plane representation of $B_j(G_{n,t}^\infty, v)$, where the children are ordered according to their original label, in $\{1, \dots, n\}$. The rooted graph $B_j(G_{n,t}^\infty, v)$ and G are isomorphic as rooted graphs, if and only if the plane representation of $B_j(G_{n,t}^\infty, v)$ is isomorphic to one of the plane representations of (G, r) , as a plane tree or forest. Therefore

$$\mathbb{P}(B_j(G_{n,t}^\infty, v) \sim (G, r)) = \sum_{(G^*, r)} \mathbb{P}(B_j^*(G_{n,t}^\infty, v) \underset{\text{plane}}{\sim} (G^*, r))$$

where the summation is taken over the plane representations of (G, r) .

Let us compute the number of injections Φ from V_G to $\{1, \dots, n\}$ preserving the order among the children of each node of G . First we choose the N_V elements of $\Phi(V_G)$, without ordering (there are $\binom{n}{N_V}$ possible choices). From there, choosing Φ is equivalent to choosing a plane forest isomorphic to G^* on N_V vertices. The number of such plane forests is:

$$\frac{N_V!}{\prod_{i=1}^{N_V} d_{w_i}^{\text{out}}!}$$

where $d_{w_i}^{\text{out}}$ is the outdegree of w_i , *i.e.* the number of children of w_i (cf. [Spe97][p.2], [CM01][p.5] or [CF03][p.12]). Therefore the number of injections Φ from V_G to $\{1, \dots, n\}$ preserving the order among the children of

each node of G is

$$\frac{n!}{(n - N_V)! \prod_{i=1}^{N_V^b} d_{w_i}^{out!}}. \quad (3)$$

It should be noted that

$$N_E = \sum_{i=1}^{N_V^b} d_{w_i}^{out} \quad (4)$$

Such an injection Φ induces an isomorphism of plane forests between (G^*, r) and $(B_j^*(G_{n,t}^\infty, v))$ if and only if:

1. For each i , the random uniform root v_i is equal to $\Phi(r_i)$;
2. For any $1 \leq i \leq N_V^b$ and $1 \leq j \leq N_V$, there is the same number of edges (either 0 or 1) between w_i and w_j in G and between $\Phi(w_i)$ and $\Phi(w_j)$ in $G_{n,t}^\infty$;
3. For any $i \leq N_V^b$ and vertex $j \in \{1, \dots, n\} \setminus \Phi(V_G)$, there is no edge between $\Phi(w_i)$ and j in $G_{n,t}^\infty$.

For a given Φ , the probability of the first property is $\frac{1}{n!}$. As the multiplicity of each edge of $G_{n,t}^\infty$ is a Poisson random variable of parameter $\frac{t}{n}$, the probability of the last two properties is:

$$\left(e^{-\frac{t}{n}} \frac{(\frac{t}{n})^1}{1!} \right)^{N_E} \left(e^{-\frac{t}{n}} \frac{(\frac{t}{n})^0}{0!} \right)^{(n-1)N_V^b - N_E} = e^{-t \frac{n-1}{n} N_V^b} \left(\frac{t}{n} \right)^{N_E}. \quad (5)$$

Combining (3), (4), (5), the fact that the probability of the first property is $\frac{1}{n!}$ and $N_v = l + N_e$, we obtain:

$$\begin{aligned} \mathbb{P} \left(B_j^*(G_{n,t}^\infty, v) \underset{plane}{\sim} (G^*, r) \right) &= \frac{n!}{(n - N_V)! \prod_{i=1}^{N_V^b} d_{w_i}^{out!}} e^{-t \frac{n-1}{n} N_V^b} \left(\frac{t}{n} \right)^{N_E} \\ &\xrightarrow{n \rightarrow \infty} \frac{e^{-t N_V^b} t^{N_E}}{\prod_{i=1}^{N_V^b} d_{w_i}^{out!}} \\ &= \prod_{i=1}^{N_V^b} \left(e^{-t} \frac{t^{d_{w_i}^{out}}}{d_{w_i}^{out!}} \right) \\ &= \mathbb{P}(GW_j^l \underset{plane}{\sim} (G^*, r)) \end{aligned}$$

where GW_j^l denotes l independent copies of a Galton-Watson tree with Poisson offspring with mean t . By summing over all the plane representations

of G , one obtains that:

$$\mathbb{P}(B_j(G_{n,t}^\infty, v) \sim (G, r)) \rightarrow \mathbb{P}(GW_j^l \sim (G, r))$$

for all G and j , ending the proof of the first part of Lemma 4.1.

By the properties of Poisson point processes, conditionally given its unlabelled version, the labels of the edges of $G_{n,t}^\infty$ are independent and uniform on $[0, t]$. Therefore, the labelled graph $G_{n,t}^\infty$ l -rooted at l independent uniform random roots converges weakly toward l independent copies of T_t^∞ . \square

By the standard properties of Poisson processes, T_t^∞ can be described as a branching tree such that the law of the set of labels of outgoing edges is a Poisson point process of intensity 1 on $[0, t]$.

Let T_∞^∞ denote the *Poisson-Weighted Infinite Tree* in dimension 1, or *PWIT*, as defined by Aldous and Steele [AS04]. The distribution of T_t^∞ can also be described as follows:

- Remove every edge of T_∞^∞ whose label is larger than t .
- Let T_t^∞ be the connected component of the root in the resulting subgraph.

It should be noted that the notion of convergence used here is not exactly the same as the one used in [AS04], as the latter allows the edges' labels to converge toward their limit, whereas the former requires the edges' labels to be eventually constant.

4.2 The forbidden degree version of infinite graphs.

4.2.a Neighborhood with boundaries

Definition 4.5. Let Ω be the set of locally finite graphs with labelled edges such that no two edges have the same label. Let $\Omega_{<\infty}$ be the set of finite graphs in Ω . For any non-negative t and graph G , let G_t be the subgraph of G restricted to its edges with label smaller than t .

For any graph $G \in \Omega_{<\infty}$, the forbidden degree version $G^k = \Phi(G)$ of G is defined by adding the edges of G in their labelling order and by removing the edges adjacent to any vertex reaching degree k . If $G \in \Omega \setminus \Omega_{<\infty}$, the labels are not necessarily well-ordered, so some care is needed to extend Φ to a larger class of graphs of Ω . This is the aim of this section.

Definition 4.6. Let e be an edge of G . (G_B, B, S) is called a *neighborhood with boundary* of e in G if:

- S and B are subset of the set of vertices of G .

- G_B is the induced subgraph of G restricted to B .
- The edge e is in G_B .
- S is a subset of B .
- S contains the boundary of B in G (*i.e.* the sets of vertices of B with a neighbor not in B in the graph G)

Remark. The last condition means that elements of $B \setminus S$ in G are never endpoints of edges between B and $G \setminus G_B$, while vertices in S may be so. Sometimes we shorten (G_B, B, S) in (G_B, S) as B can be retrieved from G_B .

Example. For any graph G , integer l , and vertex or edge x of G , $(B_l(G, x), S_l(G, x))$ is a neighborhood with boundary.

Definition 4.7. Let e be an edge of G , $\mathcal{V} = (G_B, B, S)$ a neighborhood with boundary of e in G . Let us consider the set $\mathcal{S}_{\mathcal{V}}$ of finite graphs \tilde{G} with an edge $\tilde{e} \in \tilde{G}$ and a neighborhood with boundary $(\tilde{G}_{\tilde{B}}, \tilde{B}, \tilde{S})$ of \tilde{e} in \tilde{G} such that (G_B, B, S, e) is isomorphic to $(\tilde{G}_{\tilde{B}}, \tilde{B}, \tilde{S}, \tilde{e})$. If either,

1. \tilde{e} is present in $\Phi(\tilde{G})$ for every $\tilde{G} \in \mathcal{S}_{\mathcal{V}}$,
2. or \tilde{e} is removed in $\Phi(\tilde{G})$ for every $\tilde{G} \in \mathcal{S}_{\mathcal{V}}$,

then we say that the knowledge of \mathcal{V} is sufficient to know whether e is removed in the forbidden degree version of G .

The set $\mathcal{S}_{\mathcal{V}}$ depends both on (G_B, B) and on S : the subgraph of \tilde{G} induced by \tilde{B} is isomorphic to G_B , but one can obtain \tilde{G} only by growing new edges on the vertices of \tilde{S} . That is, only the vertices of \tilde{S} may have a different degree in $\tilde{G}_{\tilde{B}}$ and in \tilde{G} . Thus S matters in the definition above, in a somehow hidden way. If finally we are able to extend Φ to some infinite graph G , we expect, obviously, that the status of e in $\Phi(G)$ (present or absent) is the same as the status of its counterpart \tilde{e} in $\Phi(\tilde{G})$.

4.2.b Locality of the forbidden version

For each edge e of G , let $l(e)$ denote the smallest integer l such that, for all t , the knowledge of $(B_l(G, e), S_l(G, e))$ is sufficient to ascertain whether e is removed in the forbidden degree version of G_t . If there is no such l , let $l(e) = \infty$.

For any vertex v of G , let $l(v)$ be the smallest integer such that the knowledge of $(B_l(G, v), S_l(G, v))$ is sufficient to know which edges adjacent to v are removed in the forbidden degree version of G_t for all t .

In this section, we shall prove that for any $T \geq 0$, a.s. $l(v) < \infty$ (resp. $l(e) < \infty$) for any vertex v (resp. any edge e) of T_T^∞ .

Definition 4.8. Let $\Omega^+ \subset \Omega$ be the set of labelled graphs G such that for all edges e of G , $l(e) < \infty$.

For any graph $G \in \Omega^+$, $\Phi(G)$ is defined as follows:

- the set of vertices of $\Phi(G)$ is V ;
- the set of edges of $\Phi(G)$ is a subset of E ;
- for each $e \in E$, e is an edge of $\Phi(G)$ if and only if e is present in $\Phi(B_{l(e)}(G, e))$.

The main result of this part is the following lemma:

Lemma 4.3. *Almost surely, for all $t \geq 0$, $T_t^\infty \in \Omega^+$.*

Lemma 4.3 will be proven in part 4.2.c. By Lemma 4.3, $\Phi(T_t^\infty)$ is well defined, and, as a subgraph of the tree T_t^∞ , $\Phi(T_t^\infty)$ is a forest. Let T_t^k be the connected component of the root \emptyset in $\Phi(T_t^\infty)$.

Corollary 4.4. *For any i , $G_{n,t}^k$ rooted at m independent uniform vertices locally converges in distribution toward m independent copies of (T_t^k, \emptyset) .*

Proof. By Lemma 4.1, $G_{n,t}^\infty$ rooted at m random uniform vertices $(v_i)_{1 \leq i \leq m}$ locally converges in distribution toward $\mathcal{T} := \cup_{1 \leq i \leq m} (T_i, r_i)_{1 \leq i \leq m}$, where (T_i, r_i) are m independent copies of (T_t^∞, \emptyset) .

Lemma 4.5. *For any positive integer m , let Ω_m^+ be the set of m -rooted elements of Ω^+ . Then the application*

$$\begin{aligned} \Omega_m^+ &\rightarrow \Omega_m^+ \\ (G, (r_i)_{1 \leq i \leq m}) &\rightarrow (\Phi(G), (r_i)_{1 \leq i \leq m}) \end{aligned}$$

is continuous for the local topology.

Lemma 4.5 entails that $(G_{n,t}^k, v)$ locally converges toward $(\Phi(\mathcal{T}), r)$, as $G_{n,t}^k = \Phi(G_{n,t}^\infty)$. As $\Phi(\mathcal{T}) = \cup_i (\Phi(T_i))$, and T_t^k is the connected component of the root in $\Phi(T_t^\infty)$, $(G_{n,t}^k, v)$ locally converges toward m independent copies of T_t^k . \square

Proof of Lemma 4.5. Let $(G^n, (r_i^n)_{1 \leq i \leq m})$ be a sequence of n -rooted graphs converging toward $(H, (s_i)_{1 \leq i \leq m}) \in \Omega_m^+$ for the local topology and let j denote a positive integer. As $H \in \Omega^+$, $\cup_i B_j(H, s_i)$ is finite, and $l(e)$ is finite for every edge e of H . Therefore, $L = \max_{e \in \cup_i B_j(H, s_i)} l(e) < \infty$. By definition of the local convergence, for n large enough, there exists an isomorphism Ψ_n between $\cup_i B_{j+L+1}(G^n, r_i^n)$ and $\cup_i B_{j+L+1}(H, s_i)$. By definition of L and $l(e)$, every edge e of $\cup_i B_j(G^n, r_i^n)$ is present in $\Phi(G^n)$ if and only if $\Psi_n(e)$ is present in $\Phi(H)$. Therefore Ψ_n induces an isomorphism between $\cup_i B_j(\Phi(G^n), r_i^n)$ and $\cup_i B_j(\Phi(H), s_i)$. As this construction works for any integer j , $(\Phi(G^n), (r_i^n)_{1 \leq i \leq m})$ locally converges toward $(\Phi(H), (s_i)_{1 \leq i \leq m})$. \square

4.2.c Propagation paths

Definition 4.9. For any element G of Ω , a *propagation path* of length $l \in \mathbb{N}$ is the data of a self-avoiding path $(v_0, e_1, v_1, e_2, \dots, e_l, v_l)$ of length l and a sequence of edges $(\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{l-1})$ of length $l - 1$ in G such that

- for all $1 \leq i \leq l - 1$, \tilde{e}_i is adjacent to v_i ;
- for all $i \neq i'$, $\tilde{e}_i \neq e_{i'}$;
- the labels of the sequence (\tilde{e}_i) are decreasing.

For any vertex v of G , let $l'(v) \in \mathbb{N} \cup \{\infty\}$ be the supremum of the lengths of the propagation paths starting at v , *i.e.* such that $v = v_0$.

Lemma 4.6. *Let $G \in \Omega$. For every vertex v in G :*

$$l(v) \leq l'(v) + 1.$$

Lemma 4.6 has a useful consequence:

Corollary 4.7. *Let Ω^- be the set of graphs $G \in \Omega$ such that for all vertex $v \in G$, $l'(v) < \infty$. Then*

$$\Omega^- \subset \Omega^+.$$

Lemma 4.3 is a consequence of Corollary 4.7 and the following lemma:

Lemma 4.8. *For all t , a.s. $T_t^\infty \in \Omega^-$.*

The rest of this subsection is devoted to the proof of Lemmata 4.6 and 4.8.

Proof of Lemma 4.6. Let G be a graph in Ω , e^* be an edge of G and l a positive integer. In order to simplify the notations, B will denote $B_l(G, e^*)$ and S will denote $S_l(G, e^*)$.

Definition 4.10. An edge e of B will be said **certain at time t** if, for all $t' \leq t$, the knowledge of (B, S) allows one to determine if e is removed in the forbidden degree version of $G_{t'}$ and **uncertain at time t** otherwise.

The edges in $G \setminus B$ are said uncertain at any time.

By definition, $l(e^*) > l$ if and only if e^* is uncertain at some time t . G is a locally finite graph, therefore B is a finite graph. Let J denote the finite set of labels of B . The application $t \rightarrow B_t$ is a right-continuous piecewise constant function jumping only at elements of J .

For every edge $e \in B$, let y_e denote the infimum of the times where e is uncertain, and t_e denote the label of e .

Lemma 4.9. *For every edge $e \in B \setminus S$, there exists:*

- an edge e' adjacent to one endpoint v of e such that $t_{e'} = y_e$;
- an edge $e'' \neq e'$ adjacent to v such that $y_{e''} < y_e$.

Proof. Let e be an edge in $B \setminus S$. y_e corresponds to a jump of $t \rightarrow B_t$, therefore there exists an edge e' such that $t_{e'} = y_e$. As e is certain at time y_e^- and uncertain at time y_e , then, \tilde{e} is removed at time y_e in some graph \tilde{G} , and it is not in some other graph \tilde{G} (according to the notations of Definition 4.7). Let e' denote the only edge with label y_e (e' is unique by definition of Ω). Only the endpoints of e' can reach the forbidden degree at time y_e , and only edges adjacent to an endpoint of e' can be removed at time y_e thus e' and e share a common endpoint v . Moreover, in order for the removal of e to be uncertain, the degree of v must be uncertain before time $y_e = t_{e'}$ in the forbidden degree version of G .², i.e. one of the edge adjacent to v is uncertain before time y_e . Let e'' denote such an edge. As an edge cannot be uncertain before its label, $e'' \neq e'$. \square

Lemma 4.10 (Necessary condition for the uncertainty to spread). *Given an edge $e \in B$ such that $y_e < \infty$, there exists a self-avoiding path $e_1, v_1, e_2, \dots, e_p, v_p$ in B with $e_1 = e$ and a sequence of edges $(\tilde{e}_i)_{1 \leq i \leq p-1}$ in B such that:*

1. for every $i \in \{1, p-1\}$, \tilde{e}_i is adjacent to v_i ;
2. for every $i \neq j$, $\tilde{e}_i \neq e_j$;
3. $t_{\tilde{e}_1} > t_{\tilde{e}_2} \cdots > t_{\tilde{e}_{p-1}}$;
4. the vertex v_p is in S .

Proof. Let Ξ denote the set of integers p such that there exists a path $e = e_1, v_1, e_2, \dots, e_p, v_p$ in B and a sequence of edges $(\tilde{e}_i)_{1 \leq i \leq p-1}$ such that:

1. for every $i \in \{1, p-1\}$, \tilde{e}_i is adjacent to v_i ;
- 2'. for every i , $\tilde{e}_i \neq e_{i+1}$;
- 3a. $y_{e_1} > y_{e_2} \cdots > y_{e_{p-1}}$;
- 3b. for every $i \in \{1, p-1\}$, $t_{\tilde{e}_i} = y_{e_i}$.

Such a path trivially exists for $p = 1$. The edges e_i in the path are distinct by condition 3a, and the graph B is finite, thus Ξ is bounded from above. Let $p_{\max} = \max \Xi$. By Lemma 4.9, if $e_{p_{\max}} \in B \setminus S$, the path can be extended one step further, by taking $e = e_p$, $\tilde{e}_p = e'$ and $e_{p+1} = e''$, and therefore p is not maximal. This entails that for any maximal path, either $v_{p_{\max}}$ or

²it is possible that e and e' share both their endpoints, e.g. if $e = e'$. In that case, the degree of at least one of their endpoints must be uncertain. Let v denote such an endpoint.

$v_{p_{\max}-1}$ is in S , and therefore there exists a path satisfying conditions 1, 2', 3a, 3b and 4 (and condition 3, as a consequence of 3a and 3b). The shortest path satisfying conditions 1, 2', 3a, 3b and 4 is self-avoiding (otherwise it would not be the shortest). If this path does not satisfy condition 2, while satisfying condition 2', there exists (i, j) such that $j \notin \{i, i+1\}$ and $\tilde{e}_i = e_j$. In that case, e_j is incident to v_i , and the path is not self-avoiding. \square

As a consequence of Lemma 4.10, if $l(e^*) > l$, then e^* is uncertain at some time t , and there exists a propagation path of length l starting at an endpoint of e^* , proving Lemma 4.6. \square

Proof of Lemma 4.8. We will now prove that for all $t \geq 0$, $T_t^\infty \in \Omega^-$ a.s. If T_t^∞ contains only the root, $T_t^\infty \in \Omega^-$. With positive probability, the root has at least one child. Let \mathcal{T} be the tree T_t^∞ conditioned on having at least one edge, and let v be one children of the root of \mathcal{T} chosen uniformly at random. The law of \mathcal{T} rerooted at v is absolutely continuous with respect to the law of \mathcal{T} . Therefore, it is sufficient to prove that almost surely $l'(\emptyset) < \infty$ in \mathcal{T} , and therefore in T_t^∞ .

Definition 4.11. A possible propagation path of length l is the data of:

- a self-avoiding path $\emptyset = v_0, e_1, v_1, e_2, v_2, \dots, e_l, v_l$ in B_l from \emptyset to a vertex v_l of S_l .
- a sequence of edges $(\tilde{e}_i)_{1 \leq i \leq l-1}$ in B_l such that:
 - for every $i \in \{1, l-1\}$, \tilde{e}_i is adjacent to v_i ;
 - for every $i \neq i'$, $\tilde{e}_i \neq e_{i'}$.

Let H_l be the set of such possible propagation paths.

A possible propagation path is a propagation path if the labels of \tilde{e} are decreasing. Let $p(l)$ denote the probability that there exists a propagation path of length l starting at the root of T_t^∞ . The set H_l only depends on the unlabelled version of the tree T_t^∞ . Conditionally on the unlabelled version of the tree T_t^∞ , the labels of the edges are i.i.d. uniform on $[0, t]$, thus the conditional probability that a given possible propagation path is actually a propagation path is $\frac{1}{(l-1)!}$. Therefore, by union bound, $p(l) \leq \frac{\mathbb{E}(|H_l|)}{(l-1)!}$.

The following lemma gives an explicit formula for the expected size of H_l :

Lemma 4.11. *Let X_t be a Poisson random variable of parameter t . Then:*

$$\mathbb{E}(|H_l|) = \mathbb{E}(X_t) \mathbb{E}(X_t^2)^{l-1} = t(t+t^2)^{l-1}$$

Proof. The proof is done by induction on l .

H_1 is the set of self-avoiding paths of length 1 starting from the root, equal to the degree of the root, distributed as X_t .

The set H_{l+1} can be constructed from H_l in the following way:

- Take an element of H_l and denote its endpoint by v_l .
- Take a child v_{l+1} of v_l and extend the self-avoiding path to v_{l+1} .
- Choose w a neighbor of v_l different from v_{l+1} (either a child of v_l , or v_{l-1}). Let \tilde{e}_l be the edge between v_l and w .

This construction gives every element of H_{l+1} once. Therefore an element of H_l gives d^2 elements of H_{l+1} with d the number of children of v_l . By the branching property of T_t^∞ , this implies that $\mathbb{E}(|H_{l+1}|) = \mathbb{E}(X_t^2)\mathbb{E}(|H_l|)$. \square

By a quick computation:

$$p(l) \leq \frac{\mathbb{E}(|H_l|)}{(l-1)!} = \frac{t(t+t^2)^{l-1}}{(l-1)!} \xrightarrow{l \rightarrow \infty} 0$$

ending the proof of Lemma 4.8. \square

Lemma 4.3 allows us to study the local limit, and therefore the convergence of l balls of radius i , with i and l fixed. In Section 5, more precise results will be needed, in which i and l can depend on n . For this reason, the following lemma will be useful:

Lemma 4.12. *There exist two sequences $b_n = o(\ln n)$ and $c_n = o(1)$ such that, asymptotically almost surely:*

1. For every v in $G_{n,t}^\infty$, $l(v) \leq b_n$.
2. For every v in $G_{n,t}^\infty$, the ball of radius b_n centered at v in $G_{n,t}^\infty$ contains at most x_n vertices, where $x_n = n^{c_n}$.

Definition 4.12 (Good event characteristic functions). To avoid dealing with problematic events of vanishing probability, we will introduce several characteristic functions, denoted GE_1, GE_2, \dots (GE stands for “Good Event”) such that GE_i is a Bernoulli variable of parameter tending to 1. Let GE_1 be the characteristic function of the events of Lemma 4.12.

Proof. By Lemma 4.6, proving that a.a.s. there is no propagation path of length b_n in $G_{n,t}^\infty$ implies the first part of Lemma 4.3. The total number of edges in $G_{n,t}^\infty$ is a Poisson variable of parameter $\binom{n}{2} \frac{t}{n} = \frac{t(n-1)}{2}$ and is smaller than tn with high probability. As a consequence, $G_{n,t}^\infty$ is a.a.s. a subgraph of $G_{tn}^\infty = G_{n,\tau_{\lfloor nt \rfloor}}^\infty$, the graph $G_{n,t}^\infty$ stopped the first time $\tau_{\lfloor tn \rfloor}$ there

are $\lfloor tn \rfloor$ edges in $G_{n,\cdot}^\infty$, thus it is sufficient to prove the absence of propagation path of length b_n in G_{nt}^∞ instead of $G_{n,t}^\infty$, and for this proof we shall rely on combinatorial arguments.

To detect a propagation path, we do not need the label of each edge, only its rank among the $\lfloor nt \rfloor$ edges of G_{nt}^∞ . It is convenient to see G_{nt}^∞ as the result of a random allocation of $\lfloor nt \rfloor$ balls in $\frac{n(n-1)}{2}$ urns (the edge $r \leq \lfloor nt \rfloor$ being incident to 2 random vertices $\{v(r), \tilde{v}(r)\}$).

Remark (Alternative description of propagation paths). A propagation path of length l can be described by a sequence of vertices (v_0, v_1, \dots, v_l) without repetition together with a l -uple without repetition (r_1, \dots, r_l) of integers smaller than tn and a strictly decreasing $l-1$ -uple $(\tilde{r}_1, \dots, \tilde{r}_{l-1})$ of integers smaller than tn such that:

$$(C1) \text{ For every } i \in \{1, \dots, l\}, \{v(r_i), \tilde{v}(r_i)\} = \{v_{i-1}, v_i\}.$$

$$(C2) \text{ For every } i \in \{1, \dots, l-1\}, v_i \in \{v(\tilde{r}_i), \tilde{v}(\tilde{r}_i)\}.$$

$$(C3) \text{ If } j \neq j', \text{ then } \tilde{r}_j \neq r_{j'}.$$

This description is equivalent to the description in Definition 4.9, where the edge e_i (resp. \tilde{e}_i) in Definition 4.9 corresponds to the edge r_i (resp. \tilde{r}_i) in this remark.

The collision set of a propagation path is the set of integers i such that $r_i = \tilde{r}_i$. The collision set is a subset of $\{1, \dots, l-1\}$.

Definition 4.13. A potential propagation path with collision set S is the data of a l -uple without repetition (r_1, \dots, r_l) of integers smaller than tn and a strictly decreasing $l-1$ -uple $(\tilde{r}_1, \dots, \tilde{r}_{l-1})$ of integers smaller than tn such that:

- $r_i = \tilde{r}_i \Leftrightarrow i \in S$.
- If $i \neq j$, then $\tilde{r}_i \neq r_j$.

A potential propagation path is only two sequences of edges (denoted by their corresponding integer), without any constraint on their endpoints: the sequence of edges described by $(r_i)_{1 \leq i \leq l}$ does not need to constitute a path nor does the edges r_i and \tilde{r}_i need to share a common endpoint. A potential propagation path will be a propagation path if and only if there exists a sequence of vertices (v_0, v_1, \dots, v_l) without repetition such that conditions (C1) and (C2) are satisfied.

The number of potential propagation paths with collision set S is:

$$\binom{\lfloor tn \rfloor}{l-1} \frac{(\lfloor tn \rfloor - l)!}{(\lfloor tn \rfloor - 2l + |S|)!} \leq \frac{(tn)^{2l-1-|S|}}{(l-1)!}$$

This formula is obtained by choosing first $(\tilde{r}_i)_{1 \leq i \leq l-1}$, and then choosing the $l - |S|$ elements of $(r_i)_{1 \leq i \leq l}$ that are still unknown.

Given two vertices v_1 and v_2 and an integer r , $\mathbb{P}(\{v(r), \tilde{v}(r)\} = \{v_1, v_2\}) = \binom{n}{2}^{-1}$, and $\mathbb{P}(v_1 \in \{v(r), \tilde{v}(r)\}) = \frac{2}{n}$. Therefore, given a potential propagation path $((r_i)_i, (\tilde{r}_i)_i)$ and a sequence of vertices (v_0, v_1, \dots, v_l) without repetition, the probability that conditions (C1) and (C2) hold is:

$$\binom{n}{2}^{-l} \left(\frac{2}{n}\right)^{l-1-|S|} = (n-1)^{-l} \left(\frac{2}{n}\right)^{2l-1-|S|}.$$

There are less than $n(n-1)^l$ possible choices for $(v_i)_{0 \leq i \leq l}$, therefore, by union bound, the probability p_l that there exists a propagation path of length b_n in G_{tn}^∞ is bounded by:

$$\begin{aligned} p_l &\leq \sum_{S \subset \{1, \dots, l-1\}} \frac{(tn)^{2l-|S|-1}}{(l-1)!} n(n-1)^l (n-1)^{-l} \left(\frac{2}{n}\right)^{2l-1-|S|} \\ &= \sum_{S \subset \{1, \dots, l-1\}} n \frac{(2t)^{2l-|S|-1}}{(l-1)!} \\ &= \frac{n}{(l-1)!} \sum_{i=1}^{l-1} \binom{l-1}{i} (2t)^{2l-i-1} \\ &= \frac{n(2t)^l (2t+1)^{l-1}}{(l-1)!} \end{aligned}$$

Taking $l = \frac{\ln n}{\sqrt{\ln \ln n}} =: b_n$, and using Stirling formula, one obtains that $p_l \rightarrow 0$.

We now need to bound the volume of the ball of radius b_n in $G_{n,t}^\infty$. For any $w \in S_l(G_{n,t}^\infty, 1)$, let $d^*(w)$ be the number of edges that connects w to some vertex of $G_{n,t}^\infty \setminus B_l(G_{n,t}^\infty, 1)$. Conditionally on $B_l(G_{n,t}^\infty, 1)$, $(d_w^*)_{w \in S_l(G_{n,t}^\infty, 1)}$ is an i.i.d. family of Poisson variables of parameter $(n - |B_l(G_{n,t}^\infty, 1)|) \frac{t}{n} \leq t$. Therefore the volume of the ball of radius b_n in $G_{n,t}^\infty$ is stochastically dominated by the volume of the first b_n generations of T_t^∞ . We assume that $t > 1$.

Let Z_l be the number of vertices in the l th generation of T_t^∞ . Conditionally on Z_i , Z_{i+1} is the sum of Z_i independent Poisson variables of parameter t , *i.e.* a Poisson variable of parameter tZ_i . Using large deviations theory (or direct computation), there exists a positive constant α such that:

$$\forall x \geq 1, \mathbb{P}(\text{Poi}(x) \geq 2x) \leq \exp(-\alpha x) \quad (6)$$

For all integers i , let $p_i = \mathbb{P}(Z_{i+1} > (2t)^{i+1} \ln^2 n \text{ and } Z_i \leq (2t)^i \ln^2 n)$.

$$\begin{aligned}
p_i &= \mathbb{E} \left(\mathbb{P}(Z_{i+1} > (2t)^{i+1} \ln^2 n | Z_i) \mathbf{1}_{Z_i \leq (2t)^i \ln^2 n} \right) \\
&= \mathbb{E} \left(\mathbb{P}(\text{Poi}(tZ_i) > (2t)^{i+1} \ln^2 n | Z_i) \mathbf{1}_{Z_i \leq (2t)^i \ln^2 n} \right) \\
&\leq \mathbb{E} \left(\mathbb{P}(\text{Poi}(t(2t)^i \ln^2 n) > (2t)^{i+1} \ln^2 n | Z_i) \mathbf{1}_{Z_i \leq (2t)^i \ln^2 n} \right) \\
&\leq \mathbb{P}(\text{Poi}(t(2t)^i \ln^2 n) > (2t)^{i+1} \ln^2 n) \\
&\leq \exp(-\alpha t (2t)^i \ln^2 n) \\
&\leq \exp(-\alpha \ln^2 n) \\
&= o\left(\frac{1}{n^2}\right)
\end{aligned}$$

Therefore $P_n := \sum_{i=0}^{b_n} p_i = o\left(\frac{1}{n}\right)$. With probability larger than $1 - P_n$, for all $i \leq b_n$, $Z_i \leq (2t)^i \ln^2 n$, and therefore $\sum_{i=0}^{b_n} Z_i \leq \frac{(2t)^{b_n+1} - 1}{2t-1} \ln^2 n =: n^{c_n}$, where $c_n = o(1)$. Therefore, $B_{b_n}(G_{n,t}^\infty, 1)$ contains less than n^{c_n} vertices with probability larger than $1 - P_n$. By union bounds, all the balls of radius b_n in $G_{n,t}^\infty$ contain less than n^{c_n} vertices with probability larger than $1 - nP_n \rightarrow 1$. \square

4.3 The local limit is a branching process

Definition 4.14. For any $G \in \Omega^+$ and $t \geq 0$, let $\Phi_t(G) = \Phi(G_t)$, where G_t still denotes the subgraph of G restricted to edges with labels smaller than or equal to t .

Given a vertex v of T_∞^∞ , different from the root, let w denote the parent of v and $T_\infty^{\infty,v}$ denote the connected component of v in $T_\infty^\infty \setminus \{v, w\}$ i.e. the subtree of T_∞^∞ starting at v , and let $\tilde{T}_\infty^{\infty,v} = (T_\infty^{\infty,v} \cup \{v, w\})$ and $\tilde{T}_t^{\infty,v} = (T_\infty^{\infty,v})_t$. See figure 1 for an example.

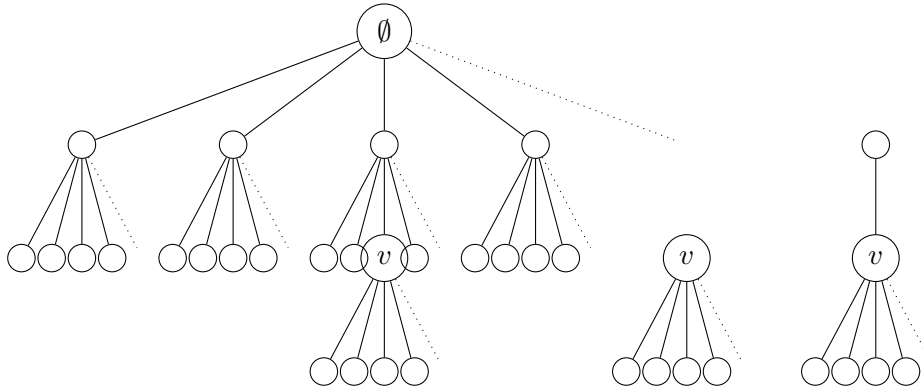


Figure 1: Example of T_∞^∞ , $T_\infty^{\infty,v}$ and $\tilde{T}_\infty^{\infty,v}$.

Lemma 4.13. For a graph $G \in \Omega^+$, an edge $e \in G$ and a non-negative t , we say that e is removed in $\Phi_t(G)$ if $e \in G_t$ and $e \notin \Phi_t(G)$.

Let t be a non-negative number and v vertex of T_∞^∞ . If $T_t^\infty \in \Omega^-$, then, at least one of the following propositions hold:

- $\Phi(T_t^\infty) \cap \tilde{T}_t^{\infty,v} = \Phi(\tilde{T}_t^{\infty,v})$;
- the edge between v and its parent w is removed in $\Phi_t(T_\infty^\infty)$.

An immediate corollary of Lemma 4.13 is:

Corollary 4.14. With the notation of Lemma 4.13, if (v, w) is removed in $\Phi(\tilde{T}_t^{\infty,v})$, then it is removed in $\Phi(T_t^\infty)$.

In other words, until (v, w) is removed in $\Phi_t(T_\infty^\infty)$, the knowledge of $\tilde{T}_t^{\infty,v}$ is sufficient to know the evolution of the subtree starting at v in the forbidden degree version of T_∞^∞ .

Proof. The graphs $\Phi(T_t^\infty) \cap \tilde{T}_t^{\infty,v}$ and $\Phi_t(\tilde{T}_t^{\infty,v})$ have the same vertex set (the vertex set of $\tilde{T}_t^{\infty,v}$), therefore any difference comes from the set of edges. Given two graphs G_1 and G_2 with the same set of vertices and an edge e , we are going to say that e separates G_1 and G_2 if e is present in one of the graphs but not the other. Let us assume that $\Phi(T_t^\infty) \cap \tilde{T}_t^{\infty,v} \neq \Phi(\tilde{T}_t^{\infty,v})$, and let e be an edge separating $\Phi(T_t^\infty) \cap \tilde{T}_t^{\infty,v}$ and $\Phi(\tilde{T}_t^{\infty,v})$.

The graph T_t^∞ is in Ω^- a.s., so is $\tilde{T}_t^{\infty,v}$, and therefore these two graphs are in Ω^+ . By definition of Ω^+ , there exists an integer l such that the knowledge of the balls of radius l centered at e and (v, w) in T_t^∞ and $\tilde{T}_t^{\infty,v}$ are sufficient to know whether e and (v, w) are present in $\Phi(T_t^\infty)$ and $\Phi(\tilde{T}_t^{\infty,v})$. Let $B^a = B_l(T_t^\infty, e) \cup B_l(T_t^\infty, (v, w))$ and $B^b = B_l(\tilde{T}_t^{\infty,v}, e) \cup B_l(\tilde{T}_t^{\infty,v}, (v, w))$. By Definition 4.7, e and (v, w) are present in $\Phi(T_t^\infty)$ (resp. $\Phi(\tilde{T}_t^{\infty,v})$) if and only if they are present in $\Phi(B^a)$ (resp. $\Phi(B^b)$). It should be noted that $B^b = B^a \cap \tilde{T}_t^{\infty,v}$. B^a is a finite graph, therefore the set of labels of B^a (and B^b) is finite. Let s be the first time such that $\Phi_s(B^a) \cap \tilde{T}_t^{\infty,v} \neq \Phi_s(B^b)$ holds, and let e' be an edge of $T_t^{\infty,v}$ that separates $\Phi_s(B^a) \cap \tilde{T}_t^{\infty,v}$ and $\Phi_s(B^b)$, i.e. e' is removed at time s in either $\Phi(B^a)$ or $\Phi(B^b)$. Let x be an endpoint of e' that reaches the forbidden degree at time s in one of the graphs, but not the other. Therefore the degree of x is different in $\Phi(B^a)$ and in $\Phi(B^b)$ strictly before time s . This implies that an edge e'' adjacent to x is present in either $\Phi_{s-}(B^a)$ or $\Phi_{s-}(B^b)$ but not in the other graph. If $x \neq w$, every edge adjacent to x in either B^a or B^b is in $\tilde{T}_t^{\infty,v}$, and e'' contradicts the definition of s and e' . Therefore $x = w$ and $e' = (v, w)$ (as (v, w) is the only edge of $\tilde{T}_t^{\infty,v}$ adjacent to w). As the degree of w in B^b is 1, w cannot reach the forbidden degree in $\Phi(B^b)$. Therefore w reaches the forbidden degree in $\Phi(B^a)$ and (v, w) is removed from $\Phi(B^a)$ i.e. (v, w) is removed in $\Phi(T_t^\infty)$. \square

Let us add to the set of rooted graphs an element \mathcal{X} , used to denote a graph that is no longer *relevant*: let $\tilde{T}_t^{k,v}$ (resp. $T_t^{k,v}$) be defined as equal to $\Phi(\tilde{T}_t^{\infty,v})$ (resp. $\tilde{T}_t^{k,v} \cap T_t^{\infty,v}$) if the edge between v and its parent has not been removed from $\Phi(\tilde{T}_t^{\infty,v})$ and equal to \mathcal{X} otherwise. Let τ_1, \dots, τ_l denote the labels of the edges adjacent to the root in T_t^∞ , ordered in increasing order, and let v_1, \dots, v_l denote the other endpoints of these edges. For any $s \leq t$, let \tilde{S}_s denote the set of integers such that $\{v_i : i \in \tilde{S}_s\}$ is the set of neighbors of the root in $\Phi_s(T_t^\infty)$ i.e. \tilde{S}_s records which edge is present in the forbidden degree version of T_t^∞ at time s . For any $i \leq l$, let $\rho_i = \inf\{s : T_s^{k,v_i} = \mathcal{X}\}$. Let $\Gamma_\tau = \{\tau_i, 1 \leq i \leq l\}$ and $\Gamma_\rho = \{\rho_i, 1 \leq i \leq l\}$.

Lemma 4.15. *Almost surely, for every $1 \leq i < j \leq l$, $\tau_i \neq \tau_j$, $\rho_i \neq \rho_j$, and either $\rho_i \neq \rho_j$ or $\rho_i = \rho_j = \infty$.*

Proof. By definition, τ_i is the label of (\emptyset, v_i) and ρ_i is the first time (\emptyset, v_i) is removed in $\Phi(\tilde{T}_t^{\infty,v_i})$, i.e. is either infinite or equal to the label of an edge adjacent to v_i . As $T^\infty \in \Omega$, no two edges have the same label. \square

Lemmata 4.13 and 4.15 allow us to consider the following dynamic set $(S_s)_{0 \leq s \leq t}$:

- at time 0, S_0 is empty;
- $s \rightarrow S_s$ is piecewise constant, and its set of jumps is included in $\Gamma_\rho \cup \Gamma_\tau$;
- for every $\tau_i \notin \Gamma_\rho$, if $|S_{\tau_i^-}| = k - 1$, then $S_{\tau_i} = \emptyset$, otherwise $S_{\tau_i} = S_{\tau_i^-} \cup \{i\}$;
- for every finite $\rho_i \notin \Gamma_\tau$, $S_{\rho_i} = S_{\rho_i^-} \setminus \{i\}$;
- for every i such that $\tau_i = \rho_i$, if $|S_{\tau_i^-}| = k - 1$, then $S_{\tau_i} = \emptyset$, otherwise $S_{\tau_i} = S_{\tau_i^-}$.

In other words, the element i is added at time τ_i , removed at time ρ_i , and S becomes empty whenever it reaches size k , even fleetingly³.

Corollary 4.16. *For any time s , $S_s = \tilde{S}_s$ and the connected component of the root is the same in $\Phi(T_s^\infty)$ and in $\cup_{i \in S_s} \Phi(\tilde{T}_s^{\infty,v_i})$.*

Proof. Let first assume that for some s , $S_s \neq \tilde{S}_s$. The set S can only change at the times $(\tau_i)_{1 \leq i \leq l}$ and $(\rho_i)_{1 \leq i \leq l}$; and the set \tilde{S} can only change at times equal to a label of an edge in $B_2(T_t^\infty, \emptyset)$. These two sets of times are finite, therefore there exists a smallest s such that $S_s \neq \tilde{S}_s$. Let $i \in S_s \Delta \tilde{S}_s$, in which Δ denotes the symmetric difference. The element i is added to S and \tilde{S} at time τ_i , therefore the difference eventually comes from the removal

³Actually, S never reaches size k , going directly from size $k - 1$ to 0, but we informally say that S reaches size k for an instant, as described in the footnote 1 page 2

times: when i is in S and \tilde{S} at time s^- and is removed from one, but not the other at time s . An element can be removed from S or \tilde{S} at time s for the following reasons:

1. S (resp. \tilde{S}) reaches size k at time s . Therefore $|S_{s^-}| = k - 1$ (resp. $|\tilde{S}_{s^-}| = k - 1$) and an element is added to S (resp. \tilde{S}) at time s . As $S_{s^-} = \tilde{S}_{s^-}$ by definition of s , and the times of addition are identical for S and \tilde{S} , S reaches size k at time s if and only if \tilde{S} reaches size k at time s .
2. $\rho_i = s$, that is (\emptyset, v_i) is removed from $\Phi(\tilde{T}_s^{\infty, v})$ at time s . By Corollary 4.14, (\emptyset, v_i) is also removed from $\Phi(T_s^\infty)$, and i is removed from \tilde{S} .
3. (\emptyset, v_i) is removed from $\Phi(T_s^\infty)$ at time s because the vertex v_i reaches degree k in $\Phi(T_s^\infty)$, *i.e.* v_i has degree $k - 1$ in $\Phi(T_{s^-}^\infty)$, and an edge is added to v_i . $i \in S_{s^-}$, therefore (\emptyset, v_i) is present in $\Phi(\tilde{T}_{s^-}^{\infty, v_i})$. By Lemma 4.13, $\Phi(T_{s^-}^\infty) \cap \tilde{T}_\infty^{\infty, v_i} = \Phi(\tilde{T}_{s^-}^{\infty, v_i})$, and therefore v_i has degree $k - 1$ in $\Phi(\tilde{T}_{s^-}^{\infty, v_i})$, and reaches degree k at time s . Therefore $\rho_i = s$, and i is removed from S at time s .

Therefore, for all s , $S_s = \tilde{S}_s$. Moreover, for any $i \in S_s$, $(\emptyset, v_i) \in \Phi(\tilde{T}_s^{\infty, v_i})$, therefore, by Lemma 4.13, $\Phi(T_s^\infty) \cap \tilde{T}_\infty^{\infty, v_i} = \Phi(\tilde{T}_s^{\infty, v_i})$, ending the proof of Corollary 4.16. □

Definition 4.15. Given a rooted graph (G, \emptyset) in Ω and a non-negative number y , we introduce the following notations:

- $\Theta^y(G, \emptyset)$ denotes the graph G rooted at \emptyset with an extra vertex w and an edge between w and \emptyset , labelled by y ;
- if $\Theta^y(G, \emptyset) \in \Omega^+$, then $\tilde{\Phi}^y(G, \emptyset)$ and $\Phi^y(G, \emptyset)$ are defined as follows: if (w, \emptyset) is removed from $\Phi(\Theta^y(G, \emptyset))$, then let $\tilde{\Phi}^y(G, \emptyset) = \mathcal{X}$ and $\Phi^y(G, \emptyset) = \mathcal{X}$; otherwise, let $\tilde{\Phi}^y(G, \emptyset) = \Phi(\Theta^y(G, \emptyset))$ and $\Phi^y(G, \emptyset) = \tilde{\Phi}^y(G, \emptyset) \cap G$;
- let $T_t^{\infty, y} = \Theta^y(T_t^\infty)$, $\tilde{T}_t^{k, y} = \tilde{\Phi}^y(T_t^\infty)$ and $T_t^{k, y} = \Phi^y(T_t^\infty)$.

Let $m(t, y) = \mathbb{P}(T_t^{k, y} \neq \mathcal{X})$. The tree T_t^∞ contains no edge with probability $e^{-t} > 0$ and in that case $T_t^{k, y} \neq \mathcal{X}$, therefore $m(t, y) > e^{-t} > 0$. Let $T_t^{k+, y}$ (resp. $\tilde{T}_t^{k+, y}$) be the random tree $T_t^{k, y}$ (resp. $\tilde{T}_t^{k, y}$) conditioned on not being equal to \mathcal{X} .

Lemma 4.17. *Conditionally on $(\tau_i)_{1 \leq i \leq l}$ and $(S_s)_{0 \leq s \leq t}$:*

- *the graphs $(T_t^{k, v_i})_{i \in S_t}$ (resp. $(\tilde{T}_t^{k, v_i})_{i \in S_t}$) are independent:*

- for each $i \in S_t$, T_t^{k,v_i} (resp. \tilde{T}_t^{k,v_i}) has the same law as T_t^{k+,τ_i} (resp. \tilde{T}_t^{k+,τ_i}).

Proof. Let \mathcal{F} be the σ -algebra generated by the sequences $(\tau_i)_{1 \leq i \leq l}$ and $(\rho_i)_{1 \leq i \leq l}$, and \mathcal{F}_1 be the σ -algebra generated by $(\tau_i)_{1 \leq i \leq l}$ and $(S_s)_{0 \leq s \leq t}$. As $\mathcal{F}_1 \subset \mathcal{F}$, it is sufficient to prove Lemma 4.17 with \mathcal{F} instead of \mathcal{F}_1 . By the branching property, conditionally on $(\tau_i)_{1 \leq i \leq l}$, the trees $(\tilde{T}_t^{\infty,v_i})_{1 \leq i \leq l}$ are independent, and for each i , \tilde{T}_t^{∞,v_i} is a copy of $\tilde{T}_t^{\infty,\tau_i}$. Each ρ_i only depends on \tilde{T}_t^{∞,v_i} , and S_t is \mathcal{F} -measurable. Therefore, conditionally on \mathcal{F} : the trees $(\tilde{T}_t^{\infty,v_i})_{1 \leq i \leq l}$ are independent and for each i , \tilde{T}_t^{∞,v_i} has the same law as $\tilde{T}_t^{\infty,\tau_i}$ conditionally on $\inf\{s : \tilde{T}_s^{k,\tau_i} = \mathcal{X}\} = \rho_i$. It follows that conditionally on \mathcal{F} : the trees $(\tilde{T}_t^{k,v_i})_{i \in S_t}$ are independent and for each i in S_t , \tilde{T}_t^{k,v_i} has the same law as T_t^{k+,τ_i} .

As $T_t^{k,v_i} = \tilde{T}_t^{k,v_i} \cap T_t^{\infty,v_i}$ and $T_t^{k,\tau_i} = \tilde{T}_t^{k,\tau_i} \cap T_t^{\infty,\tau_i}$, Lemma 4.17 with \tilde{T}_t^{k,v_i} imply Lemma 4.17 with T_t^{k,v_i} . \square

Lemma 4.17 implies the following theorem:

Theorem 4.18. *Let $B(y)$ be the law of the set of the labels of the edges adjacent to the root of $T_t^{k+,y}$. Let BP be the multitype branching process with offspring law $B(\cdot)$. Then $T_t^{k+,y}$ has same law as BP starting with a root of label y .*

Proof. $T_t^{k+,y}$ is either equal to T_t^k or to \mathcal{X} . With the notations of the proof of Lemma 4.17, the event $T_t^{k+,y} = \mathcal{X}$ is \mathcal{F}_1 -measurable, therefore, conditionally on $T_t^{k+,y} \neq \mathcal{X}$ and the sequence (τ_1, \dots, τ_l) of labels of the edges adjacent to the root, the subtrees starting at the children of the root are independent copies of T_t^{k+,τ_i} . \square

Lemma 4.17 also implies that T_t^k is a two-stages branching process:

Theorem 4.19. *Conditionally on the set $(\tau_i)_{i \in S_t}$ of labels of edges adjacent to the root in T_t^k , the subtrees starting at the root's children are independent, and copies of T_t^{k+,τ_i} .*

4.3.a Properties of the branching process

Let $\mu_{t,y}$ (resp. $\mu_{t,y}^+$, ν_t) be the law of the set of labels of edges adjacent to the root in $T_t^{k,y}$ (resp. $T_t^{k+,y}$, T_t^k). This measure can be decomposed according

to the degree of the root in these trees:

$$\begin{aligned}\mu_{t,y} &= \sum_{i=-1}^{k-2} \mu_{t,y}^i; \\ \mu_{t,y}^+ &= \sum_{i=0}^{k-2} \mu_{t,y}^{i+}; \\ \nu_t &= \sum_{i=0}^{k-1} \nu_t^i.\end{aligned}$$

in which $\mu_{t,y}^i$ (resp. $\mu_{t,y}^{i+}$, ν_t^i) only puts mass on sets of cardinality i and $\mu_{t,y}^{-1}$ is equal to $(1 - m(t, y))\delta_{\mathcal{X}}$.

Definition 4.16. For X be a finite subset of $[0, t]$, let us define the random rooted tree $T_t^{\infty, X}$ as follows:

- let P be a Poisson point process of intensity 1 on $[0, t]$;
- conditionally on P , let $(T^z)_{z \in P \cup X}$ be an i.i.d family of copies of T_t^∞ ;
- then, adding a root \emptyset to the forest $(T^z)_{z \in P \cup X}$, and, for each $z \in P \cup X$, adding an edge with label z between \emptyset and the root of T^z , one obtains $T_t^{\infty, X}$. $T^{\infty, X}$ is rooted at \emptyset .

Let E_t^X denote the event "the only edges adjacent to the root in $\Phi(T_t^{\infty, X})$ are the edges labelled by X ". If $y \notin X$, let $E_t^{y, X}$ denote the event " $\Phi^y(T_t^{\infty, X}) \neq \mathcal{X}$ and the only edges adjacent to the root in $\Phi^y(T_t^{\infty, X})$ are the edges labelled by X ".

Informally, $T_t^{\infty, X}$ is the conditional distribution of the tree T_t^∞ , given that the root is incident to edges with labels in X . This informal explanation will be justified rigorously with the help of Campbell formulas in the following pages. If $y \notin X$, a.s., $\Theta^y(T_t^{\infty, X}) \in \Omega^+$ and $\Phi^y(T_t^{\infty, X})$ is well-defined.

Definition 4.17. Let $0 \leq j \leq i \leq k - 2$ be two integers. Q_i^j is the set $\{(t, y, x_1, \dots, x_i) : 0 \leq x_1 \leq \dots \leq x_j \leq y \leq x_{j+1} \leq \dots \leq x_i \leq t\}$, and Q_i is the simplex $\{(t, x_1, \dots, x_i) : 0 \leq x_1 \leq \dots \leq x_i \leq t\}$.

Lemma 4.20. For any integer i and j , $(t, y, x_1, \dots, x_i) \rightarrow \mathbb{P}(E_t^{y, \{x_1, \dots, x_i\}})$ (resp. $(t, x_1, \dots, x_i) \rightarrow \mathbb{P}(E_t^{\{x_1, \dots, x_i\}})$) is continuous on \mathring{Q}_i^j (resp. \mathring{Q}_i), and both are larger than $\exp(-(i+1)t)$.

Lemma 4.21. $(y, t) \rightarrow m(t, y)$ is continuous on $\{0 \leq y < t\}$ and is larger than e^{-t} . For any integer i and $y < t$, $\mu_{t,y}^i$, $\mu_{t,y}^{i+}$ and ν_t^i are absolutely

continuous with respect to the Lebesgue measure on $[0, t]^i$, with respective densities:

$$\begin{aligned}\frac{\partial \mu_{t,y}^i}{\partial x} &= f_i(t, y, x) &= \mathbb{P}(E_t^{y, \{x_1, \dots, x_i\}}) \\ \frac{\partial \mu_{t,y}^{i+}}{\partial x} &= g_i(t, y, x) &= \mathbb{P}(E_t^{y, \{x_1, \dots, x_i\}}) / m(t, y). \\ \frac{\partial \nu_t^i}{\partial x} &= h_i(t, x) &= \mathbb{P}(E_t^{\{x_1, \dots, x_i\}})\end{aligned}$$

The functions f_i , g_i and h_i are not defined on the null set where two or more coordinates are equal. They are by definition symmetric in the variables $(x_j)_{1 \leq j \leq i}$. By Lemma 4.20, f_i and g_i are continuous on every set \mathring{Q}_i^j , and h_i is continuous on Q_i , and f_i , g_i and h_i are larger than $\exp(-(i+1)t)$.

The continuity in Lemma 4.20 will be proven by a coupling argument:

- Let P^{coupl} be a Poisson point process of intensity 1 on $(0, \infty)$.
- Let $(T^{z, coupl})_{z \in P^{coupl}}$ be a family of independent copies of the process T_∞^∞ .
- Let $(T^{j, coupl})_{j \leq i}$ be i independent copies of T_∞^∞ .

Let $T^{\infty, \{x_1, \dots, x_i\}}$ be constructed in the following way:

- P is the restriction of P^{coupl} to $(0, t)$.
- For any $z \in P$, $T^z = T_t^{z, coupl}$.
- For any x_j , $T^{x_j} = T_t^{j, coupl}$.

This construction allows to couple every $T_t^{\infty, \{x_1, \dots, x_i\}}$ by using the same randomness. Let T be a positive real number, and let $(t, y, x_1 \dots, x_l)$ and $(\tilde{t}, \tilde{y}, \tilde{x}_1 \dots, \tilde{x}_l)$ be two elements of \mathring{Q}_i^j such that $T \geq t$ and $T \geq \tilde{t}$. Using the notation $[a, b]$ to describe the non-empty interval $[\min(a, b), \max(a, b)]$, let $I = [\tilde{t}, t] \cup [y, \tilde{y}] \cup_i [x_i, \tilde{x}_i]$. Let l be a positive integer. Let $X = \{x_1 \dots, x_i\}$ and $\tilde{X} = \{\tilde{x}_1 \dots, \tilde{x}_i\}$

Lemma 4.22. *If the following properties hold,*

1. $P^{coupl} \cap I = \emptyset$;
2. For every $z \in P^{coupl} \cap [0, T]$, no edge in the first l generation of $T_T^{z, coupl}$ has a label in I .
3. For every j , no edge in the first l generation of $T_T^{j, coupl}$ has a label in I .

4. For every $z \in P^{\text{coupl}} \cap [0, T]$, there is no propagation path of length l starting from the root of $T_T^{z, \text{coupl}}$.
5. For every j , there is no propagation path of length l starting from the root of $T_T^{j, \text{coupl}}$.

then $E_t^{X, y}$ holds if and only if $E_t^{\tilde{X}, \tilde{y}}$ holds, and E_t^X holds if and only if $E_t^{\tilde{X}}$ holds.

Proof. The knowledge of $B^l(\Theta^y(T_t^X), \emptyset)$ (resp. $B^l(\Theta^{\tilde{y}}(T_t^{\tilde{X}}), \emptyset)$) is sufficient to know which edges are adjacent to the root in $\Phi(\Theta^y(T_t^X))$ (resp. $\Phi(\Theta^{\tilde{y}}(T_t^{\tilde{X}}))$), by properties 4 and 5. By properties 1, 2 and 3, the unlabelled versions of $B^l(\Theta^y(T_t^X), \emptyset)$ and $B^l(\Theta^{\tilde{y}}(T_t^{\tilde{X}}), \emptyset)$ are equal. For any graph $G \in \Omega^+$, the set of removed edges in $\Phi(G)$ only depends on the unlabelled graph G and the respective order of the edges' labels, not on the actual labels. By hypothesis, the elements of (t, y, x) and $(\tilde{t}, \tilde{y}, \tilde{x})$ are in the same respective order. Therefore, by properties 2 and 3, the respective order of the labels is the same in $B^l(\Theta^y(T_t^X), \emptyset)$ and $B^l(\Theta^{\tilde{y}}(T_t^{\tilde{X}}), \emptyset)$, and therefore $E_t^{x, y}$ holds if and only if $E_t^{\tilde{x}, \tilde{y}}$ holds.

The proof for E_t^X is identical upon removing Θ^y . \square

Proof of Lemma 4.20. For a given T , the probability of the properties 4 and 5 tends to 1 when l tends to ∞ . For a given T and l , the properties of properties 1, 2 and 3 tends to 1 when $(\tilde{t}, \tilde{y}, \tilde{x}_1, \dots, \tilde{x}_l)$ tends to (t, y, x_1, \dots, x_l) , proving the continuity of $\mathbb{P}(E_t^{y, X})$ and $\mathbb{P}(E_t^X)$.

If $P = \emptyset$ and every subtree starting from a children is empty, the events $E_t^{y, X}$ and E_t^X hold. $|P|$ is a Poisson variable of parameter t , and T_t^∞ is empty with probability $\exp(-t)$, therefore $\mathbb{P}(E_t^{y, X}) =$ (resp. $\mathbb{P}(E_t^X)$) is larger than $\exp(-(i+1)t)$ for all set X of size $i \leq k-2$ and any $y \notin X$ (resp. for all set X of size $i \leq k-1$). \square

Proof of Lemma 4.21. For any finite set $X \subset [0, t]$ and $y \in [0, t] \setminus X$, we consider the following construction:

- Let $(T_t^z)_{z \in X}$ be an i.i.d. family of copies of T_t^∞ .
- Let $T_t^{\infty, (X)}$ is the tree obtained by taking a vertex \emptyset , every tree $(T_t^z)_{z \in X}$ and adding an edge, labelled by z , between \emptyset and the root of T_t^z . $T_t^{\infty, (X)}$ is rooted at \emptyset .
- Let $Z_X^y \subset X$ (resp. Z_X) be equal to the random set of labels of the edges adjacent to the root of $\Phi^y(T_t^{\infty, (X)})$ (resp. $\Phi(T_t^{\infty, (X)})$), if defined.

Remark. The difference between $T_t^{\infty, X}$ and $T_t^{\infty, (X)}$ is that the latter have only edges labelled by elements of X adjacent to the root, whereas the

former have edges labelled by X and additional edges, according to the Poisson point process P .

By the branching property, conditionally on P , the set of edges adjacent to the root of T_t^∞ , T_t^∞ has same distribution as $T_t^{\infty, (P)}$, and therefore the set of edges adjacent to $\Phi^y(T_t^\infty)$ has same distribution as Z_P^y .

Let λ be a bounded positive continuous function $[0, t]^i \rightarrow [0, \infty)$.

$$\int_{[0, t]^i} \lambda(x_1, \dots, x_i) \mu_{y, t}^i(d(x_1 \dots x_i)) = \int_{[0, t]^i} \lambda(x_1, \dots, x_i) f(t, y, x_1 \dots, x_i) dx_1 \dots dx_i.$$

Let A denote the left-hand side:

$$\begin{aligned} A &= \mathbb{E} \left(\sum_{x_1, \dots, x_i \in P}^{\neq} \lambda(x_1, \dots, x_i) \mathbf{1}_{Z_P^y = \{x_1, \dots, x_i\}} \right) \\ &= \mathbb{E} \left(\mathbb{E} \left(\sum_{x_1, \dots, x_i \in P}^{\neq} \lambda(x_1, \dots, x_i) \mathbf{1}_{Z_P^y = \{x_1, \dots, x_i\}} \middle| P \right) \right) \\ &= \mathbb{E} \left(\sum_{x_1, \dots, x_i \in P}^{\neq} \lambda(x_1, \dots, x_i) \mathbb{P}(Z_P^y = \{x_1, \dots, x_i\}) \right) \\ &= \mathbb{E} \left(\sum_{x_1, \dots, x_i \in P}^{\neq} \gamma(x_1, \dots, x_i, P) \right) \end{aligned}$$

in which $\gamma(x_1, \dots, x_i, Y) = \lambda(x_1, \dots, x_i) \mathbb{P}(Z_Y^y = \{x_1, \dots, x_i\})$ for every set Y and real numbers x_1, \dots, x_i . By the reduced Campbell formula, [BB09, formula (9.12)]:

$$\mathbb{E} \left(\sum_{x_1, \dots, x_i \in P}^{\neq} \gamma(x_1, \dots, x_i, P) \right) = \int_{[0, t]^i} \mathbb{E}(\gamma(x_1, \dots, x_i, P_{x_1, \dots, x_i})) M^{(i)}(dx_1 \dots dx_i).$$

in which, for a Poisson point process of intensity Λ equal to the Lebesgue measure, $M^{(i)} = \Lambda^i$, by [BB09, Proposition 9.1.3] and P_{x_1, \dots, x_i} has same law as $P \cup \{x_1, \dots, x_i\}$ by [BB09, Corollary 9.2.5]. Therefore:

$$\begin{aligned} A &= \int_{[0, t]^i} \mathbb{E}(\gamma(x_1, \dots, x_i, P_{x_1, \dots, x_i})) dx_1 \dots dx_i \\ &= \int_{[0, t]^i} \lambda(x_1, \dots, x_i) \mathbb{P}(Z_{\{x_1, \dots, x_i\} \cup P}^y = \{x_1, \dots, x_i\}) dx_1 \dots dx_i \\ &= \int_{[0, t]^i} \lambda(x_1, \dots, x_i) \mathbb{P}(E_t^{y, X}) dx_1 \dots dx_i \end{aligned}$$

Therefore $\mu_{y, t}^i$ is absolutely continuous with respect to the Lebesgue measure with density $(x_1, \dots, x_i) \rightarrow \mathbb{P}(E_t^{y, \{x_1, \dots, x_i\}})$. This density is a probability,

and is therefore bounded by 1, and is continuous by Lemma 4.20. Therefore $(t, y) \rightarrow m(t, y) = \sum_{i=0}^{k-2} \mu_{y,t}^i(Q_i)$ is continuous.

As the random tree $T_t^{k+,y}$ is equal to the tree $T_t^{k,y}$ conditioned on not being equal to \mathcal{X} , $\mu_{y,t}^{i+}$ is absolutely continuous with respect to the Lebesgue measure with density $(x_1, \dots, x_i) \rightarrow \mathbb{P}(E_t^{y,X})/m(t, y)$.

The proof for ν_t^i is identical to the proof for $\mu_{y,t}^i$ upon replacing Φ^y by Φ , $E_t^{y,X}$ by E_t^X and Z_X^y by Z_X . □

4.3.b Link between $\mu_{\cdot,t}$ and ν_t

Lemma 4.23. *For every integer $i \geq 1$ and iuple $(x_1, \dots, x_i) \in (0, t)^i$, $h_i(t, x_1, \dots, x_i) = m(t, x_1) f_{i-1}(t, x_1, \dots, x_i)$.*

Let $X = \{x_1, \dots, x_i\}$ and $\tilde{X} = \{x_2, \dots, x_i\}$. By Lemma 4.21, the density $h_i(t, x_1, \dots, x_i)$ is equal to $\mathbb{P}(E_t^X)$, *i.e.* the probability that the only edges adjacent to the root in $\Phi(T_t^{\infty, X})$ are the edges labelled by elements of X .

Let v be the other endpoint of the edge labelled by x_1 , and T_t^{∞, x_1} the subtree of $T_t^{\infty, X}$ starting at v and rooted at v . Let $T_t^{\infty, X, \setminus x_1}$ be the tree $T_t^{\infty, X} \setminus T_t^{\infty, x_1}$. The figure 4.3.b illustrates these trees.

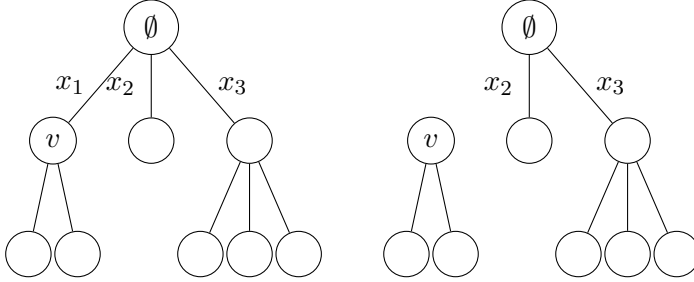


Figure 2: Example of $T_t^{\infty, X}$, T_t^{∞, x_1} and $T_t^{\infty, X, \setminus x_1}$, from left to right.

By definition of $T_t^{\infty, X}$, T_t^{∞, x_1} and $T_t^{\infty, X, \setminus x_1}$ are independent, and have same distribution as respectively T_t^∞ and $T_t^{\infty, \tilde{X}}$.

E_t^X implies that the edge labelled by x_1 is present in $\Phi(T_t^{\infty, X})$, therefore, by Lemma 4.13, E_t^X is equivalent to:

- the edge labelled by x_1 is present in $\Phi(\Theta^{x_1}(T_t^{\infty, x_1}))$, *i.e.* $\Phi^{x_1}(T_t^{\infty, x_1}) \neq \mathcal{X}$, and
- the edges adjacent to the root in $\Phi(\Theta^{x_1}(T_t^{\infty, X, \setminus x_1}))$ are the edges labelled by x_1, \dots, x_i .

This two events are independent and of probability $m(t, x_1)$ and $\mathbb{P}(E_{x_2}^{\{x_2, \dots, x_i\}})$, therefore $\mathbb{P}(E_t^X) = m(t, x_1) \mathbb{P}(E_t^{x_1, \tilde{X}})$, proving Lemma 4.23.

This formula allows to compute the density of any ν_t^i for $i \geq 1$ from the density of $\mu_{t,y}^i$. As the probability that the root of T_t^k is isolated is $1 - \sum_{i=1}^{k-1} \nu^i(Q_i)$, it is sufficient to study the measure μ to know ν . This computation also implies that $m(t, x_1) f_{i-1}(t, x_1, x_2 \dots, x_i)$ is symmetric in the x_i .

5 The equivalence between supercriticality of the local limit and the existence of the giant component

Let a_t^k be the probability that the tree T_t^k is infinite. The goal of this section is to prove Theorem 1.4.

Theorem 1.4. *For all $t \geq 0$, $\frac{C_{\max}(G_{n,t}^k)}{n}$ converges in probability to a_t^k .*

For readability, $C_{\max}(G_{n,t}^k)$ is abridged in C_{\max} .

Remark. Theorem 1.4 is meaningful both in the subcritical case and in the supercritical case.

If T_t^k is critical or subcritical, $a_t^k = 0$, and therefore the largest component's size is $o_p(n)$.

If T_t^k is supercritical, $a_t^k > 0$, so there is a giant component of size equivalent to $a_t^k n$.

This result is analogous to the result on the size of the largest component in the Erdős-Rényi graph, where a_t is the probability of survival of a Galton-Watson whose offspring is Poisson distributed with parameter t .

The local limit results imply that a_t^k is close to the expected proportion of vertices in large components. To prove Theorem 1.4, one needs to prove that a_t^k is a.s. close to the actual proportion of vertices in large components (not only in expectation), and that almost every vertices in large components are in the same component.

5.1 The subcritical or critical case

For any integer i , let N^i be the number of vertices of $G_{n,t}^k$ in components of size at least i .

Lemma 5.1. *For all $\epsilon > 0$, $\mathbb{P}(\frac{N^{\sqrt{n}}}{n} \geq a_t^k + \epsilon) \xrightarrow[n \rightarrow \infty]{} 0$.*

Lemma 5.1 allows to bound C_{\max} from above:

Corollary 5.2. *For all $\epsilon > 0$, $\mathbb{P}(\frac{C_{\max}}{n} \geq a_t^k + \epsilon) \xrightarrow[n \rightarrow \infty]{} 0$.*

For all i , if $C_{\max} \geq i$, then $N^i \geq C_{\max}$. Therefore for all integer i , $C_{\max} \leq \max(N^i, i) \leq i + N^i$. Using $i = \sqrt{n}$ gives Corollary 5.2.

Corollary 5.2 is sufficient to prove Theorem 1.4 in the subcritical or critical case, as in that case $a_t^k = 0$ and $\frac{C_{\max}}{n}$ is a positive random variable.

Proof of Lemma 5.1. Let v_1 and v_2 be two independent uniform random vertices of $G_{n,t}^k$, and i a non-negative integer. Let $C(v_1)$ (resp. $C(v_2)$) denote the component of v_1 (resp. v_2) in $G_{n,t}^k$:

$$\begin{aligned}\mathbb{P}(|C(v_1)| \geq i | G_{n,t}^k) &= \frac{N^i}{n} \\ \mathbb{P}(|C(v_1)| \geq i) &= \mathbb{E} \left(\frac{N^i}{n} \right) \\ \mathbb{P}(|C(v_1)| \geq i \text{ and } |C(v_2)| \geq i | G_{n,t}^k) &= \left(\frac{N^i}{n} \right)^2 \\ \mathbb{P}(|C(v_1)| \geq i \text{ and } |C(v_2)| \geq i) &= \mathbb{E} \left(\frac{N^i}{n} \right)^2.\end{aligned}$$

Being in a component of size at least i is a local event, therefore by the birooted local convergence proven in Lemma 4.1:

$$\begin{aligned}\mathbb{P}(|C(v_1)| \geq i) &\xrightarrow[n \rightarrow \infty]{} \mathbb{P}(|T_t^k| \geq i) \\ \mathbb{P}(|C(v_1)| \geq i \text{ and } |C(v_2)| \geq i) &\xrightarrow[n \rightarrow \infty]{} \mathbb{P}(|T_t^k| \geq i)^2.\end{aligned}$$

Therefore, for all i ,

$$\begin{aligned}\mathbb{E} \left(\frac{N^i}{n} \right) &\xrightarrow[n \rightarrow \infty]{} \mathbb{P}(|T_t^k| \geq i) \\ \text{Var} \left(\frac{N^i}{n} \right) &\xrightarrow[n \rightarrow \infty]{} 0 \\ \text{and therefore } \frac{N^i}{n} &\xrightarrow[n \rightarrow \infty]{p} \mathbb{P}(|T_t^k| \geq i).\end{aligned}$$

N^i is non-increasing in i , and $\mathbb{P}(|T_t^k| \geq i) \xrightarrow[i \rightarrow \infty]{} a_t^k$, therefore for all $\epsilon > 0$, $\mathbb{P} \left(\frac{N^{\sqrt{n}}}{n} \geq a_t^k + \epsilon \right) \xrightarrow[n \rightarrow \infty]{} 0$. \square

5.2 The supercritical case

If $a_t^k > 0$, a lower bound is needed. The following lemma gives such a lower bound.

Lemma 5.3. *Let v_1 and v_2 be two independent uniform vertices of G_n^k . Then $\liminf_n \mathbb{P}(v_1 \text{ and } v_2 \text{ are in the same component}) \geq (a_t^k)^2$.*

Let us first see why Lemma 5.3 implies Theorem 1.4. Let $(C_i)_{i \geq 1}$ be the sequence of the sizes of the component in $G_{n,t}^k$ (in any order). Conditionally on $G_{n,t}^k$, the probability that v_1 and v_2 are in the same component is $\frac{1}{n^2} \sum C_i^2$.

Let $\epsilon > 0$. Lemmata 5.1 and 5.3 imply that for n large enough:

$$\begin{aligned}
(a_t^k)^2 - \epsilon &\leq \frac{1}{n^2} \mathbb{E} \left(\sum_i C_i^2 \right) \\
&= \frac{1}{n^2} \mathbb{E} \left(\sum_{i: C_i \geq \sqrt{n}} C_i^2 + \sum_{i: C_i < \sqrt{n}} C_i^2 \right) \\
&\leq \frac{1}{n^2} \mathbb{E} \left(\sum_{i: C_i \geq \sqrt{n}} C_i^2 + \sqrt{n} \sum_{i: C_i < \sqrt{n}} C_i \right) \\
&\leq \frac{1}{n^2} \mathbb{E} \left(\sum_{i: C_i \geq \sqrt{n}} C_i^2 + \sqrt{n} \cdot n \right) \\
&\leq \mathbb{E} \left(\frac{\sum_{i: C_i \geq \sqrt{n}} C_i}{n} \right) + n^{-\frac{1}{2}} \\
&= \mathbb{E} \left(\frac{C_{\max} N \sqrt{n}}{n} \right) + n^{-\frac{1}{2}} \\
(a_t^k)^2 - \epsilon &\leq \mathbb{E} \left(\frac{C_{\max}}{n} \right) (a_t^k + \epsilon) + \epsilon \tag{7}
\end{aligned}$$

The last inequality holding for n large enough because $\frac{C_{\max}}{n} \leq 1$ a.s., $\frac{N \sqrt{n}}{n} \leq 1$ a.s. and $\mathbb{P}(\frac{N \sqrt{n}}{n} \leq a_t^k + \epsilon) \rightarrow 1$.

By taking $\epsilon \rightarrow 0$, (7) implies that $\liminf \mathbb{E}(\frac{C_{\max}}{n}) \geq a_t^k$. As $\frac{C_{\max}}{n}$ is smaller than 1 a.s. and smaller than $a_t^k + \epsilon$ with high probability by Corollary 5.2, this implies that $\frac{C_{\max}}{n} \xrightarrow{P} a_t^k = \mathbb{P}(|T_t^k| = \infty)$.

The proof of Lemma 5.3 will follow these steps:

1. With probability close to $(a_t^k)^2$, v_1 and v_2 are in components of $G_{n,t'}^k$ of size larger than a threshold (that depends on the number of vertices) at time $t' = t - \epsilon$;
2. In that case, with probability close to one, a path between w_1 and w_2 will exist at time t in G_t^k .

In order to do the first step, and use the local limit, we need to construct a graph that approximates $T_{t'}^k$ and $G_{n,t'}^k$ simultaneously.

5.2.a A few standard results on multitype branching processes

This part will summarize the results on multitype branching processes that will be used, using the results and notations of [Har63]. Let T be a multitype branching process, with type set $\mathbb{X} = [0, \tau]$ for some positive τ . For every integer i , let Z_i be the random set equal to the labels of the elements of i th generation of T . Let \mathbb{P}_x (resp. \mathbb{E}_x) denote the probability (resp. expectation) when the root of T has type x . For all $A \subset \mathbb{X}$, let $M(x, A) = \mathbb{E}_x(|A \cup Z_1|)$. We will assume that the branching processes considered always satisfy the following two conditions:

- (C1) For all $x \in \mathbb{X}$, $|Z_1| \leq k$ \mathbb{P}_x -a.s.
- (C2) There exists two positive real numbers a and b such that for all $x \in \mathbb{X}$, $M(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure, and its density $m(x, y)$ is a uniformly positive bounded function, $0 < a \leq m(x, y) \leq b < \infty$.

Conditions (C1) and (C2) will be direct consequences of Lemmata 4.20 and 4.21 for all branching processes we will consider. (C1) and (C2) implies that T satisfies technical conditions 10.1 and 13.1 with the notations of [Har63].

By [Har63, Theorem 10.1], the operator M has a real positive eigenvalue ρ , larger than any other eigenvalue. This eigenvalue ρ will be called the spectral radius associated to T . Let $q(x) = \mathbb{P}_x(|T| < \infty)$ denote the extinction probability function of T . By [Har63, Theorem 12.1, Theorem 14.1 and following remarks]:

Lemma 5.4.

1. If $\rho \leq 1$, then for all $x \in \mathbb{X}$, $q(x) = 1$
2. If $\rho > 1$:
 - For all $x \in \mathbb{X}$, $q(x) < 1$.
 - For all $x \in \mathbb{X}$, conditionally on $|T| = \infty$, $\frac{|Z_i|}{\rho^i}$ converges \mathbb{P}_x -a.s. toward a random non-zero variable W when i tends to ∞ .

For all non-negative function s , let $\varphi_x(s)$ be defined by:

$$\varphi_x(s) = \mathbb{E}_x \left(\prod_{y \in Z_1} s(y) \right).$$

Remark. With the notation of [Har63], this is actually $\Phi(-\ln s)$, but this version is more suited for our use.

By [Har63, Theorem 15.1]:

Lemma 5.5. *If $\rho > 1$,*

- *q is the only uniformly positive and uniformly less than 1 function satisfying $q(x) = \varphi_x(q)$ for all $x \in \mathbb{X}$.*
- *If q^0 is a positive and uniformly less than 1 function, and if we define the sequence of functions $(q^i)_{i \geq 1}$ by $q^{i+1}(x) = \varphi_x(q^i)$, then q^i converges everywhere toward q .*

Corollary 5.6.

- *If q^0 is a positive and uniformly less than 1 function such that for all x , $q^0(x) \geq \varphi_x(q^0)$, then for all $x \in \mathbb{X}$, $q(x) \leq q^0(x)$.*
- *If q^0 is a positive and uniformly less than 1 function such that for all x , $q^0(x) \leq \varphi_x(q^0)$, then for all $x \in \mathbb{X}$, $q(x) \geq q^0(x)$.*

Proof. We define $(q^i)_{i \geq 0}$ as in Lemma 5.5. Then by Lemma 5.5, for all x , $q^i(x)$ converges toward $q(x)$. For the first part of Corollary 5.6, $q^1(x) \leq q^0(x)$. As $\varphi_x(s)$ is an increasing function of the positive function s , the sequence $(q^i)_{i \geq 0}$ is a non-increasing sequence and is therefore larger than its limit. Similarly, in the second part, the sequence q^i is non-decreasing, and therefore smaller than its limit. \square

Let T' be a two stages-branching process, with a different law for the root, and same law as T for the remaining of the tree. Let Z'_i be the random set of labels of the i th generation of T' . Recall that q and ρ denote the extinction probability function and the spectral radius of T .

Lemma 5.7.

- $\mathbb{P}(|T'| < \infty) = \mathbb{E} \left(\prod_{y \in Z'_1} q(y) \right)$.
- *If $\rho > 1$, conditionally on $|T'| = \infty$, $\frac{|Z'_i|}{\rho^i}$ converges toward a random non-zero variable W .*

For such a two-stages branching process T' , ρ will be called the spectral radius associated to T' .

Proof. Conditionally on the first generation Z_1 , the subtrees starting at elements of Z_1 are independent, so the probability of extinction is $\prod_{y \in Z'_1} q(y)$. The first part of Lemma 5.7 is obtained by taking the expectation. The second part is obtained by using Lemma 5.4 on the surviving subtrees. \square

For all t , let ρ_t be the spectral radius associated to T_t^{k+} or T_t^k .

Lemma 5.8. *The spectral radius ρ_t is a upper semi-continuous function of t .*

Lemma 5.8 will be proven in part 5.6.

5.3 Approximating $G_{n,t}^k$ by an idealised graph

Let t be such that $a_t^k > 0$, i.e. $\rho_t > 1$. Let $\frac{1}{2} > \epsilon_1 > 0$ and $t' = t - \epsilon_1$. Let $\epsilon_2 > 0$. Let $K_n = \frac{(1-\epsilon_2)\ln n}{\ln(\rho_{t'})}$. Using Lemma 5.8, we can choose ϵ_1 small enough so $\rho_{t'} > 1$.

The graph G^{mod} will be constructed dynamically while exploring $G_{n,t'}^\infty$. G^{mod} will denote the growing graph process and $G_{\text{end}}^{\text{mod}}$ the graph G^{mod} at the end of its construction. At any point, G^{mod} will satisfy the following properties:

- G^{mod} is either empty, one planar tree with labelled edges or two planar trees with labelled edges, such that no two edges have the same label.
- In each planar tree, the children of a given vertex are ordered, for the planar tree order, in increasing order of the labels of the outgoing edges.

The graph $G_{\text{end}}^{\text{mod}}$ will have the same law as two independent copies of $T_{t'}^\infty$. Informally, $G_{\text{end}}^{\text{mod}}$ will be built in such a way that the components of the roots in $\Phi(G_{\text{end}}^{\text{mod}})$ are close to the components of v_1 and v_2 in $G_{n,t'}^k = \Phi(G_{n,t'}^\infty)$. In order to achieve this, we define an exploration process, that will only look at the parts of the graph $G_{n,t'}^\infty$ that are useful to construct the component of v_1 and v_2 in $G_{n,t'}^k$. At the beginning of its exploration, $G_{n,t'}^\infty$ (seen from v_1 and v_2) looks like two independent copies of $T_{t'}^\infty$. But, as the exploration continues, these two graphs differ: $G_{n,t'}^\infty$ can have cycles or multiple edges, and the numbers of neighbors of the vertices decreases (in law), as a growing fraction of $G_{n,t'}^\infty$ is known. The following construction explains how to couple $G_{n,t'}^\infty$ with its idealized branching version $G_{\text{end}}^{\text{mod}}$ in such a way that the differences between $\Phi(G_{\text{end}}^{\text{mod}})$ and $G_{n,t'}^k$ are small.

Let V^G denote the set of vertices of $G_{n,t}^\infty$. In order to construct G^{mod} , the process will use $G_{n,t'}^\infty$ and some extra randomness. Initially, G^{mod} is empty, and will be created dynamically along with the exploration of $G_{n,t'}^\infty$. The process will add vertices to G^{mod} , from V^G and extra vertices. The extra vertices will be called *dummy vertices*.

As $G_{\text{end}}^{\text{mod}}$ is two planar trees, we can define the following notions:

Definition 5.1. At any point, G^{mod} is a rooted planar tree or two rooted planar trees. For any vertex w of G^{mod} and integer i , the i -children of w are defined as the vertices at distance i of w in the subtree starting at w .

The set of vertices of $G_{\text{end}}^{\text{mod}}$ is ordered with the breadth-first order, denoted by $<$. For any two vertices w and w' of $G_{\text{end}}^{\text{mod}}$, $w < w'$ if either:

- w is in the first component of $G_{\text{end}}^{\text{mod}}$ and w' is in the second component.
- w and w' are in same component, and the distance between w and the root is strictly smaller than the distance between w' and the root.

- w and w' are in the same component, in the same generation, and the father of w is strictly smaller than the father of w' for the breadth-first order.
- w and w' are two children of the same vertex w'' , and the label of the edge between w and w'' is smaller than the label of the edge between w' and w'' .

The construction of $G_{\text{end}}^{\text{mod}}$ will use the following increasing sets:

- $A^G \subset V^G$ will denote the set of vertices of $G_{n,t'}^\infty$ that have already been discovered.
- $B^G \subset A^G$ will denote the set of the vertices whose neighbors are known. These vertices will be called *used*.
- $A^C \subset A^G$ will denote the set of *corrupted* vertices. They indicate where G^{mod} and $G_{n,t'}^\infty$ are different.
- $B^C \subset A^C$ will denote the set of corrupted vertices whose children in $G_{\text{end}}^{\text{mod}}$ have already been constructed.
- A^{Dum} will denote the set of dummy vertices. At any point, $A^{\text{Dum}} \cap V^G = \emptyset$.
- $B^{\text{Dum}} \subset A^{\text{Dum}}$ will denote the subset of dummy vertices whose children in $G_{\text{end}}^{\text{mod}}$ have already been constructed.
- $A = A^G \cup A^{\text{Dum}}$ is the set of the vertices of G^{mod} . The sets A^G and A^{Dum} are disjoint.
- $B = B^G \cup B^C \cup B^{\text{Dum}}$ is the set of the vertices of G^{mod} whose children in G^{mod} have already been constructed. The sets $B^G \cup B^C$ and B^{Dum} are disjoint, but B^G and B^C might not be disjoint.

During the process, all these sets will only increase.

$G^{\text{mod}-}$ is the isomorphism class of the graph G^{mod} , with respect to the isomorphism of graphs with labelled edges and unlabelled vertices. This allows for example to consider $G^{\text{mod}-}$ without knowing if a given vertex is a dummy vertex. As the labels of the edges of G^{mod} are all different, the only automorphism of G^{mod} , seen as a graph with labelled edges and unlabelled vertices, is the identity, and therefore a unique way to map $G^{\text{mod}-}$ back to G^{mod} , allowing us to consider the vertex of G^{mod} associated to a given vertex of $G^{\text{mod}-}$.

Let \mathcal{F} denote the increasing σ -algebra, generated by $A^G, B^G, A^C, B^C, A^{\text{Dum}}, B^{\text{Dum}}$, the set of labelled edges with at least one endpoint in B in the graph $G_{n,t'}^\infty$ and the set of labelled edges with at least one endpoint in B

in $G_{\text{end}}^{\text{mod}}$. \mathcal{F} represents the current knowledge obtained with the exploration of $G_{n,t'}^{\infty}$ and the construction of G^{mod} .

The following tools will be used in the construction of G^{mod} .

5.3.a Split a vertex/edge:

There is no multiple edge in $T_{t'}^{\infty}$, whereas there can be multiple edges in $G_{n,t'}^{\infty}$. To solve this issue, any multiple edge will be substituted upon discovery by the appropriate number of simple edges. If a multiple edge e of multiplicity l is discovered between w and w' while looking at the neighbors of w , add w' to G^{mod} and only one edge between w and w' , labelled by the smallest label of e . For any label y among the $l - 1$ other labels, considered in increasing order, add a dummy vertex w_y to G^{mod} and A^{Dum} , and add an edge between w and w_y labelled by y in G^{mod} . An example can be found in figure 3. This operations allows to avoid adding multiple edges to G^{mod} without altering the degree of w (it does alter the degree of w').

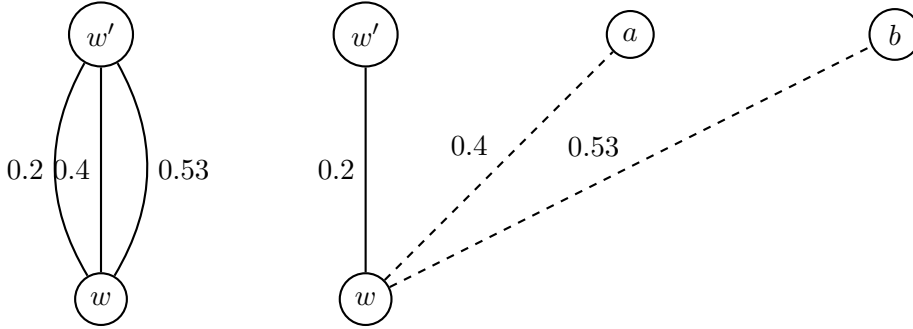


Figure 3: The triple edge between w and w' in $G_{n,t'}^{\infty}$ is split in G^{mod} by adding two dummy vertices a and b .

5.3.b Probing a vertex w :

This operation will be done for a vertex $w \in A^G \setminus B$. The goal is to approximate the set of labelled edges adjacent to w in $G_{n,t'}^{\infty}$ by a Poisson point process of intensity 1 on $[0, t']$.

- Let μ_w^{dummy} be an independent Poisson point process of intensity $\frac{|A^G|}{n}$ on $[0, t']$. For each point s of μ_w , in increasing order, add a dummy vertex w' and an edge labelled by s between w and w' to G^{mod} . Add w' to A^{Dum} . If there is at least one such edge, add w to A^C and B^C .
- For every vertex $w' \in V^G \setminus A^G$, look at the edge between w' and w :
 - If there is a single edge between w and w' , add this edge and w' to G^{mod} . Add w' to A^G .

- If there is a multiple edge between w and w' , split it. Add w' to A^G and A^C . Add w to A^C and B^C .
- If there is no edge between w and w' , do not do anything.
- For every vertex w' in $A^G \setminus B^G$, do not add the edges between w and w' to G^{mod} . If there is at least one such edge, add w to A^C and B^C and add w' to A^C .
- Add w to B^G .

5.3.c Fake-probing a vertex w :

1. If w is a vertex of $A^{Dum} \setminus B^{Dum}$, fake-probing w is the following operations:
 - Add w to B^{Dum} .
 - Let Y_w be an independent Poisson point process of intensity 1 on $[0, t']$. For each point y of Y_w , add a dummy vertex w_y to G^{mod} and A^{Dum} , and add an edge labelled by y between w and w_y to G^{mod} .
2. If w is a vertex of $A^G \setminus B$, fake-probing w is the following operations:
 - Add w to A^C and B^C .
 - Let Y_w be an independent Poisson point process of intensity 1 on $[0, t']$. For each point y of Y_w , add a dummy vertex w_y to G^{mod} and A^{Dum} , and add an edge labelled by y between w and w_y to G^{mod} .

For any vertex w , and any set, σ -algebra or graph X , let X_w denote the value of X before the probing or the fake-probing of w . Similarly, let X_{w+} denote the value of X after the probing or fake-probing of w .

The following lemma summarizes the properties that will hold through the construction of $G_{\text{end}}^{\text{mod}}$:

Lemma 5.9.

1. For every vertex $w' \in G_w^{\text{mod}}$, $w' \neq w$, the set of edges adjacent to w' in G^{mod} is not modified during the probing or fake-probing of w .

If the following properties hold before the probing or the fake-probing of a vertex w in $A_w \setminus B_w$, then the following properties will also hold after the probing or fake-probing of w .

2. The vertex set of G^{mod} is A .

3. For every vertex $w' \in B^G \setminus B^C$, the set of edges adjacent to w' is the same in G^{mod} and $G_{n,t'}^\infty$.
4. For every vertex $w' \in A \setminus B$, w' has no child in G^{mod} .
5. If $w' \in A^G \setminus A^C$, the set of edges between w' and elements of B^G is the same in G^{mod} and in $G_{n,t'}^\infty$.
6. If $w' \in V^G \setminus A$ then there is no edge between w' and any vertex of B^G in $G_{n,t'}^\infty$.

We assume that the choice of w and of whether to do a probing or a fake-probing are \mathcal{F} -measurable. If the following property holds before the probing or fake-probing of w , then it holds after:

7. Let $E^B \subset E$ be the set of edges with at least one endpoint in B^G . For any $e \in E^B$, the set of labels of e in $G_{n,t'}^\infty$ is \mathcal{F} -measurable. Conditionally on \mathcal{F} , the set of labels of edges in $E \setminus E^B$ is an i.i.d. family of Poisson point processes of intensity $\frac{1}{n}$ on $[0, t']$.

If the properties 2-7 hold before the probing of w and if one of these two properties holds before the probing of a vertex, they still holds after:

- 8a. G^{mod} is a tree rooted at \tilde{v}_1 .
- 8b. G^{mod} is two trees rooted at \tilde{v}_1 and \tilde{v}_2 .

If the properties 2-7 hold before the probing of w , then

9. conditionally on \mathcal{F}_w , the law of set of all labels of edges between w and the children of w in G^{mod} , including dummy vertices, is a Poisson point process of intensity 1 on $[0, t']$.

Proof. The properties 1-6 and 8a/8b are direct consequences of the description of the probing and fake-probing of a vertex.

If $w \in A^G \setminus B^G$ is probed, the only vertex added to B^G during the probing of w is w , and the edges of $G_{n,t'}^\infty$ looked at are the edges adjacent to w . If w is fake-probed, no vertex is added to B^G and no edge is looked at during the fake-probing of w . Therefore property 7 holds after the probing if it holds before the probing.

If $w \in A$ is probed, by property 7, conditionally on \mathcal{F}_w , the set of labels of edges between w and elements of $V^G \setminus |A^G|$ is a Poisson point process of intensity $\frac{n-|A^G|}{n}$. Adding an independent Poisson point process of intensity $\frac{|A^G|}{n}$ gives the wanted distribution for the property 9. If w is fake-probed, property 9 is a direct consequence of the definition of the fake-probing. \square

Each time several vertices are probed (or fake-probed) at the same time by the algorithm, they are probed in breadth-first order.

5.3.d The initialisation of a component:

This tool is used at the beginning of the construction of each of the two components of $G_{\text{end}}^{\text{mod}}$. Let $j \in \{1, 2\}$. The aim of this part is to construct the ball of radius b_n centered at \tilde{v}_j in G^{mod} .

The description assume that, before the initialisation, the properties 2-7 of Lemma 5.9 hold and that either G^{mod} is empty (if $j = 1$) or 8a holds (if $j = 2$).

For every i, j , $S_{j,i}$ is the sphere of radius i centered at \tilde{v}_j in $G_{\text{end}}^{\text{mod}}$. Let $\mathcal{B}_{1,i} = \cup_{i' \leq i} S_{1,i'}$ and $\mathcal{B}_{2,i} = \cup_{i' \leq i} S_{2,i'} \cup G_{\text{end},1}^{\text{mod}}$. For any $w \in G^{\text{mod}}$, $\text{generation}(w)$ is the ordered pair (j, i) such that $w \in S_{j,i}$.

Definition 5.2 (Initialisation failed). If one of the following events happen, the initialisation is said to have failed:

- A previous initialisation has failed.
- Before the initialisation, v_j was already in A^G .
- At any point of the construction of a $S_{j,i}$, a vertex is added to A^C .

If $v_j \in A^G$ or if a previous initialisation has failed, then add to G^{mod} and A^{Dum} a dummy vertex \tilde{v}_j , as the root of the second component. Otherwise add v_j to A^G and to G^{mod} and let $\tilde{v}_j = v_j$.

Let $0 \leq i < b_n$. We assume that $S_{j,i}$ has already been built. If the initialisation has not yet failed, probe every vertex w of $S_{j,i}$. If the initialisation has failed, fake-probe every vertex w of $S_{j,i}$. In both cases, $S_{j,i+1}$ is the set of children of elements of $S_{j,i}$ in G^{mod} , *i.e.* the sphere of radius $i+1$ in G^{mod} centered at \tilde{v}_j .

Starting with $S_{j,0} = \{\tilde{v}_j\}$, the sets $S_{j,i}$ are built until S_{j,b_n} has been built. For any set, σ -algebra of graph X , let $X_{j,i}$ denote the value of X after the construction of $S_{j,i}$.

Lemma 5.10.

- After the construction of $S_{j,i}$, if the initialisation has not yet failed, $S_{j,i} \subset A^G \setminus (B^G \cup A^C)$.
- After the construction of $S_{j,i}$, for every $i' < i$, $S_{j,i'} \subset B$.

These properties are direct consequences of the algorithm: if at any point a vertex is added to A^C , the initialisation is said to have failed. The first property allows us to probe the elements of $S_{j,i}$ when the initialisation has not yet failed.

5.3.e Subsequent construction of the component of v_j in G^{mod} :

If the initialisation has not failed, the balls of radius b_n centered at v_j are the same in G^{mod} and in $G_{n,t'}^\infty$, by property 3 of Lemma 5.9. Therefore, if $GE_1 = 1$, it is possible to determine the neighbors of v_j in $G_{n,t'}^k$ (by Lemma 4.12). Let $D_{j,0} = \{v_j\}$, $E_{j,0} = \{v_j\}$ and $D_{j,1}$ be the set of neighbors of v_j in $G_{n,t'}^k$. If the initialisation has failed or if it is not possible to determine the neighbors of v_j in $G_{n,t'}^k$, let $D_{j,1} = D_{j,0} = E_{j,0} = \emptyset$.

After the initialisation, the component of \tilde{v}_j in G^{mod} is a planar tree rooted at \tilde{v}_j of depth at most b_n , the vertices of the $b_n - 1$ first generations belonging to B and the vertices of the b_n th generation belonging to $A \setminus B$.

5.3.f One iteration of the main part of the algorithm:

We now describe the iteration i , constructing the sets $E_{j,i}$, $D_{j,i+1}$. For all j, i , $E_{j,i} \subset D_{j,i} \subset S_{j,i}$. For any element w of $D_{j,i}$, if we are able to determine the children of w in the forbidden degree version of G^{mod} , w is added to $E_{j,i}$ and its children are added to $D_{j,i+1}$. More precisely, assuming $D_{j,i}$ and \mathcal{B}_{j,b_n+i-1} are known, the iteration i is the following operations:

- Initially, $E_{j,i}$ and $D_{j,i+1}$ are empty.
- For each $w \in S_{j,i+b_n-1}$, in the breadth-first order, if w is a $b_n - 1$ -children of an element of $D_{j,i}$, probe w . Otherwise, fake-probe w .
- Consider every w in $D_{j,i}$ such that no $(b_n - 1)$ -child or b_n -child of w is in A^C . If the knowledge of G^{mod} is sufficient to determine the set N of neighbors of w in $G_{n,t'}^k$ add w to $E_{j,i}$, and add every element of $N \setminus p(w)$ to $D_{j,i+1}$ where $p(w)$ is the parent of w in G^{mod} .

Remark. If $GE_2 = 1$, the knowledge of the ball of radius b_n centered at w in $G_{n,t'}^\infty$ is sufficient to know the set N . Lemma 5.11 will prove that the balls of radius b_n centered at w in $G_{\text{end}}^{\text{mod}}$ and $G_{n,t'}^\infty$ are identical and therefore that the knowledge of G^{mod} is sufficient to know the set N if $GE_2 = 1$. Lemma 5.11 will also prove that for every b_n -children of elements of $D_{j,i}$ is in $A_{j,i+b_n}^G$, and therefore can be probed.

An iteration is the execution of the algorithm above, for a given value of i . For any set, σ -algebra or graph X , let $X_{j,i+b_n}$ denote the value of X after the i th iteration in the construction of the component of \tilde{v}_j , *i.e.* the construction of $E_{j,i}$ and $D_{j,i+1}$.

5.3.g Finalisation of a component:

After K_n iterations of this algorithm are done, and \mathcal{B}_{j,K_n+b_n} is constructed, graft to every vertex of S_{j,b_n+K_n} an independent copy of $T_{t'}^\infty$. Add every

added vertex to A^{Dum} and B^{Dum} . Add every vertex of S_{j,b_n+K_n} to A^C and B^C .

The value of the sets and σ -algebra X after the finalisation of the j th component will be denoted by $X_{j,\infty}$. If the properties 2-7 and either 8a or 8b hold before the finalisation of a component, they hold after the finalisation of a component, as no edge of $G_{n,t'}^k$ is looked at, and the copies are independent.

5.3.h Construction of $G_{\text{end}}^{\text{mod}}$:

The construction of $G_{\text{end}}^{\text{mod}}$ is done by starting with $A^G = B^G = A^C = B^C = A^{Dum} = B^{Dum} = \emptyset$ and doing the following operations:

1. First initialisation, for the component of \tilde{v}_1 .
2. Do the iteration i of the algorithm for i from 1 to K_n , to construct \mathcal{B}_{1,b_n+i} and the sets $E_{1,i}$ and $D_{1,i+1}$ for every $i \leq K_n$.
3. Finalise the construction of the first component.
4. Second initialisation, for the component of \tilde{v}_2 .
5. Do the iteration i of the algorithm for i from 1 to K_n , to construct $\mathcal{B}_{2,i+b_n}$ and the sets $E_{2,i}$ and $D_{2,i+1}$ for every $i \leq K_n$.
6. Finalise the construction of the second component.

For any graph, σ -algebra of set X , X_{end} denotes the value of X after the complete algorithm.

5.3.i First properties of G^{mod}

Lemma 5.11. *For any $i < b_n$, $j \in \{1, 2\}$, after the construction of $S_{j,i}$ in the initialisation:*

1. *the properties 2-7 of Lemma 5.9 hold;*
2. *if $j = 1$, 8a holds; if $j = 2$, 8b holds.*

For every $i \leq K_n$ and $j \in \{1, 2\}$, after the iteration i constructing $E_{j,i}$ and $D_{j,i+1}$:

3. *for every vertex w of $E_{j,i}$, the ball of radius b_n centered at w is the same in G^{mod} and $G_{n,t'}^\infty$;*
4. *every vertex of $D_{j,i+1}$ is a child of a vertex of $E_{j,i}$ in G^{mod} ;*
5. *the set of vertices of $G_{j,i+b_n}^{\text{mod}}$ is $\mathcal{B}_{j,i+b_n}$;*
6. *$A_{j,i+b_n} = \mathcal{B}_{j,i+b_n}$;*

7. $B_{j,i+b_n} = \mathcal{B}_{j,i+b_n-1}$;
8. the properties 2-7 of Lemma 5.9 hold;
9. if $j = 1$, 8a holds; if $j = 2$, 8b holds.

After the construction of G_{end}^{mod} .

10. G_{end}^{mod-} has same law as two independent copies of $T_{t'}^\infty$. Moreover the second component of G_{end}^{mod-} is independent of $\mathcal{F}_{1,\infty}$.

Proof. The properties 1, 2, 8 and 9 are consequences of Lemma 5.9 and the fact that the properties 2-7 of Lemma 5.9 are preserved by the initialisation or the finalisation of a component. The properties 4, 5, 6 and 7 are consequences of the description of the algorithm, as every vertex of $S_{j,i+b_n-2}$ is either probed or fake-probed for the construction of $D_{j,i}$.

The property 3 is proven by induction. As noted in the subsection 5.3.e, property 3 holds for $i = 0$ and $j \in \{1, 2\}$. Let us assume that the property 3 holds for $i \geq 0$ and $j \in \{1, 2\}$. Let w be a vertex of $E_{j,i+1} \subset D_{j,i+1}$. By property 4, w is a children of a vertex w' of $E_{j,i}$. As the ball of radius b_n centered at w' is the same in G^{mod} and $G_{n,t'}^\infty$, the only possible difference between the balls of radius b_n centered at w is the neighbors of the $b_n - 1$ -children of w . On one hand, if none of the $b_n - 1$ -children or the b_n -children of w are added to A^C , then every ball of radius 1 centered at such children is the same in G^{mod} and in $G_{n,t'}^\infty$ and therefore the ball of radius b_n centered in w is the same in G^{mod} and $G_{n,t'}^\infty$. On the other hand, if one such vertex is added to A^C , w is not added to $E_{j,i}$.

To prove property 10, it is sufficient to show that for any i and j , conditionally on $\mathcal{B}_{j,i}$, the law of the set of labels of the edges outgoing of the vertices of $S_{j,i}$ is an i.i.d. family of Poisson point processes of intensity 1 on $[0, t']$. If $i < b_n$, conditionally on $\mathcal{F}_{j,i}$, if the initialisation have failed at an earlier stage, every fake-probing gives the correct law. If the initialisation has not yet failed, the algorithm is going to probe each vertex of $S_{j,i}$. By Lemma 5.9, the conditional law of the set of exiting edges is also a Poisson point process of intensity 1 on $[0, t']$ (some of them might point to dummy vertices). As $\mathcal{B}_{j,i}$ is measurable with respect to $\mathcal{F}_{j,i}$, the conditional law is the same conditioned on $\mathcal{B}_{j,i}$.

For all $i \in \{b_n, \dots, K_n + b_n - 1\}$, $\mathcal{B}_{j,i}$ is measurable with respect to $\mathcal{F}_{j,i}$. Each vertex of the ball of radius i is either probed or fake-probed. Therefore, by property 7 of Lemma 5.9, conditionally on $\mathcal{F}_{j,i}$, or on $\mathcal{B}_{j,i}$, the set of labels of edges between each vertex of the ball of radius i and their children is an i.i.d. family of Poisson point processes of intensity 1 on $[0, t']$.

If $i \geq K_n + b_n$, the law of the subsequent generations comes from the grafting of independent copies of $T_{t'}^\infty$, and gives the correct conditional law by the branching properties of $T_{t'}^\infty$. \square

$G_{\text{end}}^{\text{mod}}$ has the law of two independent copies of $T_{t'}^{\infty}$. Therefore $G_{\text{end}}^{\text{mod}} \in \Omega^+$ a.s. Let $G_{\text{end}}^{\text{mod}k}$ denote the components of \tilde{v}_1 and \tilde{v}_2 in $\Phi(G_{\text{end}}^{\text{mod}})$, the forbidden degree version of G^{mod} . For any $i \leq K_n$, $j \in \{1, 2\}$, let $\mathcal{D}_{j,i}^-$ be the union of the sets $D_{j',i'}$ where (j', i') is strictly smaller than (j, i) for the lexicographic order. The set $\mathcal{D}_{j,i}^-$ is the union of the sets $D_{j',i'}$ constructed before $D_{j,i}$. Let $\mathcal{D}_{j,i}^+ = \mathcal{D}_{j,i}^- \cup D_{j,i}$. The following lemma gives a few useful observations about the sets $D_{j,i}$ and $E_{j,i}$ and the graph G^{mod} produced by the algorithm and allows to link them with the component of v_1 and v_2 in $G_{n,t'}^k$:

Lemma 5.12.

1. Every pair of vertices of \mathcal{D}_{2,K_n}^+ that are neighbors in $G_{\text{end}}^{\text{mod}}$ are also neighbors in $G_{n,t'}^k$ and in $G_{\text{end}}^{\text{mod}k}$, and the edge between them has the same label in $G_{n,t'}^k$ and $G_{\text{end}}^{\text{mod}k}$.
2. Every vertex of $D_{j,i}$ belongs to the component of v_j in $G_{n,t'}^k$ and in $G_{\text{end}}^{\text{mod}k}$.
3. For $i \leq K_n$ and $j \in \{1, 2\}$, every neighbor of vertices of $E_{j,i}$ in $G_{n,t'}^k$ (resp. in $G_{\text{end}}^{\text{mod}k}$) belong to either $D_{j,i+1}$ or $E_{j,i-1}$.

The first and third observations come from the definition of $D_{j,i}$. The second observation is consequence of the first observation of Lemma 5.12 and the property 4 of Lemma 5.11.

The first two observations guarantee that the subgraph of $G_{\text{end}}^{\text{mod}}$ restricted to vertices of \mathcal{D}_{2,K_n}^+ can be seen as a subgraph of $G_{n,t'}^k$. The last property guarantees that the degree of vertices of $E_{j,i}$ is the same in this subgraph and in $G_{n,t'}^k$.

5.4 Finding a path in $G_{n,t}^k$

Definition 5.3. For every $i < K_n$ and $j \in \{1, 2\}$, $\tilde{E}_{j,i}$ is the set of vertices w of $E_{j,i}$ such that no edge is added to w between t' and t in $G_{n,t'}^{\infty}$. Let $\tilde{E}_{j,K_n} = E_{j,K_n}$.

The sets $\hat{E}_{j,i}$ are defined by induction:

- For $j \in \{1, 2\}$, $\hat{E}_{j,0} = \tilde{E}_{j,0}$.
- For $j \in \{1, 2\}$, $i \in \{1, \dots, K_n\}$, $\hat{E}_{j,i} = \{u \in \tilde{E}_{j,i} : p(u) \in \hat{E}_{j,i-1}\}$, where $p(u)$ denotes the parent of u in $G_{\text{end}}^{\text{mod}k}$.

The set $\hat{E}_{j,i}$ is the subset of vertices u of $\tilde{E}_{j,i}$ such that no edge is added between t' and t to any vertex in the path from \tilde{v}_j to u (including u if $i < K_n$, excluding u if $i = K_n$). By Lemma 5.12, any neighbor in $G_{\text{end}}^{\text{mod}k}$ of a vertex v of $E_{j,i}$ is either in $D_{j,i+1}$ or in $D_{j,i-1}$. Therefore the following lemma holds:

Lemma 5.13. *Starting from G_{end}^{modk} , remove every elements of $\cup D_{j,i} \setminus \tilde{E}_{j,i}$. Let $G_{end}^{\hat{mod}k}$ be the union of the components of v_1 and v_2 in the resulting graph. Then the set of vertices in the balls of radius K_n centered at v_1 and v_2 in $G_{end}^{\hat{mod}k}$ is $\cup \hat{E}_{j,i}$.*

Lemma 5.14.

1. For $j \in \{1, 2\}$, every vertex of the ball of radius $K_n - 1$ in $G_{end}^{\hat{mod}k}$ is in the component of v_j in the graph $G_{n,u}^k$ for any $u \in [t', t]$.
2. For $j \in \{1, 2\}$, if w of \hat{E}_{j,K_n} is such that
 - the degree of w in G_{end}^{modk} is smaller than $k - 2$, and
 - at most one edge is added to w in $G_{n,\cdot}^\infty$ between t' and t

then w is in the component of v_j in $G_{n,u}^k$ for any $u \in [t', t]$.

By Lemma 5.12, all the elements of $\cup_i \tilde{E}_{j,i}$ and the edges between them in G_{end}^{modk} are present in $G_{n,t'}^k$. Therefore $G_{end}^{\hat{mod}k}$ is a subgraph of $G_{n,t'}^k$. As by definition of $\tilde{E}_{j,i}$ no edge is added to any vertex in the balls of radius $K_n - 1$ between t' and t , no vertex of the balls of radius K_n reaches the forbidden degree between t' and t and therefore the ball of radius $K_n - 1$ is a subgraph of $G_{n,u}^k$ for any $u \in [t', t]$.

If w is a non-saturated vertex of $G_{end}^{\hat{mod}k}$ such that at most one edge is added to w in $G_{n,\cdot}^\infty$ between t' and t , the edge between w and its parent in G_{end}^{modk} is present in $G_{n,u}^k$ for any $u \in [t', t]$.

Theorem 5.15. *For any $\eta > 0$, for ϵ_1 and ϵ_2 small enough and n large enough, with probability at least $(a_t^k)^2 - \eta$, there exists a vertex w_1 in \hat{E}_{1,K_n} and a vertex w_2 in \hat{E}_{2,K_n} such that:*

- The vertices w_1 and w_2 are non-saturated in $G_{n,t'}^k$.
- There is an edge added between time t' and time t between w_1 and w_2 in $G_{n,\cdot}^\infty$.
- No other edge is added between time t' and time t to either w_1 or w_2 in $G_{n,\cdot}^\infty$.

Theorem 5.15 implies Lemma 5.3, as by Lemma 5.14, w_1 and w_2 are respectively in the component of v_1 and v_2 in $G_{n,t}^k$ and the edge between v_1 and v_2 is present in $G_{n,t}^k$.

To prove this result, the following facts will be proven:

1. For ϵ_2 small enough, with probability larger than $(a_t^k)^2 - \eta$, for all j , there are at least $n^{\frac{2}{3}}$ non saturated vertices in $|\hat{E}_{j,K_n}|$.

2. The probability that the previous condition happens without the existence of w_1 and w_2 satisfying the conditions of Theorem 5.15 converges to 0.

It will be done by finding a subgraph of $G_{\text{end}}^{\widehat{\text{modk}}}$ that is a branching process with offspring law close to the law of $T_{t'}^k$. Therefore, the probability of survival of each component is close to $a_{t'}^k$, and conditionally on survival, the components will be large enough.

Lemma 5.16. *Let $z_n = n^{1-\frac{\epsilon_2}{2}}$. With high probability, at any point of the algorithm $|A^G| \leq z_n$.*

Every element of A^G belongs to a ball of radius b_n centered at an element of $\cup_{j,i} D_{j,i}$ in $G_{n,t'}^\infty$. Moreover, by Lemma 4.12, if $GE_1 = 1$, there exists $x_n = n^{o(1)}$ such that every ball of radius b_n in $G_{n,t'}^\infty$ contains less than x_n vertices. Therefore,

$$GE_1 |A_{\text{end}}^G| \leq x_n \sum_{1 \leq j \leq 2, 0 \leq i \leq K_n} |D_{j,i}|.$$

As every element of $D_{j,i}$ belongs to the ball of radius i centered at \tilde{v}_j in $G_{\text{end}}^{\text{modk}}$, and $G_{\text{end}}^{\text{modk}}$ has the same law as $T_{t'}^k$:

$$\mathbb{E} \left(\sum_{1 \leq j \leq 2, 0 \leq i \leq K_n} |D_{j,i}| \right) \leq 2\mathbb{E}(|B_{K_n}(T_{t'}^k, \emptyset)|)$$

where $B_{K_n}(T_{t'}^k, \emptyset)$ is the ball of radius K_n centered at the root \emptyset . As $T_{t'}^k$ is a (two-stages) supercritical branching process, by Lemma 5.7:

$$\frac{|B_i(T_{t'}^k, \emptyset)|}{\rho_{t'}^i} \xrightarrow{i \rightarrow \infty} W.$$

with W a random variable with a finite expectation. As $K_n = \frac{(1-\epsilon_2)\ln n}{\ln(\rho_{t'})}$, this limit implies that

$$\begin{aligned} \sum_{1 \leq j \leq 2, 0 \leq i \leq K_n} \mathbb{E}|D_{j,i}| &\leq n^{1-\epsilon_2+o(1)} \\ \mathbb{E}(GE_1 |A_{\text{end}}^G|) &\leq x_n n^{1-\epsilon_2+o(1)} \\ &= n^{1-\epsilon_2+o(1)} \end{aligned}$$

Therefore, with high probability, $|A_{\text{end}}^G| \leq z_n$. Let GE_2 be the random variable equal to 1 if this inequality holds and $GE_1 = 1$. Otherwise, let $GE_2 = 0$. As A^G is an increasing subset, $|A^G| \leq |A_{\text{end}}^G|$ at any point of the algorithm.

Lemma 5.17. *With high probability, all the vertices in the balls of radius $K_n + b_n$ centered at \tilde{v}_1 and \tilde{v}_2 in $G_{\text{end}}^{\text{mod}}$ have less than $\log n$ children. Let GE_3 be the random variable equal to 1 if this property holds and $GE_2 = 1$. Otherwise, let $GE_3 = 0$.*

Proof. The expected number of vertices in the sphere of radius i centered at \tilde{v}_j in G^{mod} is t^i . Therefore the expected total volume of each ball of radius $K_n + b_n$ centered at \tilde{v}_1 and \tilde{v}_2 is equal to $\frac{t^{K_n+b_n}-1}{t-1} = o(n^{\alpha_1})$ for some $\alpha_1 > 0$. Therefore it is smaller than n^{α_1} with high probability by Markov's inequality. We assume that n is large enough such that $\log n > 2t'$. The law of the degree of a given vertex is a Poisson random variable of parameter t' , therefore the probability that one of the first n^{α_1} vertices of each component on $G_{\text{end}}^{\text{mod}}$ (in the breadth-first order) has degree larger than $\log n$ is smaller than

$$\begin{aligned}
2n^{\alpha_1} \sum_{i=\lfloor \log n \rfloor}^{\infty} e^{t'} \frac{t'^i}{i!} &\leq 2n^{\alpha_1} e^{-t'} \frac{t'^{\lfloor \log n \rfloor}}{\lfloor \log n \rfloor!} \underbrace{\sum_{i \geq 0} \frac{t'^i}{(2t')^i}}_{=2} \\
&= 4n^{\alpha_1} e^{-t'} \frac{t'^{\lfloor \log n \rfloor}}{\lfloor \log n \rfloor!} \\
&\sim 4e^{-t'} n^{\alpha_1} n^{\log t'} \frac{1}{\sqrt{2\pi \lfloor \log n \rfloor}} \left(\frac{e}{\lfloor \log n \rfloor} \right)^{\lfloor \log n \rfloor} \\
&\leq \alpha_2 n^{\alpha_3 - \log \log n} \\
&\rightarrow 0
\end{aligned}$$

With α_2 and α_3 two constants. The Stirling's approximation is used for the factorial. \square

Lemma 5.18. *With high probability, no initialisation fails.*

As A is initially empty, it is not possible that $v_1 \in A$ before the first initialisation. If $GE_2 = 1$, the size of the set A is smaller than z_n before the second initialisation. As v_2 is a uniform vertex independent of $\mathcal{F}_{1,\infty}$, the probability that v_2 belongs to A before its initialisation, conditionally on $\mathcal{F}_{1,\infty}$ before the second initialisation is equal to $\frac{|A^G|}{n}$. This quantity is smaller than $\frac{z_n}{n}$ if $GE_2 = 1$, therefore

$$\begin{aligned}
\mathbb{P}(v_2 \in A_{1,\infty}) &\leq \mathbb{P}(GE_2 = 0) + \mathbb{P}(GE_2 = 1 \text{ and } v_2 \in A_{1,\infty}) \\
&= \mathbb{P}(GE_2 = 0) + \mathbb{E}(GE_2 \frac{|A_{1,\infty}|}{n}) \\
&\leq \mathbb{P}(GE_2 = 0) + \frac{z_n}{n} \\
&= o(1)
\end{aligned}$$

A vertex is added to A^C for the first time during the construction of $S_{j,i+1}$ if one of the following event happens:

1. Two vertices of $S_{j,i}$ are linked by an edge in $G_{n,t'}^\infty$.
2. A vertex of $S_{j,i}$ is linked to a vertex of $A_{j,i}^G \setminus B_{j,i}^G$ by an edge in $G_{n,t'}^\infty$.
3. Two vertices of $S_{j,i}$ are linked by an edge to the same vertex $w \notin S_{j,i-1}$ in $G_{n,t'}^\infty$.
4. There is a multiple edge in $G_{n,t'}^\infty$ between a vertex w of $S_{j,i}$ and a vertex in $V^G \setminus A_{j,i}$.
5. When a vertex w is probed, the Poisson point process of intensity $\frac{|A^G|}{n}$ creating dummy vertices is non-empty.

By Lemma 5.11, conditionally on the σ -algebra $\mathcal{F}_{j,i}$, the labels of the edges in $G_{n,t'}^{\text{mod}}$ with no endpoint in $B_{j,i}$ are an i.i.d. family of Poisson point processes if intensity $\frac{1}{n}$ on $[0, t']$. Therefore, conditionally on $\mathcal{F}_{j,i}$:

1. the probability of the first event is smaller than $\binom{|S_{j,i}|}{2} \mathbb{P}(\text{Poi}(\frac{t'}{n}) \geq 1)$;
2. The probability of the second event is smaller than $|S_{j,i}| |A_{i,j}^G| \mathbb{P}(\text{Poi}(\frac{t'}{n}) \geq 1)$;
3. the probability of the third event is smaller than $n \binom{|S_{j,i}|}{2} \mathbb{P}(\text{Poi}(\frac{t'}{n}) \geq 1)^2$ (there are less than n possible choices for w);
4. the probability of the fourth event is smaller than $|S_{j,i}| n \mathbb{P}(\text{Poi}(\frac{t'}{n}) \geq 2)$;
5. the probability of the last event is smaller than $\frac{t' |S_{j,i}| |A_{j,i+1}^G|}{n}$.

Recall that $GE_3 = 1$ implies that $|S_{j,i}| \leq x_n$ and $|A_{j,i+1}^G| \leq z_n$. Let $y_n = n^{-\frac{\epsilon_2}{3}}$. Either $GE_3 = 0$ or the sum of these probabilities is smaller than $\frac{y_n}{2b_n}$, for n large enough. By union bound, the probability than one of these events happens for $i \leq b_n$ during one of the initialisations and $GE_2 = 3$ is smaller than $y_n = o(1)$.

Let GE_4 be the random variable equals to 1 if no initialisation fails and $GE_3 = 1$. Otherwise, let $GE_4 = 0$.

5.5 Construction of an included branching process

The goal of this part is to find a branching process simultaneously included in $G_{\text{end}}^{\text{mod}}$, $G_{n,t'}^k$ and $G_{n,t}^k$ with law close to T_t^k .

5.5.a Construction of the included branching process for time t'

The idea is to find a two-stages branching process whose set of vertices is a subset of $\cup \tilde{E}_{j,i}$ and whose law will be close to T_t^k . In order to do this, some bounds of the probability that a given vertex w is in $D_{j,i} \setminus \tilde{E}_{j,i}$ will be needed. For technical reasons, $w \in D_{j,i} \setminus E_{j,i}$ and $w \in E_{j,i} \setminus \tilde{E}_{j,i}$ will be treated separately. Let us add the following families of random variables, that will be used as extra randomness when needed, such that, conditionally on \mathcal{F}_{end} , these families are i.i.d. families of uniform random variables on $[0, 1]$.

- $(U_w^a)_{w \in G_{\text{end}}^{\text{mod}}}$;
- $(U_{w,w'}^b)_{w \leq w' \in G_{\text{end}}^{\text{mod}}}$;
- $(U_{w,w'}^c)_{w \leq w' \in G_{\text{end}}^{\text{mod}}}$;
- $(U_w^d)_{w \in G_{\text{end}}^{\text{mod}}}$;

These random variables will be used to construct independent variables, by using the following straightforward construction:

Lemma 5.19. *If U and V are random variables, \mathcal{F} a σ -algebra and $p \in [0, 1)$ such that:*

- $V \in \{0, 1\}$ a.s.
- $\mathbb{P}(V = 1 | \mathcal{F}) < p$ a.s.
- U is a uniform random variable on $[0, 1]$, independent of \mathcal{F} .

Then the random variable V^{ind} equals to 1 if either $V = 1$ or $U < \frac{p - \mathbb{P}(V=1|\mathcal{F})}{\mathbb{P}(V=0|\mathcal{F})}$ is a Bernoulli random variable of parameter p , larger than V , independent of \mathcal{F} .

For any $w \in G_{\text{end}}^{\text{mod}}$, let us define the following sets:

- $\Gamma_w^a = \{w' \in G_{\text{end}}^{\text{mod}}; w' < w\}$.
- $\Gamma_{w+}^a = \{w' \in G_{\text{end}}^{\text{mod}}; w' \leq w\} = \Gamma_w^a \cup \{w\}$.
- $\Gamma_w^b = \{(w', w'') \in G_{\text{end}}^{\text{mod}^2}; w' \leq w'' < w\}$.
- $\Gamma_{w+}^b = \{(w', w'') \in G_{\text{end}}^{\text{mod}^2}; w' \leq w'' \leq w\}$.
- $\Gamma_w^c = \{(w', w'') \in G_{\text{end}}^{\text{mod}^2}; w' \leq w'', w' < w, \text{generation}(w'') \leq \text{generation}(w)\}$.
- $\Gamma_{w+}^c = \{(w', w'') \in G_{\text{end}}^{\text{mod}^2}; w' \leq w'', w' \leq w, \text{generation}(w'') \leq \text{generation}(w)\}$.

- $\Gamma_w^d = \{w' \in G_{\text{end}}^{\text{mod}} : \text{generation}(w') + (0, b_n) \leq \text{generation}(w)\}$.

These sets will denote the set of uniform variables that have been used via Lemma 5.19 when w is probed or fake-probed. For every $w \in B_{\text{end}}^G$, let $\mathcal{F}_w^{\text{aug}}$ be the σ -algebra generated by \mathcal{F}_w , $(U_{w'}^a)_{w' \in \Gamma_w^a}$, $(U_{(w', w'')}^b)_{(w', w'') \in \Gamma_w^b}$, $(U_{(w', w'')}^c)_{(w', w'') \in \Gamma_w^c}$ and $(U_{w'}^d)_{w' \in \Gamma_w^d}$.

For every (j, i) , $i \geq b_n$, let $\mathcal{F}_{j,i}^{\text{aug}}$ be the σ -algebra generated by $\mathcal{F}_{j,i}$, $(U_w^a)_{w \in \mathcal{B}_{j,i-1}}$, $(U_{(w, w')}^b)_{(w, w') \in \mathcal{B}_{j,i-1}}$, $(U_{(w, w')}^c)_{(w, w') \in \mathcal{B}_{j,i-1}}$ and $(U_w^d)_{w \in \mathcal{B}_{j,i-b_n}}$.

For every vertex $w \in B_{\text{end}}^G$, let $\mathcal{F}_{w+}^{\text{aug}}$ be the σ -algebra generated by \mathcal{F}_w , $(U_{w'}^a)_{w' \in \Gamma_{w+}^a}$, $(U_{(w', w'')}^b)_{(w', w'') \in \Gamma_{w+}^b}$, $(U_{(w', w'')}^c)_{(w', w'') \in \Gamma_{w+}^c}$ and $(U_{w'}^d)_{w' \in \Gamma_{w+}^d}$.

For any w probed or fake-probed during the construction of $E_{j,i}$ (therefore such that $\text{generation}(w) = (j, i + b_n - 1)$), $\mathcal{F}_{j,i+b_n-1}^{\text{aug}} \subset \mathcal{F}_w^{\text{aug}} \subset \mathcal{F}_{w+}^{\text{aug}} \subset \mathcal{F}_{j,i+b_n}^{\text{aug}}$.

Lemma 5.20. *For every $w \in B_{\text{end}}$, every \mathcal{F}_{end} -measurable random variable X , the distribution of X is the same conditionally on \mathcal{F}_w (resp. \mathcal{F}_{w+}) and conditionally on $\mathcal{F}_w^{\text{aug}}$ (resp. $\mathcal{F}_{w+}^{\text{aug}}$).*

For every $i \in \{a, b, c, d\}$, every $w \in B_{\text{end}}^G$, Γ_w^i is \mathcal{F}_w -measurable and Γ_{w+}^i is \mathcal{F}_{w+} -measurable. Lemma 5.20 comes from the fact that the law of every U -variables is independent of \mathcal{F}_{end} . Lemma 5.20 simply means that any event that can be expressed without the extra randomness is independent of the extra randomness.

In order to construct the included branching process, one needs to study when a vertex belongs to $\cup D_{j,i} \setminus E_{j,i}$. Assuming $GE_1 = 1$, a vertex w_1 is in $D_{j,i} \setminus E_{j,i}$ for some (i, j) if a $b_n - 1$ -child or a b_n -child of w_1 is added to A^C during the construction of $D_{j,i+1}$. Let w_2 be a $b_n - 1$ -child of w_1 . During the construction of $D_{j,i+1}$, only $b_n - 1$ -children of elements of $D_{j,i}$ are probed.

- The vertex w_2 is added to A^C while probed if one of the following happens:
 - There exists a multiple edge between w_2 and another vertex in $G_{n,t'}^\infty$.
 - When w_2 is probed, $\mu_{w_2}^{\text{dummy}}$ is not empty.
 - There is an edge between w_2 and a vertex w_3 of $A_{w_2}^G \setminus B_{w_2}^G$.
- The vertex w_2 is added to A^C while another vertex w_3 is probed if there is an edge between w_2 and w_3 in $G_{n,t'}^\infty$.
- A child w_3 of w_2 is added to A^C while w_2 is probed if there is a multiple edge between w_2 and w_3 in $G_{n,t'}^\infty$.
- A child w_3 of w_2 is added to A^C while another vertex w_4 is probed if there is an edge between w_4 and w_3 in $G_{n,t'}^\infty$.

For any $(i, j) \in \{0, \dots, K_n\} \times \{1, 2\}$, let $\Delta_{j, i+b_n-1}$ be the set of b_n-1 -children of elements of $D_{j, i}$. For any i, j , $\Delta_{j, i+b_n-1}$ is a subset of $S_{j, i+b_n-1}$. Then w_2 or one of its children is added to A^C during the construction of $D_{j, i+1}$ if at least one of the following events happens:

1. while w_2 is probed, $\mu_{w_2}^{dummy}$ is not empty;
2. there is a multiple edge in $G_{n, t'}^\infty$ between w_2 and vertex not in $A_{w_2}^G$;
3. there is an edge between w_2 and a vertex of A^G not in $\Delta_{j, i+b_n-1}$ and not a children of an element of $\Delta_{j, i+b_n-1}$;
4. there is an edge between w_2 and a vertex $w_3 \in \Delta_{j, i+b_n-1}$;
5. there are a vertex $w_3 \neq w_2$ in $\Delta_{j, i+b_n-1}$ and $w_4 \notin A_{j, i+b_n-1}^G$, such that there are edges between w_2 and w_4 and between w_3 and w_4 in $G_{n, t'}^\infty$.

For any (i, j) and any vertices w_2 and w_3 in $\Delta_{j, i+b_n-1}$, let $X_{w_2}^a$ be equal to 1 if any of the first three events occurs, X_{w_2, w_3}^b be equal to 1 if the fourth event occurs and X_{w_2, w_3}^c be equal to 1 if the fifth event occurs. This classification corresponds to the following criteria:

- a. one vertex of $\Delta_{j, i+b_n-1}$ is added to A^C ;
- b. two vertices $w_2 < w_3$ of $\Delta_{j, i+b_n-1}$ are added to A^C , when w_2 is considered;
- c. two vertices $w_2 < w_3$ of $\Delta_{j, i+b_n-1}$ are added to A^C , when w_3 is considered.

\mathcal{F}_w^a be the σ -algebra generated by \mathcal{F}_w and $G_{w+}^{\text{mod-}}$. \mathcal{F}_w^a represents the knowledge obtained by the algorithm before w is probed, and the set of labels of edges outgoing from w (without knowing their other endpoints). Let $\mathcal{F}_w^{\text{aug}, a}$ be the σ -algebra generated by $\mathcal{F}_w^{\text{aug}}$ and $G_{w+}^{\text{mod-}}$, i.e. \mathcal{F}_w^a with the knowledge of the appropriate extra-randomness.

For any $w \in \cup_{j, i} \Delta_{j, i+b_n-1}$, let GE_w^c be equal to 1 if for every $w' < w$ the degree of w' in $G_{\text{end}}^{\text{mod}}$ is smaller than $\log n$, and let GE_w^a be equal to 1 if the degree of w is smaller than $\log n$ and $|A_w^G| \leq z_n$. For any w , $GE_w^a \geq GE_4$ and $GE_w^c \geq GE_4$. GE_w^a is \mathcal{F}_w^a -measurable, and GE_w^c is \mathcal{F}_w -measurable.

Let $\eta_n^a := \frac{(\ln n + t')z_n + \ln^2 n}{n}$, $\eta_n^b := \frac{t'}{n}$ and $\eta_n^c := \frac{t' \ln n}{n}$. For n large enough, all these variables are in $[0, 1]$.

Lemma 5.21. *For any $w \in \cup \Delta_{j, i+b_n-1}$, conditionally on $\mathcal{F}_w^{\text{aug}, a}$, the families (X_w^a) , $(X_{w, w'}^b)_{w' \in \Delta_{j, i+b_n-1}, w < w'}$, $(X_{w', w}^c)_{w' \in \Delta_{j, i+b_n-1}, w' < w}$ are independent Bernoulli variables. Moreover:*

- $\mathbb{P}(GE_w^a X_w^a = 1 | \mathcal{F}_w^{\text{aug}, a}) \leq \eta_n^a$ a.s.;
- For all $w' \in \Delta_{j, i+b_n-1}$, $w < w'$, $\mathbb{P}(X_{w, w'}^b = 1 | \mathcal{F}_w^{\text{aug}, a}) \leq \eta_n^b$ a.s.;
- For all $w' \in \Delta_{j, i+b_n-1}$, $w' < w$, $\mathbb{P}(GE_w^c X_{w, w'}^c = 1 | \mathcal{F}_w^{\text{aug}, a}) \leq \eta_n^c$ a.s.;

Proof. By Lemma 5.20, it is sufficient to prove Lemma 5.21 with \mathcal{F}_w^a instead of $\mathcal{F}_w^{aug,a}$.

$\mathcal{F}_{j,i+b_n-1}$, $G_w^{\text{mod-}}$, GE_w^c are measurable with respect to \mathcal{F}_w . The only difference between $G_w^{\text{mod-}}$ and $G_{w+}^{\text{mod-}}$ is the set of edges outgoing from w . Let denote in this proof by e_w the set of labels of outgoing edges. Therefore \mathcal{F}_w^a is the σ -algebra generated by e_w and \mathcal{F}_w .

$V^G \setminus B_w^G$ can be decomposed in these disjoint subsets:

- $V_w^{a1} = V^G \setminus A_w^G$;
- V_w^{a2} is the subset of vertices in A_w^G , but not in $\Delta_{j,i+b_n-1}$ or children of elements of $\Delta_{j,i+b_n-1}$;
- $V_w^b = \Delta_{j,i+b_n-1} \cap A_w^G$;
- For any $w' \in \Delta_{j,i+b_n-1}$, $w' < w$, $V_{w,w'}^c$ is the subset of vertices that are in A_w^G and are children of w' .

Conditionally on \mathcal{F}_w :

- μ_w^{dummy} is a Poisson point process of intensity $\frac{|A_w^G|}{n}$ on $[0, t']$,
- $(\Pi_{(w,w')})_{w' \in V_w^G \setminus B_w^G}$ is an i.i.d. family of Poisson point processes of intensity $\frac{1}{n}$ on $[0, t']$, independent of μ_w^{dummy} .

e_w is the union of μ_w^{dummy} and the $n - |A_w^G|$ processes associated to elements on V_w^{a1} . Therefore, conditionally on \mathcal{F}_w , e_w is a Poisson point process of intensity 1 on $[0, t']$.

- The variable $(X_w^a, G_{w+}^{\text{mod-}})$ depends only on μ_w^{dummy} and on $(\Pi_{(w,w')})_{w' \in V_w^{a1} \cup V_w^{a2}}$;
- for $w' \in \Delta_{j,i+b_n-1}$, $w < w'$, $X_{w,w'}^b$ depends only on the $\Pi_{(w,w')}$;
- for $w' \in \Delta_{j,i+b_n-1}$, $w > w'$, $X_{w,w'}^c$ depends only on the $(\Pi_{(w,w'')})_{w'' \in V_{w,w'}^c}$,

therefore conditionally on \mathcal{F}_w , the following variables are independent:

- $(X_w^a, G_{w+}^{\text{mod-}})$,
- $(X_{w,w'}^b)$ for $w' \in \Delta_{j,i+b_n-1}$, $w < w'$
- $(X_{w,w'}^c)$ for $w' \in \Delta_{j,i+b_n-1}$.

This implies that conditionally on \mathcal{F}_w and $G_{w+}^{\text{mod-}}$ (*i.e.* conditionally on \mathcal{F}_w^a), the following variables are independent:

- X_w^a ,
- $(X_{w,w'}^b)$ for $w' \in \Delta_{j,i+b_n-1}$, $w < w'$

- $(X_{w,w'}^c)$ for $w' \in \Delta_{j,i+b_n-1}$.

To conclude the proof, we now prove that the parameters of these variables are respectively smaller than η_n^a , η_n^b and η_n^c .

For any element y of e_w , let α_y be equal to 0 if $y \in \mu_w^{dummy}$ and equal to w' if $y \in \Pi_{(w,w')}$. Conditionally on \mathcal{F}_w^a , the family $(\alpha_y)_{y \in e_w}$ is an i.i.d. family, with $\mathbb{P}(\alpha_y = 0 | \mathcal{F}_w^a) = \frac{|A_w^G|}{n}$ and $\mathbb{P}(\alpha_y = v | \mathcal{F}_w^a) = \frac{1}{n}$ for any $v \in V^G \setminus A_w^G$. Therefore, conditionally on \mathcal{F}_w and G_{w+}^{mod-} , the probability that there is a multiple edge between w and an element of $V^G \setminus A_w^G$ is smaller than $(n - |A_w^G|)\mathbb{P}(\text{Binom}(|e_w|, \frac{1}{n}) \geq 2) \leq \frac{|e_w|^2}{n}$ and the probability that there is an element y of e_w such that $\alpha_y = 0$ is smaller than $\frac{|e_w||A_w^G|}{n}$. If $GE_w^a = 1$, then $|e_w| \leq \log n$ and $|A_w^G| \leq z_n$. Conditionally on \mathcal{F}_w^a , the probability that there is at least an edge between w and an element of V_w^{a2} , conditionally on \mathcal{F}_w , is smaller than $\frac{t'|V_w^{a2}|}{n} \leq \frac{t'|A_w^G|}{n}$. If $GE_w^a = 1$, this quantity is smaller than $\frac{t'z_n}{n}$. By summing the previous results, we obtain that $\mathbb{P}(GE_w^a X_w^a = 1 | \mathcal{F}_w^a) \leq \eta_n^a$.

$X_{w,w'}^b$ is equal to 1 if $\Pi_{(w,w')} \neq \emptyset$. Therefore $\mathbb{P}(X_{w,w'}^b | \mathcal{F}_w^a) = e^{-\frac{t'}{n} \frac{t'}{n}} = \eta_n^b$.

$X_{w',w}^c$ is equal to 1 if $\cup_{w'' \in V_{w,w'}^c} \Pi_{(w,w'')} \neq \emptyset$. The probability of this event is smaller than $\frac{t'|V_{w,w'}^c|}{n}$. If $GE_w^c = 1$, then $|V_{w,w'}^c| \leq \log n$ and therefore $\mathbb{P}(GE_w^c X_{w',w}^c) \leq \frac{t' \log n}{n} = \eta_n^c$. \square

The following constructions use the tool explained in Lemma 5.19.

For any $w \in \cup_{j,i} \Delta_{j,i+b_n-1}$, let $X_w^{a,indep} = 1$ if either $GE_w^a X_w^a = 1$ or $U_w^a \leq \frac{\eta_n^a - \mathbb{P}(GE_w^a X_w^a = 1 | \mathcal{F}_w^{aug,a})}{\mathbb{P}(GE_w^a X_w^a = 0 | \mathcal{F}_w^{aug,a})}$. Otherwise, let $X_w^{a,indep} = 0$.

For any $(w, w') \in \cup_{j,i} \Delta_{j,i+b_n-1}^2$ with $w < w'$, let $X_{w,w'}^{b,indep} = 1$ if either $X_{w,w'}^b = 1$ or $U_{w,w'}^b \leq \frac{\eta_n^b - \mathbb{P}(X_{w,w'}^b = 1 | \mathcal{F}_w^{aug,a})}{\mathbb{P}(X_{w,w'}^b = 0 | \mathcal{F}_w^{aug,a})}$. Otherwise, let $X_{w,w'}^{b,indep} = 0$.

For any $(w, w') \in \cup_{j,i} \Delta_{j,i+b_n-1}^2$ with $w' < w$, let $X_{w',w}^{c,indep} = 1$ if either $GE_w^c X_{w',w}^c = 1$ or $U_{w',w}^c \leq \frac{\eta_n^c - \mathbb{P}(GE_w^c X_{w',w}^c = 1 | \mathcal{F}_w^{aug,a})}{\mathbb{P}(GE_w^c X_{w',w}^c = 0 | \mathcal{F}_w^{aug,a})}$. Otherwise let $X_{w',w}^{c,indep} = 0$.

Corollary 5.22. *For all $w \in \Delta_{j,i+b_n-1}$ and $\bar{w} \in \{w, w+\}$, conditionally on $\mathcal{F}_{j,i-1}^{aug}$ and $G_{\bar{w}}^{mod-}$, the families of random variables $(X_{w'}^{a,indep})_{w' \in \Delta_{j,i+b_n-1} \cap \Gamma_{\bar{w}}^1}$, $(X_{w',w''}^{b,indep})_{(w',w'') \in \Gamma_{\bar{w}}^2 \cap (\Delta_{j,i+b_n-1})^2}$, $(X_{w',w''}^{c,indep})_{(w',w'') \in \Gamma_{\bar{w}}^c \cap (\Delta_{j,i+b_n-1})^2}$ are independent i.i.d. families of Bernoulli random variables of respective parameters η_n^a , η_n^b and η_n^c .*

The version with $w+$ is equivalent to the version with w by substituting w by the next vertex of $\Delta_{j,i+b_n-1}$. Corollary 5.22 is proven by induction. Corollary 5.22 holds for w if w is the first element of $\Delta_{j,i+b_n-1}$ (for the breadth-first order), as the families are empty. Assuming that Corollary 5.22 holds for a vertex w , let F be the σ -algebra generated by the

random variables $(X_{w'}^{a,indep})_{w' \in \Delta_{j,i+b_n-1} \cap \Gamma_w^1}$, $(X_{w',w''}^{b,indep})_{(w',w'') \in \Gamma_w^2 \cap (\Delta_{j,i+b_n-1})^2}$, $(X_{w',w''}^{c,indep})_{(w',w'') \in \Gamma_w^c \cap (\Delta_{j,i+b_n-1})^2}$. All these variables are \mathcal{F}_w^{aug} -measurable, therefore conditionally on $\mathcal{F}_{j,i+b_n-1}^{aug}$, F and G_{w+}^{mod-} :

- (X_w^a) , $(X_{w,w'}^b)_{w' \in \Delta_{j,i}, w < w'}$, $(X_{w',w}^c)_{w' \in \Delta_{j,i}, w' < w}$ are independent Bernoulli variables, of respective parameters smaller than η_n^a , η_n^b and η_n^c , by Lemma 5.21;
- $(X_w^{a,indep})$, $(X_{w,w'}^{b,indep})_{w' \in \Delta_{j,i}, w < w'}$, $(X_{w',w}^{c,indep})_{w' \in \Delta_{j,i}, w' < w}$ are independent Bernoulli variables, of respective parameters equal to η_n^a , η_n^b and η_n^c , by Lemma 5.19.

And therefore Corollary 5.22 holds for $w+$. By applying Corollary 5.22 to the last element of $\Delta_{j,i}$, one obtains:

Corollary 5.23. *Conditionally on the σ -algebra $\mathcal{F}_{j,i+b_n-1}$ and the unlabelled graph $G_{j,i+b_n}^{mod-}$, the families of random variables $(X_{w \in \Delta_{j,i+b_n-1}}^{a,indep})$, $(X_{w,w' < w' \in \Delta_{j,i+b_n-1}}^{b,indep})$, $(X_{w,w' < w' \in \Delta_{j,i+b_n-1}}^{c,indep})$ are independent i.i.d. families of Bernoulli random variables of respective parameters η_n^a , η_n^b and η_n^c .*

For any $w \in D_{j,i}$, let $X_w^{a,parent} = 1$ if there exists w' , a $b_n - 1$ -child of w such that $X_{w'}^{a,indep} = 1$. For any $w_1, w_2 \in D_{j,i}$, let $X_{w_1, w_2}^{bc,parent} = 1$ if there exists w'_1 , a $b_n - 1$ -child of w_1 and w'_2 a $b_n - 1$ -child of w_2 such that $X_{w'_1, w'_2}^{b,indep} = 1$ or $X_{w'_1, w'_2}^{c,indep} = 1$. Let $GE_{j,i}$ be the random variable equal to 1 if all the balls of radius $b_n - 1$ centered at vertices of $D_{j,i}$ in G^{mod} contains at most x_n vertices and $|A_{j,i}^G| \leq z_n$. Otherwise, let $GE_{j,i} = 0$. $GE_{j,i}$ is $\mathcal{F}_{j,i+b_n-1}$ -measurable, and $GE_{j,i} \geq GE_4$.

Corollary 5.24. *Conditionally on $\mathcal{F}_{j,i+b_n-1}^{aug}$ and $G_{j,i+b_n}^{mod-}$, the families of random variables $(X_w^{a,parent})_{w \in D_{j,i}}$ and $(X_{w,w'}^{bc,parent})_{w,w' \in D_{j,i}}$ are independent. Moreover,*

$$\begin{aligned} \mathbb{E}(GE_{j,i} X_w^{a,parent} | \mathcal{F}_{j,i+b_n-1}^{aug}, G_{j,i+b_n}^{mod-}) &\leq x_n \eta_n^a \\ \mathbb{E}(GE_{j,i} X_{w,w'}^{bc,parent} | \mathcal{F}_{j,i+b_n-1}^{aug}, G_{j,i+b_n}^{mod-}) &\leq x_n^2 (\eta_n^b + \eta_n^c) =: \eta_n^{bc,parent}. \end{aligned}$$

Lemma 5.25. *Let I be an ordered finite set, and let $(X_{i,i'})_{i,i' \in I, i \leq i'}$ be a family of independent Bernoulli random variables of parameters smaller than η . For $i < i'$, let $X_{i',i} = X_{i,i'}$. Let \mathcal{F}_2 be a σ -algebra. Then there exists a family of random variables $(X_i^{bound})_{i \in I}$ such that:*

- $(X_i^{bound})_{i \in I}$ is an i.i.d. family of Bernoulli random variables of parameter $\min(1, 2\sqrt{\eta|I|})$.
- For any $i, i' \in I$, $X_{i,i'} \leq X_i^{bound}$.

For any Y \mathcal{F}_2 -measurable, $\mathbb{E}(Y|(X_{i,i'})_{i,i' \in I}) = \mathbb{E}(Y|(X_{i,i'})_{i,i' \in I}, (X_i^{bound})_{i \in I})$.

Lemma 5.25 allows to take an i.i.d. family of Bernoulli variables indexed by two elements of I and bound it by an i.i.d. indexed family of Bernoulli variables indexed by one element of I . The bound $2\sqrt{\eta|I|}$ does not seem to be optimal, but is sufficient for our use. As we need a specific version of Lemma 5.25, the proof will be done in our particular case. A proof of Lemma 5.25 can be found in [Section 5.7](#).

Let $\eta_n^{parent} = x_n \eta_n^a + 2\sqrt{z_n \eta_n^{bc,parent}} = o(1)$. We assume that n is large enough such that $\eta_n^{parent} \leq 1$. As a consequence, $z_n \eta_n^{bc,parent} \leq 1$.

For any $w \in D_{j,i}$, let $X_w^{parent} = 1$ if either $GE_{j,i}^{parent} X_w^{a,parent} = 1$ or there exists $w' \in D_{j,i}$ such that $GE_{j,i}^{parent} X_{w,w'}^{bc,parent} = 1$. Otherwise, let $X_w^{parent} = 0$. Let us define $(X_w^{parent,indep})_{w \in D_{j,i}}$ as $X_w^{parent,indep} = 1$ if either

- $X_w^{parent} = 1$;
- $U_w^d \leq \frac{\eta_n^{parent} - \mathbb{P}(X_w^{parent}=1|\mathcal{F}_{j,i-1}^{aug}, G_{j,i}^{mod-}, (X_{w'}^{parent,indep})_{w' < w, w' \in D_{j,i}})}{\mathbb{P}(X_w^{parent}=0|\mathcal{F}_{j,i+b_n-1}^{aug}, G_{j,i+b_n}^{mod-}, (X_{w'}^{parent,indep})_{w' < w, w' \in D_{j,i}})}$.

If neither condition happens, let $X_w^{parent,indep} = 0$.

This family of variables is defined by induction on w . Let $\mathcal{F}_w^{parent,indep}$ be the σ -algebra generated by $\mathcal{F}_{j,i-1}^{aug}$, $G_{j,i}^{mod-}$ and $(X_{w'}^{parent,indep})_{w' < w, w' \in D_{j,i}}$.

Lemma 5.26. For all $w \in D_{j,i}$,

$$\mathbb{P}(X_w^{parent} = 1 | \mathcal{F}_w^{parent,indep}) \leq x_n \eta_n^a + \sqrt{z_n \eta_n^{bc,parent}}.$$

Proof. Let $p_w = \frac{\eta_n^{parent} - \mathbb{P}(X_w^{parent}=1|\mathcal{F}_w^{parent,indep})}{\mathbb{P}(X_w^{parent}=0|\mathcal{F}_w^{parent,indep})}$.

Lemma 5.26 is proven by induction. We assume that Lemma 5.26 holds for every $w' \in D_{j,i} < w$. Therefore for every $w' < w$:

$$\begin{aligned} p_{w'} &\geq \frac{\sqrt{z_n \eta_n^{bc,parent}}}{\mathbb{P}(X_w^{parent} = 0 | \mathcal{F}_w^{parent,indep})} \\ &\geq \sqrt{z_n \eta_n^{bc,parent}} \end{aligned} \tag{8}$$

$$\begin{aligned} X_w^{parent} &\leq X_w^{a,parent} + \sum_{w' \in D_{j,i}, w' < w} X_{w',w}^{bc,parent} + \\ &\quad + \sum_{w' \in D_{j,i}, w' > w} X_{w,w'}^{bc,parent} \end{aligned}$$

$$\begin{aligned} \mathbb{E}(X_w^{parent} | \mathcal{F}_w^{parent,indep}) &\leq \mathbb{E}(X_w^{a,parent} | \mathcal{F}_w^{parent,indep}) + \\ &\quad + \sum_{w' \in D_{j,i}, w' < w} \mathbb{E}(X_{w',w}^{bc,parent} | \mathcal{F}_w^{parent,indep}) + \\ &\quad + \sum_{w' \in D_{j,i}, w' > w} \mathbb{E}(X_{w,w'}^{bc,parent} | \mathcal{F}_w^{parent,indep}) \end{aligned} \tag{9}$$

As $D_{j,i}$ is $\mathcal{F}_w^{\text{parent}, \text{indep}}$ -measurable. Each term of the right-hand-side of the inequality (9) will be bound separately.

$$\begin{aligned}
x_n \eta_n^a &\geq \mathbb{E} \left(X_w^{a, \text{parent}} \middle| \mathcal{F}_{j,i}^{\text{aug}}, G_{j,i}^{\text{mod-}} \right) \\
&= \mathbb{E} \left(X_w^{a, \text{parent}} \middle| \mathcal{F}_{j,i}^{\text{aug}}, G_{j,i}^{\text{mod-}}, (X_{w'}^{a, \text{parent}})_{w' < w, w' \in D_{j,i}}, \right. \\
&\quad \left. (X_{w', w''}^{bc, \text{parent}})_{w' < w'' \in D_{j,i}} \right) \\
&= \mathbb{E} \left(X_w^{a, \text{parent}} \middle| \mathcal{F}_{j,i}^{\text{aug}}, G_{j,i}^{\text{mod-}}, (X_{w'}^{a, \text{parent}})_{w' < w \in D_{j,i}}, \right. \\
&\quad \left. (X_{w', w''}^{bc, \text{parent}})_{w' < w'' \in D_{j,i}}, (U_{w'}^d)_{w' \in D_{j,i}} \right). \quad (10)
\end{aligned}$$

The first inequality comes from Corollary 5.24, the second equality comes from the conditional independence of the variables $X_w^{a, \text{parent}}$ and $X_{w, w'}^{bc, \text{parent}}$ in Corollary 5.24 and the last equality comes from the conditional independence of $(U_w^d)_{w \in D_{j,i+1}}$. As $\mathcal{F}_w^{\text{parent}, \text{indep}}$ is included in the σ -algebra used in (10), we have:

$$\mathbb{E}(X_w^{a, \text{parent}} | \mathcal{F}_w^{\text{parent}, \text{indep}}) \leq x_n \eta_n^a. \quad (11)$$

We are now going to prove that for all w, w' :

$$\mathbb{E}(X_{w, w'}^{bc, \text{parent}} | \mathcal{F}_w^{\text{parent}, \text{indep}}) \leq \sqrt{\frac{\eta_n^{bc, \text{parent}}}{z_n}} \text{ a.s.} \quad (12)$$

Let $w' < w \in D_{j,i}$. Let $\mathcal{F}_{w', w}^{\text{parent}}$ be the σ -algebra generated by $\mathcal{F}_{j,i-1}^{\text{aug}}$, $G_{j,i}^{\text{mod-}}$, $X_{w^1 \in D_{j,i}}^{a, \text{parent}}$, $(X_{w^1, w^2}^{bc, \text{parent}})_{(w^1, w^2) \neq (w', w), w^1 < w^2 \in D_{j,i}}$, $(U_{w^1}^d)_{w^1 \in D_{j,i} \setminus \{w'\}}$. As $\mathcal{F}_w^{\text{parent}, \text{indep}}$ is included in the σ -algebra generated by $\mathcal{F}_{w', w}^{\text{parent}}$ and $X_{w'}^{\text{parent}, \text{indep}}$, we will prove:

$$\mathbb{E}(X_{w', w}^{bc, \text{parent}} | \mathcal{F}_{w', w}^{\text{parent}}, X_{w'}^{\text{parent}, \text{indep}}) \leq \sqrt{\frac{\eta_n^{bc, \text{parent}}}{z_n}} \text{ a.s.} \quad (13)$$

as (12) is a consequence of (13). There are two possibilities. If $X_{w'}^{\text{parent}, \text{indep}} = 0$, then $X_{w', w}^{bc, \text{parent}} = 0$ and (13) holds. Otherwise, if $X_{w'}^{\text{parent}, \text{indep}} = 1$:

$$\begin{aligned}
\mathbb{P}(X_{w', w}^{bc, \text{parent}} = 1 | \mathcal{F}_{w', w}^{\text{parent}}, X_{w'}^{\text{parent}, \text{indep}} = 1) &= \frac{\mathbb{P}(X_{w', w}^{bc, \text{parent}} = 1, X_{w'}^{\text{parent}, \text{indep}} = 1 | \mathcal{F}_{w', w}^{\text{parent}})}{\mathbb{P}(X_{w'}^{\text{parent}, \text{indep}} = 1 | \mathcal{F}_{w', w}^{\text{parent}})} \\
&= \frac{\mathbb{P}(X_{w', w}^{bc, \text{parent}} = 1 | \mathcal{F}_{w', w}^{\text{parent}})}{\mathbb{P}(X_{w'}^{\text{parent}, \text{indep}} = 1 | \mathcal{F}_{w', w}^{\text{parent}})} \\
&\leq \frac{\mathbb{P}(X_{w', w}^{bc, \text{parent}} = 1 | \mathcal{F}_{w', w}^{\text{parent}})}{\mathbb{P}(U_{w'}^d \leq p_{w'} | \mathcal{F}_{w', w}^{\text{parent}})} \\
&= \frac{\mathbb{P}(X_{w', w}^{bc, \text{parent}} = 1 | \mathcal{F}_{w', w}^{\text{parent}})}{\mathbb{E}(p_{w'} | \mathcal{F}_{w', w}^{\text{parent}})} \quad (14)
\end{aligned}$$

By using Corollary 5.24 and the conditional independence of $(U_w^d)_{w \in D_{j,i}}$ we have:

$$\begin{aligned} \mathbb{P}(X_{w',w}^{bc,parent} = 1 | \mathcal{F}_{w',w}^{parent}) &= \mathbb{P}(X_{w',w}^{bc,parent} = 1 | \mathcal{F}_{j,i+b_n-1}^{aug}, G_{j,i+b_n}^{mod-}) \\ &\leq \eta_n^{bc,parent} \end{aligned} \quad (15)$$

By using (8) and (15) in (14), the equation (13) holds, and therefore equation (12) is proven for all $w' < w$.

Let $w' > w$. By Corollary 5.24 and the conditional independence of the $(U_v^d)_{v \in D_{j,i}}$:

$$\begin{aligned} \mathbb{E} \left(X_{w,w'}^{bc,indep} \left| \begin{array}{l} \mathcal{F}_{j,i+b_n-1}^{aug}, G_{j,i+b_n}^{mod-}, X_{w^1 \in D_{j,i}}^{a,parent}, \\ (X_{w^1,w^2}^{bc,parent})_{w^1 \leq w, w^1 \leq w^2 \in D_{j,i}}, \\ (U_{w^1}^d)_{w^1 \neq w \in D_{j,i}} \end{array} \right. \right) &= \mathbb{E}(X_{w,w'}^{bc,indep} | \mathcal{F}_{j,i+b_n-1}^{aug}, G_{j,i+b_n}^{mod-}) \\ &\leq \eta_n^{bc,parent} \\ &\leq \sqrt{\frac{\eta_n^{bc,parent}}{z_n}} \end{aligned}$$

as by hypothesis $\eta_n^{bc,parent} z_n \leq 1$. As $\mathcal{F}_w^{parent,indep}$ is measurable with respect to this σ -algebra, the bound also holds conditionally on \mathcal{F}_w^{parent} , proving inequality (12) for $w' > w$.

By summing the bounds obtained in (11) and (12), we obtain:

$$\begin{aligned} \mathbb{E}(GE_{j,i} X_w^{parent} | \mathcal{F}_w^{parent,indep}) &\leq x_n \eta_n^a + z_n \sqrt{\frac{\eta_n^{bc,parent}}{z_n}} \\ &= x_n \eta_n^a + \sqrt{z_n \eta_n^{bc,parent}}. \end{aligned}$$

□

Corollary 5.27. *Conditionally on $\mathcal{F}_{j,i+b_n-1}^{aug}$ and $G_{j,i+b_n}^{mod-}$, $(X_w^{parent,indep})_{w \in D_{j,i+b_n}}$ is an i.i.d. family of Bernoulli variables of parameter η_n^{parent} .*

Corollary 5.27 uses the construction of Lemma 5.19 to obtain i.i.d. variables.

Definition 5.4. For every $i \in \mathbb{N}$, $j \in \{1, 2\}$, $w \in S_{j,i} \setminus D_{j,i}$, let $X_w^{parent,indep} = 1$ if $U_w^d \leq \eta_n^{parent}$. Otherwise, let $X_w^{parent,indep} = 0$.

Definition 5.4 allows us to extend the family $(X_w^{parent,indep})_{w \in D_{j,i}}$ to vertices not in a $D_{j,i}$, in such a way that the family is still i.i.d:

Lemma 5.28. *Conditionally on $\mathcal{F}_{j,i+b_n-1}^{aug}$ and $G_{j,i+b_n}^{mod-}$, $(X_w^{parent,indep})_{w \in S_{j,i}}$ is an i.i.d. family of Bernoulli variables of parameter η_n^{parent} .*

By construction, $(X_w^{parent,indep})_{w \in S_{j,i} \setminus D_{j,i}}$ is an i.i.d. family of Bernoulli variables of parameter η_n^{parent} , independent of $(X_w^{parent,indep})_{w \in D_{j,i}}$ and $G_{j,i+b_n}^{mod-}$, conditionally on $F_{j,i+b_n-1}^{aug}$.

Lemma 5.29. *Conditionally on G_{end}^{mod-} , $(X_w^{parent,indep})_{w \in G_{end}^{mod-}}$ is an i.i.d. family of Bernoulli variables of parameter η_n^{parent} .*

The proof is done by proving by induction the following two claims:

- $C_{j,i}^1$: Conditionally on $G_{j,i+b_n}^{mod-}$, $(X_w^{parent,indep})_{w \in \mathcal{B}_{j,i}}$ is an i.i.d. family of Bernoulli variables of parameter η_n^{parent} .
- $C_{j,i}^2$: Conditionally on $G_{j,i+b_n+1}^{mod-}$, $(X_w^{parent,indep})_{w \in \mathcal{B}_{j,i}}$ is an i.i.d. family of Bernoulli variables of parameter η_n^{parent} .

Lemma 5.30. *Conditionally on $\mathcal{F}_{j,i+b_n}^{aug}$, the law of $G_{j,i+b_n+1}^{mod-}$ can be described as follows:*

- Let $(P_w)_{w \in S_{j,i+b_n}}$ be an i.i.d. family of Poisson point processes of intensity 1 on $[0, t']$.
- Starting with G^{mod-} , for each vertex $w \in S_{j,i+b_n}$, for each element y of P_w graft an edge labelled by y between w and a new vertex. The resulting graph is $G_{j,i+b_n+1}^{mod-}$.

Proof. By Lemma 5.9, this construction gives the law of $G_{j,i+b_n+1}^{mod}$ conditionally on $\mathcal{F}_{j,i+b_n}$. By Lemma 5.20, this construction therefore gives the law of $G_{j,i+b_n+1}^{mod}$ conditionally on $\mathcal{F}_{j,i+b_n}$. \square

1. $C_{1,0}^1$ comes from Lemma 5.28 for $j = 1$ and $i = 0$.
2. Let $(j, i) \in \{1, 2\} \times \mathbb{N}$. The graph $G_{j,i+b_n}^{mod-}$ and the variables $(X_w^{parent,indep})_{w \in \mathcal{B}_{j,i}}$ are $\mathcal{F}_{j,i+b_n}^{aug}$ -measurable. Therefore, Lemma 5.30 entails that, $C_{j,i}^1$ implies $C_{j,i}^2$.
3. For any j, i , $C_{j,i}^2$ implies $C_{j,i+1}^1$ by Lemma 5.28.
4. The graph $G_{j,\infty}^{mod-}$ is obtained from $G_{j,K_n+b_n}^{mod-}$ by grafting i.i.d. copies of $T_{t'}^\infty$ to every vertex of S_{j,K_n+b_n} . The family $(X_w^{parent,indep})_{w \in G_{j,\infty}^{mod-} \setminus G_{j,K_n+b_n}^{mod-}}$ is constructed as an i.i.d. family of Bernoulli variables of parameter η_n^{parent} via Definition 5.4. Therefore for all j , C_{j,K_n}^2 implies $C_{j,\infty}^2$.
5. $C_{1,\infty}^1$ implies $C_{2,0}^1$ from Lemma 5.28 for $j = 2$ and $i = 0$.

To summarize the result so far:

- GE^4 is Bernoulli variable, such that $\mathbb{P}(GE^4 = 0) \xrightarrow[n \rightarrow \infty]{} 0$.
- Conditionally on $G_{\text{end}}^{\text{mod-}}$, $(X_w^{\text{parent, indep}})_{w \in G_{\text{end}}^{\text{mod-}}}$ is an i.i.d. family of Bernoulli variables of parameter η_n^{parent} .
- $\eta_n^{\text{parent}} \xrightarrow[n \rightarrow \infty]{} 0$.
- If $GE^4 = 1$, then $w \in D_{j,i} \setminus E_{j,i}$ implies $X_w^{\text{parent, indep}} = 1$.
- If $GE^4 = 1$, no initialisation fails.

This summary ends the first part of the proof of the construction of the included branching process, dealing with the vertices in $D_{j,i} \setminus E_{j,i}$. We now need to deal with the vertices in $E_{j,i} \setminus \tilde{E}_{j,i}$, *i.e.* the vertices that might reach the forbidden degree between time t' and t .

5.5.b Extension of the included branching process to the time t

As $G_{n,t'}^\infty$ and $G_{n,[t',t]}^\infty$ are independent, conditionally on $\mathcal{F}_{\text{end}}^{\text{aug}}$ the sets of labelled edges in $G_{n,[t',t]}^\infty$ is an i.i.d. family of Poisson point processes of intensity $\frac{1}{n}$ on $[t', t]$. We now remove remaining vertices adjacent to an edge added between t' and t , except if this edge is added between two vertices of the K_n th generation.

- Let $E^{t'-}$ be the set of vertices w of $\cup_{i < K_n: j \in \{1,2\}} D_{j,i}$ such that $X_w^{\text{parent, indep}} = 0$;
- let E^{t', K_n} be the set of vertices w of $D_{1, K_n} \cup D_{2, K_n}$ such that $X_w^{\text{parent}} = 0$;
- let $E^{t'+} = E^{t'-} \cup E^{t', K_n}$.
- For each w in $E^{t'-}$, let $X_w^{[t', t]} = 1$ if at least on edge is added to w in $G_{n,t}^\infty$ between t' and t .
- For each w in E^{t', K_n} , let $X_w^{[t', t]} = 1$ if there is at least one edge added between w and an element of $V^G \setminus E^{t', K_n}$.

The variable $X_w^{[t', t]}$ is used to know if w might be removed because of an edge added to w between time t' and t .

We split the set of edges according to how many endpoints belong to $E^{t'+}$.

- For any $(w, w') \in (E^{t'+})^2 \setminus (E^{t', K_n})^2$, let $X_{w, w'}^{[t', t], a} = 1$ if at least one edge is added between w and w' between t' and t . Otherwise, let $X_{w, w'}^{[t', t], a} = 0$.

- For any $(w, w') \in (E^{t', K_n})^2$ let $X_{w, w'}^{[t', t], a} = 0$.
- For any $w \in E^{t'+}$, let $X_w^{[t', t], b} = 1$ if at least one edge is added between w and an element of $V^G \setminus E^{t'+}$ between time t' and t . Otherwise, let $X_w^{[t', t], b} = 0$.

For any $w \in E^{t'+}$,

$$X^{[t', t]} = \max_{w' \in E^{t'+}} (X_{w, w'}^{[t', t], a}, X_w^{[t', t], b}).$$

Conditionally on $\mathcal{F}_{\text{end}}^{\text{aug}}$, $(X_{w, w'}^{[t', t], a})_{(w, w') \in E^{t'+}}$ is a family of independent Bernoulli variables of parameter $1 - \exp(-\frac{t-t'}{n}) \leq \frac{\epsilon_1}{n}$ and is independent of $(X_w^{[t', t], b})_{w \in E^{t'+}}$. The set $E^{t'+}$ is included in A_{end}^G , therefore if $GE_4 = 1$, then $|E^{t'+}| \leq z_n$. As GE_4 is $\mathcal{F}_{\text{end}}^{\text{aug}}$ -measurable, Lemma 5.25 allows to construct $(X_w^{[t', t], a})_{w \in E^{t'+}}$ such that, if $GE^4 = 1$:

- For all w, w' , $X_{w, w'}^{[t', t], a} \leq X_w^{[t', t], a}$;
- Conditionally on $\mathcal{F}_{\text{end}}^{\text{aug}}$, $(X_w^{a, [t', t]})_{w \in E^{t'+}}$ is an i.i.d. family of Bernoulli variables of parameter $2\sqrt{\frac{\epsilon_1}{n} z_n}$, independent of $((X_w^{[t', t], b})_{w \in E^{t'+}}, \mathcal{F}_{\text{end}}^{\text{aug}}, G_{n, t'}^\infty)$.

Conditionally on $\mathcal{F}_{\text{end}}^{\text{aug}}$, $(X_w^{[t', t], b})_{w \in E^{t'+}}$ is an independent i.i.d. family of Bernoulli variables of parameter smaller than $\epsilon_1 \frac{n - |E^{t'+}|}{n} \leq \epsilon_1$. Therefore conditionally on $\mathcal{F}_{\text{end}}^{\text{aug}}$, $\max(X_w^{[t', t], a}, X_w^{[t', t], b}) \geq GE_4 X_w^{[t', t]}$ is an independent family of Bernoulli variables of parameter smaller than $2\sqrt{\frac{\epsilon_1}{n} z_n} + \epsilon_1 =: \eta_n^{[t', t]}$. It should be noticed that $\eta_n^{[t', t]} = \epsilon_1 + o(1)$.

Let $(U_w^e)_{w \in G_{\text{end}}^{\text{mod}}}$ be a family of random variables such that conditionally on $(\mathcal{F}_{\text{end}}^{\text{aug}}, G_{n, t'}^\infty, (X_w^{[t', t], a})_{w \in E^{t'+}})$, the family $(U_w^e)_{w \in G_{\text{end}}^{\text{mod}}}$ is i.i.d. uniform variables on $[0; 1]$. For all $w \in G_{\text{end}}^{\text{mod}} \setminus E^{t'+}$, let $X_w^{[t', t], a} = X_w^{[t', t], b} = 0$.

Let $X_w^{[t', t], \text{indep}} = 1$ if either:

- $X_w^{[t', t], a} = 1$;
- $X_w^{[t', t], b} = 1$;
- $U_w^e \leq \frac{\eta_n^{[t', t]} - p'_w}{1 - p'_w}$ with $p'_w = \mathbb{P}(X_w^{[t', t], a} = 1 \text{ or } X_w^{[t', t], b} = 1 | \mathcal{F}_{\text{end}}^{\text{aug}})$.

Then conditionally on $\mathcal{F}_{\text{end}}^{\text{aug}}$, $(X_w^{[t', t], \text{indep}})_{w \in G_{\text{end}}^{\text{mod}}}$ is an i.i.d. family of Bernoulli variables of parameter $\eta_n^{[t', t]}$. Therefore conditionally on $G_{\text{end}}^{\text{mod}-}$, $(X_w^{\text{parent}, \text{indep}}, (X_w^{[t', t], \text{indep}})$ are two independent families of Bernoulli variables of parameters η_n^{parent} and $\eta_n^{[t', t]}$, and therefore their maximum is an i.i.d. family of Bernoulli variables of parameter $\eta_n^{[0, t]} = 1 - (1 - \eta_n^{[t', t]})(1 - \eta_n^{\text{parent}}) = \epsilon_1 + o(1)$.

5.5.c Use of the included branching process

The subgraph of $G_{\text{end}}^{\text{modk}}$ with only vertices w such that $X_w^{\text{parent,indep}} = X_w^{[t',t],\text{final}} = 0$ has same law as an independent percolation of parameter $1 - \eta_n^{[0;t]}$ on two independent copies of $T_{t'}^k$.

For n large enough, $\eta_n^{[0;t]} \leq 2\epsilon_1$. For this reason, in the following part, we will study the percolation of parameter $1 - 2\epsilon_1$. The resulting graph is also a two-stages multitype branching process.

Lemma 5.31. *Let $G_{\text{end}}^{\text{modk},\epsilon_1}$ be the two-stages multitype branching process obtained by doing a percolation of parameter $1 - 2\epsilon_1$ on $G_{\text{end}}^{\text{modk}}$. Let a_{t',ϵ_1}^k be the probability of survival, and ρ_{t',ϵ_1} the spectral radius associated to $G_{\text{end}}^{\text{modk},\epsilon_1}$. Then:*

$$a_{t',\epsilon_1}^k \xrightarrow{\epsilon_1 \rightarrow 0, t' \rightarrow t} a_t^k; \quad (16)$$

$$\liminf_{\epsilon_1 \rightarrow 0, t' \rightarrow t} \rho_{t',\epsilon_1} \geq \rho_t. \quad (17)$$

The proof of Lemma 5.31 will be done in Section 5.6.

Recall that $K_n = \frac{(1-\epsilon_2)\ln n}{\ln(\rho_{t'})}$. Let $\epsilon > 0$. When ϵ_1 and ϵ_2 tends to 0, $(\rho_{t',\epsilon_1})^{\frac{1}{1-\epsilon_2}}$ converges to $\rho_t > (\rho_t)^{\frac{2}{3}}$. We assume that ϵ_1 and ϵ_2 are small enough so that $(\rho_{t',\epsilon_1})^{1-\epsilon_2} > (\rho_{t'})^{\frac{2}{3}}$, and that $a_{t',\epsilon_1}^k \geq a_t^k - \epsilon$.

The graph $G_{\text{end}}^{\text{modk},\epsilon_1}$ is a two-stages branching process. Therefore, conditionally on survival, the size of the i th generation grows as $(\rho_{t',\epsilon_1})^i$, by Lemma 5.4:

$$\frac{|S^i(G_{\text{end}}^{\text{modk},\epsilon_1})|}{(\rho_{t',\epsilon_1})^i} \xrightarrow{i \rightarrow \infty} W \text{ a.s.}$$

where S^i denotes the sphere of radius i and W is a random variable, a.s. positive if $G_{\text{end}}^{\text{modk},\epsilon_1}$ is infinite. Therefore, for any $1 < \alpha < \rho_{t',\epsilon_1}$:

$$\mathbb{P}(|S^i(G_{\text{end}}^{\text{modk},\epsilon_1})| \geq \alpha^i) \xrightarrow{i \rightarrow \infty} a_{t',\epsilon_1}^k.$$

By choosing α close enough to ρ_{t',ϵ_1} , $n^{\frac{2}{3}} = o(\alpha^{K_n})$, and therefore the probability that the K_n th generation of $G_{\text{end}}^{\text{modk},\epsilon_1}$ contains at least $n^{\frac{2}{3}}$ is larger than $a_{t',\epsilon_1}^k - \epsilon$.

Conditionally on the first K_n generations of $G_{\text{end}}^{\text{modk},\epsilon_1}$, the offspring of each vertex w of the K_n th generation of $G_{\text{end}}^{\text{modk},\epsilon_1}$, seen as a subgraph of $G_{\text{end}}^{\text{modk}}$ is described by the functions f_i . By Lemma 4.20, for $t' \leq t$, these density functions are bounded from below, let say by $\delta > 0$. Therefore, by the law of large numbers, with high probability when n tends to infinity, if the K_n th generation of $G_{\text{end}}^{\text{modk},\epsilon_1}$ contains at least $n^{\frac{2}{3}}$ vertices, then at least $\frac{\delta}{2}n^{\frac{2}{3}}$ of them have no children in $G_{\text{end}}^{\text{modk}}$. As a consequence, for n large enough and ϵ_1 and ϵ_2 small enough, the following events happen simultaneously with probability larger than $(a_t^k - \epsilon)^2 - \epsilon$:

- The first and second component of $G_{\text{end}}^{\text{modk}, \epsilon_1}$ are infinite;
- The K_n th generation of these components contains at least $n^{\frac{2}{3}}$ vertices.
- Among these vertices, at least $2n^{\frac{3}{5}}$ have no children in $G_{\text{end}}^{\text{modk}}$, and are therefore not saturated in $G_{n,t}^k$.

If these conditions are satisfied, let H_1 (resp. H_2) be the set of the first $2n^{\frac{3}{5}}$ (in the breadth-first order) vertices of the generation K_n of the first (resp. second) component of $G_{\text{end}}^{\text{modk}, \epsilon_1}$ with no children in $G_{\text{end}}^{\text{modk}}$. By construction, any element w of H_j satisfies the following properties:

- There is a path from v_j to w in $G_{n,t'}^k$, such that no edge is added to any vertex of this path between t' and t , except possibly to w .
- No edge is added between w and an element of $V^G \setminus E^{t'-}$ between t' and t .

Let $\mathcal{F}^{\text{aug}2}$ be the σ -algebra generated by \mathcal{F}^{aug} , $(U_w^e)_{w \in G^{\text{modend}}}$ and $(\Pi_{(w,w')} \cap [t', t])_{(w,w') \in (V^G)^2 \setminus (E^{t', K_n})^2}$.

H_1 and H_2 are $\mathcal{F}_{\text{end}}^{\text{aug}2}$ -measurable. Conditionally on $\mathcal{F}_{\text{end}}^{\text{aug}2}$, the sets of labels of edges between elements of E^{t', K_n} with labels in $[t', t]$ are independent Poisson point processes of intensity $\frac{1}{n}$ on $[t', t]$. Remove any vertex w in $H_1 \cup H_2$, such that at least an edge is added between w and element of $E^{t', K_n} \setminus (H_1 \cup H_2)$. These events are independent over the vertices w , and of probability smaller than ϵ_1 . Let \tilde{H}_1 (resp. \tilde{H}_2) be the set of remaining vertices w of H_1 (resp. H_2). Assuming $\epsilon_1 < \frac{1}{2}$, by the law of large numbers, with high probability $|\tilde{H}_1| \geq n^{\frac{3}{5}}$ and $|\tilde{H}_2| \geq n^{\frac{3}{5}}$.

For any $(w_1, w_2) \in \tilde{H}_1 \times \tilde{H}_2$, exactly one edge is added between t' and t between w_1 and w_2 with probability $e^{-\frac{\epsilon_1}{n}} \frac{\epsilon_1}{n} =: a_n$. These events are independent, therefore conditionally on $|\tilde{H}_1| \geq n^{\frac{3}{5}}$, $|\tilde{H}_2| \geq n^{\frac{3}{5}}$, the probability that no simple edge is added between an element of \tilde{H}_1 and an element of \tilde{H}_2 is smaller than:

$$(1 - a_n)^{n^{\frac{6}{5}}} \xrightarrow{n \rightarrow \infty} 0.$$

Moreover, the probability that there exists a vertex $w_1 \in H_1 \cup H_2$ such that two edges toward two different vertices of $H_1 \cup H_2$ are added between t' and t is smaller than $(4n^{\frac{3}{5}})^3 (\frac{\epsilon_1}{n})^2 = o(1)$ by union bound.

All these results implies that if $|\tilde{H}_1| \geq 2n^{\frac{3}{5}}$ and $|\tilde{H}_2| \geq 2n^{\frac{3}{5}}$, then with high probability there exist $w_1 \in \tilde{H}_1$ and $w_2 \in \tilde{H}_2$ such that:

- An edge is added between w_1 and w_2 between t' and t .
- No other edge is added to either w_1 or w_2 between t' and t .

In that case there exists a path from v_1 to v_2 in $G_{n,t}^k$, through w_1 and w_2 , and this concludes the proof of Lemma 5.3.

5.6 Proof of Lemmata 5.8 and 5.31

For all $s \leq t$ and $\epsilon \geq 0$, let $T_{s,\epsilon}^k$ be the component of the root obtained by taking a percolation of parameter $1-2\epsilon$ on T_s^k . $T_{s,\epsilon}^k$ is a two-stages multitype branching process, with one offspring law for the root, and another offspring law for all the other vertices. By definition, the spectral radius of a two-stages branching process is the spectral radius of the associated one-stage branching process. Let $(T_{s,\epsilon}^{k+,y})_{y \in [0,t]}$ denote the one-stage branching process associated to $T_{s,\epsilon}^k$ starting at a non root vertex, where y denotes the type of the first vertex. Let $T_{s,\epsilon}^{k+,\epsilon,y}$ be the branching process obtained by doing a percolation of parameter $1-2\epsilon$ on $T_s^{k+,y}$. With the notation of section 5.2.a, let $M_{s,\epsilon}$ be the operator M for the branching process $T_{s,\epsilon}^{k+,\epsilon,\cdot}$. If $\epsilon = 0$, M_s will be used to denote $M_{s,0}$. As M corresponds to the expected number of vertices of each type, $M_{s,\epsilon} = (1-2\epsilon)M_s$ and

$$\rho_{s,\epsilon} = (1-2\epsilon)\rho_s.$$

Therefore Lemma 5.8 implies the limit (17) in Lemma 5.31.

For $s \leq t$, let $m_s(y, z)$ denote the expected density of the number of children of type z that $T_s^{k+,y}$ has. With the functions g_i introduced in Lemma 4.21, for all $y, z \in [0, s]$, :

$$m_s(y, z) = \sum_{i=1}^{k-2} \frac{1}{i-1!} \underbrace{\int_0^s \int_0^s \dots \int_0^s}_{i-1 \text{ times}} g_i(t, y, z, x_2, x_3, \dots, x_i) dx_2 dx_3 \dots dx_i.$$

By the boundedness and continuity property of the functions g_i shown in Lemma 4.20, this means that $s \rightarrow M_s$ is a continuous application, and therefore that its spectral radius is a upper semi-continuous function of s , by [New51], proving Lemma 5.8.

For any $s \leq t$ and $\epsilon \geq 0$, let $q_{s,\epsilon}(x)$ be the extinction probability of the one-stage multitype branching process $T_{s,\epsilon}^{k+,\epsilon,x}$ starting with a vertex of type x . As previously, to simplify notations, let $q_s = q_{s,0}$. By Lemma 5.5, $q_{s,\epsilon}$ is the smaller positive solution of:

$$\varphi_{s,\epsilon}(f) = f$$

where $\varphi_{s,\epsilon}$ is the operator defined by

$$\varphi_{s,\epsilon}f(y) = \mathbb{E}_y\left(\prod_{i=1}^{Z^\epsilon} f(X_i^\epsilon)\right)$$

where $(X_1^\epsilon, \dots, X_{Z^\epsilon}^\epsilon)$ has the law of the types of the vertices of the first generation of $T_{s,\epsilon}^{k+,\epsilon,x}$. By the percolation properties, $\varphi_{s,\epsilon}$ can be computed as:

$$\varphi_{s,\epsilon}f(x) = \mathbb{E}_x\left(\prod_{i=1}^Z f(X_i)^{B_i}\right)$$

where (X_1, \dots, X_Z) has the law of the types of the vertices in the first generation of $T_s^{k+,x}$, and $(B^i)_{i \in \mathbb{N}}$ is an i.i.d. family of Bernoulli variables equal to 0 with probability $1 - 2\epsilon$. Therefore, for $s = t$ and $\epsilon = 0$:

$$\begin{aligned} \varphi_{t,0}f(y) &= g^0(t, y) + \int_{x_1=0}^t g^1(t, y, x_1)f(x_1)dx_1 + \dots + \\ &+ \frac{1}{k-2!} \int_{x_1, x_2, \dots, x_{k-2}=0}^t g^{k-2}(t, y, x_1, \dots, x_{k-2})f(x_1) \dots f(x_{k-2})dx_1, \dots, dx_{k-2}. \end{aligned} \quad (18)$$

For any $\delta \in [0, 1]$, let $q^\delta = \delta \mathbf{1} + (1 - \delta)q_t$. For any $\epsilon \geq 0$, $s \in [0, t]$, $\delta \in [0, 1]$, $x \in [0, t]$, let

$$l(\epsilon, s, \delta, x) = (\varphi_{s,\epsilon}q^\delta)(x) - q^\delta(x).$$

As T_t^k is supercritical by hypothesis, q_t^k is not uniformly equal to 1, as we are working in the supercritical case. As all the functions g are positive and continuous by Lemma 4.20, q_t^k is continuous and not equal to 1 at any point. For any x , $\delta \rightarrow q^\delta(x)$ is therefore a strictly increasing positive linear function. Therefore for any $i \geq 2$ and any $x_1 \dots x_i$, $\delta \rightarrow q^\delta(x_1)q^\delta(x_2) \dots q^\delta(x_i)$ is a strictly convex function of $\delta \in [0, 1]$, and therefore so is:

$$\delta \rightarrow h^i(t, y, \delta) := \int_{x_1, x_2, \dots, x_i=0}^t g^i(t, y, x_1, \dots, x_i)q^\delta(x_1) \dots q^\delta(x_i)dx_1, \dots, dx_i.$$

By equation (18), for a fixed y , $l(0, t, \delta, y)$ is a polynomial in δ , of degree at most $k - 2$, and its coefficients can be expressed as integrals of the functions q_t^k and g_i . As these functions are continuous, the coefficient of the polynomial $\delta \rightarrow l(0, t, \delta, y)$ are continuous in y and its derivative $(\delta, y) \rightarrow \frac{\partial l(0, t, \delta, y)}{\partial \delta}$ is a continuous function of (δ, y) and is negative for $\delta = 0$ for all y . As $[0, 1] \times [0, t]$ is a compact set, there exists $\eta > 0$ and $\delta_0 > 0$ such that, for all $y \in [0, t]$ and all $0 \leq \delta \leq \delta_0$,

$$\begin{aligned} \frac{\partial l(0, t, \delta, y)}{\partial \delta} &< -\eta \\ l(0, t, \delta, y) &< -\eta\delta \quad \text{as } l(0, t, 0, y) = 0 \end{aligned}$$

Let $\delta \in (0, \delta_0)$. By Lemma 4.20, $(\epsilon, s, y) \rightarrow l(\epsilon, s, \delta, y)$ is a continuous function on the compact set $\{(\epsilon, s, \delta, y) : \epsilon \in [0, \frac{1}{2}], s \in [0, t], y \in [0, s]\}$, and therefore uniformly continuous. There exists $\epsilon_{\max} > 0$ and $s_{\min} < t$ such that, for all $\epsilon \in [0, \epsilon_{\max}]$, $s \in [s_{\min}, t]$, $y \in [0, s]$:

$$\begin{aligned} |l(\epsilon, s, \delta, y) - l(0, t, \delta, y)| &\leq \eta\delta \\ l(\epsilon, s, \delta, y) &\leq 0 \\ (\varphi_{s,\epsilon}q^\delta)(y) &\leq q^\delta(y) \end{aligned}$$

As this inequality holds for all y , by Corollary 5.6, $q^\delta \geq q_{s,\epsilon}$. As the law of the set of labels of the vertices of the first generation of $T_{s,\epsilon}^k$ converges to the law of the set of labels of the vertices of the first generation of T_t^k , and as δ can be chosen arbitrarily small, this inequality implies that

$$\liminf_{\epsilon \rightarrow 0, s \rightarrow t} \tilde{q}_{t',\epsilon}^k \leq \tilde{q}_{t,0}^k$$

where $\tilde{q}_{s,\epsilon}^k$ is the probability of extinction of $T_t^{k,\epsilon}$. The other side of the limits is a consequence of the following facts:

- by definition, $\tilde{q}_{s,\epsilon}^k$ is the limit the non-decreasing sequence $(\mathbb{P}(|T_s^{k,\epsilon}| \leq n))_{n \geq 0}$,
- the application $(s, \epsilon) \rightarrow T_s^{k,\epsilon}$ is continuous for the local limit,
- The event $|T| \leq n$ is continuous for the local limit.

Therefore the equation (16) of Lemma 5.31 holds (the probability of survival $a_{s,\epsilon}^k$ is equal to $1 - \tilde{q}_{s,\epsilon}^k$).

5.7 Proof of Lemma 5.25

Proof. If $2\sqrt{\eta|I|} \geq 1$, Lemma 5.25 is straightforward, so w.l.o.g. we can assume that $2\sqrt{\eta|I|} \leq 1$. In particular, $\eta|I| \leq 1$.

For any i , let $X_i = \max_j X_{i,j}$.

Let $(U_i)_{i \in I}$ be an independent i.i.d. family of uniform variables. For any i , let $X_i^{indep} \in \{0, 1\}$ be defined by $X_i^{indep} = 1$ if and only if either:

- $X_i = 1$,
- or $U_i \leq \frac{2\sqrt{\eta|I|} - \mathbb{P}(X_i=1|X_1^{indep}, \dots, X_{i-1}^{indep})}{\mathbb{P}(X_i=0|X_1^{indep}, \dots, X_{i-1}^{indep})}$.

Lemma 5.32. *For any $i \in I$, $P(X_i = 1 | (X_{i'}^{indep})_{i' < i}) \leq \sqrt{\eta|I|}$.*

If Lemma 5.32 holds, by Lemma 5.19, $(X_i^{indep})_{i \in I}$ is an i.i.d. family of Bernoulli variables of parameter $2\sqrt{\eta|I|}$.

Proof. Lemma 5.32 is proven by induction. By union bound,

$$\begin{aligned} \mathbb{E}(X_i | (X_{i'}^{indep})_{i' < i}) &\leq \sum_{i' \in I} \mathbb{E}(X_{i,i'} | (X_{i'}^{indep})_{i' < i}) \\ &= \sum_{j \in I, j < i} \mathbb{E}(X_{i,j} | (X_{i'}^{bound})_{i' < i}) + \sum_{j \in I, j \geq i} \mathbb{E}(X_{i,j} | (X_{i'}^{bound})_{i' < i}) \end{aligned}$$

For any $j \in I$, let $\mathcal{F}_{i,j}$ be the σ -algebra generated by $(X_{i_1, i_2})_{i_1, i_2 \in I, (i_1, i_2) \neq (i, j)}$ and $(U_{i_1})_{i_1 \in I, i_1 \neq j}$. For all $i_1 \notin \{i, j\}$, $X_{i_1}^{indep}$ is $\mathcal{F}_{i,j}$ -measurable. Conditionally on $\mathcal{F}_{i,j}$, U_j and $X_{i,j}$ are independent, U_j is a uniform variable and $X_{i,j}$ is a Bernoulli random variable of parameter smaller than η . If $j < i$:

$$\mathbb{E}(X_{i,j} | (X_{i'}^{indep})_{i' < i}) = \mathbb{E}(\mathbb{E}(X_{i,j} | \mathcal{F}_{i,j}, X_j^{indep}) | (X_{i'}^{indep})_{i' < i}).$$

There are two possibilities. If $X_j^{indep} = 0$, then $X_{i,j} = 0$. Otherwise, if $X_j^{indep} = 1$:

$$\begin{aligned} \mathbb{E}(X_{i,j} | \mathcal{F}_{i,j}, X_j^{indep} = 1) &= \frac{\mathbb{P}(X_{i,j} = 1, X_j^{indep} = 1 | \mathcal{F}_{i,j})}{\mathbb{P}(X_j^{indep} = 1 | \mathcal{F}_{i,j})} \\ &\leq \frac{\mathbb{P}(X_{i,j} = 1 | \mathcal{F}_{i,j})}{\mathbb{P}(U_j \leq p_j | \mathcal{F}_{i,j})} \\ &= \frac{\mathbb{P}(X_{i,j} = 1 | \mathcal{F}_{i,j})}{\mathbb{E}(p_j | \mathcal{F}_{i,j})} \end{aligned}$$

where $p_j = \frac{2\sqrt{\eta|I|} - \mathbb{P}(X_j = 1 | X_1^{indep}, \dots, X_{j-1}^{indep})}{\mathbb{P}(X_j = 0 | X_1^{indep}, \dots, X_{j-1}^{indep})}$. By induction,

$$\begin{aligned} \sqrt{\eta|I|} &\leq 2\sqrt{\eta|I|} - \mathbb{P}(X_j = 1 | X_1^{indep}, \dots, X_{j-1}^{indep}) \\ &\leq p_j \\ \mathbb{E}(X_{i,j} | \mathcal{F}_{i,j}, X_j^{indep} = 1) &\leq \frac{\eta}{\sqrt{\eta|I|}} \\ &= \sqrt{\frac{\eta}{|I|}} \end{aligned}$$

Therefore, almost surely $\mathbb{E}(X_{i,j} | \mathcal{F}_{i,j}, X_j^{indep}) \leq \sqrt{\frac{\eta}{|I|}}$, and therefore $\mathbb{E}(X_{i,j} | (X_{i'}^{indep})_{i' < i}) \leq \sqrt{\frac{\eta}{|I|}}$.

If $j \geq i$, $\mathbb{E}(X_{i,j} | \mathcal{F}_{i,j}) \leq \eta \leq \frac{\eta}{\sqrt{\eta|I|}} = \sqrt{\frac{\eta}{|I|}}$, and therefore $\mathbb{E}(X_{i,j} | (X_{i'}^{indep})_{i' < i}) \leq \sqrt{\frac{\eta}{|I|}}$, as all the $X_{i'}^{indep}$, for $i' < i$ are $\mathcal{F}_{i,j}$ -measurable.

By summing over I , one obtains Lemma 5.25. □

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