Conformal metrics in R^{2m} with constant Q-curvature and arbitrary volume

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CONFORMAL METRICS IN $\mathbb{R}^{2m}$ WITH CONSTANT Q-CURVATURE AND ARBITRARY VOLUME

XIA HUANG AND DONG YE

ABSTRACT. We study the polyharmonic problem $\Delta^m u = \pm e^u$ in $\mathbb{R}^{2m}$, with $m \geq 2$. In particular, we prove that *for any* $V > 0$, there exist radial solutions of $\Delta^m u = -e^u$ such that

$$\int_{\mathbb{R}^{2m}} e^u dx = V.$$

It implies that for m odd, given any $Q_0 > 0$ and arbitrary volume $V > 0$, there exist conformal metrics g on $\mathbb{R}^{2m}$ with constant Q-curvature equal to Q_0 and $\mathrm{vol}(g) = V$. This answers some open questions in Martinazzi's work [12].

1. INTRODUCTION

In $\mathbb{R}^{2m}$, if the conformal metric $g_u = e^{2u}|dx|^2$ satisfies $(-\Delta)^m u = Q(x)e^{2mu}$, it is well known that (see for instance [1]) $Q(x)$ is just the Q-curvature of the metric g_u. Here $|dx|^2$ is the Euclidean metric.

One interesting question in conformal geometry is to understand the metrics with constant Q-curvature, i.e. to understand solutions of $(-\Delta)^m u = Qe^{2mu}$ in $\mathbb{R}^{2m}$ with $Q \in \mathbb{R}$. Considering $v = u - \ln\lambda$ with $\lambda > 0$, $(-\Delta)^m u = Qe^{2mu}$ is equivalent to $(-\Delta)^m v = \lambda^{2m} Q e^{2mv}$. Therefore the precise value of Q is not important and we can reduce the study to $Q \in \{0, \pm 1\}$.

For $Q > 0$ case, let $(\mathbb{S}^{2m}, g_{\mathbb{S}^{2m}})$ be the standard unit sphere in $\mathbb{R}^{2m+1}$, $\pi : \mathbb{S}^{2m} \backslash \{(0, \dots, 0, 1)\} \to \mathbb{R}^{2m}$ be the stereographic projection. We know that $Q_{g_{\mathbb{S}^{2m}}} = (2m-1)!$ and

$$(\pi^{-1})^* g_{\mathbb{S}^{2m}} = \frac{4|dx|^2}{(1+|x|^2)^2} =: g_\pi.$$

Consequently $Q(g_\pi) = (2m-1)!$. By scaling and translation, for any $x_0 \in \mathbb{R}^{2m}$, $\lambda > 0$,

$$u_{x_0,\lambda} = \ln \frac{2\lambda}{1+\lambda|x-x_0|^2} \tag{1.1}$$

satisfies

$$Q\big(g_{u_{x_0,\lambda}}\big) = (2m-1)! \quad \text{and} \quad \mathrm{vol}\big(g_{u_{x_0,\lambda}}\big) = \int_{\mathbb{R}^{2m}} e^{2mu_{x_0,\lambda}} dx = \mathrm{vol}(\mathbb{S}^{2m}).$$

For $m = 1$, Chen & Li [3] proved that any solution to

$$-\Delta u = e^{2u} \text{ in } \mathbb{R}^2, \quad \mathrm{vol}(g_u) = \int_{\mathbb{R}^2} e^{2u} dx < \infty \tag{1.2}$$

is given by the formula (1.1), hence $\mathrm{vol}(g_u) = \mathrm{vol}(\mathbb{S}^2)$, i.e. any conformal metric g in $\mathbb{R}^2$ with positive constant Gauss curvature and finite volume is provided by the stereographic projection of $\mathbb{S}^2$ into $\mathbb{R}^2$, up to translation and dilation.

Remark that without the assumption of finite volume, Liouville [10] showed that there are many other entire solutions to $-\Delta u = e^{2u}$ in $\mathbb{R}^2$.

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The situation is very different for $m > 1$. Consider the problem

$$(-\Delta)^m u = (2m-1)!e^{2mu} \text{ in } \mathbb{R}^{2m}, \quad \text{vol}(g_u) = \int_{\mathbb{R}^{2m}} e^{2mu} dx < \infty. \tag{1.3}$$

Chang & Chen [2] proved the existence of non-spherical solutions: When $m > 1$, for any $0 < V < \text{vol}(\mathbb{S}^{2m})$, there exists a solution to (1.3) such that $\text{vol}(g_u) = V$.

The condition $\text{vol}(g_u) < \text{vol}(\mathbb{S}^{2m})$ was not only suggested by technical reasons, but it is also necessary when $m = 2$. Indeed, let $m = 2$, Lin showed in [9] that any solution to (1.3) verifies $\text{vol}(g_u) \leq \text{vol}(\mathbb{S}^4)$, and the equality holds if and only if the solution is spherical (i.e. given by (1.1)). Moreover, when $m = 2$, Wei & Ye [14] proved the existence of solution with any asymptotic behavior at infinity predicted by Lin. In particular, it means that for $m = 2$ and any volume V less than $\text{vol}(\mathbb{S}^4)$, there exists a very rich family of non radial solutions of (1.3) with $\text{vol}(g_u) = V$. Thus the situation in $\mathbb{R}^4$ is somehow well understood for $Q > 0$.

Recently, Martinazzi [12] found striking and new phenomena for $m \geq 3$: The solutions to (1.3) can have volume larger than $\text{vol}(\mathbb{S}^{2m})$. More precisely,

(i) for $m = 3$, there exists $V^* > 0$ such that for any $V \geq V^*$, we have a solution u of (1.3) in $\mathbb{R}^6$ such that $\text{vol}(g_u) = V$;
(ii) for any $m \geq 3$ odd, there exists $V_m > \text{vol}(\mathbb{S}^{2m})$ such that for every $V \in (0, V_m]$, there is a solution u of (1.3) satisfying $\text{vol}(g_u) = V$.

However, he could not rule out that $V_3 < V^*$ in (i)-(ii) (when $m = 3$) and he asked if a gap phenomenon is possible, that is, could it be a volume V in (V_3, V^*) such that the problem (1.3) has no solution verifying $\text{vol}(g_u) = V$? He asked also if the result in (i) could be generalized for $m \geq 5$ odd.

In this work, we generalize completely (i)-(ii) by proving that for $m \geq 3$ odd, there exist solutions to (1.3) with arbitrary volume.

Theorem 1.1. *For every $m \geq 3$ odd, and every $V \in (0, +\infty)$, there exists a conformal metric in $\mathbb{R}^{2m}$ satisfying $Q_g \equiv (2m-1)!$ and $\text{vol}(g) = V$.*

The result for m even is less complete, but still suggests that no gap phenomenon exists for (1.3).

Theorem 1.2. *For every $m \geq 2$ even, let*

$$\mathcal{V} = \left\{ \int_{\mathbb{R}^{2m}} e^{2mu} dx, \quad \text{with radial function } u \text{ satisfying } (1.3) \right\}.$$

Then $\mathcal{V}$ is an interval.

Our approach is to study respectively entire radial solutions of the following polyharmonic equations ($m \geq 2$):

$$\Delta^m u = -e^u \quad \text{in } \mathbb{R}^{2m} \tag{1.4}$$

and

$$\Delta^m u = e^u \quad \text{in } \mathbb{R}^{2m}. \tag{1.5}$$

The main results are

Theorem 1.3. *Let $m \geq 2$. Then for any $V \in (0, +\infty)$, there exists a radial solution u to (1.4) such that*

$$\int_{\mathbb{R}^{2m}} e^u dx = V.$$

Theorem 1.4. *Let $m \geq 2$. If there exists an entire radial solution u_0 to (1.5), then for any $0 < V < \|e^{u_0}\|_{L^1(\mathbb{R}^{2m})}$, there exists a radial solution u of (1.5) such that*

$$\int_{\mathbb{R}^{2m}} e^u dx = V.$$

Notice that given a solution u to (1.4) or (1.5), the function $v := \frac{1}{2m}[u - \ln(2m)!]$ solves

$$(-\Delta)^m v = \pm(-1)^{m+1}(2m-1)!e^{2mv} \text{ in } \mathbb{R}^{2m}, \quad \int_{\mathbb{R}^{2m}} e^{2mv}dx = \frac{1}{(2m)!}\int_{\mathbb{R}^{2m}} e^u dx.$$

Hence, Theorems 1.1 and 1.2 are just direct consequence of Theorems 1.3 and 1.4 respectively. So we need just to prove Theorems 1.3 and 1.4.

Furthermore, for the negative constant Q-curvature case, i.e. when $Q < 0$, there is no entire solution to $\Delta u = e^{2u}$ in $\mathbb{R}^N$ for $m = 1$ and any dimension $N \geq 1$ (see for example Theorem 1 in [13]). Here again, we find a completely different situation for $m > 1$. Recently, Hyder & Martinazzi showed that for any $m \geq 2$, $V > 0$, and any polynomial $P(x)$ of degree $\leq (2m-2)$ verifying $\lim_{\|x\|\to\infty} x\cdot\nabla P(x) = \infty$, there exists u such that $(-\Delta)^m u = -(2m-1)!e^{2mu}$ in $\mathbb{R}^{2m}$ and

$$\int_{\mathbb{R}^{2m}} e^{2mu}dx = V, \quad u(x) = -P(x) + \frac{2V}{\mathrm{vol}(\mathbb{S}^{2m})}\ln\|x\| + C + o(1) \quad \text{as } \|x\| \to \infty.$$

The above result is a direct consequence of Theorem 1.2 in [7] combined with Theorem C there, which was previously proved in [11].

2. Proof of Theorem 1.3

2.1. **Preliminaries.** Consider the following initial value problem in $\mathbb{R}^N$ for general dimensions $N \geq 3$.

$$\begin{cases} \Delta^m u = -e^u, \\ u^{(2i+1)}(0) = 0, \quad \forall\, 0 \leq i \leq m-1, \\ \Delta^i u(0) = a_i, \quad \forall\, 0 \leq i \leq m-1. \end{cases} \tag{2.1}$$

Here $u(x) = u(r)$ is a radial function, the Laplacian Δ is seen as $\Delta u = r^{1-N}\left(r^{N-1}u'\right)'$ and a_i are constants in $\mathbb{R}$. We will denote $u_{(a_i)}$ the radial solution to (2.1).

Clearly, there exist suitable constants α_i such that $\Phi_\alpha(r) = \sum_{0\leq j\leq m-1}\alpha_j r^{2j}$ verifies

$$\Delta^i\Phi_\alpha(0) = a_i, \quad \forall\, 0 \leq i \leq m-1.$$

As $\Delta^m(u_{(a_i)} - \Phi_\alpha) = -e^{u_{(a_i)}} < 0$, it's easy to check that $u_{(a_i)}(r) \leq \Phi_\alpha(r)$ whenever $u_{(a_i)}$ exists. Therefore $-e^{u_{(a_i)}}$ is locally bounded whenever $u_{(a_i)}$ exists. Applying standard ODE theory, we can claim that for any $(a_i) \in \mathbb{R}^m$, the unique radial solution of (2.1) is defined globally in $\mathbb{R}_+$, in other words, an entire radial solution to $\Delta^m u = -e^u$ exists in $\mathbb{R}^N$ for any (a_i).

Remark 2.1. For $N \neq 2m$, if u is a solution to (2.1) with $e^u \in L^1(\mathbb{R}^N)$, we can get solution with arbitrary L^1 norm by the scaling $u_\lambda(x) = u(\lambda x) + 2m\ln\lambda$, since

$$\Delta^m u_\lambda = -e^{u_\lambda}, \quad \int_{\mathbb{R}^N} e^{u_\lambda}dx = \lambda^{2m-N}\int_{\mathbb{R}^N} e^u dx.$$

So our main concern here is only relevant for $N = 2m$. We should mention that Farina & Ferrero provide recently in [4] many precise studies for radial solutions of $\Delta^m u = \pm e^u$ in $\mathbb{R}^N$ with general $m, N \in \mathbb{N}^*$.

The following Lemma is inspired by [4]. It's a simple but important fact for our proof.

Lemma 2.2. *Let $m \geq 3$ and u be a radial solution to (2.1), if $a_{m-2} = \Delta^{m-2}u(0) = 0$. Then $\lim_{r\to+\infty}\Delta^{m-1}u(r) < 0$.*

Proof. Let $v = \Delta^{m-1}u$. As $\Delta v = -e^u < 0$, $v(r)$ is decreasing in $\mathbb{R}_+$, so $\lim_{r\to+\infty} v(r) = \ell \in \mathbb{R}\cup\{-\infty\}$ exists. Assume that $\ell \geq 0$, then $v(r) > 0$ in $\mathbb{R}_+$ and $\Delta^{m-2}u(r)$ is increasing in r, which implies $\lim_{r\to+\infty}\Delta^{m-2}u(r) = \ell_1 > 0$, since $\Delta^{m-2}u(0) = 0$. By iterations, we conclude that

$$\lim_{r\to+\infty}\Delta^k u(r) = \infty, \quad \forall\, 0 \leq k \leq m-3.$$

Therefore $\lim_{r\to+\infty}\Delta v(r)=-\infty$. Again, by integrations, we get $\lim_{r\to+\infty}v(r)=-\infty$, which contradicts $\ell\geq 0$, hence there holds $\ell<0$. ∎

A useful consequence of Lemma 2.2 is the following continuity result.

Proposition 2.3. *Let $m\geq 3$ and $\Sigma_0:=\mathbb{R}^{m-2}\times\{0\}\times\mathbb{R}$. Then for any $(a_i)_{0\leq i\leq m-1}\in\Sigma_0$, i.e. $a_{m-2}=0$, the radial solution $u_{(a_i)}$ to equation (2.1) satisfies*

$$V(a_i):=\int_{\mathbb{R}^N}e^{u_{(a_i)}}dx<\infty.$$

Moreover, the function V is continuous in Σ_0.

Proof. Given $(a_i)\in\Sigma_0$, $\lim_{r\to+\infty}\Delta^{m-1}u_{(a_i)}(r)<0$ by Lemma 2.2, hence there is $R>0$ large such that $\Delta^{m-1}u_{(a_i)}(R)<0$.

By ODE theory, the radial solution $u_{(a_i)}$ to equation (2.1) is continuous with respect to (a_i) in $C^k_{loc}(\mathbb{R}^N)$ for any $k\in\mathbb{N}$, see for example the proof of Proposition A3 in [5]. Consequently, there exists $\delta>0$ small such that for any $|(a_i')-(a_i)|\leq\delta$, there holds

$$\left\|u_{(a_i')}-u_{(a_i)}\right\|_{C^{2m}(\overline{B}_R)}\leq 1\quad\text{and}\quad\Delta^{m-1}u_{(a_i')}(R)\leq\frac{\Delta^{m-1}u_{(a_i)}(R)}{2}=:-M<0.$$

As $\Delta^{m-1}u$ is decreasing in r for any radial solution to $\Delta^m u=-e^u$, we have $\Delta^{m-1}u_{(a_i')}(r)\leq -M$ if $r\geq R$ and $|(a_i')-(a_i)|\leq\delta$. Therefore, for $r\geq R$ and $|(a_i')-(a_i)|\leq\delta$,

$$\begin{aligned}
&\Delta^{m-2}u_{(a_i')}(r)\\
&=\Delta^{m-2}u_{(a_i')}(R)+\int_R^r\frac{1}{\rho^{N-1}}\left[R^{N-1}\left(\Delta^{m-2}u_{(a_i')}\right)'(R)+\int_R^\rho s^{N-1}\Delta^{m-1}u_{(a_i')}(s)ds\right]d\rho\\
&\leq C_1+\int_R^r\left(-\frac{M}{N}\rho+C_2\rho^{1-N}\right)d\rho\\
&\leq -\frac{Mr^2}{2N}+\frac{MR^2}{2N}+C_1+\frac{C_2R^{2-N}}{N-2}=-\frac{Mr^2}{2N}+C_3.
\end{aligned}$$

Here C_i are some constants independent of (a_i') verifying $|(a_i')-(a_i)|\leq\delta$. We get then $M'>0$ and $R'\geq R$ such that

$$\Delta^{m-2}u_{(a_i')}(r)\leq -M'<0,\quad\text{for all } r\geq R',\ |(a_i')-(a_i)|\leq\delta.$$

By iterations, we can conclude that there exist $M_0>0$ and R_0 large such that

$$u_{(a_i')}(r)\leq -M_0r^{2m-4},\quad\text{for all } r\geq R_0,\ |(a_i')-(a_i)|\leq\delta.$$

Clearly $V(a_i)<\infty$ by the above estimate. It's not difficult to deduce the continuity of V in (a_i) using the continuity of $u_{(a_i)}$ in $C^0_{loc}(\mathbb{R}^N)$ with respect to (a_i), and the uniform estimate out of a compact set, we omit the details. ∎

If $m=2$, we consider radial solutions to the following biharmonic equation

$$\begin{cases}\Delta^2u=-e^u\\ u'(0)=u'''(0)=0,\\ \Delta u(0)=a,u(0)=-b.\end{cases}\tag{2.2}$$

Corresponding to Lemma 2.2 for $m\geq 3$, we have

Lemma 2.4. *For any $a,\ b\in\mathbb{R}$, the radial solution to (2.2) satisfies $\lim_{r\to+\infty}\Delta u(r)<0$.*

Proof. Let $v = \Delta u$, as $\Delta v = -e^u < 0$, v is decreasing in $r \in (0, +\infty)$. So $\lim_{r\to+\infty} v(r) = \ell$ exists. If $\ell \geq 0$, we have $v(r) > 0$ in $\mathbb{R}_+$, then u is increasing in r and $\Delta v = -e^u \leq -e^{u(0)} = -e^{-b}$ in $\mathbb{R}^N$. Then $\lim_{r\to+\infty} v(r) = -\infty$ since

$$v(r) - a = \int_0^r \frac{1}{\omega_{N-1}\rho^{N-1}} \int_{B_\rho} \Delta v dx d\rho \leq -e^{-b}\frac{r^2}{2N}.$$

This contradicts the assumption $\ell \geq 0$. So $\ell < 0$. ∎

Here and after, ω_{N-1} denotes the volume of the standard sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$. Denote $u_{a,b}$ the radial solution to (2.2) and

$$\widetilde{V}(a, b) := \int_{\mathbb{R}^N} e^{u_{a,b}} dx.$$

Using Lemma 2.4, we can prove the continuity of $\widetilde{V}$ very similarly as for Proposition 2.3, so we omit the proof.

Proposition 2.5. *For any $(a, b) \in \mathbb{R}^2$, $\widetilde{V}(a, b) < \infty$. Moreover, $\widetilde{V}$ is continuous in $\mathbb{R}^2$.*

2.2. **Solutions with large volume for** (1.4)**.** Here we prove the existence of radial solutions to (1.4) with any large volume. As mentioned in Remark 2.1, the problem is relevant only in the conformal dimension. From now on, we fix $N = 2m$, even though similar result holds true for any $N \geq 3$. The crucial point is to consider some special initial conditions.

More precisely, for $m \geq 3$ and $N = 2m$, let

$$c_0 := 4^{m-1} \times \prod_{k=1}^{m-1} \big[k(m - 1 + k)\big] \tag{2.3}$$

and consider (2.1) with $(a_i) = (-b, 0, \dots 0, c_0) \in \Sigma_0 = \mathbb{R}^{m-2} \times \{0\} \times \mathbb{R}$.

$$\begin{cases} \Delta^m u = -e^u, \\ u^{(2i+1)}(0) = \Delta^k u(0) = 0, \quad \forall\, i = 0 \dots m-1;\ k = 1, \dots m-2, \\ \Delta^{m-1} u(0) = c_0, \\ u(0) = -b < 0. \end{cases} \tag{2.4}$$

Theorem 2.6. *Let $m \geq 3$, denote by u_b the radial solution to equation* (2.4)*. Then*

$$\lim_{b\to+\infty} \int_{\mathbb{R}^{2m}} e^{u_b} dx = +\infty. \tag{2.5}$$

Similarly, let $a = 8$ in (2.2) *for $m = 2$, there holds*

$$\lim_{b\to+\infty} \widetilde{V}(8, b) = +\infty. \tag{2.6}$$

Proof. We handle the cases $m \geq 3$ and $m = 2$ together. For simplicity and without confusion, we denote by u the solution to (2.4) or the solution to (2.2) with $a = 8$.

For any $m \geq 2$, let $\Phi(x) = \Phi(r) = r^{2m-2} - b$. Hence $\Delta^m \Phi = 0$ in $\mathbb{R}^{2m}$ and $\Delta^i \Phi(0) = \Delta^i u(0)$ for any $0 \leq i \leq m - 1$. Set $w = u - \Phi$, then $\Delta^m w = \Delta^m u = -e^u < 0$ and $\Delta^i w(0) = 0$ for $0 \leq i \leq m - 1$. By iterations, we deduce easily that $\Delta^i w \leq 0$ in $\mathbb{R}^{2m}$ for $0 \leq i \leq m - 1$. In particular, $w \leq 0$ in $\mathbb{R}^{2m}$, i.e. $u \leq \Phi$ in $\mathbb{R}^{2m}$.

Let $R_0 := b^{\frac{1}{2m-2}}$, the unique zero of Φ in $(0, \infty)$. To prove (2.5) or (2.6), we proceed by three steps.

Step 1. Estimate of $\Delta^{m-1} w(R_0)$.

As $\Delta^m w = -e^u$, we have, for any $r > 0$,

$$\begin{aligned}\Delta^{m-1}w(r) = -\int_0^r \frac{1}{\rho^{2m-1}}\int_0^\rho e^{u(s)}s^{2m-1}dsd\rho &\geq -\int_0^r \rho^{1-2m}\int_0^\rho e^{\Phi(s)}s^{2m-1}dsd\rho\\ &= -\int_0^r e^{s^{2m-2}-b}s^{2m-1}ds\int_s^r \rho^{1-2m}d\rho\\ &= -\frac{1}{2m-2}\int_0^r e^{s^{2m-2}-b}s\left[1-\left(\frac{s}{r}\right)^{2m-2}\right]ds\\ &\geq -\int_0^r e^{s^{2m-2}-b}\left(1-\frac{s}{r}\right)sds\\ &= -\frac{r^2}{2}\int_0^1 e^{r^{2m-2}t^{m-1}-b}\big(1-\sqrt{t}\big)dt.\end{aligned}$$

For the second inequality, we used the convexity of the function $h(x) = x^{2m-2}$ in $\mathbb{R}_+$ and we applied the change of variable $t = r^{-2}s^2$ for the last line. Therefore,

$$\Delta^{m-1}w(R_0) \geq -\frac{R_0^2}{2}\int_0^1 e^{b(t^{m-1}-1)}\big(1-\sqrt{t}\big)dt =: \xi(b).$$

Moreover, there exists $\lambda > 0$ depending on m such that $\lambda(1-t) \leq 1-t^{m-1}$ for any $t \in [0,1]$. So we get

$$\begin{aligned}\int_0^1 e^{b(t^{m-1}-1)}(1-\sqrt{t})dt &\leq \int_0^1 e^{\lambda b(t-1)}(1-t)dt\\ &= \frac{1}{\lambda^2 b^2} - \left(\frac{1}{\lambda b}+\frac{1}{\lambda^2 b^2}\right)e^{-\lambda b} = O\left(\frac{1}{b^2}\right) \quad \text{as } b\to\infty.\end{aligned}$$

As $\Delta^{m-1}w(r)$ is decreasing in r, there holds

$$\Delta^{m-1}w(r) \geq \Delta^{m-1}w(R_0) \geq \xi(b) = O\left(b^{\frac{1}{m-1}-2}\right) \quad \text{for } r \leq R_0. \tag{2.7}$$

Step 2. Estimates of $\Delta^i u(r)$, $i = 0, ..., m-1$ for $r > R_0$.

Define $r_0 = \inf\{r > 0,\ u(r) = 0\} \in (R_0, +\infty]$. We claim that

$$\lim_{b\to+\infty}(r_0 - R_0) = 0. \tag{2.8}$$

Remark that $u \leq 0$ in $[R_0, r_0]$, so $\Delta^m w = -e^u \geq -1$ if $R_0 \leq r \leq r_0$. Therefore, if $r \in [R_0, r_0]$,

$$\begin{aligned}\Delta^{m-1}w(r) &\geq \Delta^{m-1}w(R_0) - \int_{R_0}^r \frac{1}{\omega_{2m-1}\rho^{2m-1}}\left(\int_{B_{R_0}} e^\Phi dx + \int_{B_\rho\setminus B_{R_0}} dx\right)d\rho,\\ &= \Delta^{m-1}w(R_0) - \int_{R_0}^r \rho^{1-2m}\left(\int_0^{R_0} e^{s^{2m-2}-b}s^{2m-1}ds + \int_{R_0}^\rho s^{2m-1}ds\right)d\rho.\end{aligned}$$

When $b \to +\infty$, there holds

$$\eta(b) := \int_0^{R_0} e^{s^{2m-2}-b}s^{2m-1}ds = R_0^{2m}\int_0^1 e^{-b(1-t^{2m-2})}t^{2m-1}dt = O\left(\frac{R_0^{2m}}{b}\right).$$

We obtain that for $r \in [R_0, r_0]$,

$$\Delta^{m-1}w(r) \geq \Delta^{m-1}w(R_0) - \eta(b)R_0^{1-2m}(r-R_0) - \frac{1}{2m}\int_{R_0}^r \rho\left[1-\left(\frac{R_0}{\rho}\right)^{2m}\right]d\rho. \tag{2.9}$$

On the other hand, by the convexity of $h(x) = x^{2m}$ in $\mathbb{R}_+$,

$$\frac{1}{2m}\int_{R_0}^r \rho\left[1-\left(\frac{R_0}{\rho}\right)^{2m}\right]d\rho \leq \int_{R_0}^r \rho\left(1-\frac{R_0}{\rho}\right)d\rho = \frac{(r-R_0)^2}{2}, \quad \forall\, r \geq R_0.$$

Denote $\widetilde{r_0} := \min\{r_0, R_0+1\}$. Combining (2.7) and (2.9), for $r \in [R_0, \widetilde{r_0}]$, we have (as $m \geq 2$)

$$\Delta^{m-1} w(r) \geq O\left(b^{\frac{1}{m-1}-2}\right) - O\left(\frac{R_0}{b}\right)(r-R_0) - \frac{(r-R_0)^2}{2} \geq O(1). \tag{2.10}$$

Using again (2.7), we obtain

$$\Delta^{m-1} w(r) \geq O\left(b^{\frac{1}{m-1}-2}\right) + O(1)\chi_{[R_0,\widetilde{r_0}]}, \quad \forall\, r \in [0, \widetilde{r_0}].$$

Here and in the following, χ_A denotes the characteristic function of a subset A and $O(1)$ denotes a quantity uniformly bounded for b sufficiently large.

By iterations, for $0 \leq j \leq m-2$ and $r \in [0, \widetilde{r_0}]$, we get

$$\Delta^{j} w(r) \geq O\left(b^{\frac{1}{m-1}-2}\right) r^{2(m-1-j)} + O(1)\chi_{[R_0,\widetilde{r_0}]}. \tag{2.11}$$

In particular, let $j=0$, there holds

$$u(r) \geq \Phi(r) + O\left(b^{\frac{1}{m-1}-2}\right) r^{2(m-1)} + O(1)\chi_{[R_0,\widetilde{r_0}]}, \quad \forall\, r \in [0, \widetilde{r_0}].$$

Using the convexity of Φ, we have then

$$u(r) \geq (2m-2)R_0^{2m-3}(r-R_0) + O\left(b^{\frac{1}{m-1}-2}\right) r^{2(m-1)} + O(1) \quad \text{in } [R_0, \widetilde{r_0}].$$

Fix any $\varepsilon \in (0,1)$, suppose that $r_0 > R_0 + \varepsilon$. Then

$$0 > u(R_0+\varepsilon) \geq (2m-2) b^{\frac{2m-3}{2m-2}} \varepsilon + O\left(b^{\frac{1}{m-1}-1}\right) + O(1),$$

which is impossible for b large enough, since

$$\frac{2m-3}{2m-2} - \frac{1}{m-1} + 1 = \frac{4m-7}{2(m-1)} > 0, \quad \forall\, m \geq 2.$$

In other words, when b is sufficiently large, we have $r_0 \leq R_0 + \varepsilon$, so the claim (2.8) is proved. An immediate consequence of (2.8) is

$$\liminf_{b\to+\infty} \Delta^{m-1} u(r_0) \geq c_0. \tag{2.12}$$

Indeed, applying the first inequality in (2.10),

$$\Delta^{m-1} w(r_0) \geq O\left(b^{\frac{1}{m-1}-2}\right) - O\left(\frac{R_0}{b}\right)(r_0-R_0) - \frac{(r_0-R_0)^2}{2},$$

we get $\liminf_{b\to+\infty} \Delta^{m-1} w(r_0) \geq 0$ by (2.8), hence (2.12) holds true as $\Delta^{m-1}\Phi \equiv c_0$.

Step 3. The proof of (2.5) and (2.6).

Consider first $m \geq 3$. Recall that we denote by u, the radial solution of (2.4). Let

$$V(b) := \int_{\mathbb{R}^{2m}} e^u dx = \omega_{2m-1} \int_0^\infty e^{u(s)} s^{2m-1} ds.$$

By equation $\Delta^m u = -e^u$, we get

$$r^{2m-1}(\Delta^{m-1}u)'(r) = -\int_0^r e^{u(s)} s^{2m-1} ds \geq -\frac{V(b)}{\omega_{2m-1}}.$$

For any $r > r_0$, using the above inequality on $[r_0, r]$, there holds

$$\begin{aligned}\Delta^{m-1}u(r_0) &\leq \Delta^{m-1}u(r) + \frac{V(b)}{\omega_{2m-1}} \int_{r_0}^r s^{1-2m} ds \leq \Delta^{m-1}u(r) + \frac{V(b)}{\omega_{2m-1}} \int_{r_0}^\infty s^{1-2m} ds \\ &= \Delta^{m-1}u(r) + \frac{V(b) r_0^{2-2m}}{(2m-2)\omega_{2m-1}}.\end{aligned}$$

Tending r to $+\infty$, we conclude by Lemma 2.2 that

$$V(b) > (2m-2)\omega_{2m-1} r_0^{2m-2} \Delta^{m-1} u(r_0).$$

Hence $\lim_{b\to+\infty} V(b) = +\infty$ by (2.12) and $\lim_{b\to+\infty} r_0 = +\infty$.

The proof of (2.6) is completely similar, so we omit it. ∎

Remark 2.7. The formula (2.6) gives a positive answer to a question in [12], page 981. Assume that u solves (2.2) with $a = 8$ in $\mathbb{R}^4$. Let $v(x) = u(\lambda x) + 4\ln\lambda$ with $\lambda = e^{b/4}$, then $\Delta^2 v = -e^v$, $v(0) = 0$ and $v''(0) = e^{b/2}u''(0) = 2e^{b/2}$. Hence $v''(0) \to +\infty$ is equivalent to $b \to +\infty$.

2.3. **Proof of Theorem 1.3 completed.** Let $m \geq 3$ and $\widetilde{u}_b$ be the radial solution to equation (2.1) with $(a_i) = (-b, 0, ...0, -c_0) \in \Sigma_0$. As above, there holds $\widetilde{u}_b \leq \Psi(r) := -r^{2m-2} - b$ in $\mathbb{R}^{2m}$. Hence

$$\lim_{b\to+\infty} \int_{\mathbb{R}^{2m}} e^{\widetilde{u}_b} dx \leq \lim_{b\to+\infty} \int_{\mathbb{R}^{2m}} e^{\Psi} dx = \lim_{b\to+\infty} \int_{\mathbb{R}^{2m}} e^{-|x|^{2m-2}-b} dx = 0.$$

By Proposition 2.3, Theorem 2.6 and the above estimate, we get readily that $V(\Sigma_0) = (0, \infty)$, so we are done.

The argument for $m = 2$ is completely similar. Considering the radial solution $\widetilde{u}_b$ to (2.2) with $a = -8$, we prove easily that $\inf_{\mathbb{R}^2} \widetilde{V}(a, b) = 0$. Using (2.6) and Proposition 2.5, there holds $\widetilde{V}(\mathbb{R}^2) = (0, \infty)$. ∎

3. Proof of Theorem 1.4

For (1.5), we use a different approach, which is based on the following well-known comparison result (see for instance Proposition A.2 in [4])

Lemma 3.1. *Let $u, v \in C^{2m}([0, R))$ be two radial functions such that $\Delta^m u - e^u \geq \Delta^m v - e^v$ in $[0, R)$ and*

$$\Delta^k u(0) \geq \Delta^k v(0),\ (\Delta^k u)'(0) \geq (\Delta^k v)'(0), \quad \forall\ 0 \leq k \leq m-1. \tag{3.1}$$

Then we have $u \geq v$ in $[0, R)$.

Let u_0 be an entire radial solution of (1.5) with $V_0 := \|e^{u_0}\|_{L^1(\mathbb{R}^{2m})} \in (0, \infty]$, consider u_α the solution to the following initial value problem

$$\begin{cases} \Delta^m u = e^u, \\ u^{(2i+1)}(0) = 0, & \forall\ 0 \leq i \leq m-1, \\ \Delta^i u(0) = \Delta^i u_0(0), & \forall\ 0 \leq i \leq m-2, \\ \Delta^{m-1} u(0) = \Delta^{m-1} u_0(0) - \alpha, & \alpha > 0. \end{cases} \tag{3.2}$$

For any $\alpha > 0$, by Lemma 3.1, we have $u_\alpha \leq u_0$ whenever it exists. On the other hand, we have $u_\alpha \geq \Phi_\alpha = \sum_{0\leq j\leq m-1} \alpha_j r^{2j}$ with $\alpha_j \in \mathbb{R}$ verifying $\Delta^i \Phi_\alpha(0) = \Delta^i u_\alpha(0)$ for $i = 0, \ldots, m-1$. Then no blow-up will occur for u_α in any compact set, which means that u_α is globally defined and $\alpha \mapsto u_\alpha$ is a decreasing family of functions in $\mathbb{R}^{2m}$ by Lemma 3.1. We claim that

$$\lim_{\alpha\to\infty} u_\alpha(r) = -\infty, \quad \text{for any } r \geq 0. \tag{3.3}$$

Let $v_\alpha = \Delta^{m-1} u_\alpha$, as $0 \leq \Delta v_\alpha = e^{u_\alpha} \leq e^{u_0}$ and

$$v_\alpha(r) = \Delta^{m-1} u_0(0) - \alpha + \frac{1}{2m-2} \int_0^r \Delta v_\alpha(s) s \left[1 - \left(\frac{s}{r}\right)^{2m-2}\right] ds,$$

we get readily that v_α tends uniformly to $-\infty$ in any compact set of $\mathbb{R}_+$ as $\alpha \to \infty$. By iterations, we obtain that $\Delta^{m-2} u_\alpha, \ldots \Delta u_\alpha$ tend to $-\infty$ uniformly in any compact set of $\mathbb{R}_+$, hence (3.3) is satisfied.

Moreover, by Lemmas 7.6 and 7.8 in [4], for any $\alpha > 0$, there holds $\lim_{r\to\infty} \Delta^{m-1} u_\alpha(r) < 0$ hence $u_\alpha(r) \leq -C_\alpha r^{2m-2}$ for r large enough with some $C_\alpha > 0$. Therefore $e^{u_\alpha} \in L^1(\mathbb{R}^{2m})$ for

any $\alpha > 0$, and $\alpha \mapsto \|e^{u_\alpha}\|_{L^1(\mathbb{R}^{2m})}$ is continuous in $(0,\infty)$, combining with the monotonicity w.r.t. α. Furthermore, the claim (3.3) implies then

$$\lim_{\alpha\to\infty}\int_{\mathbb{R}^{2m}} e^{u_\alpha}dx = 0.$$

As the monotonicity of u_α gives also

$$\lim_{\alpha\to 0^+}\int_{\mathbb{R}^{2m}} e^{u_\alpha}dx = \int_{\mathbb{R}^{2m}} e^{u_0}dx = V_0,$$

the proof is completed. ∎

4. Further remarks and open questions

By the proof of Proposition 2.3, for $m \geq 3$ and $(a_i) \in \mathbb{R}^m$, if the solution of (2.1) verifies $\lim_{r\to+\infty}\Delta^{m-1}u_{(a_i)}(r) < 0$, then $\mathrm{vol}(g_u)$ is finite and the function V with the initial data as variables is continuous at the point (a_i). However we have no answer for the following question.

Question 1: For any $(a_i) \in \mathbb{R}^m$ with $m \geq 3$, let u be the solution of (2.1), is the total volume $\mathrm{vol}(g_u)$ finite? If the answer is yes, is the volume function V continuous in whole $\mathbb{R}^m$?

Another natural question comes from Theorem 1.2. Our approach is to study radial solutions of (1.5). By Hyder & Martinazzi's result in [7] on the negative constant Q-curvature situation, for $m \geq 3$ *odd*, the radial solutions of (1.5) can provide arbitrary volume. However, this is not always true for m *even*, since when $m = 2$, $\mathcal{V} = (0, \mathrm{vol}(\mathbb{S}^4)]$ by [9, 2, 14].

Question 2: Let m be even and $m \geq 4$, do we have $\mathcal{V} = (0,\infty)$ for radial solutions of (1.5)?

Consider the radial solutions to (1.5) as a initial value problem with $u(0) = 0$, $\Delta^k u(0) = \beta_k$, $1 \leq k \leq m-1$. By Theorem 2.2 in [4], there exists a function $\Phi : \mathbb{R}^{m-2} \to (-\infty, 0)$ such that the solution u is globally defined in $\mathbb{R}^{2m}$ if and only if $\beta_{m-1} \leq \Phi(\beta_1, \dots \beta_{m-2})$. On the other hand, for given $\beta_1, \dots \beta_{m-2}$, the solution $u_{(\beta_i)}$ is increasing w.r.t. β_{m-1} by Lemma 3.1, so

$$\sup \mathcal{V} = \sup\left\{\int_{\mathbb{R}^{2m}} e^{2mu}dx,\ \ u = u_{(\beta_i)} \text{ with } \beta_{m-1} = \Phi(\beta_1, \dots \beta_{m-2}).\right\}.$$

Therefore, to answer the above question, we need just to understand the radial solutions with (β_i) on the boundary hypersurface for the global existence. Unfortunately we have few information for these borderline entire radial solutions. For instance, we don't know the asymptotic decay of such solutions as $r \to \infty$, see Theorem 2.5 and Problem 2.1 (ii) in [4].

A last question concerns the infinite volume entire solutions. When $m = 1$, Liouville proved that given a holomorphic function h in $\Omega \subset \mathbb{C}$, the function

$$u(z) := \ln \frac{2|h'(z)|}{1 + |h(z)|^2}$$

satisfies $-\Delta u = e^{2u}$ in $\Omega\backslash\{z \in \Omega, h'(z) = 0\}$. The conformal metrics in $\mathbb{R}^2$ with $\mathrm{vol}(g_u) < \infty$, i.e. the solutions to (1.2) correspond to $h(z) = az + b$ with $a, b \in \mathbb{C}$. So we can describe many entire solutions of $-\Delta u = e^{2u}$ in $\mathbb{R}^2$ with infinite volume.

For $m \geq 2$, of course we can use entire radial solutions v of $(-\Delta)^m v = e^{2mv}$ in $\mathbb{R}^N$ with $3 \leq N \leq 2m-1$ to construct constant Q-curvature conformal metrics in $\mathbb{R}^{2m}$ with infinite volume, for example by considering $u(x) := v(x_1, \dots x_N)$. However, we wonder if other examples exist.

Question 3: For $m \geq 2$, are there entire solutions of $(-\Delta)^m u = e^{2mu}$ in $\mathbb{R}^{2m}$ such that u does not allow any symmetry and $e^{2mu} \notin L^1(\mathbb{R}^{2m})$?

It is worthy to mention that similar problem was studied also in odd space dimensions, with fractional Laplacian. Let $n \geq 3$ be odd, consider

$$(-\Delta)^{\frac{n}{2}} u = (n-1)! e^{nu} \text{ in } \mathbb{R}^n, \quad V := \int_{\mathbb{R}^n} e^{nu} dx < \infty. \tag{4.4}$$

Recently, Jin *et al.* proved in [8] that for $n = 3$, (4.4) has a solution *if and only if* $V \in (0, \mathrm{vol}(\mathbb{S}^3)]$. This is similar to Lin's result in $\mathbb{R}^4$ with $m = 2$ in (1.3). Moreover, for arbitrary $n \geq 5$ odd, Hyder proved in [6] the existence of solution to (4.4) with any $V \in (0, \mathrm{vol}(\mathbb{S}^n)]$, without excluding the possibility that $V > \mathrm{vol}(\mathbb{S}^n)$. This would be an interesting problem to understand if similar phenomenon to [12] or Theorem 1.1 here exists.

Question 4: For $n \geq 5$ odd, are there solutions of (4.4) such that $V > \mathrm{vol}(\mathbb{S}^n)$?

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