Constructive formulations of resonant Maxwell's equations

Martin Campos Pinto, Bruno Després

Constructive formulations of resonant Maxwell's equations*

M. Campos Pinto† and B. Després‡

February 25, 2016

Abstract

In this article we propose new constructive weak formulations for resonant time-harmonic wave equations with singular solutions. Our approach follows the limiting absorption principle and combines standard weak formulations of PDEs with properties of elementary special functions adapted to the singularity of the solutions, called manufactured solutions. We show the well-posedness of several formulations obtained by these means for the limit problem in dimension one, and propose a generalization in dimension two.

2010 Mathematics Subject Classification: 35Q60, 35B34, 65N20.

Keywords: Maxwell's equations; Singular solutions; Resonant dielectric tensor; Resonant heating; Limit absorption principle; Manufactured solutions.

1 Introduction

Resonant time-harmonic wave equations are found in the modeling of electromagnetic waves in magnetized plasmas [21, 13], in the modeling of metamaterials [4] and in aeroacoustics in recirculating flows [3]. The list is non exhaustive. In all cases, the mathematical solutions of these linear time-harmonic equations with varying coefficients may display highly singular solutions inside the domain. This is comparable, but different, to the singular solutions encountered at the boundary of domains with reentrant corners [15, 11]. In our case functions u with bounds like $\|x^\beta u\|_{L^2} < \infty$ for various positive and negative $\beta \in \mathbb{R}$ are the rule, see [10, 20]. For $\beta > 0$, it expresses a singular behavior near $x = 0$, or even worse, the possibility of a Dirac mass inside the domain. In this work we focus on Maxwell's equations involving a cold plasma dielectric tensor, which is a set of equations that models the propagation of a time harmonic electromagnetic wave in a plasma. It is known [13] that the solution may indeed take the form of a Dirac mass plus a principal value. It is also known [13] that the analysis of such singular solutions following the limit absorption principle exhibits a non standard behavior called resonant heating, involving a non zero energy loss in the vanishing limit of the small regularization parameter. These singular solutions and phenomena question the usual tools of the mathematical theory of linear partial differential equations, since most of the usual techniques are no longer applicable. For instance the usual $H(\text{curl})$ setting [19, 18] is not enough when the Maxwell equations involve such resonant cold plasma tensors unless some coercivity remains, the latter case being addressed in the only work [2] we know about on the mathematical theory of Maxwell's equation with a cold plasma tensor. Thus, resonant equations offer a large number of open problem from the perspective of mathematical analysis.

This work addresses new weak formulations for resonant time-harmonic wave equations with singular solutions, having in perspective that these formulations must be constructive. By constructiveness we understand that the weak formulations are well posed (existence and uniqueness of the solution) and could be discretized in a straightforward manner within a standard finite element solver. Our main idea, to achieve constructiveness, is

*The support of ANR under contract ANR-12-BS01-0006-01 is acknowledged. This work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training program 2014-2018 under grant agreement AWP15-ENR-01/CEA-05. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

†CNRS, Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, 4, place Jussieu 75005, Paris, France

‡Sorbonne Universités, UPMC Univ Paris 06, CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, 4, place Jussieu 75005, Paris, France

to mix standard weak formulations of PDEs with elementary special functions which are used to characterize the singularity of the solutions. It will be also visible that dissipative formulations discussed below are reminiscent of entropy techniques which are standard for non linear equations [16] and have recently been extended to Friedrichs systems [14]. It is possible to think that dissipative formulations are distant cousins to singularity extraction techniques [1, 11]. The comparison of our ideas with T-coercivity techniques [9, 6] is an open problem.

General strategy

Our strategy to address resonant equations is hereafter explained in general dimension. We believe that it can be easily generalized to many wave equations with singular solutions.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain in dimension $1 \leq d \leq 3$. The boundary $\Gamma = \partial\Omega$ is smooth with outgoing normal $\mathbf{n} = \mathbf{n}(\mathbf{x})$ for $\mathbf{x} \in \Gamma$. The generic model problem consists of the time harmonic ($\partial_t = -i\omega$, $i^2 = -1$) Maxwell equations with a non standard dielectric tensor $\underline{\underline{\varepsilon}}$,

$$\begin{cases} \nabla \times \nabla \times \mathbf{E} - \underline{\underline{\varepsilon}}\mathbf{E} = 0, & \mathbf{x} \in \Omega, \\ -\left((\nabla \times \mathbf{E}) \times \mathbf{n}\right) \times \mathbf{n} + i\lambda \mathbf{E} \times \mathbf{n} = \mathbf{f} \times \mathbf{n}, & \mathbf{x} \in \Gamma. \end{cases} \tag{1}$$

The mixed boundary condition with coefficient $\lambda > 0$ is written here for the completeness of the presentation. We will use the notation that $\mathbf{B} = \nabla \times \mathbf{E}$ is called the magnetic field. The mathematical theory is nowadays comprehensive for standard dielectric tensors [18, 17, 8]. Our interest is in non standard hermitian differentiable dielectric tensors [21]

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^* \in \left[\mathcal{C}^q(\Omega)\right]^{d\times d}, \qquad q \geq 1.$$

An example of a singular solution is easily obtained with the cold plasma tensor at the hybrid resonance which may be written in non dimensional variable $\mathbf{x} = (x, y, z)$ as

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}(\mathbf{x}) = \begin{pmatrix} x & i & 0 \\ -i & x & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This will be illustrated with the example of the Budden problem where singular solutions are encountered at $x = 0$, where the extra diagonal part of the 2×2 block tensor dominates the corresponding diagonal part [13, 12]. Since such a problem may be ill-posed in standard functional spaces like $L^2(\Omega)^3$ or $H(\text{curl}, \Omega) = \left\{\mathbf{F} \in L^2(\Omega)^3,\ \nabla \times \mathbf{F} \in L^2(\Omega)^3\right\}$, it is usual to regularize it in the context of the limit absorption principle. The approximate solution is $\mathbf{E}^\nu$, $\nu > 0$, solution to

$$\begin{cases} \nabla \times \nabla \times \mathbf{E}^\nu - \left(\underline{\underline{\varepsilon}} + i\nu \underline{\underline{I}}\right) \mathbf{E}^\nu = 0, & \mathbf{x} \in \Omega, \\ -\left((\nabla \times \mathbf{E}^\nu) \times \mathbf{n}\right) \times \mathbf{n} + i\lambda \mathbf{E}^\nu \times \mathbf{n} = \mathbf{f} \times \mathbf{n}, & \mathbf{x} \in \Gamma. \end{cases} \tag{2}$$

Here $\underline{\underline{I}} = (\delta_{ij})_{1 \leq i,j \leq d}$ is the identity matrix. Contrary to Problem (1), the latter problem is endowed with a natural coercive inequality [18] in the space

$$H(\text{curl}, \Omega, \Gamma) = \left\{\mathbf{F} \in H(\text{curl}, \Omega),\ \mathbf{F} \times \mathbf{n} \in L^2(\Gamma)^3\right\},$$

as recalled in the appendix. In the context of the so-called limit absorption principle, the objective is to pass to the limit $\nu \to 0^+$ and to study the limit electric field. The major difficulty is when the coefficients of the dielectric tensor are such that

$$\mathbf{E}^+ := \lim_{\nu \to 0^+} \mathbf{E}^\nu \notin L^1_{\text{loc}}(\Omega). \tag{3}$$

Our goal in this work is to explain in what sense the limit electric field $\mathbf{E}^+$ is nevertheless a solution of the initial problem (1). In this direction we will construct equalities and inequalities satisfied by the limit $\mathbf{E}^+$, and to reach this aim we will use what we call *manufactured solutions*. The principle of a manufactured solution is that it should satisfy the following properties.

(P1) It is known analytically and its limit for $\nu \to 0^+$ is trivial to determine.

(P2) It satisfies the same (or a similar) equation than $\mathbf{E}^\nu$, but the right hand side may be non zero.

(P3) Some of its products against the exact solution admit limits in L^1 as $\nu \to 0^+$.

In view of (3), the latter property is a severe one which can be reached only by a convenient design of good manufactured solutions adapted to $\mathbf{E}^+$.

It will be convenient to consider the first order version of (2) obtained by introducing the field $\mathbf{B}^\nu = \nabla \times \mathbf{E}^\nu$, which will be called magnetic field (for the sake of simplicity, indeed the physical magnetic field should rather be defined as $(i\omega)^{-1}\nabla \times \mathbf{E}^\nu$). The resulting first order system reads then

$$\begin{cases} \mathbf{B}^\nu - \nabla \times \mathbf{E}^\nu = 0 \\ \nabla \times \mathbf{B}^\nu - \left(\underline{\underline{\varepsilon}} + i\nu \underline{\underline{I}}\right)\mathbf{E}^\nu = 0 \end{cases} \quad \text{on } \Omega. \tag{4}$$

As for the manufactured solutions we will consider two constructions, which will lead to two different formulations for the limit problem. One construction relies on dissipative inequalities which are natural from the viewpoint of energy considerations. The other construction, which relies on weak manipulations, needs less notations and for that reason it will serve as a guideline in the text.

Thus, the first class of manufactured solutions $(\mathbf{F}^\nu, \mathbf{C}^\nu)$ will be constructed as solutions to some non-homogeneous version of system (4), namely

$$\begin{cases} \mathbf{C}^\nu - \nabla \times \mathbf{F}^\nu = \mathbf{q}^\nu \\ \nabla \times \mathbf{C}^\nu - \left(\underline{\underline{\varepsilon}} + i\nu \underline{\underline{I}}\right)\mathbf{F}^\nu = \mathbf{g}^\nu \end{cases} \quad \text{on } \Omega \tag{5}$$

which specifies Property (P2) in a first way. Here the right-hand sides are non-zero to allow for explicit solutions in general configurations, and the limit formulation will follow by considering the energy dissipation of the difference field $(\mathbf{E}^\nu - \mathbf{F}^\nu, \mathbf{B}^\nu - \mathbf{C}^\nu)$. In both cases indeed, we will require the following property specifying (P1) above.

Condition 1. *The functions $\mathbf{F}^\nu$ and $\mathbf{C}^\nu$ are known analytically, as well as their pointwise limits as $\nu \to 0^+$, denoted $\mathbf{F}^+$, $\mathbf{C}^+$. The same holds for the right hand sides $\mathbf{q}^\nu$, $\mathbf{g}^\nu$, with limits denoted $\mathbf{q}^+$, $\mathbf{g}^+$.*

Energy exchanges can then be measured by introducing the Poynting vector

$$\Pi^\nu := \operatorname{Im}(\mathbf{E}^\nu \times \overline{\mathbf{B}^\nu}) = (2i)^{-1}(\mathbf{E}^\nu \times \overline{\mathbf{B}^\nu} - \overline{\mathbf{E}^\nu} \times \mathbf{B}^\nu)$$

(usually it is defined as the real part of $\mathbf{E}^\nu \times \overline{\mathbf{B}}^\nu$, but here the physical magnetic field is $(i\omega)^{-1}\mathbf{B}^\nu$). Indeed, using Equation (125) we compute

$$\nabla \cdot \left(\mathbf{E}^\nu \times \overline{\mathbf{B}^\nu}\right) = \overline{\mathbf{B}^\nu} \cdot \nabla \times \mathbf{E}^\nu - \mathbf{E}^\nu \cdot \nabla \times \overline{\mathbf{B}^\nu} = |\mathbf{B}^\nu|^2 - \mathbf{E}^\nu \cdot \left(\overline{(\underline{\underline{\varepsilon}} + i\nu \underline{\underline{I}})\mathbf{E}^\nu}\right) = |\mathbf{B}^\nu|^2 - (\underline{\underline{\varepsilon}}\mathbf{E}^\nu) \cdot \overline{\mathbf{E}^\nu} + i\nu\, |\mathbf{E}^\nu|^2$$

where we note that $(\underline{\underline{\varepsilon}}\mathbf{E}^\nu) \cdot \overline{\mathbf{E}^\nu}$ is a real number due to the hermitianity of the dielectric tensor. It follows that

$$\nabla \cdot \Pi^\nu = \operatorname{Im}\left(\nabla \cdot \left(\mathbf{E}^\nu \times \overline{\mathbf{B}^\nu}\right)\right) = \nu\, |\mathbf{E}^\nu|^2$$

that is, the divergence of the Poynting vector represents the dissipation of energy. Next for a manufactured function solution to (5), we consider the Poynting vector of the difference field. We have

$$\begin{aligned} \nabla \cdot \left((\mathbf{E}^\nu - \mathbf{F}^\nu) \times \overline{(\mathbf{B}^\nu - \mathbf{C}^\nu)}\right) &= \overline{(\mathbf{B}^\nu - \mathbf{C}^\nu)} \cdot (\nabla \times (\mathbf{E}^\nu - \mathbf{F}^\nu)) - (\mathbf{E}^\nu - \mathbf{F}^\nu) \cdot \left(\nabla \times \overline{(\mathbf{B}^\nu - \mathbf{C}^\nu)}\right) \\ &= R + \overline{(\mathbf{B}^\nu - \mathbf{C}^\nu)} \cdot \mathbf{q}^\nu + (\mathbf{E}^\nu - \mathbf{F}^\nu) \cdot \overline{\mathbf{g}^\nu} + i\nu\, |\mathbf{E}^\nu - \mathbf{F}^\nu|^2 \end{aligned}$$

where $R = |\mathbf{B}^\nu - \mathbf{C}^\nu|^2 - \left(\underline{\underline{\varepsilon}}(\mathbf{E}^\nu - \mathbf{F}^\nu)\right) \cdot \overline{(\mathbf{E}^\nu - \mathbf{F}^\nu)}$ is again a real number. Thus we obtain

$$\operatorname{Im}\left(\nabla \cdot \left((\mathbf{E}^\nu - \mathbf{F}^\nu) \times \overline{(\mathbf{B}^\nu - \mathbf{C}^\nu)}\right)\right) = \operatorname{Im}\left(\overline{(\mathbf{B}^\nu - \mathbf{C}^\nu)} \cdot \mathbf{q}^\nu + (\mathbf{E}^\nu - \mathbf{F}^\nu) \cdot \overline{\mathbf{g}^\nu}\right) + \nu\, |\mathbf{E}^\nu - \mathbf{F}^\nu|^2. \tag{6}$$

This will allow us to write dissipative inequalities for the limit solutions. In particular, for any non-negative cut-off function $\varphi \in C^1_{0,+}(\Omega)$ we have

$$\int_\Omega \operatorname{Im}\left(\overline{(\mathbf{E}^\nu - \mathbf{F}^\nu)} \times (\mathbf{B}^\nu - \mathbf{C}^\nu)\right) \cdot \nabla\varphi + \int_\Omega \operatorname{Im}\left(\overline{(\mathbf{E}^\nu - \mathbf{F}^\nu)} \cdot \mathbf{g}^\nu - \overline{(\mathbf{B}^\nu - \mathbf{C}^\nu)} \cdot \mathbf{q}^\nu\right)\varphi = \int_\Omega \nu\, |\mathbf{E}^\nu - \mathbf{F}^\nu|^2 \varphi \geq 0.$$

To derive a relation for the limit solution $\mathbf{E}^+, \mathbf{B}^+$ we then state an additional condition specifying Property (P3).

Condition 2. *The limits $\nu \to 0^+$ of the functions in (4) and (5) satisfy in L^1 the identities*

$$\begin{cases} Im\left(\overline{(\mathbf{E}^\nu - \mathbf{F}^\nu)} \times (\mathbf{B}^\nu - \mathbf{C}^\nu)\right) \to Im\left(\overline{(\mathbf{E}^+ - \mathbf{F}^+)} \times (\mathbf{B}^+ - \mathbf{C}^+)\right), \\ Im\left(\overline{(\mathbf{E}^\nu - \mathbf{F}^\nu)} \cdot \mathbf{g}^\nu\right) \to Im\left(\overline{(\mathbf{E}^+ - \mathbf{F}^+)} \cdot \mathbf{g}^+\right), \\ Im\left(\overline{(\mathbf{B}^\nu - \mathbf{C}^\nu)} \cdot \mathbf{q}^\nu\right) \to Im\left(\overline{(\mathbf{B}^+ - \mathbf{C}^+)} \cdot \mathbf{q}^+\right). \end{cases} \tag{7}$$

As a direct consequence, the limit solution $(\mathbf{E}^+, \mathbf{B}^+)$ satisfies the following dissipative relation.

Proposition 1. *If* $(\mathbf{F}^\nu, \mathbf{C}^\nu)$ *is a solution to the non homogeneous system* (5) *such that Conditions 1 and 2 hold, then the inequality*

$$\int_\Omega Im\left(\overline{(\mathbf{E}^+ - \mathbf{F}^+)} \times (\mathbf{B}^+ - \mathbf{C}^+)\right) \cdot \nabla\varphi + \int_\Omega Im\left(\overline{(\mathbf{E}^+ - \mathbf{F}^+)} \cdot \mathbf{g}^+ - \overline{(\mathbf{B}^+ - \mathbf{C}^+)} \cdot \mathbf{q}^+\right)\varphi \geq 0 \tag{8}$$

holds with the limit manufactured solution $(\mathbf{F}^+, \mathbf{C}^+)$*, and all non-negative cut-off function* $\varphi \in C^1_{0,+}(\Omega)$*.*

This inequality is the basis of the constructive method presented in section 4.

A second formulation for the limit problem can be obtained with manufactured solutions satisfying a symmetrized system. Using the same notations for simplicity, we then consider that $(\mathbf{F}^\nu, \mathbf{C}^\nu)$ are solutions to

$$\begin{cases} \mathbf{C}^\nu - \nabla \times \mathbf{F}^\nu = \mathbf{q}^\nu \\ \nabla \times \mathbf{C}^\nu - \left(\underline{\underline{\varepsilon}} + i\nu\underline{\underline{I}}\right)^t \mathbf{F}^\nu = \mathbf{g}^\nu \end{cases} \quad \text{on } \Omega \tag{9}$$

where again the non-zero right-hand sides allow us to construct explicit solutions. Using again (125), we compute

$$\begin{aligned} \nabla \cdot (\mathbf{E}^\nu \times \mathbf{C}^\nu - \mathbf{F}^\nu \times \mathbf{B}^\nu) &= \mathbf{C}^\nu \cdot \nabla \times \mathbf{E}^\nu - \mathbf{E}^\nu \cdot \nabla \times \mathbf{C}^\nu - \mathbf{B}^\nu \cdot \nabla \times \mathbf{F}^\nu + \mathbf{F}^\nu \cdot \nabla \times \mathbf{B}^\nu \\ &= \mathbf{C}^\nu \cdot \mathbf{B}^\nu - \mathbf{E}^\nu \cdot \left(\mathbf{g}^\nu + (\underline{\underline{\varepsilon}} + i\nu\underline{\underline{I}})^t \mathbf{F}^\nu\right) - \mathbf{B}^\nu \cdot (\mathbf{C}^\nu - \mathbf{q}^\nu) + \mathbf{F}^\nu \cdot \left((\underline{\underline{\varepsilon}} + i\nu\underline{\underline{I}})\mathbf{E}^\nu\right) \\ &= \mathbf{B}^\nu \cdot \mathbf{q}^\nu - \mathbf{E}^\nu \cdot \mathbf{g}^\nu. \end{aligned} \tag{10}$$

This leads to considering another condition specifying Property (P3) for these manufactured solutions.

Condition 3. *The limits $\nu \to 0^+$ of the solutions to (4) and (9) satisfy in L^1 the identities*

$$\begin{cases} \mathbf{E}^\nu \times \mathbf{C}^\nu \to \mathbf{E}^+ \times \mathbf{C}^+, \\ \mathbf{F}^\nu \times \mathbf{B}^\nu \to \mathbf{F}^+ \times \mathbf{B}^+, \\ \mathbf{B}^\nu \cdot \mathbf{q}^\nu \to \mathbf{B}^+ \cdot \mathbf{q}^+, \\ \mathbf{E}^\nu \cdot \mathbf{g}^\nu \to \mathbf{E}^+ \cdot \mathbf{g}^+. \end{cases} \tag{11}$$

As a straightforward consequence, the limit solution $(\mathbf{E}^+, \mathbf{B}^+)$ now satisfies the following weak (integral) relation.

Proposition 2. *If* $(\mathbf{F}^\nu, \mathbf{C}^\nu)$ *is a solution to the symmetrized system* (9) *such that Conditions 1 and 3 hold, then the integral relation*

$$\int_\Omega \left(\mathbf{E}^+ \times \mathbf{C}^+ - \mathbf{F}^+ \times \mathbf{B}^+\right) \cdot \nabla\varphi = \int_\Omega \left(\mathbf{E}^+ \cdot \mathbf{g}^+ - \mathbf{B}^+ \cdot \mathbf{q}^+\right)\varphi \tag{12}$$

holds with the limit manufactured solution $(\mathbf{F}^+, \mathbf{C}^+)$*, and all cut-off functions* $\varphi \in C^1_0(\Omega)$*.*

At the end of the analysis, most in this work depends on the possibility of designing good manufactured solutions, if they exist.

Main results

We show that manufactured functions make sense for the resonant Maxwell equations. Specifically, we construct manufactured solution in dimension one that satisfy the conditions listed above. This allows us to derive three original weak formulations for the limit Maxwell problem, see Problems 1, 2 and 3, that are proven to be well-posed, as stated in the corresponding Theorems 1, 2 and 3. That is, they admit the same unique solution which coincides with the one obtained by vanishing dissipation, thus also proving its uniqueness. The proof is a combination of elementary a priori estimates easily obtained in dimension one and of analytic properties of the weak formulations involving our manufactured solutions. We make use of the Hardy inequality to prove the correctness of the functional setting. An important asset of the dissipative formulation (Problem 3) is a direct measure of the limit resonant heating defined in [13, 12], see Remark 12.

In dimension two, the examination of manufactured solutions shows new technical difficulties, and for this reason we concentrate mainly on constructive issues. We obtain through manufactured solutions a characterization of singular solutions which is completely new to our knowledge. We show how to derive some weak limits which can be used to construct weak formulations. We have to assume certain bounds on $\mathbf{E}^\nu$, $\mathbf{B}^\nu$. Even if these bounds are very natural since there are already satisfied by the manufactured solutions, these regularity assumptions are an important restriction for the moment. We finally show that the weak identities obtained with manufactured solutions can be completely reinterpreted as strong bounds in standard norms for new variables obtained by suitable linear combinations of the electric and magnetic fields.

Organization of the work
The plan is as follows. Section 2 is devoted to the Budden problem in dimension $d = 1$, for which a particular explicit solution is available and for which some manufactured solutions are given. Next we develop in Section 3 weak formulations in dimension $d = 1$: the manufactured solutions correspond to (9) and the weak formulations need less notations; this is the reason we start with these formulations. In a second stage, Section 4 presents dissipative inequalities as an alternative: they need more notations than the previous section but the principles are perhaps more natural from energy considerations. Other manufactured solutions are briefly discussed in Section 5. The construction of manufactured in dimension 2 is developed in Section 6. Finally we show how to rewrite the weak formulations obtained with manufactured solutions as strong bounds for suitable linear combinations of the components of the electromagnetic field.

2 The Budden problem in 1D

The Budden problem is the reduction of (1) or (2) to planar (slab) geometry and for the Transverse Electric (TE) mode, called X-mode (for eXtraordinary) in the plasma physics community [7, 22]. In the TE mode the electric field has the form $\mathbf{E} = (E_1, E_2, 0)$ and the (ad-hoc) magnetic field is $\mathbf{B} := \nabla \times \mathbf{E} = (0, 0, B_3)$ with $B_3 = \partial_x E_2 - \partial_y E_1$. Here we concentrate mostly on solutions that only depend of the first variable x, as they offer a simpler framework that is representative of most of the technicalities and difficulties. The resulting 1D problem will be used as a testbed for the different tools developed in this work. It has also the asset that the singularity of the solution shows up explicitly. For the non regularized problem, the dielectric tensor reads

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^* = \begin{pmatrix} \alpha(x) & i\delta(x) & 0 \\ -i\delta(x) & \alpha(x) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x \in \Omega = (-1, 1). \tag{13}$$

In the 1D TE setting, Maxwell's equations (1) restated as a first-order problem of the form (4) then reduce to

$$\begin{cases} B_3 - (E_2)' = 0, \\ -\alpha E_1 - i\delta E_2 = 0, \\ -(B_3)' + i\delta E_1 - \alpha E_2 = 0, \end{cases} \tag{14}$$

and one readily sees from the second equation that the component E_1 will be singular in the points where α vanishes, unless δ or E_2 also vanish there. For the simplicity of the presentation, we assume the following [13].

Assumption 1 (Regularity of coefficients)**.** *The plasma-dependent parameters α and δ satisfy the following properties: $\alpha \in W^{2,\infty}(\Omega)$, $x = 0$ is the only root of α and $r := \alpha'(0) \neq 0$. Moreover, $\delta \in W^{1,\infty}(\Omega)$ and $\delta > 0$ on $\Omega = (-1, 1)$.*

2.1 An analytical solution

As proposed in [13], a simple explicit solution to System (14) is available for well chosen coefficients α and δ. Eliminating E_1 and B_3, one obtains

$$-E_2'' + \left(\frac{\delta^2}{\alpha} - \alpha\right) E_2 = 0.$$

We choose $\alpha = -x$ which opens the possibility of a resonant solution at $x = 0$, and to comply with Assumption 1 we take $\frac{\delta^2}{\alpha} - \alpha = \frac{1}{4} - \frac{1}{x}$. The positive root is $\delta(x) = \sqrt{1 - \frac{1}{4}x + x^2} > 0$ for all $x \in \mathbb{R}$. Therefore one has the decomposition $E_2 = au + bv$, $a, b \in \mathbb{R}$, where u and v are solutions of the Whittaker equation

$$u'' + \left(\frac{1}{x} - \frac{1}{4}\right) u = 0. \tag{15}$$

Proposition 3. *The solutions of the Whittaker equation (15) are spanned by*

$$u(x) = xe^{-\frac{x}{2}} \qquad \text{and} \qquad v(x) = -e^{\frac{x}{2}} + \left(\log|x| + \int_1^x \frac{e^y - 1}{y} dy\right) xe^{-\frac{x}{2}}.$$

Proof. Indeed successive derivation yields

$$u'(x) = \left(1 - \frac{x}{2}\right) e^{-\frac{x}{2}} \text{ and } u''(x) = \left(-1 + \frac{x}{4}\right) e^{-\frac{x}{2}}$$

which shows that u is solution of the Whittaker equation. The other solution is obtained by usual variation of the constant. One gets $v = \left(\int \frac{dx}{u^2}\right) u$, that is with a convenient determination of the bounds

$$v(x) = \left(\int_1^x \frac{e^y dy}{y^2} - e\right) xe^{-\frac{x}{2}} = \left(-\frac{e^x}{x} + \int_1^x \frac{e^y dy}{y}\right) xe^{-\frac{x}{2}} = \left(-\frac{e^x}{x} + \log|x| + \int_1^x \frac{(e^y - 1)dy}{y}\right) xe^{-\frac{x}{2}}.$$

It ends the proof. $\square$

The representation formula $E_2 = au + bv$ shows that E_2 is locally bounded since $u, v \in L^\infty_{\text{loc}}(\mathbb{R})$, that is

$$E_2 \in L^\infty_{\text{loc}}(\mathbb{R}).$$

It is not the case for E_1 since

$$E_1 = -\frac{i\delta}{\alpha} E_2 = i\left(1 - \frac{1}{4}x + x^2\right)^{\frac{1}{2}} \left[ae^{-\frac{x}{2}} + b\left(-\frac{1}{x}e^{\frac{x}{2}} + \left(\log|x| + \int_0^x \frac{e^y - 1}{y} dy\right) e^{-\frac{x}{2}}\right)\right]. \tag{16}$$

This formula shows that E_1 has, as a solution of (14), a generic singularity at $x = 0$ since $E_1 + \frac{ib}{x} \in L^\infty_{\text{loc}}(\mathbb{R})$. It is striking to note that in general E_1 is not integrable

$$E_1 \notin L^1_{\text{loc}}(\mathbb{R}) \quad \text{ for } \ b \neq 0.$$

Finally the magnetic field reads $B_3 = E_2' = au' + bv'$. Since $u' \in L^\infty_{\text{loc}}(\mathbb{R})$ and clearly $v'(x) - \log|x| \in L^\infty_{\text{loc}}(\mathbb{R})$, one gets that the magnetic field is not locally bounded in general, that is $B_3 - b\log|x| \in L^\infty_{\text{loc}}(\mathbb{R})$. One has nevertheless that

$$B_3 \in L^p_{\text{loc}}(\mathbb{R}) \qquad 1 \leq p < \infty.$$

2.2 A priori bounds on regularized solutions

We now consider the system (14) regularized as

$$\begin{cases} B_3^\nu - (E_2^\nu)' = 0, \\ -(\alpha + i\nu)E_1^\nu - i\delta E_2^\nu = 0, \\ -(B_3^\nu)' + i\delta E_1^\nu - (\alpha + i\nu)E_2^\nu = 0, \end{cases} \tag{17}$$

with non zero collisionality $\nu \neq 0$, and plasma-dependent parameters α, δ satisfying Assumption 1. As above we consider the academic domain $\Omega = (-1, 1)$ and supplement (17) with boundary conditions derived from (2)

$$B_3^\nu(-1) + i\lambda E_2^\nu(-1) = f_- \quad \text{ and } \quad B_3^\nu(1) - i\lambda E_2^\nu(1) = f_+, \tag{18}$$

where $f_\pm \in \mathbb{C}$.

Proposition 4. *Assume $\nu > 0$ and $\lambda > 0$. There exists a unique solution $(E_1^\nu, E_2^\nu, B_3^\nu) \in L^2(\Omega)^3$ of (17)-(18). Moreover there exists a constant $C > 0$ which does not depend on ν, such that*

$$\|E_2^\nu\|_{L^2(\Omega)} + \|B_3^\nu\|_{L^2(\Omega)} \leq C. \tag{19}$$

Proof. As explained in the appendix the well posedness is a general property of such coercive systems with $\nu > 0$. So we concentrate on the a priori bounds. Multiply the second equation by $\overline{E_1^\nu}$, the third one by $\overline{E_2^\nu}$ and integrate in Ω

$$-\int_\Omega (B_3^\nu)'\overline{E_2^\nu}dx + \int_\Omega \left(-\alpha|E_1^\nu|^2 - \alpha|E_2^\nu|^2 + i\delta E_1^\nu\overline{E_2^\nu} - i\delta E_2^\nu\overline{E_1^\nu}\right)dx - i\nu\int_\Omega \left(|E_1^\nu|^2 + |E_2^\nu|^2\right)dx = 0.$$

Multiply the first equation of (17) by $\overline{B_3^\nu}$, integrate in Ω and conjugate: $\int_\Omega |B_3^\nu|^2dx - \int_\Omega B_3^\nu\overline{E_2^\nu}'dx = 0$. Add the two identities and take the imaginary part: $-\mathrm{Im}\left(\int_\Omega \left(B_3^\nu\overline{E_2^\nu}\right)' dx\right) - \nu\int_\Omega \left(|E_1^\nu|^2 + |E_2^\nu|^2\right)dx = 0$. Integrate and use the two boundary conditions (18)

$$\lambda\left(|E_2^\nu(-1)|^2 + |E_2^\nu(1)|^2\right) + \nu\int_\Omega \left(|E_1^\nu|^2 + |E_2^\nu|^2\right)dx = -\mathrm{Im}\left(f_-\overline{E_2^\nu}(-1) + f_+\overline{E_2^\nu}(1)\right).$$

Together with (18) and a Cauchy-Schwarz inequality, this identity shows a uniform bound for the boundary datas

$$|E_2^\nu(-1)| + |B_3^\nu(-1)| + |E_2^\nu(1)| + |B_3^\nu(1)| \le C_1$$

where $C_1 > 0$ is a function of f_-, f_+, λ but is independent of ν. This bound is now propagated from the boundary to the center to obtain the last part of the claim as follows. Eliminating E_1^ν in (17) and using Assumption 1, we observe that

$$|(E_2^\nu)'(x)| = |B_3^\nu(x)| \qquad \text{and} \qquad |(B_3^\nu)'(x)| = \left|\left(\frac{\delta^2}{(\alpha+i\nu)} - (\alpha+i\nu)\right)E_2^\nu(x)\right| \le \frac{C_2}{|x|}|E_2^\nu(x)| \tag{20}$$

for some constant $C_2 > 0$ depending only on α and δ. We then consider the function g defined on $[-1,0)$ by

$$g(-1) = |E_2^\nu(-1)|, \qquad g'(-1) = |B_3^\nu(-1)| \qquad \text{and} \qquad g''(x) = \frac{C_2}{|x|}|E_2^\nu(x)|, \qquad -1 \le x < 0 \tag{21}$$

(a symmetric argument can be used on $(0,1]$). Clearly, g is increasing and nonnegative. Moreover straightforward computations, see below, show that g dominates $|E_2^\nu|$: using (20) we have $g'' \ge |(B_3^\nu)'| \ge |B_3^\nu|'$ hence (again with (20))

$$g'(x) = g'(-1) + \int_{-1}^x g''(y)\,dy \ge |B_3^\nu(-1)| + \int_{-1}^x |B_3^\nu|'(y)\,dy = |B_3^\nu|(x) \ge |(E_2^\nu)'|(x) \ge |E_2^\nu|'(x).$$

Repeating the argument gives $g(x) = g(-1) + \int_{-1}^x g'(y)\,dy \ge |E_2^\nu(-1)| + \int_{-1}^x |E_2^\nu|'(y)\,dy \ge |E_2^\nu|(x)$ which shows that g dominates $|E_2^\nu|$ indeed. In particular, g satisfies from (21)

$$g''(x) \le \frac{C_2 g(x)}{|x|}, \qquad -1 \le x < 0$$

which is easily integrated by observing that $(\log g)'' \le \frac{g''}{g}$ on any interval where $g > 0$. Specifically, we find that g, and hence E_2^ν, is bounded on $(-1,0)$ uniformly in ν. Using again (20) this shows that $|B_3^\nu|'(x) \le \frac{C_3}{|x|}$, hence

$$|E_2^\nu(x)| + |B_3^\nu(x)| \le C_4(1 + |\log|x||) \tag{22}$$

with constants independent of ν. This ends the proof since the logarithm is in every L^p_{loc} for $1 \le p < \infty$. □

Other similar bounds are easy to obtain for E_1^ν, but we do not pursue in this direction. Instead, we emphasize the following important property of the regularized solutions, as it helps to understand the main goal of this work.

Corollary 1. *Up to the extraction of a subsequence, the functions E_2^ν, B_3^ν admit a strong limit in $L^2(\Omega)$ as $\nu \to 0^+$, that we denote by E_2^+, B_3^+. This limit is a solution of the system*

$$\left\{\begin{array}{ll} \displaystyle\int_\Omega (B_3^+\varphi_1 + E_2^+\varphi_1')dx = 0, & \forall \varphi_1 \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \left(B_3^+\varphi_2' + \left(\frac{\delta^2}{\alpha} - \alpha\right)E_2^+\varphi_2\right)dx = 0, & \forall\varphi_2 \in H_{0,0}^1(\Omega), \end{array}\right. \tag{23}$$

where the latter space is defined as

$$H_{0,0}^1(\Omega) := \left\{u \in H_0^1(\Omega),\ u(0) = 0\right\}. \tag{24}$$

Remark 1. *The second equation in (23) makes sense as a consequence of the Hardy inequality for functions which vanish at the origin, see for example the proof of proposition 8.*
Although we use a specific notation E_2^+, B_3^+ for the above limit, we should keep in mind that it is only defined up to a subsequence and may not be unique. A related question is that of the well-posedness of System (23)*, and there the answer is known to be negative. Indeed, it is known that the regularized solutions may have different limits depending on the side from which the limit $\nu \to 0$ is taken [13]. Since this information has been lost in the limit equations* (23)*, one sees that the resulting system must be ill-posed. One important contribution of this work will be to complement System* (23) *in such a way that it becomes well-posed, and that is amenable to numerical approximations. One already sees that the additional constraint will have to depend on the side from which the limit $\nu \to 0$ is taken. Note that such results can also be expressed in terms of Fredholm indices [4].*

Proof. The bound (19) show that there exists a weak limit E_2^+, B_3^+ in $L^2(\Omega)$, up to a subsequence. This weak limit can be proved to be a strong limit using two facts: a) with (17) and the uniform L^2 bound (19), one has for $0 < \varepsilon < 1$

$$\|E_2^\nu\|_{H^1(\Omega)\backslash(-\epsilon,\epsilon)} + \|B_3^\nu\|_{H^1(\Omega)\backslash(-\epsilon,\epsilon)} \le C_\varepsilon$$

where C_ε is uniform with respect to ν; b) the point wise bound (22) yields that

$$\|E_2^\nu\|_{L^2(-\epsilon,\epsilon)} + \|B_3^\nu\|_{L^2(-\epsilon,\epsilon)} \le 2C_4(1 + |\log \varepsilon|)\varepsilon^{\frac{1}{2}}.$$

A combination of the compactness of $H^1(\Omega)\backslash(-\epsilon,\epsilon)$ in $L^2(\Omega)\backslash(-\epsilon,\epsilon)$ (using a)) and a control of the remainder (using b)) yields the first result.

To show that this limit satisfies (23) we observe that the solutions to (17) clearly satisfy, for all $\nu \neq 0$,

$$\left\{ \begin{aligned} &\int_\Omega (B_3^\nu \varphi_1 + E_2^\nu \varphi_1')dx = 0, \qquad \forall \varphi_1 \in H_0^1(\Omega), \\ &\int_\Omega \left(B_3^\nu \varphi_2' + \left(\frac{\delta^2}{\alpha + i\nu} - \alpha \right) E_2^\nu \varphi_2 \right) dx = 0, \qquad \forall \varphi_2 \in H_{0,0}^1(\Omega). \end{aligned} \right. \tag{25}$$

To pass to the limit we then observe that $\frac{\varphi_2}{x} \in L^2(\Omega)$, according to Hardy's inequality and the fact that $\varphi_2(0) = 0$. The properties of α stated in Assumption 1 give then $\left|\frac{\varphi_2}{\alpha+i\nu}\right| \le \left|\frac{\varphi_2}{\alpha}\right| \le C\left|\frac{\varphi_2}{x}\right| \in L^2(\Omega)$, hence the pointwise limit $\frac{\varphi_2}{\alpha+i\nu} \to \frac{\varphi_2}{\alpha}$ holds in $L^2(\Omega)$. This allows to pass to the limit in (25) and completes the proof. ☐

The second component of the electric field is actually more regular.

Proposition 5. *There exists a constant $C > 0$ independent of ν such that*

$$\|E_2^\nu\|_{H^1(\Omega)} \le C. \tag{26}$$

Proof. Indeed the first equation of the system (17) immediately gives a control of the gradient $\|(E_2^\nu)'\|_{L^2(\Omega)} \le C$ thanks to (19). ☐

So the main point is to to find some means to characterize the limit and to show its uniqueness. To this end, we introduce manufactured solutions which will be used to test the limit.

2.3 Manufactured solutions for the 1D case

As explained in the introduction, the idea of a manufactured solution is threefold: it exhibits some parts of the singular behavior of the solutions, and for this one can compare with the structure of the explicit solutions in the previous section; it is given by an analytic expression; and it is a non homogeneous solution of either the original system (5) or the symmetrized one (9), with a right hand side bounded in convenient functional spaces. In this section we detail the construction of such manufactured solutions for the 1D case. In the setting of the regularized Budden problem (17), we thus consider the non homogeneous system

$$\left\{ \begin{aligned} C_3^\nu - (F_2^\nu)' &= g_3^\nu \\ -(\alpha + i\nu)F_1^\nu + i\delta F_2^\nu &= g_1^\nu \\ -(C_3^\nu)' - i\delta F_1^\nu - (\alpha + i\nu)F_2^\nu &= g_2^\nu. \end{aligned} \right. \tag{27}$$

Here we have considered the symmetrized system, i.e., (9), since our analysis will mostly address weak constraints of the form (12) which are simpler in their expression. We note however that in 1D, the manufactured solutions designed for the original system (5) only differ by a couple of sign changes, see Section 4.

Because the horizontal component E_1 of the electric field has a singular limit, we first construct manufactured solutions satisfying $g_1^\nu = 0$ so that testing the (exact or manufactured) electric fields against $\mathbf{g}^\nu$ to satisfy the limit conditions 2 and 3 will not suffer from this singular limit. Later we will also consider righ-hand sides g_1^ν and g_1^+ which are not zero but vanish at $x = 0$. Inspired by the existence of two kinds of solutions for the Whittaker equation (15), one regular and one singular, we have constructed two kinds of manufactured solutions. What our analysis reveals is that the limits of these manufactured solutions as $\nu \to 0^+$ satisfy two key properties, see Remark 4 below.

Definition 1 (A regular manufactured solution)**.** *A first manufactured solution is*

$$F_1^\nu = -1, \qquad F_2^\nu = i\frac{\alpha + i\nu}{\delta}, \qquad C_3^\nu = i\frac{\alpha'(0)}{\delta(0)} + \nu\frac{\delta'(0)}{\delta(0)^2} + i\delta(0)x \tag{28}$$

and the right hand sides defined by the symmetrized system (27) *read*

$$g_1^\nu \equiv 0, \qquad g_2^\nu = i(\delta - \delta(0)) + \frac{(\alpha + i\nu)^2}{i\delta}, \qquad q_3^\nu = i\left(\frac{\alpha'(0)}{\delta(0)} - \frac{\alpha'}{\delta}\right) + \nu\left(\frac{\delta'(0)}{\delta(0)^2} - \frac{\delta'}{\delta^2}\right) + i\delta(0)x + i\frac{\alpha\delta'}{\delta^2}.$$

They satisfy $q_3^\nu(0) = g_2^\nu(0) = 0$, moreover $q_3^\nu \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and $g_2^\nu \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ hold with bounds independent of ν.

Proof. Evident from the definition. □

Definition 2 (A singular manufactured solution)**.** *A second manufactured solution is*

$$F_1^\nu = -\frac{1}{\alpha + i\nu}, \quad F_2^\nu = \frac{i}{\delta}, \quad C_3^\nu = i\frac{\delta(0)}{r}\left(\frac{1}{2}\log\left(r^2x^2 + \nu^2\right) - i\arctan\left(\frac{rx}{\nu}\right)\right). \tag{29}$$

The right hand sides defined by the symmetrized system (27) *read $g_1^\nu \equiv 0$ and*

$$g_2^\nu = i\frac{\delta - \delta(0)}{rx + i\nu} + i\delta\left(\frac{1}{\alpha + i\nu} - \frac{1}{rx + i\nu}\right) - i\frac{\alpha + i\nu}{\delta}, \quad q_3^\nu = i\frac{\delta(0)}{r}\left(\frac{1}{2}\log\left(r^2x^2 + \nu^2\right) - i\arctan\left(\frac{rx}{\nu}\right)\right) + i\frac{\delta'}{\delta^2}.$$

In particular, it satisfies $q_3^\nu \in L^2_{\mathrm{loc}}(\mathbb{R})$ and $g_2^\nu \in L^2_{\mathrm{loc}}(\mathbb{R})$ with bounds uniform with respect to ν.

Proof. The only non trivial part concerns $g_2^\nu = -\big((C_3^\nu)' + i\delta F_1^\nu\big) - (\alpha + i\nu)F_2^\nu$ which comes from the third line of (27). One has by definition that

$$\begin{aligned}
(C_3^\nu)' + i\delta F_1^\nu &= \delta(0)\left(i\frac{rx}{r^2x^2 + \nu^2} + \frac{1}{\nu}\frac{1}{(rx/\nu)^2 + 1}\right) - i\delta\frac{1}{\alpha + i\nu} = i\delta(0)\frac{rx - i\nu}{r^2x^2 + \nu^2} - i\delta\frac{1}{\alpha + i\nu} \\
&= i\delta(0)\frac{1}{rx + i\nu} - i\delta\frac{1}{\alpha + i\nu} = i\frac{\delta(0) - \delta}{rx + i\nu} + i\delta\left(\frac{1}{rx + i\nu} - \frac{1}{\alpha + i\nu}\right).
\end{aligned}$$

Therefore

$$|(C_3^\nu)' + i\delta F_1^\nu| \le \frac{|\delta - \delta(0)|}{|rx|} + |\delta|\frac{|rx - \alpha|}{|rx\alpha|}.$$

Using Assumption 1 one obtains that $(C_3^\nu)' + i\delta F_1^\nu \in L^\infty_{\mathrm{loc}}(\Omega)$, and the estimate for g_2^ν follows. □

Remark 2. *The form of C_3^ν in the second manufactured solution* (29) *is motivated by the fact that $(C_3^\nu)' + i\delta F_1^\nu$ should contain no singularity, and as shown above this is achieved by letting C_3^ν be an antiderivative of $\frac{i\delta(0)}{rx+i\nu}$. An alternative to* (29) *is to use the principal value of the logarithm in the complex plane, defined as*

$$\log(rx + i\nu) = \frac{1}{2}\log(r^2x^2 + \nu^2) + i\arg(rx + i\nu) \tag{30}$$

with arg *the principal argument of a complex number, taken in $(-\pi, \pi]$. Indeed, one has $\log(rx + i\nu)' = \frac{r}{rx+i\nu}$ so that setting $C_3^\nu = \frac{i\delta(0)}{r}\log(rx + i\nu)$ would lead to a valid construction. When generalizing our constructions to the 2D case we will find that using the logarithm is actually more natural, as it is involved in the definition of the complex power*

$$(rx + i\nu)^\lambda = e^{\lambda\log(rx + i\nu)}, \qquad \lambda \in \mathbb{C}.$$

Because we restrict ourselves to $\nu > 0$ the different complex logarithms visible in (29) and (30) only differ by a constant. Indeed, the angle $\arg(rx + i\nu)$ is always in $(0, \pi)$ which yields $\arctan\left(\frac{rx}{\nu}\right) = \frac{\pi}{2} - \arg(rx + i\nu)$. In particular, the choice in (29) *corresponds to setting $C_3^\nu = \frac{i\delta(0)}{r}\log(rx + i\nu) + \frac{\pi\delta(0)}{2r}$.*

2.4 Limits of manufactured solutions

Manufactured solutions have natural limits in $L^2(\Omega)$ as $\nu \to 0^+$, except of course the first component F_1^ν of the second manufactured solutions which is singular.

Notation 1 (Limits of the first manufactured solution)**.** *The pointwise limits of the regular solution* (28) *are*

$$F_1^+ = -1, \qquad F_2^+ = i\frac{\alpha}{\delta}, \qquad C_3^+ = i\frac{\alpha'(0)}{\delta(0)} + i\delta(0)x \tag{31}$$

and those of the right hand sides are

$$g_1^+ = 0, \qquad g_2^+ = i(\delta - \delta(0)) + \frac{\alpha^2}{i\delta}, \qquad q_3^+ = i\left(\frac{\alpha'(0)}{\delta(0)} - \frac{\alpha'}{\delta}\right) + i\delta(0)x + i\frac{\alpha\delta'}{\delta^2}.$$

They satisfy $g_2^+(0) = q_3^+(0) = 0$. Moreover $g_2^+ \in W^{1,\infty}_{\rm loc}(\mathbb{R})$ and $q_3^+ \in W^{1,\infty}_{\rm loc}(\mathbb{R})$.

Notation 2 (L^2-limits of the second manufactured solution)**.** *As $\nu \to 0^+$, the components (F_2^ν, C_3^ν) of the second manufactured solution* (29) *admit limits in L^2 which are*

$$F_2^+ = \frac{i}{\delta}, \qquad C_3^+ = i\frac{\delta(0)}{r}\left(\log(|rx|) - i\frac{\pi}{2}sign(rx)\right). \tag{32}$$

The right hand sides also admit limits in L^2 as $\nu \to 0^+$, denoted

$$g_1^+ = 0, \qquad g_2^+ = i\frac{\delta - \delta(0)}{rx} + i\delta\left(\frac{1}{\alpha} - \frac{1}{rx}\right) - i\frac{\alpha}{\delta}, \qquad q_3^+ = i\frac{\delta(0)}{r}\left(\log(|rx|) - i\frac{\pi}{2}sign(rx)\right) + i\frac{\delta'}{\delta^2}.$$

Remark 3. *The "+" exponent used in the above notations is a reminder of the fact that these limits are obtained with positive values of ν, as some of them would be different if taken on the negative side. Obviously this is not the case for the first manufactured solution* (31) *but we observe that for the second one* (29) *we have*

$$C_3^- := \lim_{\nu \to 0^-} C_3^\nu = \frac{i\delta(0)}{r}\left(\log(|rx|) + i\frac{\pi}{2}sign(rx)\right) \neq C_3^+.$$

In this article we restrict ourselves to positive values of ν.

It is important to notice that in the second manufactured solution, $C_3^+ \notin L^\infty(\Omega)$ is the sum of a logarithmic unbounded part plus a discontinuous part. We also note that only the discontinuous part depends on the side from which the limit is taken.
The limit of F_1^ν is very singular since it is the sum of a Dirac mass plus a principal value [13]. We do not desire to use it since this quantity is extremely singular and quite difficult to control in numerical methods. Moreover the weak equations that will be designed and studied below do not involve the limit F_1^+.

Remark 4. *Clearly, one could construct a wide range of manufactured solutions. As our analysis will show, a key property of the ones proposed above is that*

i) for the first (regular) manufactured solution (28) *the limit field C_3^+ is continuous and non-zero at $x = 0$, i.e. $C_3^+(0^-) = C_3^+(0^+) \neq 0$,*

ii) for the second (singular) manufactured solution(29)*, it is the limit field F_2^+ that is non-zero at $x = 0$, $F_2^+(0) \neq 0$.*

A useful technical result is as follows.

Proposition 6. *Consider an additional real test function $\phi \in H^1(\Omega)$ with $\phi \leq 0$. Then the limit $\nu = 0^+$ of the second manufactured solution (32) satisfies*

$$Re\left(-\int_\Omega C_3^+\phi' dx + \int_\Omega g_2^+\phi dx + \left[C_3^+\phi\right]_{-1}^{1}\right) \leq 0. \tag{33}$$

Proof. For $\nu > 0$ compute

$$
\begin{aligned}
-\int_\Omega C_3^\nu \phi' dx + \int_\Omega g_2^\nu \phi dx + \left[C_3^\nu \phi\right]_{-1}^1 &= \int_\Omega (C_3^\nu)' \phi dx + \int_\Omega g_2^\nu \phi dx \\
&= \int_\Omega \Big(i\delta F_1^\nu - (\alpha + i\nu) F_2^\nu \Big) \phi dx \\
&= \int_\Omega \Big(\frac{\delta^2}{\alpha + i\nu} - (\alpha + i\nu) \Big) F_2^\nu \phi dx \\
&= -\int_\Omega \Big(\frac{\alpha(\alpha^2 + \nu^2 - \delta^2) + i\nu(\alpha^2 + \nu^2 + \delta^2)}{\alpha^2 + \nu^2} \Big) i \left(\frac{\phi}{\delta} \right) dx.
\end{aligned}
$$

Taking the real part of the above equality we obtain

$$
\mathrm{Re}\left(-\int_\Omega C_3^\nu \phi' dx + \int_\Omega g_2^\nu \phi dx + \left[C_3^\nu \phi\right]_{-1}^1 \right) = \nu \int_\Omega \frac{\alpha^2 + \nu^2 + \delta^2}{\alpha^2 + \nu^2} \times \frac{\phi}{\delta} dx \le 0. \tag{34}
$$

Since this inequality passes to the limit for $\nu \to 0^+$, the proof is ended. □

Remark 5. *It is easy to pass to the limit in the right hand side (34) using the fact that*

$$
\nu \int_{\mathbb{R}} \frac{dx}{\alpha(x)^2 + \nu^2} = \nu \int_{\mathbb{R}} \frac{dx}{\alpha'(0)^2 x^2 + \nu^2} + \varepsilon(\nu) = \frac{1}{|\alpha'(0)|} \int_{\mathbb{R}} \frac{dx}{x^2 + 1} + \varepsilon(\nu) \underset{\nu \to 0^+}{\longrightarrow} \frac{\pi}{|\alpha'(0)|} \tag{35}
$$

with $\varepsilon \in C^0(\mathbb{R}^+)$ and $\varepsilon(0) = 0$. Using that $\frac{\phi}{\delta} \in H^1(\Omega)$ is a smooth function, one readily obtains

$$
\lim_{\nu \to 0^+} \left(\nu \int_\Omega \frac{\alpha^2 + \nu^2 + \delta^2}{\alpha^2 + \nu^2} \times \frac{\phi}{\delta} dx \right) = \left(\frac{\pi \delta(0)}{|\alpha'(0)|} \right) \phi(0).
$$

One obtains a formula, more precise than (33)

$$
Re\left(-\int_\Omega C_3^+ \phi' dx + \int_\Omega g_2^+ \phi dx + \left[C_3^+ \phi\right]_{-1}^1 \right) = \left(\frac{\pi \delta(0)}{|\alpha'(0)|} \right) \phi(0). \tag{36}
$$

In this formula the test function $\phi \in H^1(\Omega)$ is necessarily real valued.

3 Weak formulations for the 1D case

Our goal is now to obtain some information on the limit solutions to the regularized Budden problem (17) when $\nu \to 0^+$, as discussed e.g. in Remark 1 We start with integral formulations of the form (12), indeed they yield simpler models for the limit problem. Since the manufactured solutions satisfy naturally the two first properties detailed in the introduction, it is necessary to verify wether the additional property (P3), namely Condition 3, is satisfied. This in fact almost done. Indeed, in the one dimensional case studied in this section, and using the fact that $g_1^\nu = 0$ we observe that neither E_1^ν nor F_1^ν appear in the limits (11). Since we have seen in Section 2.4 that all the components of the fields involved in these limits are bounded uniformly in $L^2(\Omega)$, we verify that Condition 3 holds indeed. We can thus apply Proposition 2 on the manufactured solution corresponding to the symmetrized system (27).

Proposition 7. *The weak limits E_2^+, B_3^+, described in corollary 1 satisfy*

$$
\int_\Omega \left(F_2^+ B_3^+ - E_2^+ C_3^+ \right) \varphi' dx = \int_\Omega \left(q_3^+ B_3^+ - g_2^+ E_2^+ \right) \varphi dx, \quad \forall \varphi \in C_0^1(\Omega) \tag{37}
$$

where $(F_2^+, B_3^+, q_3^+, g_2^+)$ denote the limits of either the first manufactured solution (31) *or the second one* (32).

Proof. The proof can be performed in two ways. Either by performing basic combinations of (17) and (27) and passing to the limit $\nu = 0+$. Or by considering Proposition 2: for a test function which depends only of the variable x, the claim (37) is indeed just the opposite of (12) restricted to a one dimensional domain. The proof is ended. □

Formulas like (37) can be used in many ways to complete the limit equations (23) as discussed in Remark 1. We privilege weak formulations in L^2-based spaces since such spaces are ultimately quite convenient for numerical methods. We define the space

$$L^2_x(\Omega) = \left\{u \in \mathcal{D}'(\Omega) \text{ such that } xu \in L^2(\Omega)\right\}$$

and the space

$$H^1_{0,0}(\Omega) = \left\{u \in H^1_0(\Omega),\ u(0) = 0\right\}.$$

A remark is that $\left(E_1^+, E_2^+, B_3^+\right) \in L^2_x(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ is solution in the sense of distribution
- of the system

$$\left\{\begin{array}{r} B_3^+ - (E_2^+)' = 0, \\ -\alpha E_1^+ - i\delta E_2^+ = 0, \\ -(B_3^+)' + i\delta E_1^+ - \alpha E_2^+ = 0, \end{array}\right. \quad \text{separately in } \mathcal{D}'(-1,0) \text{ and } \mathcal{D}'(-1,0); \tag{38}$$

- of the boundary conditions

$$B_3^+(-1) + i\lambda E_2^+(-1) = f_- \text{ and } B_3^+(1) - i\lambda E_2^+(1) = f_+; \tag{39}$$

- and of one single integral relation (37) with the second manufactured solution, for one single test function φ such that $\varphi(0) \neq 0$.
Our next goal is to show that these three conditions define a well posed system. One can eliminate the electric field E_1^+ provided some convenient weak form of the equation is used.

Proposition 8. *Assume that* $\left(E_1^+, E_2^+\right) \in L^2_x(\Omega) \times L^2(\Omega)$ *satisfy*

$$\int_\Omega \alpha E_1^+ \varphi(x)dx + \int_\Omega i\delta E_2^+ \varphi(x)dx = 0, \quad \forall \varphi \in L^2(\Omega). \tag{40}$$

Then

$$\int_\Omega i\delta E_1^+(x)\varphi_2(x)dx = \int_\Omega \frac{\delta^2}{\alpha} E_2^+ \varphi_2(x)dx, \quad \forall \varphi_2 \in H^1_{0,0}(\Omega). \tag{41}$$

Proof. The key estimate is the Hardy inequality written in the form

$$\int_\Omega \frac{u^2(x)}{x^2}dx \leq 4\int_\Omega |u'(x)|^2 dx, \quad \forall u \in H^1(\Omega) \text{ such that } u(0) = 0.$$

It is used to show that $\frac{1}{x}\varphi_2 \in L^2(\Omega)$ for any $\varphi_2 \in H^1_{0,0}(\Omega)$.
The verifications go as follows. Since all integrals in (40) and (41) are integrable due to the hypotheses, it is sufficient to notice that

$$\mathcal{A} = \int_\Omega i\delta E_1^+(x)\varphi_2(x)dx = \int_\Omega \alpha E_1^+(x)\left(\frac{i\delta}{\alpha}\varphi_2(x)\right)dx$$

where $\frac{i\delta}{\alpha}\varphi_2(x) = \frac{ix\delta}{\alpha}\left(\frac{1}{x}\varphi_2\right)(x)$ is the product of a bounded function times $\frac{1}{x}\varphi_2 \in L^2(\Omega)$, so is in $L^2(\Omega)$, and αE_1^+ is also in $L^2(\Omega)$ since $|\alpha(x)| \leq C|x|$ for some constant $C > 0$ by hypothesis. So one has using (40)

$$\mathcal{A} = \int_\Omega (-i\delta E_2^+)(x)\left(\frac{i\delta}{\alpha}\varphi_2(x)\right)dx = \int_\Omega \frac{\delta^2}{\alpha}E_2^+\varphi_2(x)dx, \quad \forall \varphi_2 \in H^1_{0,0}(\Omega)$$

The proof is ended. □

The first formulation that we consider is as follows.

Problem 1. *Find* $(e_2, b_3) \in L^2(\Omega) \times L^2(\Omega)$ *which satisfy three conditions:*

i) *they satisfy the weak formulations*

$$\left\{\begin{array}{ll} \displaystyle\int_\Omega (b_3\varphi_1 + e_2\varphi_1')dx = 0, & \forall \varphi_1 \in H^1_0(\Omega), \\ \displaystyle\int_\Omega \left(b_3\varphi_2' + \left(\frac{\delta^2}{\alpha} - \alpha\right)e_2\varphi_2\right)dx = 0, & \forall \varphi_2 \in H^1_{0,0}(\Omega), \end{array}\right. \tag{42}$$

ii) *they satisfy the boundary conditions*

$$b_3(-1) + i\lambda e_2(-1) = f_- \quad \text{and} \quad b_3(1) - i\lambda e_2(1) = f_+, \tag{43}$$

in the sense of distributions

iii) *they satisfy one single integral relation (37) with the second singular manufactured solution, for one single test function $\varphi \in H_0^1(\Omega)$ such that $\varphi(0) \neq 0$.*

Remark 6. *Notice that the derivative in the sense of distribution of e_2 is $b_3 \in L^2(\Omega)$. So $e_2 \in H^1(\Omega)$, which yields that e_2 is continuous at $x = 0$. It is related to proposition 5.*

Remark 7. *Using* (42) *one easily sees that both e_2 and b_3 are H^1 and hence continuous far from 0. In particular the boundary conditions* (43) *hold in a pointwise sense.*

Remark 8 (Integrability)**.** *Notice again that*

$$\left(\frac{\delta^2}{\alpha} - \alpha\right) e_2\varphi_2 = \left[\frac{x}{\alpha}\right] \left[\left(\delta^2 - \alpha^2\right) e_2\right] \left[\frac{1}{x}\varphi_2\right]$$

is the product of three terms where the first is bounded in $L^\infty(\Omega)$, the second is controlled by the L^2 norm of e_2 and the third is controlled by the $H_0^1(\Omega)$ norm due to the Hardy inequality. Therefore the second weak identity in Problem 1 is integrable and makes sense. We do not comment the boundary conditions (48) since they pose no problem because the solution is easily shown to be continuous at $x = \pm 1$ (see below).

Remark 9. *As in (20) the weak equations (42) can be recast for $x \neq 0$ as*

$$\frac{d}{dx}\mathbf{G}(x) = M(x)\mathbf{G}(x), \quad \mathbf{G}(x) = \begin{pmatrix} e_2(x) \\ b_3(x) \end{pmatrix}, \qquad x \neq 0, \tag{44}$$

where the matrix $M(x) \in \mathbb{C}^{2\times 2}$ satisfies the bound $|M(x)| \leq \frac{C}{|x|}$. So (with basic arguments) e_2 and b_3 belongs to $H^1(\Omega)\backslash(-\varepsilon, \varepsilon)$, for all $0 < \varepsilon < 1$. Therefore (e_2, b_3) belongs, for $0 < x \leq 1$ to vectorial space of dimension two with two basis functions, and for $-1 \leq x < 0$ to the vectorial space of dimension two with two basis functions. It means that the problem can be reduced to a linear equation in dimension $n = 4$, where the degrees of freedom are the components along the basis functions on both sides.
With this in mind, the number of linear constraints is: 2 boundary conditions (48); one integral relation (37); and one continuity relation for e_2 at $x = 0$; that is four linear constraints. One can expect that these four linear constraints are linear independent, which will prove the existence and uniqueness of the solution of problem 1.

The first main result of this section which confirms the relevance of our approach is the following.

Theorem 1. *For all $(f_-, f_+) \in \mathbb{C}^2$, there exists a unique solution (e_2, b_3) to Problem 1 and it coincides with the limit solution (E_2^+, B_3^+) defined in Corollary 1.*

Corollary 2. *A direct by-product of our analysis (namely Corollary 1, Proposition 7 and Theorem 1) is that the limit (E_2^+, B_3^+) considered in Corollary 1 is unique.*

The proof of the theorem is based on the preliminary result.

Proposition 9. *Let (e_2, b_3) be an homogeneous solution to Problem 1, i.e., with $f_- = f_+ = 0$. Define*

$$\Lambda(e_2, \psi) := i\lambda(F_2^+ e_2\psi)(1) + i\lambda(F_2^+ e_2\psi)(-1) - \left[e_2 C_3^+ \psi\right]_{-1}^{1}$$

with F_2^+, C_3^+ the limits of the second manufactured solution. Then one has the identity

$$\int_\Omega \left(F_2^+ b_3 - e_2 C_3^+\right) \psi' dx = \int_\Omega \left(q_3^+ b_3 - g_2^+ e_2\right) \psi dx + \Lambda(e_2, \psi) \qquad \forall \psi \in H^1(\Omega). \tag{45}$$

Proof. The proof proceeds by several steps.
• **First step:** We first observe that (45) holds for the particular $\psi = \varphi \in H_0^1(\Omega)$ involved in iii). Indeed in this case (45) is just a reformulation of (37) since $\Lambda(e_2, \varphi) = 0$.

• **Second step:** we now consider a general $\psi \in H_0^1(\Omega)$ such that $\psi(0) \neq 0$ but $\psi \neq \varphi$. Decompose by linearity $\psi = \varphi + \phi$ with $\phi \in H_{0,0}^1(\Omega)$. It is actually easy to show the claim (45) for $\phi \in H_{0,0}^1(\Omega)$ as a consequence of i). Indeed compute

$$\begin{aligned}\int_\Omega \left(F_2^+ b_3 - e_2 C_3^+\right)\phi' dx &= -\int_\Omega \Big(- b_3 (F_2^+\phi)' + b_3 (F_2^+)'\phi + e_2 C_3^+ \phi' \Big) dx \\ &= -\int_\Omega \left(\Big(\frac{\delta^2}{\alpha} - \alpha\Big) e_2 F_2^+ \phi + b_3 (C_3^+ - q_3^+)\phi + e_2 C_3^+ \phi' \right) dx\end{aligned}$$

where we used (42) with $\varphi_2 = F_2\phi \in H_{0,0}^1$ and the properties (27) of the manufactured solution. As already noticed the first relation in i) yields that $b_3 = e_2'$: one finds

$$\int_\Omega (b_3 C_3^+\phi + e_2 C_3^+\phi')dx = \int_\Omega C_3^+ (e_2\phi)' dx = \int_{x<0} C_3^+ (e_2\phi)' dx + \int_{x>0} C_3^+ (e_2\phi)' dx.$$

An integration by part in the two integrals yields

$$\int_\Omega (b_3 C_3^+\phi + e_2 C_3^+\phi')dx = -\int_{x<0} (C_3^+)' e_2 \phi dx - \int_{x>0} (C_3^+)' e_2 \phi dx$$

where it can be checked that both integrals are correctly defined because $\phi \in H_{0,0}^1(\Omega)$ vanishes at the origin. So one can write

$$\int_\Omega (b_3 C_3^+\phi + e_2 C_3^+\phi')dx = \int_\Omega \left(g_2^+ - \Big(\frac{\delta^2}{\alpha} - \alpha\Big) F_2^+ \right) e_2 \phi dx$$

that is

$$-\int_\Omega \left(\Big(\frac{\delta^2}{\alpha} - \alpha\Big) e_2 F_2^+ \varphi + b_3 (C_3^+ - q_3^+)\varphi + e_2 C_3^+ \varphi' \right) dx = \int_\Omega (q_3^+ b_3 - g_2^+ e_2)\varphi$$

which shows the claim (45) for $\phi \in H_{0,0}^1(\Omega)$ and so by additivity for all $\psi \in H_0^1(\Omega)$. Note that one still has $\Lambda(e_2, \psi) = 0$.

• **Third step:** Next we extend to functions which do not necessarily vanish at the boundary. We remind that (e_2, C_3^+) are continuous at the boundary thanks to remark 9. Let us consider a small parameter $\varepsilon > 0$. Decompose $\psi \in H^1(\Omega)$ as

$$\psi(x) = \psi_\varepsilon(x) + \psi(-1)\left(\frac{-x-1+\varepsilon}{\varepsilon}\right)_+ + \psi(1)\left(\frac{x-1+\varepsilon}{\varepsilon}\right)_+,$$

that is $\psi_\varepsilon(x) = \psi(x) - \psi(-1)\left(\frac{-x-1+\varepsilon}{\varepsilon}\right)_+ - \psi(1)\left(\frac{x-1+\varepsilon}{\varepsilon}\right)_+$. Note that $\psi_\varepsilon \in H_0^1(\Omega)$ which therefore satisfies the claim (45). One has the decomposition

$$\begin{aligned}&\int_\Omega \left(F_2^+ b_3 - e_2 C_3^+\right)\psi' dx - \int_\Omega \left(q_3^+ b_3 - g_2^+ e_2\right)\psi dx \\ &= \int_\Omega \left(F_2^+ b_3 - e_2 C_3^+\right)\psi_\varepsilon' dx - \int_\Omega \left(q_3^+ b_3 - g_2^+ e_2\right)\psi_\epsilon dx \\ &+\psi(-1)\left(\int_\Omega \left(F_2^+ b_3 - e_2 C_3^+\right)\left(\frac{-x-1+\varepsilon}{\varepsilon}\right)_+' dx - \int_\Omega \left(q_3^+ b_3 - g_2^+ e_2\right)\left(\frac{-x-1+\varepsilon}{\varepsilon}\right)_+ dx\right) \\ &+\psi(1)\left(\int_\Omega \left(F_2^+ b_3 - e_2 C_3^+\right)\left(\frac{x-1+\varepsilon}{\varepsilon}\right)_+' dx - \int_\Omega \left(q_3^+ b_3 - g_2^+ e_2\right)\left(\frac{x-1+\varepsilon}{\varepsilon}\right)_+ dx\right) \\ &= 0 - \frac{\psi(-1)}{\varepsilon}\int_{-1}^{-1+\varepsilon} \left(F_2^+ b_3 - e_2 C_3^+\right) dx - \psi(-1)\int_{-1}^{-1+\varepsilon} \left(q_3^+ b_3 - g_2^+ e_2\right)\left(\frac{-x-1+\varepsilon}{\varepsilon}\right) dx \\ &+\frac{\psi(1)}{\varepsilon}\int_{1-\varepsilon}^{1} \left(F_2^+ b_3 - e_2 C_3^+\right) dx - \psi(1)\int_{1-\varepsilon}^{1} \left(q_3^+ b_3 - g_2^+ e_2\right)\left(\frac{x-1+\varepsilon}{\varepsilon}\right) dx.\end{aligned}$$

The third and last terms tend to zero with $\varepsilon \to 0$. The second term admits a limit by continuity

$$\lim_{\varepsilon=0^+}\left(-\frac{\psi(-1)}{\varepsilon}\int_{-1}^{-1+\varepsilon} \left(F_2^+ b_3 - e_2 C_3^+\right) dx\right) = -\psi(-1)\left(F_2^+ b_3 - e_2 C_3^+\right)(-1) = \left(i\lambda F_2^+ e_2 \psi + e_2 C_3^+ \psi\right)(-1)$$

using the boundary condition ii) at $x=-1$ (read in a classical sense according to Remark 7). The fourth term has the limit

$$\lim_{\varepsilon=0^+}\left(\frac{\psi(1)}{\varepsilon}\int_{1-\varepsilon}^{1}\left(F_2^+b_3-e_2C_3^+\right)dx\right)=\psi(1)\left(F_2^+b_3-e_2C_3^+\right)(1)=\left(i\lambda F_2^+e_2\psi-e_2C_3^+\psi\right)(1)$$

using the other boundary condition ii) at $x=1$, again in a classical sense. □

Proof of theorem 1. The existence of the solution $(e_2,b_3)=(E_2^+,B_3^+)$ has been shown in Corollary 1, so only the uniqueness remains to prove. Moreover problem 1 is a linear problem in finite dimension as explained in Remark 9, for which existence is a consequence of uniqueness. In both cases the important part is the uniqueness for which one takes $f_+=f_-=0$. To prove uniqueness we will rely on an identity for which proposition 6 yields an important sign information.
Notice first the simple identity

$$\begin{aligned}\int_\Omega e_2'(F_2^+\varphi)'dx &= \int_\Omega\left(e_2'(F_2^+)'\varphi+e_2'F_2^+\varphi'\right)dx\\ &= \int_\Omega\left(e_2'(C_3^+-q_3^+)\varphi+b_3F_2^+\varphi'\right)dx \qquad\qquad \text{(then use identity (45))}\\ &= \int_\Omega e_2'(C_3^+-q_3^+)\varphi dx+\int_\Omega\Big(e_2C_3^+\varphi'+(q_3^+b_3-g_2^+e_2)\varphi\Big)dx+\Lambda(e_2,\varphi)\\ &= \int_\Omega C_3^+(e_2\varphi)'dx-\int_\Omega g_2^+e_2\varphi dx+\Lambda(e_2,\varphi).\end{aligned}\tag{46}$$

Let us take the particular test function $\varphi=\frac{\overline{e_2}}{F_2^+}=-i\delta\overline{e_2}$: this is is legit since by construction, the second manufactured solution satisfies $|F_2^+|\geq c>0$, see Remark 4. Then define $\phi:=-ie_2\varphi=-\delta|e_2|^2$ which is such that $\phi\in H^1(\Omega)$ and $\phi\leq 0$. One obtains from (46)

$$\int_\Omega |e_2|'^2dx=\int_\Omega iC_3^+\phi'dx-\int_\Omega ig_2^+\phi dx+\Lambda\left(e_2,\frac{\overline{e_2}}{F_2^+}\right)$$

that is

$$\int_\Omega |e_2|'^2dx=\int_\Omega iC_3^+\phi'dx-\int_\Omega ig_2^+\phi dx+i\lambda|e_2|^2(-1)+i\lambda|e_2|^2(1)-[iC_3\phi]_{-1}^1.$$

Rearrange

$$i\lambda|e_2|(-1)^2+i\lambda|e_2|(1)^2=\int_\Omega |e_2|'^2dx-\int_\Omega iC_3^+\phi'dx+\int_\Omega ig_2^+\phi dx+[iC_3\phi]_{-1}^1.$$

Take the imaginary part

$$\lambda|e_2|(-1)^2+\lambda|e_2|(1)^2=\Re\left(-\int_\Omega C_3^+\phi'dx+\int_\Omega g_2^+\phi dx+[C_3^+\phi]_{-1}^1\right).$$

The crux of the proof is that this identity is exactly (33) with $\phi\in H^1(\Omega)$ and $\phi\leq 0$. Therefore

$$\lambda|e_2|(-1)^2+\lambda|e_2|(1)^2\leq 0.$$

Since $\lambda>0$ by hypothesis, this shows that $e_2(-1)=e_2(1)=0$. The boundary conditions yield also $C_3^+(-1)=C_3^+(1)=0$. The ODE argument developed in remark 9 shows the solution (e_2,C_3^+) vanish almost everywhere. The proof of the uniqueness, and so the proof of the theorem, is ended. □

Remark 10. *As announced, the above proof relies on a key property of the manufactured solution involved in Problem 1 (which is the second one), namely the fact that $F_2^+(0)\neq 0$.*

A natural reformulation of Problem 1 where the test functions are weakened in the first formulation but with two integral relations of the form (37) is a follows.

Problem 2. *Find $(e_2,b_3)\in L^2(\Omega)\times L^2(\Omega)$ which satisfy three conditions:*

i) *they satisfy the weak formulations*

$$\begin{cases} \displaystyle\int_\Omega (b_3\varphi_1 + e_2\varphi_1')dx = 0, & \forall \varphi_1 \in H^1_{0,0}(\Omega), \\ \displaystyle\int_\Omega \left(b_3\varphi_2' + \left(\frac{\delta^2}{\alpha} - \alpha\right) e_2\varphi_2 \right) dx = 0, & \forall \varphi_2 \in H^1_{0,0}(\Omega), \end{cases} \tag{47}$$

where the test functions are in the same space,

ii) *they satisfy the boundary conditions*

$$b_3(-1) + i\lambda e_2(-1) = f_- \text{ and } b_3(1) - i\lambda e_2(1) = f_+, \tag{48}$$

in the sense of distributions

iii) *they satisfy two integral relations for the same test function $\varphi \in H^1_0(\Omega)$ such that $\varphi(0) \neq 0$: one integral relation (37) with the first (regular) manufactured solution* (31)*; another integral relation (37) with the second (singular) manufactured solution* (32)*.*

The following result completes Theorem 1 and highlights the role of the first manufactured solution.

Theorem 2. *For all $(f_-, f_+) \in \mathbb{C}^2$, there exists a unique solution (e_2, b_3) to Problem 2 and it coincides with the limit solution (E_2^+, B_3^+) defined in Corollary 1.*

Proof. Once again, the fact that the limit (E_2^+, B_3^+) is a solution of Problem 2 is a consequence of Corollary 1. Our method of proof is to show that a solution of Problem 2 is also a solution of Problem 1 for which uniqueness holds. In this direction we observe that the weak identity

$$\int_\Omega (b_3\varphi_1 + e_2\varphi_1')dx = 0 \quad \forall \varphi_1 \in H^1_{0,0}(\Omega)$$

yields that $e_2 \in H^1(-1,0)$ and $e_2 \in H^1(0,1)$ separately. Therefore e_2 admits a continuous limit on the right and on the left. So if one proves the continuity at $e_2(0^-) = e_2(0^-)$, then it will show that $e_2 \in H^1(\Omega)$. In this case one can write

$$\int_\Omega (b_3\varphi_1 + e_2\varphi_1')dx = 0 \quad \forall \varphi_1 \in H^1_0(\Omega)$$

which shows that solution of Problem 2 is also a solution of Problem 1. In summary it is sufficient to show the continuity at zero of e_2. One can proceed as follows.

• Firstly it is easy to extend Proposition 9 for the first (non singular) manufactured solution (31). Thus, starting from hypothesis iii) one obtains that

$$\int_\Omega \left(F_2^+ b_3 - e_2 C_3^+\right)\psi' dx = \int_\Omega \left(q_3^+ b_3 - g_2^+ e_2\right)\psi dx \qquad \forall \psi \in H^1_0(\Omega). \tag{49}$$

Notice that here we consider boundary conditions f_-, f_+ that may be nonzero but this is compensated by the fact that ψ vanishes at the boundary (which will be sufficient to our needs since the continuity of e_2 at 0 is ultimately a local property).

• Secondly take

$$\begin{cases} \psi_\varepsilon(x) = \frac{x+\varepsilon}{\varepsilon} & -\varepsilon \le x \le 0, \\ \psi_\varepsilon(x) = \frac{-x+\varepsilon}{\varepsilon} & 0 \le x \le \varepsilon, \\ \psi_\varepsilon(x) = 0 & \text{elsewhere,} \end{cases}$$

and rewrite (49) as

$$\begin{aligned} &-\frac{1}{\varepsilon}\int_{-\varepsilon}^0 e_2(x)C_3^+(x)dx + \frac{1}{\varepsilon}\int_0^\varepsilon e_2(x)C_3^+(x)dx \\ &= \int_\Omega \left(q_3^+ b_3 - g_2^+ e_2\right)\psi_\varepsilon dx - \frac{1}{\varepsilon}\int_{-\varepsilon}^0 F_2^+(x)b_3(x)dx + \frac{1}{\varepsilon}\int_0^\varepsilon F_2^+(x)b_3(x)dx. \end{aligned} \tag{50}$$

The continuity of e_2 on both sides and the definition of C_3^+ (31) yields the limit

$$\lim_{\varepsilon=0^+}\left(-\frac{1}{\varepsilon}\int_{-\varepsilon}^0 e_2(x)C_3^+(x)dx + \frac{1}{\varepsilon}\int_0^\varepsilon e_2(x)C_3^+(x)dx\right) = C_3^+(0)\left[e_2(0^+) - e_2e(0^-)\right], \quad C_3^+(0) = i\frac{\alpha'(0)}{\delta(0)} \neq 0.$$

The limit on the right hand side in (50) are easy to determine. The limit of the first integral vanishes

$$\left|\int_\Omega \left(q_3^+ b_3 - g_2^+ e_2\right) \psi_\varepsilon dx\right| \le \int_{-\varepsilon}^{\varepsilon} \left|q_3^+ b_3 - g_2^+ e_2\right| dx \underset{\varepsilon\to 0^+}{\longrightarrow} 0.$$

Using that $F_2^+(0) = 0$ by definition (31), one gets that $|F_2^+(x)| \le C\varepsilon$ for $x \in [-\varepsilon, \varepsilon]$, for some $C > 0$. So the second integral is bounded as

$$\left|\frac{1}{\varepsilon}\int_{-\varepsilon}^{0} F_2^+(x) b_3(x) dx\right| \le C \int_{-\varepsilon}^{0} |b_3(x)| dx \underset{\varepsilon\to 0^+}{\longrightarrow} 0.$$

The last term tends to zero for similar reasons. Therefore one gets that

$$C_3^+(0)\left[e_2(0^+) - e_2(0^-)\right] = 0 \tag{51}$$

which turns into $e_2(0^+) = e_2(0^-)$ since $C_3^+(0) \ne 0$. So e_2 is continuous at $x = 0$, that is $e_2 \in H^1(\Omega)$. The proof is ended. □

Remark 11. *As announced, the above proof relies on a key property of the additional manufactured solution involved in Problem 2 (which is the first one), namely the fact that $C_3^+(0^-) = C_3^+(0^+) \ne 0$.*

4 Weak formulations via dissipative inequalities

We now consider the other use of manufactured solutions that was described in the introduction, which is called dissipative inequalities. Dissipative inequalities are fundamentally comparison inequalities based on energy considerations. Additional references [16, 14] also show that dissipative inequalities are reminiscent of entropy inequalities in hyperbolic equations. This is clear by noticing that we use non negative cut-off functions $\varphi \in C^1_{0,+}(\Omega)$. We observe that Condition 2 is verified for the same reasons than Condition 3 in the above section.
The non homogeneous system (5) written in the context of the regularized Budden equation (17) writes

$$\begin{cases} C_3^\nu - (F_2^\nu)' = q_3^\nu \\ -(\alpha + i\nu)F_1^\nu - i\delta F_2^\nu = g_1^\nu \\ -(C_3^\nu)' + i\delta F_1^\nu - (\alpha + i\nu) F_2^\nu = g_2^\nu. \end{cases} \tag{52}$$

We notice that the mapping

$$(F_1^\nu, F_2^\nu, C_3^\nu, g_1^\nu, g_2^\nu, q_3^\nu) \mapsto (-F_1^\nu, F_2^\nu, C_3^\nu, -g_1^\nu, g_2^\nu, q_3^\nu) \tag{53}$$

transforms a solution of (52) into a solution of the symmetrized system (27), and vice-versa. Therefore we can reuse the manufactured solutions constructed and analysed above. In this whole section we will denote by F_2^ν, C_3^ν and F_2^+, C_3^+ the second (singular) manufactured solution (29) and their limits (32) as $\nu \to 0^+$.

Proposition 10. *The limits of the Budden system (17) described in Corollary 1 satisfy the dissipative inequalities*

$$-Im \int_\Omega (E_2^+ - F_2^+)\overline{(B_3^+ - C_3^+)}\varphi' dx - Im \int_\Omega \left(q_3^+ \overline{(B_3^+ - C_3^+)} - g_2^+ \overline{(E_2^+ - F_2^+)}\right)\varphi dx \ge 0, \quad \forall \varphi \in C^1_{0,+}(\Omega). \tag{54}$$

Proof. It is sufficient to apply Proposition 1 in a one-dimensional domain. Another proof is possible by direct and more pedestrian manipulations of (17) and (52). The proof is ended. □

It is natural to define the functional $\mathcal{J} : L^2(\Omega) \times L^2(\Omega) \times \mathbb{C} \to \mathbb{R}$

$$\mathcal{J}(e_2, b_3, k) := -\mathrm{Im} \int_\Omega (e_2 - kF_2^+)\overline{(b_3 - kC_3^+)}\varphi' dx - \mathrm{Im} \int_\Omega \left(kq_3^+ \overline{(b_3 - kC_3^+)} - kg_2^+ \overline{(e_2 - kF_2^+)}\right)\varphi dx \tag{55}$$

where $\varphi \in C^1_{0,+(\Omega)}$ is one cut-off function with $\varphi(0) > 0$, (F_2^+, C_3^+) is the second manufactured function (31), and k is a complex number introduced to scale the second manufactured function. We note that $\mathcal{J}$ is quadratic with respect to (e_2, b_3, k) and that it must be non negative considering (54). It is therefore natural to write the Euler-Lagrange conditions for the minimum. This will be done with the Lagrangian $\mathcal{L}$ defined below in (64). Before that, we study $\mathcal{J}$.

Proposition 11. *One has the expansion in ascending powers of k*

$$\mathcal{J}(e_2, b_3, k) = -Im \int_\Omega e_2 \overline{b_3} \varphi' dx - Im \left[k \int_\Omega \left(\left(C_3^+ \overline{e_2} - F_2^+ \overline{b_3} \right) \varphi' + \left(q_3^+ \overline{b_3} - g_2^+ \overline{e_2} \right) \varphi \right) dx \right] + \frac{\pi \varphi(0)}{|r|} |k|^2. \tag{56}$$

Proof. The expansion (55) is easily rearranged in three terms, the two first ones being evident. The third quadratic term comes from the simplification/identity

$$-\mathrm{Im} \int_\Omega F_2^+ \overline{C_3^+} \varphi' dx + \mathrm{Im} \int_\Omega \left(q_3^+ \overline{C_3^+} - g_2^+ \overline{F_2^+} \right) \varphi dx = \frac{\pi \varphi(0)}{|r|} \tag{57}$$

which we need to prove. It is a corollary of the identity (36) with the real valued function $\phi = \overline{-iF_2^+ \varphi} = \frac{\varphi}{\delta} \in H_0^1(\Omega)$: indeed with this choice (36) recasts as

$$\mathrm{Re} \left(- \int_\Omega C_3^+ \overline{(-iF_2^+ \varphi)}' dx + \int_\Omega g_2^+ \overline{(-iF_2^+ \varphi)} dx \right) = \left(\frac{\pi \delta(0)}{|\alpha'(0)|} \right) \overline{(-iF_2^+ \varphi)}(0).$$

One gets $\mathrm{Im} \left(\int_\Omega C_3^+ \overline{(F_2^+ \varphi)}' dx - \int_\Omega g_2^+ \overline{(F_2^+ \varphi)} dx \right) = \left(\frac{\pi}{|\alpha'(0)|} \right) \varphi(0)$ rewritten as

$$\mathrm{Im} \left(- \int_\Omega \overline{C_3^+} (F_2^+ \varphi)' dx - \int_\Omega g_2^+ \overline{F_2^+} \varphi dx \right) = \left(\frac{\pi}{|\alpha'(0)|} \right) \varphi(0) = \frac{\pi \varphi(0)}{|r|}. \tag{58}$$

One has that

$$\overline{C_3^+} (F_2^+ \varphi)' = \overline{C_3^+} F_2^+ \varphi' + \overline{C_3^+} (F_2^+)' \varphi = \overline{C_3^+} F_2^+ \varphi' - \overline{C_3^+} q_3^+ \varphi + |C_3^+|^2 \varphi.$$

Therefore

$$\mathrm{Im} \left(- \int_\Omega \overline{C_3^+} (F_2^+ \varphi)' dx \right) = \mathrm{Im} \left(- \int_\Omega \overline{C_3^+} F_2^+ \varphi' dx + \int_\Omega \overline{C_3^+} q_3^+ \varphi dx \right). \tag{59}$$

With (59), one can transform (58) into (57). So the proof is ended. $\square$

Proposition 12. *One has the identity*

$$\mathcal{J}(E_2^+, B_3^+, k) = \frac{\pi \delta(0)^2 \varphi(0)}{|r|} \left| E_2^+(0) - k F_2^+(0) \right|^2. \tag{60}$$

Proof. Let us restart from the equations with $\nu > 0$. Since $g_1^\nu = 0$, the identity (6) is rewritten in dimension one as

$$\mathrm{Im} \left[(E_2^\nu - kF_2^\nu) \overline{(B_3^\nu - kC_3^\nu)} \right]' - \mathrm{Im} \left[k q_3^\nu \overline{(B_3^\nu - kC_3^\nu)} - k g_2^\nu \overline{(E_2^\nu - kF_2^\nu)} \right] = \nu \left(|E_1^\nu - kF_1^\nu|^2 + |E_2^\nu - kF_2^\nu|^2 \right). \tag{61}$$

Defining $\mathcal{J}^\nu$ as in (55) but with the manufactured solution corresponding to $\nu > 0$, this yields

$$\mathcal{J}^\nu(E_2^\nu, B_3^\nu, k) = \nu \int_\Omega \left(|E_1^\nu - kF_1^\nu|^2 + |E_2^\nu - kF_2^\nu|^2 \right) \varphi dx.$$

Since $(\alpha + i\nu)(E_1^\nu - kF_1^\nu) + i\delta (E_2^\nu - kF_2^\nu) = 0$, one has also

$$\begin{aligned} \mathcal{J}^\nu(E_2^\nu, B_3^\nu, k) &= \nu \int_\Omega \left(\frac{\delta^2}{\alpha^2 + \nu^2} + 1 \right) |E_2^\nu - kF_2^\nu|^2 \varphi dx \\ &= \nu \int_\Omega \left(\frac{\delta^2}{\alpha^2 + \nu^2} + 1 \right) \varphi dx \times |E_2^\nu(0) - kF_2^\nu(0)|^2 \\ &+ \nu \int_\Omega \left(\frac{\delta^2}{\alpha^2 + \nu^2} + 1 \right) \left(|E_2^\nu - kF_2^\nu|^2 - |E_2^\nu(0) - kF_2^\nu(0)|^2 \right) \varphi dx. \end{aligned} \tag{62}$$

Since E_2^ν and F_2^ν are uniformly bounded in $H^1(\Omega)$ which is compactly embedded in $C^0(\overline{\Omega})$ by the Rellich-Kondrachov theorem [5], one has

$$\lim_{\nu \to 0^+} (E_2^\nu(0) - kF_2^\nu(0)) = E_2^+(0) - kF_2^+(0).$$

Using (35), one gets

$$\lim_{\nu\to 0^+}\left(\nu\int_\Omega\left(\frac{\delta^2}{\alpha^2+\nu^2}+1\right)dx\times|E_2^\nu(0)-kF_2^\nu(0)|^2\right)=\frac{\pi\delta(0)^2}{|r|}\left|E_2^+(0)-kF_2^+(0)\right|^2$$

which is actually the result.
The remaining term in (62) can be bounded as follows due to uniform $H^1(\Omega)$ boundedness. There exists $C>0$ such that

$$\left||E_2^\nu-kF_2^\nu|^2-|E_2^\nu(0)-kF_2^\nu(0)|^2\right|\le C\min\left(\sqrt{|x|},1\right),$$

so for all $\varepsilon>0$, one has

$$\begin{aligned}&\nu\int_\Omega\left(\frac{\delta^2}{\alpha^2+\nu^2}+1\right)\left(|E_2^\nu-kF_2^\nu|^2-|E_2^\nu(0)-kF_2^\nu(0)|^2\right)dx\\
&\le\nu\int_{\Omega\setminus(-\varepsilon,\varepsilon)}\left(\frac{\delta^2}{\alpha^2+\nu^2}+1\right)\left(|E_2^\nu-kF_2^\nu|^2-|E_2^\nu(0)-kF_2^\nu(0)|^2\right)dx\\
&\quad+\nu\int_{(-\varepsilon,\varepsilon)}\left(\frac{\delta^2}{\alpha^2+\nu^2}+1\right)\left(|E_2^\nu-kF_2^\nu|^2-|E_2^\nu(0)-kF_2^\nu(0)|^2\right)dx\\
&\le C\nu\int_{\Omega\setminus(-\varepsilon,\varepsilon)}\left(\frac{\delta^2}{\alpha^2+\nu^2}+1\right)dx+\sqrt{\varepsilon}\nu\int_{(-\varepsilon,\varepsilon)}\left(\frac{\delta^2}{\alpha^2+\nu^2}+1\right)dx\\
&\le C\nu\left(\int_{\Omega\setminus(-\varepsilon,\varepsilon)}\left(\frac{\delta^2}{\alpha^2}+1\right)dx\right)+\sqrt{\varepsilon}\left(\nu\int_\Omega\left(\frac{\delta^2}{\alpha^2+\nu^2}+1\right)dx\right).\end{aligned}$$

By using the properties of α stated in Assumption 1, the second integral in parentheses is uniformly bounded with respect to ν: there exists $\widetilde{C}>0$ such that $\nu\int_\Omega\left(\frac{\delta^2}{\alpha^2+\nu^2}+1\right)dx\le\widetilde{C}$ for all $\nu>0$. So the second term can be made as small as required by taking a small ε. It remains to take a small ν such that the first contribution is also as small as desired. It shows that

$$\lim_{\nu\to 0^+}\left(\nu\int_\Omega\left(\frac{\delta^2}{\alpha^2+\nu^2}+1\right)\left(|E_2^\nu-kF_2^\nu|^2-|E_2^\nu(0)-kF_2^\nu(0)|^2\right)dx\right)=0.$$

Finally the limit $\mathcal{J}^\nu(E_2^\nu,B_3^\nu,k)\to\mathcal{J}(E_2^+,B_3^+,k)$ follows again from Condition 2, and the proof is ended. $\square$

The formulas (56) and (60) are polynomials in k of the same degree, and they are equal for $(e_2,b_3)=(E_2+,B_3^+)$. In this case the coefficients of the polynomials are equal, that is

$$\begin{cases}-\mathrm{Im}\displaystyle\int_\Omega E_2^+\overline{B_3^+}\varphi'dx & =\dfrac{\pi\delta(0)^2\varphi(0)}{|r|}\left|E_2^+(0)\right|^2,\\
\displaystyle\int_\Omega\left(\left(C_3^+\overline{E_2^+}-F_2^+\overline{B_3^+}\right)\varphi'+\left(q_3^+\overline{B_3^+}-g_2^+\overline{E_2^+}\right)\varphi\right)dx & =2i\dfrac{\pi\delta(0)^2\varphi(0)}{|r|}F_2^+(0)\overline{E_2^+(0)},\\
\dfrac{\pi\varphi(0)}{|r|} & =\dfrac{\pi\delta(0)^2\varphi(0)}{|r|}\left|F_2^+(0)\right|^2.\end{cases}\tag{63}$$

The last identity is actually a triviality. Some simplifications come from $F_2^+(0)=\frac{i}{\delta(0)}$.
Define the Lagrangian as the sum of the potential and of the linear constraints with convenient Lagrange multipliers λ_1 and λ_2

$$\mathcal{L}(e_2,b_3,k;\lambda_1,\lambda_2)=\mathcal{J}(e_2,b_3,k)+\mathrm{Im}\int_\Omega(\overline{b_3}\lambda_1+\overline{e_2}\lambda_1')dx+\mathrm{Im}\int_\Omega\left(\overline{b_3}\lambda_2'+\left(\frac{\delta^2}{\alpha}-\alpha\right)\overline{e_2}\lambda_2\right)dx\in\mathbb{R}\tag{64}$$

with complex valued domain

$$(e_2,b_3,k,\lambda_1,\lambda_2)\in L^2(\Omega)\times L^2(\Omega)\times\mathbb{C}\times H_0^1(\Omega)\times H_{0,0}^1(\Omega).\tag{65}$$

The conjugations introduced in the weak formulations of the constraints are only of the simplicity of notations in problem 3. Taking the variations with respect to the unknowns and the Lagrange multipliers, one obtains the formal extremality conditions

$$\begin{cases} \left\langle \dfrac{\partial \mathcal{L}}{\partial e_2}, u \right\rangle & = 0, \qquad \forall u \in L^2(\Omega), \\ \left\langle \dfrac{\partial \mathcal{L}}{\partial b_3}, v \right\rangle & = 0, \qquad \forall v \in L^2(\Omega), \\ \dfrac{\partial \mathcal{L}}{\partial k} & = 0, \\ \left\langle \dfrac{\partial \mathcal{L}}{\partial \lambda_1}, \varphi_1 \right\rangle & = 0, \qquad \forall \varphi_1 \in H_0^1(\Omega), \\ \left\langle \dfrac{\partial \mathcal{L}}{\partial \lambda_2}, \varphi_2 \right\rangle & = 0, \qquad \forall \varphi_1 \in H_{0,0}^1(\Omega). \end{cases}$$

The weak form of these expressions is immediate for the two first and two last equations. And the third one admits a strong form which is easily obtained by a linear-quadratic simplification of $\mathcal{J}$: indeed consider $j(k) = a|k|^2 - \mathrm{Im}(kb)$ with $a = \frac{\pi\varphi(0)}{|r|} > 0$ and

$$b = \int_\Omega \left(\left(C_3^+ \overline{e_2} - F_2^+ \overline{b_3} \right) \varphi' + \left(q_3^+ \overline{b_3} - g_2^+ \overline{e_2} \right) \varphi \right) dx \in \mathbb{C}.$$

One has the algebra

$$j(k) = a|k|^2 + \mathrm{Re}(ikb) = a|k|^2 + \frac{1}{2}\mathrm{Re}(kib) + \frac{1}{2}\mathrm{Re}(\overline{kib}) = a \left[\left(k - i\frac{\overline{b}}{2a} \right) \overline{\left(k - i\frac{\overline{b}}{2a} \right)} - \frac{|b|^2}{4a} \right].$$

So $\partial_k j = 0 \Longleftrightarrow \overline{b} + 2iak = 0$. So the minimum of j is $k = i\frac{\overline{b}}{2a}$. Identifying a and b with the coefficients in (63), one obtains problem 3.

Problem 3. *Let $\varphi \in C_{0,+}^1(\Omega)$ with $\varphi(0) > 0$. Find $(e_2, b_3, k, \lambda_1, \lambda_2) \in L^2(\Omega) \times L^2(\Omega) \times \mathbb{C} \times H_0^1(\Omega) \times H_{0,0}^1(\Omega)$ such that*

$$\begin{cases} \displaystyle\int_\Omega \left((b_3 - kC_3^+)\varphi' + kg_2^+\varphi + \lambda_1' + \left(\frac{\delta^2}{\alpha} - \alpha \right) \lambda_2 \right) u dx = 0, & \forall u \in L^2(\Omega), \\ \displaystyle\int_\Omega \left(-(e_2 - kF_2^+)\varphi' - kq_3^+\varphi + \lambda_1 + \lambda_2' \right) v dx = 0, & \forall v \in L^2(\Omega), \\ \displaystyle\int_\Omega \left(\left(\overline{C_3^+} e_2 - \overline{F_2^+} b_3 \right) \varphi' + \left(\overline{q_3^+} b_3 - \overline{g_2^+} e_2 \right) \varphi \right) dx + 2i\frac{\pi\varphi(0)}{|r|} k = 0, & \\ \displaystyle\int_\Omega (b_3\varphi_1 + e_2{\varphi_1}') dx = 0, & \forall \varphi_1 \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \left(b_3{\varphi_2}' + \left(\frac{\delta^2}{\alpha} - \alpha \right) e_2\varphi_2 \right) dx = 0, & \forall \varphi_2 \in H_{0,0}^1(\Omega) \end{cases} \tag{66}$$

with the boundary conditions in the sense of distributions

$$b_3(-1) + i\lambda e_2(-1) = f_- \text{ and } b_3(1) - i\lambda e_2(1) = f_+, \tag{67}$$

An interesting a priori estimate satisfied by solutions to this problem is the following. It will serve to show the well-posedness of Problem 3, and will also highlight the physical status of the parameter k, see Remark 12 below.

Proposition 13. *A solution to Problem 3 satisfies the a priori estimate*

$$\frac{\pi\varphi(0)}{|r|}|k|^2 = -Im \int_\Omega e_2 \overline{b_3} \varphi' dx. \tag{68}$$

Proof. The proof is purely algebraic. Take the conjugate of third equation in (66) and rearrange

$$2i\frac{\pi\varphi(0)}{|r|}\overline{k} = \int_\Omega \left(\left(C_3^+\overline{e_2} - F_2^+\overline{b_3} \right) \varphi' + \left(q_3^+\overline{b_3} - g_2^+\overline{e_2} \right) \varphi \right) dx.$$

Multiply by k

$$2i\frac{\pi\varphi(0)}{|r|}|k|^2 = \int_\Omega \left(\left(kC_3^+\overline{e_2} - kF_2^+\overline{b_3} \right) \varphi' + \left(kq_3^+\overline{b_3} - kg_2^+\overline{e_2} \right) \varphi \right) dx. \tag{69}$$

Take $u = \overline{e_2}$ in the first equation

$$0 = \int_\Omega \left((b_3 - kC_3^+)\varphi' + kg_2^+\varphi + \lambda_1' + \left(\frac{\delta^2}{\alpha} - \alpha \right) \lambda_2 \right) \overline{e_2} dx = 0. \tag{70}$$

Take $v = \overline{b_3}$ in the second equation

$$0 = \int_\Omega \left(-(e_2 - kF_2^+)\varphi' - kq_3^+\varphi + \lambda_1 + \lambda_2' \right) \overline{b_3} dx. \tag{71}$$

Add (70-71) to (69)

$$\begin{aligned} 2i\frac{\pi\varphi(0)}{|r|}|k|^2 &= -\int_\Omega \left(e_2\overline{b_3} - b_3\overline{e_2} \right) \varphi' dx + \int_\Omega \left[\left(\lambda_1' + \left(\frac{\delta^2}{\alpha} - \alpha \right) \lambda_2 \right) \overline{e_2} + (\lambda_1 + \lambda_2') \overline{b_3} \right] dx \\ &= -2i \ \mathrm{Im} \int_\Omega e_2\overline{b_3}\varphi' dx + \int_\Omega \left[\lambda_1'\overline{e_2} + \lambda_1\overline{b_3} \right] dx + \int_\Omega \left[\left(\frac{\delta^2}{\alpha} - \alpha \right) \lambda_2\overline{e_2} + \lambda_2'\overline{b_3} \right] dx. \end{aligned}$$

The two last integral vanish in view of the two last equations in (66). After simplification by a factor 2, the proof is ended. □

Theorem 3. *For all $(f_-, f_+) \in \mathbb{C}^2$, there exists a unique solution to Problem 3, which is $(e_2, b_3) = \left(E_2^+, B_3^+ \right)$, $k = \frac{E_2^+(0)}{F_2^+(0)} = -i\delta(0)E_2^+(0)$ and $(\lambda_1, \lambda_2) = \left(-B_3^+ + kC_3^+, E_2^+ - kF_2^+ \right) \varphi$.*

Proof. The proof is a matter of elementary verifications.
• We first show that

$$(e_2, b_3, k, \lambda_1, \lambda_2) = \left(E_2^+, B_3^+, \frac{E_2^+(0)}{F_2^+(0)} = -i\delta(0)E_2^+(0), -\left(B_3^+ - kC_3^+ \right) \varphi, \left(E_2^+ - kF_2^+ \right) \varphi \right) \tag{72}$$

is a solution to (66), which will prove the existence after we have checked that (72) belongs to the proper space, see below.
- The two last equations in (66) have already been proved in Corollary 1.
- Considering the second relation in (63), the third relation of (66) is satisfied for e_2, b_3 and k given by (72).
- Next one can insert directly (72) in (66) and check that the two first identities of (66) hold trivially.
• Let us then check the embeddings $\left(-B_3^+ + kC_3^+ \right) \varphi \in H_0^1(\Omega)$ and $\left(E_2^+ - kF_2^+ \right) \varphi \in H_{0,0}^1(\Omega)$. The second one holds since both functions $E_2^+\varphi$ and $F_2^+\varphi$ are in $H_0^1(\Omega)$ and k is chosen such that $E_2^+ - kF_2^+$ vanishes at the origin. Finally we observe that the first weak equation in (66) can be recast in strong form

$$(B_3^+ - kC_3^+)\varphi' + kg_2^+\varphi + \lambda_1' + \left(\frac{\delta^2}{\alpha} - \alpha \right) \lambda_2 = 0.$$

Since $\lambda_2 = \left(E_2^+ - kF_2^+ \right) \varphi \in H_{0,0}^1(\Omega)$, the Hardy's inequality yields one more time that $\left(\frac{\delta^2}{\alpha} - \alpha \right) \lambda_2 \in L^2(\Omega)$. The other terms being in also square integrable, one gets that $\lambda_1' \in L^2(\Omega)$. So

$$\lambda_1 = -(B_3^+ - kC_3^+)\varphi \in H_0^1(\Omega).$$

It finishes the verification that (72) is solution to (66).
• The uniqueness can be proved considering the solution of the homogeneous equations with $f_- = f_+ = 0$. One has first the a priori estimate (68). Due to the last equations of (66) one has away from the resonance $x = 0$

$$b_3 - (e_2)' = 0 \quad \text{and} \quad -(b_3)' + \left(\frac{\delta^2}{\alpha} - \alpha \right) e_2 = 0 \qquad x \neq 0.$$

So

$$2\mathrm{Im}(e_2\overline{b_3})' = 2\mathrm{Im}\left(|b_3|^2 + \left(\frac{\delta^2}{\alpha} - \alpha\right)|e_2|^2\right) = 0.$$

One can multiply this expression by a test function $\phi \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ and $\phi(0) = 0$, and integrate by parts. Adding the result to (68), one gets that

$$i\frac{\pi\psi(0)}{|r|}|k|^2 = -\mathrm{Im}\int_\Omega e_2\overline{b_3}\psi' dx + \mathrm{Im}\left(e_2\overline{b_3}\psi\right)(1) - \mathrm{Im}\left(e_2\overline{b_3}\psi\right)(-1), \quad \psi = \varphi + \phi.$$

Take $\psi \equiv 1$ and use the homogeneous boundary conditions. It yields $i\frac{\pi}{|r|}|k|^2 = -i\lambda|e_2(1)|^2 - i\lambda|e_2(-1)|^2$. Since $\lambda > 0$, one obtains $k = e_2(1) = e_2(-1) = 0$. The boundary condition yields $b_3(1) = b_3(-1) = 0$. This is propagated inside the domain by the equation: so the solution (e_2, b_3) vanishes everywhere. The Lagrange multipliers are solutions of the homogeneous equations $\lambda_1' + \left(\frac{\delta^2}{\alpha} - \alpha\right)\lambda_2 = \lambda_1 + \lambda_2' = 0$. Since the Lagrange multipliers vanish at the boundary by definition, integration of these equations with the Cauchy-Lipschitz theorem cancels the Lagrange multipliers. The proof is ended. □

Remark 12. *An asset of problem 3 is the unknown k. Indeed one has from (6) and (68)*

$$\lim_{\nu\to 0^+}\int_\Omega \nu|E_1^\nu|^2\varphi dx = \lim_{\nu\to 0^+}\mathrm{Im}\int_\Omega \overline{\mathbf{E}^\nu}\times\mathbf{B}^\nu\varphi dx = -\mathrm{Im}\int_\Omega E_2^+\overline{B_3^+}\varphi dx = \frac{\pi\varphi(0)}{|r|}|k|^2.$$

For a test function $\varphi \geq 0$ such that $\varphi(0) = 1$, one gets that $|k|^2$ is a measure of the resonant heating. The resonant heating is positive for $E_2^+(0) \neq 0$. On physical grounds the resonant heating is the amount of energy communicated by the electromagnetic field to a bath of underlying static ions, see [13] for a proof that the resonant heating is positive.

5 Other manufactured solutions for the 1D case

Most of the properties established so far are actually independent of the precise form of the manufactured solutions, as long as:

i) they satisfy the conditions 1 to 3, in order to provide integral constraints for the limit resonant field

ii) and they have the properties highlighted in Remark 4, in order that the resulting formulations are well-posed.

In particular, one can think of other manufactured solutions to improve some aspects of the method. We may consider for instance the situation where the extra diagonal δ of the dielectric tensor (13) is

$$\delta \in W^{1,\infty}(\Omega), \quad \delta \geq 0, \quad \delta(0) > 0.$$

Typically δ may vanish away from the singular point $x = 0$. In this case the functions (28-29) just blow up where $\delta = 0$. In view of an effective definition of the manufactured solutions, it seems attractive to use frozen coefficients. For the simplicity of the presentation we also "freeze" the slope $r := \alpha'(0)$.
We consider more precisely

$$F_1 = 0, \quad F_2 = 0, \quad C_3 = 1. \tag{73}$$

which is now independent of ν. The right hand sides of the symmetrized system (27) associated with this new family satisfy $g_1^\nu \equiv 0$ $g_2^\nu \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and $q_3^\nu \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$, with (trivial) bounds uniform with respect to ν. This family is extremely simple, and it satisfies the key property that C_3 is continuous and non zero at the origin, see Remark 11. Hence the proposed family.
The new second manufactured solution that we consider is

$$F_1^\nu = -\frac{1}{rx + i\nu}, \quad F_2^\nu = \frac{i}{\delta(0)}, \quad C_3^\nu = i\frac{\delta(0)}{r}\log(rx + i\nu) \tag{74}$$

where log denotes the principal value of the logarithm in the complex plane, see Remark 2. Again this solution satisfies the key property that F_2^+ is non zero at the origin, see Remark 10, so that it only remains to verify that the right hand sides associated with this new family via the symmetrized system (27) are integrable enough, according to Conditions 2 and 3. In this right hand side we have $q_3^\nu \in W^{1,\infty}_{\rm loc}(\mathbb{R})$ and $g_2^\nu \in W^{1,\infty}_{\rm loc}(\mathbb{R})$ with bounds uniform with respect to ν: it does not need further comments. Now, unless the linearized coefficients are equal to the true ones, one has $g_1^\nu \neq 0$. Instead, (27) gives

$$g_1^\nu = \frac{\alpha + i\nu}{rx + i\nu} - \frac{\delta}{\delta(0)}.$$

We define the space

$$L^2_{1/x}(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) \text{ such that } \frac{1}{x} u \in L^2(\Omega) \right\}$$

which is dual of $L^2_x(\Omega)$ in which E_1^ν and its limit E_1^+ lay.

Proposition 14. *The function g_1^ν admits a limit in $L^2_{1/x}(\Omega)$ as ν tends to zero.*

Proof. Indeed

$$g_1^\nu = \left(\frac{\alpha + i\nu}{rx + i\nu} - 1 \right) - \left(\frac{\delta}{\delta(0)} - 1 \right).$$

Both terms are in $L^2_{1/x}(\Omega)$ with a uniform bound, and they admit a limit in $L^2_{1/x}(\Omega)$. □

Now, the Poynting-like equality (10) reads

$$(E_2^\nu C_3^\nu - F_2^\nu B_3^\nu)' = B_3^\nu q_3^\nu - E_1^\nu g_1^\nu - E_2^\nu g_2^\nu$$

with the product $E_1^\nu g_1^\nu$ composed of terms which are naturally in dual spaces. Therefore one can pass to the limit and obtain the integral relation

$$\int_\Omega \left(F_2^+ B_3^+ - E_2^+ C_3^+\right) \varphi dx = \int_\Omega \left(B_3^+ q_3^+ - E_1^+ g_1^+ - E_2^+ g_2^+\right) dx. \tag{75}$$

The central term in the right hand side can also written as $\int_\Omega (xE_1^+) \left(\frac{1}{x} g_1^+\right) dx$ where both terms between parentheses are in $L^2(\Omega)$. It yields for example the following problem which is in the same vein of problem 2.

Remark 13. *Using the original system (17) it is in fact possible to express $E_1^\nu g_1^\nu$ as a linear combination of E_2^ν and B_3^ν with weights satisfying uniform L^2 bounds. The resulting formulation appears naturally in the alternative approach proposed in Section 6.4.*

Problem 4. *Find $(e_1, e_2, b_3) \in L^2_x(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ which satisfy three conditions:*

i) *they satisfy the weak formulations*

$$\begin{cases} \displaystyle\int_\Omega (b_3 \varphi_1 + e_2 \varphi_1')dx = 0, & \forall \varphi_1 \in H^1_{0,0}(\Omega), \\ \displaystyle\int_\Omega (\alpha e_1 + i\delta e_2)\, \phi dx = 0, & \forall \phi \in L^2(\Omega), \\ \displaystyle\int_\Omega (b_3 \varphi_2' + i\delta e_1 \varphi_2 - \alpha e_2 \varphi_2)\, dx = 0, & \forall \varphi_2 \in H^1_{0,0}(\Omega), \end{cases} \tag{76}$$

ii) *they satisfy the boundary conditions in the sense of distributions,*

$$b_3(-1) + i\lambda e_2(-1) = f_- \qquad \text{and} \qquad b_3(1) - i\lambda e_2(1) = f_+, \tag{77}$$

iii) *they satisfy two integral relations for the same test function $\varphi \in H^1_0(\Omega)$ with $\varphi(0) \neq 0$: one integral relation (75) with the regular manufactured solution (73); and the same integral relation (75) but with the limit of singular manufactured solution (74).*

The existence and uniqueness of the solution can easily be studied with the methods used before for problems 1 and 2. The detailed proofs would add little to the material already exposed, so we discard them.

6 MultiD formulations

Multidimensional formulations of resonant Maxwell's equations pose formidable difficulties in terms of deciding an appropriate functional setting. Our goal hereafter is to show that a natural generalization of manufactured solutions is possible. The rigorous justification of the functional setting is not fully addressed in this work. We refer nevertheless to [20] on the behavior of resonant solutions of a scalar equation for metamaterial modeling. However it will be evident that that sole behavior of singular manufactured solutions, which are known analytically, is already a strong indication of the nature of the singularities. We concentrate on constructive issues and restrict the presentation to essential ideas.
For simplicity, we start from the Maxwell's equation in dimension 2 with dissipation $\nu > 0$ in a domain $\Omega \subset \mathbb{R}^2$

$$\left\{ \begin{array}{cccc} B_3^\nu & +\partial_y E_1^\nu & -\partial_x E_2^\nu & = 0, \\ \partial_y B_3^\nu & -(\alpha + i\nu)E_1^\nu & -i\delta E_2^\nu & = 0, \\ -\partial_x B_3^\nu & +i\delta E_1^\nu & -(\alpha+i\nu)E_2^\nu & = 0. \end{array} \right. \tag{78}$$

We assume that the coefficients $\alpha(x,y)$ and $\delta(x,y)$ are smooth in $C^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. The function δ is uniformly positive: $0 < \delta_- \leq \delta$. The function α vanishes on the vertical line and only there, i.e.,

$$\left\{ \begin{array}{ll} \alpha(x,y) < 0, & x < 0, \\ \alpha(x,y) = 0, & x = 0, \\ \alpha(x,y) > 0, & x > 0. \end{array} \right.$$

Assumption 2. *We will make the assumption that B_3^ν, E_2^ν are bounded uniformly in $L^2_{\text{loc}}(\Omega)$ and that they admit a limit $B_3^+, E_2^+ \in L^2_{\text{loc}}(\Omega)$. The important assumption is for the horizontal part of the electric field. We admit that xE_1^ν is bounded uniformly in $L^2_{\text{loc}}(\Omega)$, and admits a limit which is noted as $xE_1^+ \in L^2_{\text{loc}}(\Omega)$.*
This assumption is natural: with respect to the results already obtained in dimension 1, see Proposition 4 and Corollary 1; it can also be justified by comparison with the singular behavior of the 2D manufactured solutions, see proposition 17 below; and it can be considered as a extension to systems of similar results obtained in the literature [10, 20].

In the next sections we will construct two families of manufactured functions that generalize the weak techniques of Section 3 to this particular setting. We already note that the exact form of the manufactured solutions (and in particular the singular ones) will make Assumption 2 very natural. To simplify the presentation we do not develop the dissipative approach here, and leave that generalization to the reader.
In view of Assumption 2, we see that one term may pose a problem in the integral relation (10), that is $E_1^\nu C_3^\nu$ which is the product of a function in $L^2_{y,\text{loc}}(\Omega) = \left\{u \text{ such that } xu \in L^2_{\text{loc}}(\Omega)\right\}$ by a function in $L^2_{\text{loc}}(\Omega)$. This product is ultimately non integrable and so might ruin the whole construction. Therefore we propose to make it disappear by using a test function invariant in the vertical direction, i.e.,

$$\varphi(x,y) = \varphi(x), \qquad \varphi(0) \neq 0. \tag{79}$$

To design local relations in $\Omega \subset \mathbb{R}^2$ we thus need to construct manufactured solutions that are localized in the vertical dimension by a smooth cut-off w with compact support in y, and we signify this dependance with a super index, $F_1^{w,\nu}$, $F_2^{w,\nu}$, $C_3^{w,\nu}$. In 2D, the inhomogeneous symmetrized system (9) reads then

$$\left\{ \begin{array}{cccc} C_3^{w,\nu} & +\partial_y F_1^{w,\nu} & -\partial_x F_2^{w,\nu} & = q_3^{w,\nu}, \\ \partial_y C_3^{w,\nu} & -(\alpha + i\nu)F_1^{w,\nu} & +i\delta F_2^{w,\nu} & = g_1^{w,\nu}, \\ -\partial_x C_3^{w,\nu} & -i\delta F_1^{w,\nu} & -(\alpha+i\nu)F_2^{w,\nu} & = g_2^{w,\nu}. \end{array} \right. \tag{80}$$

6.1 A family of non singular manufactured solutions

After inspection of the various possibilities offered by the local analytical expansions needed for the design of manufactured solutions, we generalize the non singular family under the form

$$F_1^{w,\nu} = 0, \quad F_2^{w,\nu}(x,y) = -\frac{w'(y)}{i\delta(x)} \quad \text{and} \quad C_3^{w,\nu}(x,y) = w(y) \qquad \text{where} \quad w \in H_0^1(\mathbb{R}). \tag{81}$$

The right hand sides $(q_3^\nu, g_1^\nu, g_2^\nu)$ defined by (80) are then

$$q_3^{w,\nu} = w + w'\partial_x\left(\frac{1}{i\delta}\right), \quad g_1^{w,\nu} = 0 \quad \text{and} \quad g_2^{w,\nu} = \frac{w'(\alpha+i\nu)}{i\delta}$$

and the Poynting-like equality (10) reads

$$\partial_x \left(E_2^\nu C_3^{w,\nu} - B_3^\nu F_2^{w,\nu}\right) - \partial_y \left(E_1^\nu C_3^{w,\nu} - B_3^\nu F_1^{w,\nu}\right) = q_3^{w,\nu} B_3^\nu - g_2^{w,\nu} E_2^\nu.$$

Here we verify that under Assumption 2 all terms are naturally bounded in $L^1_{\mathrm{loc}}(\Omega)$ except $E_1^\nu C_3^{w,\nu}$, but as noted above the use of a test function of the form (79) will make that term disappear. One obtains after integration

$$\int_\Omega \left(B_3^\nu F_2^{w,\nu} - E_2^\nu C_3^{w,\nu}\right)(x,y)\varphi'(x)dxdy = \int_\Omega \left(q_3^{w,\nu} B_3^\nu - g_2^{w,\nu} E_2^\nu\right)(x,y)\varphi(x)dxdy. \tag{82}$$

The product $\varphi(x)w(y)$ has compact support in Ω provided the support of φ in the x direction and the support of w in the y direction are conveniently restricted. One can now pass formally to the limit $\nu \to 0^+$ and obtain

$$\int_\Omega \left(B_3^+ F_2^{w,+} - E_2^+ C_3^{w,+}\right)(x,y)\varphi'(x)dxdy = \int_\Omega \left(q_3^{w,+} B_3^+ - g_2^{w,+} E_2^+\right)\varphi(x)dxdy \tag{83}$$

with

$$F_2^{w,+} = -\frac{w'(y)}{i\delta(x)}, \quad C_3^{w,+}(x,y) = w(y), \quad q_3^{w,+} = w + w'\partial_x\left(\frac{1}{i\delta}\right) \quad \text{and} \quad g_2^{w,+} = \frac{w'\alpha}{i\delta}. \tag{84}$$

Note the non intuitive structure of the test functions: one $w(y)$ is in the manufactured solution; the other one $\varphi(x)$ is in the integration by parts.

6.2 A family of singular manufactured solutions

Inspection of the different possibilities show a strong dependance with respect to the variation of the coefficients in the vertical direction. As pointed in Remark 14 below, this can be viewed as a consequence of the following property.

Proposition 15 (Two divergence identities)**.** *The solutions to (78) verify two divergence identities. The first one is standard*

$$\partial_x \left((\alpha + i\nu)E_1^\nu + i\delta E_2^\nu\right) + \partial_y \left((\alpha + i\nu)E_2^\nu - i\delta E_1^\nu\right) = 0. \tag{85}$$

The second one writes (where $\delta > 0$)

$$\partial_x \left(\frac{\alpha + i\nu}{i\delta}E_1^\nu + B_3^\nu \partial_y\left(\frac{1}{i\delta}\right)\right) + \partial_y \left(\frac{\alpha + i\nu}{i\delta}E_2^\nu - B_3^\nu \partial_x\left(\frac{1}{i\delta}\right)\right) + B_3^\nu = 0. \tag{86}$$

Proof. The solutions to (78) satisfy (where $\delta > 0$)

$$\begin{cases} E_1^\nu = \frac{1}{i\delta}\left(+(\alpha + i\nu)E_2^\nu + \partial_x B_3^\nu\right), \\ E_2^\nu = \frac{1}{i\delta}\left(-(\alpha + i\nu)E_1^\nu + \partial_y B_3^\nu\right). \end{cases}$$

Insert in the first equation of (78)

$$B_3^\nu + \partial_y \left(\frac{1}{i\delta}(\alpha + i\nu)E_2^\nu + \frac{1}{i\delta}\partial_x B_3^\nu\right) - \partial_x \left(-\frac{1}{i\delta}(\alpha + i\nu)E_1^\nu + \frac{1}{i\delta}\partial_y B_3^\nu\right) = 0.$$

One obtains the claim (86) after use of the identity

$$-\partial_y \left(\frac{1}{i\delta}\partial_x B_3^\nu\right) + \partial_x \left(\frac{1}{i\delta}\partial_y B_3^\nu\right) = \partial_y \left(B_3^\nu \partial_x\left(\frac{1}{i\delta}\right)\right) - \partial_x \left(B_3^\nu \partial_y\left(\frac{1}{i\delta}\right)\right). \tag{87}$$

The proof is ended. □

Remark 14. *The identity (86) has no counterparts in dimension 1 since it is essentially sensitive to the y-derivatives of δ. Both identities (85) and (86) express that a certain vector field has a bounded divergence (assuming B_3^ν is uniformly bounded in $L^2_{\mathrm{loc}}(\Omega)$). The difference is that the first vector*

$$\underline{\underline{\varepsilon}}\mathbf{E}^\nu = (\alpha + i\nu)\mathbf{E}^\nu - i\delta(\mathbf{E}^\nu)^\perp = \left((\alpha + i\nu)E_1^\nu + i\delta E_2^\nu, (\alpha + i\nu)E_2^\nu - i\delta E_1^\nu\right)$$

is not uniformly bounded in $L^2_{\mathrm{loc}}(\Omega)^2$ due to the term $i\delta E_1^\nu$ in the second component. On the contrary, the second vector

$$\frac{\alpha + i\nu}{i\delta}\mathbf{E}^\nu + \left(\nabla \times \frac{1}{i\delta}\right) B_3^\nu = \left(\frac{\alpha + i\nu}{i\delta}E_1^\nu + B_3^\nu \partial_y\left(\frac{1}{i\delta}\right), \frac{\alpha + i\nu}{i\delta}E_2^\nu - B_3^\nu \partial_x\left(\frac{1}{i\delta}\right)\right)$$

is uniformly bounded in $L^2_{\mathrm{loc}}(\Omega)^2$ as a consequence of Assumption 2. Hence it is in $H_{\mathrm{loc}}(div, \Omega)$ and continuity of the normal component on the line $x = 0$ is expected. It is the basis of the manufactured design below.

In view of singular behavior of the horizontal part of the electric field which is a feature of all singular manufactured solutions constructed so far, the latter remark gives the intuition that when generalizing the singular family (74) to the 2D case, one should replace $\frac{rx+i\nu}{i\sigma}E_1^\nu$ by $\frac{rx+i\nu}{i\sigma}E_1^\nu - \frac{\sigma'}{i\sigma^2}B_3^\nu$ where we have denoted

$$r(y) = \partial_x \alpha(0,y) \qquad \text{and} \qquad \sigma(y) = \delta(0,y).$$

Following this intuition and aiming at integrable right-hand sides in the symmetrized equations (80), we propose to construct the singular solutions as the solution of the following system (as above, w is a smooth test function of the y variable)

$$\begin{cases} F_1^{w,\nu} - \dfrac{\sigma'}{(rx+i\nu)\sigma} C_3^{w,\nu} = -\dfrac{w}{rx+i\nu} \\ \partial_y C_3^{w,\nu} - (rx+i\nu)F_1^{w,\nu} + i\sigma F_2^{w,\nu} = 0 \\ -\partial_x C_3^{w,\nu} - i\sigma F_1^{w,\nu} = 0. \end{cases} \tag{88}$$

Note that the first equation of (80) has not been taken directly into account here, but only indirectly through the design of the singular field $F_1^{w,\nu} - \frac{\sigma'}{(rx+i\nu)\sigma}C_3^{w,\nu}$ suggested by the above observations. We shall see below that this construction actually leads to a remainder $q_3^{w,\nu}$ that is uniformly integrable.

Proposition 16. *A solution to the constraints* (88) *that generalizes the singular family* (74) *is*

$$\begin{cases} F_1^{w,\nu}(x,y) = -\left(\dfrac{w(y)}{r(y)x+i\nu}\right)(r(y)x+i\nu)^{-i\frac{\sigma'(y)}{r(y)}}, \\ F_2^{w,\nu}(x,y) = \dfrac{i}{\sigma(y)}\left(w(y)(r(y)x+i\nu)^{-i\frac{\sigma'(y)}{r(y)}} + \partial_y\left(w(y)\sigma(y)\left(\dfrac{1-(r(y)x+i\nu)^{-i\frac{\sigma'(y)}{r(y)}}}{\sigma'(y)}\right)\right)\right), \\ C_3^{w,\nu}(x,y) = w(y)\sigma(y)\left(\dfrac{1-(r(y)x+i\nu)^{-i\frac{\sigma'(y)}{r(y)}}}{\sigma'(y)}\right) \end{cases} \tag{89}$$

where we remind that $w \in H_0^1(\mathbb{R})$ is an arbitrary cutoff in the y dimension. This solution can be extended by continuity on $y = y^*$ such that $\sigma'(y^*) = 0$, as

$$\begin{cases} \lim\limits_{y\to y^*} F_1^{w,\nu}(x,y) = -\dfrac{w}{rx+i\nu}, \\ \lim\limits_{y\to y^*} F_2^{w,\nu}(x,y) = w\left(\dfrac{i}{\sigma} + \dfrac{r'}{r^2}\left(\log(rx+i\nu) - \dfrac{rx}{rx+i\nu}\right) + \dfrac{\sigma''}{2}\dfrac{i}{r^2}\log(rx+i\nu)^2\right) - \dfrac{w'}{r}\log(rx+i\nu), \\ \lim\limits_{y\to y^*} C_3^{w,\nu}(x,y) = \dfrac{iw\sigma}{r}\log(rx+i\nu) \end{cases} \tag{90}$$

where all the functions of y are to be evaluated at y^*.

Remark 15. *If the plasma coefficients α and δ depend only on x, the above solution further simplifies into*

$$F_1^{w,\nu}(x,y) = -\frac{w(y)}{rx+i\nu}, \quad F_2^{w,\nu}(x,y) = \frac{iw(y)}{\delta(0)} - \frac{w'(y)}{r}\log(rx+i\nu), \quad C_3^{w,\nu}(x,y) = \frac{iw(y)\delta(0)}{r}\log(rx+i\nu).$$

This indeed corresponds to the 1D singular solution (74)*, combined with a smooth cutoff w in the y dimension, and reduces exactly to* (74) *if one takes $w \equiv 1$.*

Proof. Writing $u(x) = C_3^{w,\nu}(x,y)$ for some fixed valued of y, the third identity of (88) recasts as

$$\frac{d}{dx}u + \frac{a}{rx+i\nu}u = \frac{b}{rx+i\nu}$$

with $a = i\sigma'(y)$ and $b = i\sigma(y)w(y)$ independent of x. The general solution is

$$u = \frac{b}{a}\left(k(rx+i\nu)^{-\frac{a}{r}} + 1\right) = \sigma w\left(\frac{k(rx+i\nu)^{-i\frac{\sigma'}{r}} + 1}{\sigma'}\right), \quad k \in \mathbb{C}.$$

It is amenable to set k so that u admits a limit for vanishing σ'. This is why we take $k=-1$ which yields $C_3^{w,\nu}$. The values of $F_1^{w,\nu}$ and $F_2^{w,\nu}$ are then derived from the first and second lines of (88), respectively. Finally to prove the limits (90), we rewrite

$$M(x,y) := \left(\frac{1-(r(y)x+i\nu)^{-i\frac{\sigma'(y)}{r(y)}}}{\sigma'(y)} \right) = \frac{1}{s(y)} \int_0^{s(y)} m_x(y,z)dz \tag{91}$$

where we have denoted $m_x(y,z) := \frac{i}{r(y)} \log(r(y)x+i\nu)(r(y)x+i\nu)^{-i\frac{z}{r(y)}}$ and $s(y)=\sigma'(y)$. This readily gives

$$\lim_{y\to y^*} M(x,y) = m_x(y^*,0) = \frac{i}{r(y^*)} \log(r(y^*)x+i\nu). \tag{92}$$

We also need to estimate the limit of the partial derivative. We have

$$\begin{aligned}
\partial_y M(x,y) &= \partial_y \left(\frac{1}{s(y)} \int_0^{s(y)} m_x(y,z)dz \right) \\
&= \frac{1}{s(y)} \int_0^{s(y)} \partial_1 m_x(y,z)dz + \frac{s'(y)}{s(y)} m_x(y,s(y)) - \frac{s'(y)}{s(y)^2} \int_0^{s(y)} m_x(y,z)dz \\
&= \frac{1}{s(y)} \int_0^{s(y)} \partial_1 m_x(y,z)dz + \frac{s'(y)}{s(y)} \int_0^{s(y)} \left(\frac{m_x(y,s(y))-m_x(y,z)}{s(y)} \right) dz \\
&= \frac{1}{s(y)} \int_0^{s(y)} \partial_1 m_x(y,z)dz + \frac{s'(y)}{s(y)^2} \int_0^{s(y)} \int_z^{s(y)} \partial_2 m_x(y,u)dudz,
\end{aligned} \tag{93}$$

hence $\lim_{y\to y^*} \partial_y M(x,y) = \partial_1 m_x(y^*,0) + \frac{\sigma''(y^*)}{2} \partial_2 m_x(y^*,0)$ A straightforward computation gives then

$$\partial_1 m_x(y,z) = \left(\frac{i}{r} \frac{r'x}{rx+i\nu} - \frac{ir'}{r^2} \log(rx+i\nu) \right) \left(1 - \frac{iz}{r} \log(rx+i\nu) \right) (rx+i\nu)^{-i\frac{z}{r}} \tag{94}$$

(with an implicit dependence of r on y), and

$$\partial_2 m_x(y,z) = \frac{1}{r^2} \log(rx+i\nu)^2 (rx+i\nu)^{-i\frac{z}{r}}, \tag{95}$$

so that

$$\lim_{y\to y^*} \partial_y M(x,y) = -\frac{ir'}{r^2} \left(\log(rx+i\nu) - \frac{rx}{rx+i\nu} \right) + \frac{\sigma''(y^*)}{2} \frac{1}{r^2} \log(rx+i\nu)^2 \tag{96}$$

with r and r' evaluated at y^*. $\square$

This manufactured solution satisfies the following estimates.

Proposition 17. *Under Assumption 2, the singular manufactured solution (89) satisfies the bound*

$$\|F_1^{w,\nu}\|_{L^2_{x,\mathrm{loc}}(\Omega)} + \|F_2^{w,\nu}\|_{L^2_{\mathrm{loc}}(\Omega)} + \|C_3^{w,\nu}\|_{L^2_{\mathrm{loc}}(\Omega)} \le C < \infty, \tag{97}$$

with a constant independent of $\nu \in (0,1)$. Moreover, the following limits hold in the same spaces as $\nu \to 0^+$:

$$\begin{cases}
F_1^{w,+}(x,y) = -\left(\dfrac{w(y)}{r(y)x} \right) (r(y)x+i0^+)^{-i\frac{\sigma'(y)}{r(y)}}, \\
F_2^{w,+}(x,y) = \dfrac{i}{\sigma(y)} \left(w(y)(r(y)x+i0^+)^{-i\frac{\sigma'(y)}{r(y)}} + \partial_y \left(\sigma(y)w(y) \left(\dfrac{1-(r(y)x+i0^+)^{-i\frac{\sigma'(y)}{r(y)}}}{\sigma'(y)} \right) \right) \right), \\
C_3^{w,+}(x,y) = \sigma(y)w(y) \left(\dfrac{1-(r(y)x+i0^+)^{-i\frac{\sigma'(y)}{r(y)}}}{\sigma'(y)} \right)
\end{cases} \tag{98}$$

where we remind that the complex powers are defined according to the principal value of the logarithm, and where the notation $(a+i0^+)^\lambda$ expresses a positive absorption limit, i.e.,

$$(a+i0^+)^\lambda := \lim_{\nu\to 0^+} (a+i\nu)^\lambda = \begin{cases} e^{\lambda \log|a|} & \text{if } a>0, \\ e^{\lambda(\log|a|+i\pi)} & \text{if } a<0, \end{cases} \qquad \text{for } \; a \in \mathbb{R}^*, \; \lambda \in \mathbb{C}.$$

Remark 16. *By letting $\nu \to 0^+$ in* (90)*, one finds that if $\sigma'(y^*) = 0$ the limit solution becomes*

$$\begin{cases} F_1^{w,+}(x,y^*) = -\dfrac{w}{rx}, \\ F_2^{w,+}(x,y^*) = w\left(\dfrac{i}{\sigma} + \dfrac{r'}{r^2}\left(\log(rx+i0^+) - 1\right) + \dfrac{\sigma''}{2}\dfrac{i}{r^2}\log(rx+i0^+)^2\right) - \dfrac{w'}{r}\log(rx+i0^+), \\ C_3^{w,+}(x,y^*) = \dfrac{iw\sigma}{r}\log(rx+i0^+) \end{cases} \tag{99}$$

where again, all the functions of y (namely σ, r, w and their derivatives) are to be evaluated at y^*.

Remark 17. *Considering the singular limit $F_1^{w,+}$, we have*

$$F_1^{w,+}(x,y) = \begin{cases} -\frac{w(y)}{r(y)x}|r(y)x|^{-i\frac{\sigma'(y)}{r(y)}} & \text{if} \quad r(y)x > 0, \\ -\frac{w(y)}{r(y)x}|r(y)x|^{-i\frac{\sigma'(y)}{r(y)}} e^{\frac{\pi\sigma'(y)}{r(y)}} & \text{if} \quad r(y)x < 0. \end{cases} \tag{100}$$

Here $|r(y)x|^{i\frac{\sigma'(y)}{r(y)}}$ has modulus one. Therefore this formula shows two effects for $\sigma'(y) \neq 0$. One is a phase shift phenomenon $|r(y)x|^{-i\frac{\sigma'(y)}{r(y)}}$. The other one is a damping/growth phenomenon along the resonant curve (here the vertical line) with rate $e^{\frac{\pi\sigma'(y)}{r(y)}}$, on one side of that curve. An illustration is given in Figures 1, 2 and 3, for simple plasma parameters $\alpha(x,y) = -x$, $\delta(x,y) = 1 + y^2$.

Proof. We begin with the uniform bounds. The first term is bounded as

$$\begin{array}{rcccl} |xF_1^{w,\nu}| & = & \left|\frac{x}{rx+i\nu}\right| & \left|(x+i\nu)^{i\frac{\sigma'}{r}}\right| & |w(y)| \\ & \leq & \frac{1}{|r(y)|} & e^{\frac{|\sigma'(y)|\pi}{|r|}} & |w(y)| \in L^2_{\mathrm{loc}}(\Omega). \end{array}$$

So $F_1^{w,\nu}$ is bounded in $L^2_{x,\mathrm{loc}}(\Omega)$ uniformly with respect to $\nu \in (0,1)$. Similar estimates used in combination with the limit (92) for small values of $\sigma'(y)$ show that $C_3^{w,\nu}$ is bounded in $L^2_{\mathrm{loc}}(\Omega)$ uniformly with respect to $\nu \in (0,1)$. To prove the bound for $F_2^{w,\nu}$ it is sufficient to obtain a good bound for $\partial_y M(x,y)$, with $M := (1 - (r(y)x + i\nu)^{-i\frac{\sigma'(y)}{r(y)}})/\sigma'(y)$, see (91). Now, it is easily seen from (94)-(95) that for bounded values of (x,y), both $|\partial_1 m_x(y,z)|$ and $|\partial_2 m_x(y,z)|$ are controlled by $C\big(1 + |\log(r(y)^2x^2)|\big)^2$ with a constant independent of $\nu \in (0,1)$. Using (93) this yields that $\partial_y M$, and hence $F_2^{w,\nu}$, is in $L^2_{\mathrm{loc}}(\Omega)$ uniformly with respect to $\nu \in (0,1)$. Finally the limits (98) are easily obtained in a pointwise sense, and the result follows by dominated convergence. $\square$

Denoting by $q_3^{w,\nu}$, $g_1^{w,\nu}$ and $g_2^{w,\nu}$ the right hand sides in System (80) associated with the singular manufactured solution (89), we rewrite the Poynting-like equality (10) as

$$\partial_x\left(E_2^\nu C_3^{w,\nu} - B_3^\nu F_2^{w,\nu}\right) - \partial_y\left(E_1^\nu C_3^{w,\nu} - B_3^\nu F_1^{w,\nu}\right) = q_3^{w,\nu}B_3^\nu - g_1^{w,\nu}E_1^\nu - g_2^{w,\nu}E_2^\nu.$$

Integrating against a test function of the form (79) we then obtain

$$\int_\Omega \left(B_3^\nu F_2^{w,\nu} - E_2^\nu C_3^{w,\nu}\right)(x,y)\varphi'(x)dxdy = \int_\Omega \left(q_3^{w,\nu}B_3^\nu - g_1^{w,\nu}E_1^\nu - g_2^{w,\nu}E_2^\nu\right)(x,y)\varphi(x)dxdy. \tag{101}$$

We thus need to verify that the functions $q_3^{w,\nu}$, $g_1^{w,\nu}$ and $g_2^{w,\nu}$ have enough integrability to pass to the limit.

Proposition 18. *The right hand sides in* (80) *corresponding to the singular manufactured solution* (89) *satisfy*

$$\|q_3^{w,\nu}\|_{L^2_{\mathrm{loc}}(\Omega)} + \|g_1^{w,\nu}\|_{L^2_{1/x,\mathrm{loc}}(\Omega)} + \|g_2^{w,\nu}\|_{L^2_{\mathrm{loc}}(\Omega)} \leq C < \infty \tag{102}$$

uniformly with respect to $\nu \in (0,1)$. Moreover, the following limits hold in the same spaces as $\nu \to 0^+$:

$$\begin{cases} g_1^{w,+}(x,y) = (r(y)x - \alpha(x,y))F_1^{w,+}(x,y) + i(\delta(x,y) - \sigma(y))F_2^{w,+}(x,y), \\ g_2^{w,+}(x,y) = i(\sigma(y) - \delta(x,y))F_1^{w,+}(x,y) - \alpha(x,y)F_2^{w,+}(x,y), \\ q_3^{w,+}(x,y) = C_3^{w,+}(x,y). \end{cases} \tag{103}$$

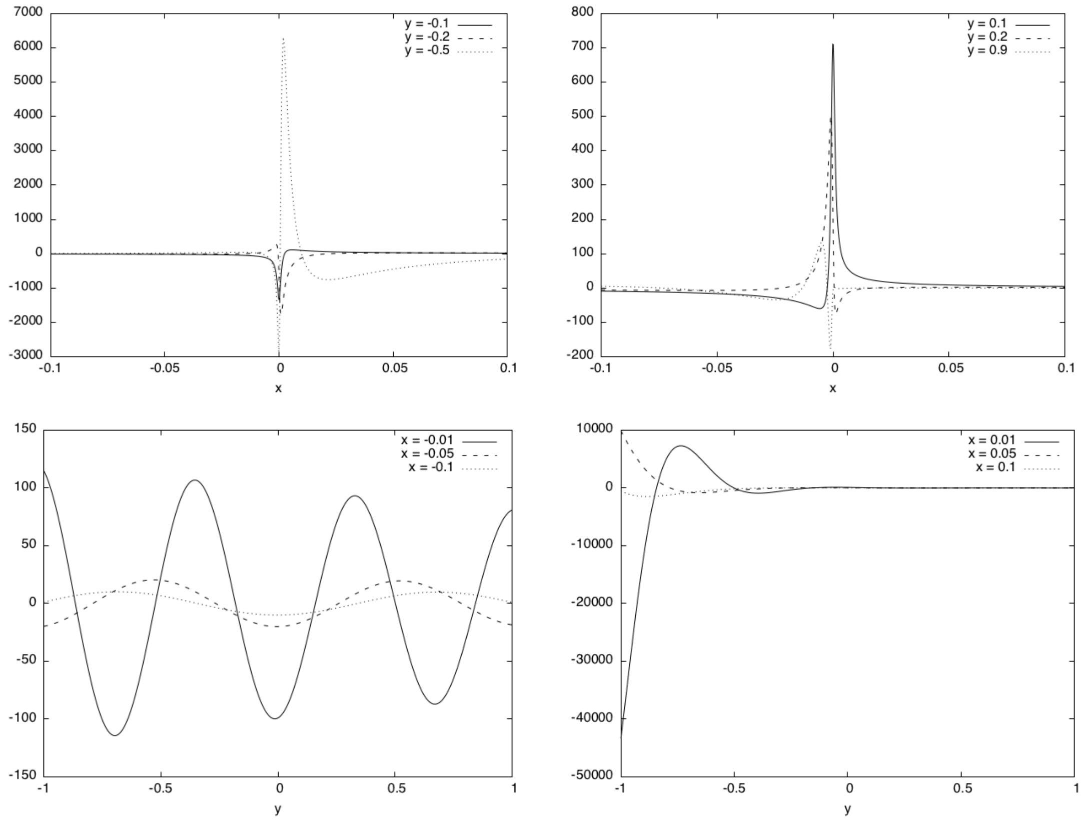

Figure 1: Real part of the singular manufactured solution $F_1^{w,\nu}$ as given by (89) on a 2D domain $\Omega = (-1,1)^2$, with dissipation $\nu = 0.001$ and weight $w \equiv 1$. Here the plasma-dependent parameters are $\alpha(x,y) = -x$ and $\delta(x,y) = y^2 + 1$. Upper plots show cuts in the x direction at specified values of y for $y = -0.1$, $y = -0.2$ and $y = -0.5$ on the left and $y = 0.1$, $y = 0.2$ and $y = 0.9$ on the right. Lower plots show cuts in the y direction at specified values of x, for $x = -0.01$, $x = -0.05$ and $x = -0.1$ on the left and $x = 0.01$, $x = 0.05$ and $x = 0.1$ on the right. The cuts $x = 0$ or $y = 0$ do not bring additional material so are not represented: they can be identified directly on Figure 2.
One sees two different effects: a change of amplitude on the different cuts for different y on the top representations which corresponds to the developping singularity at $x = 0$, and a change of both phase and amplitude on the different cuts for different x on the bottom representations. As discussed in Remark 17, these effects are due to the fact that $\sigma'(y) := \partial_y \delta(y,0) \neq 0$. Finally we note that this solution corresponds to the symmetrized system (80). A solution to the original system would be obtained by changing the sign of δ.

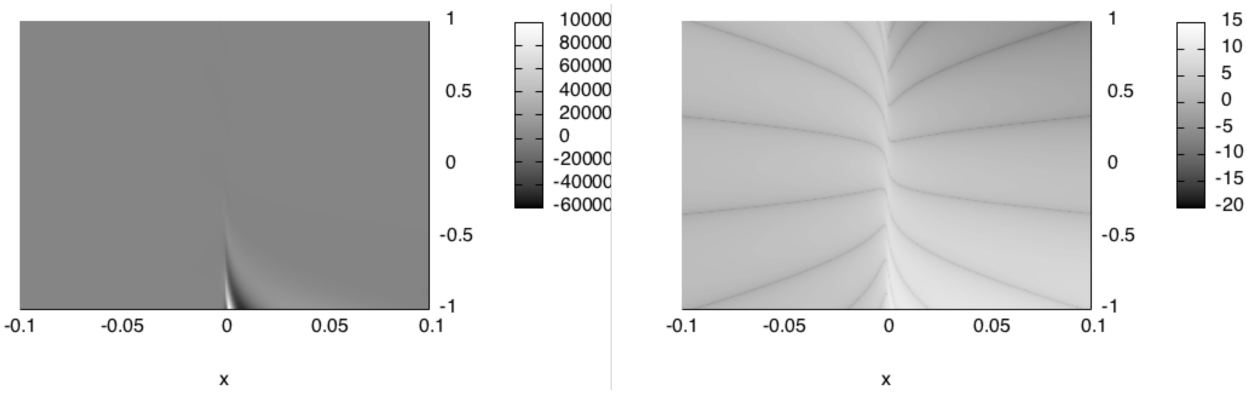

Figure 2: Real part of the singular manufactured solution $F_1^{w,\nu}$ as given by (89) on a 2D domain $\Omega = (-1, 1)^2$, with dissipation $\nu = 0.001$ and weight $w \equiv 1$. As in Figure 1 above, the plasma-dependent parameters are $\alpha(x, y) = -x$ and $\delta(x, y) = y^2 + 1$. The right plot shows the same function in logscale, to improve the visibility of the oscillations along y outside the bottom-lower region where the negative sign of $\sigma'(y) := \partial_y \delta(y, 0)$ creates an exponential growths in the $-y$ direction, see Remark 17.

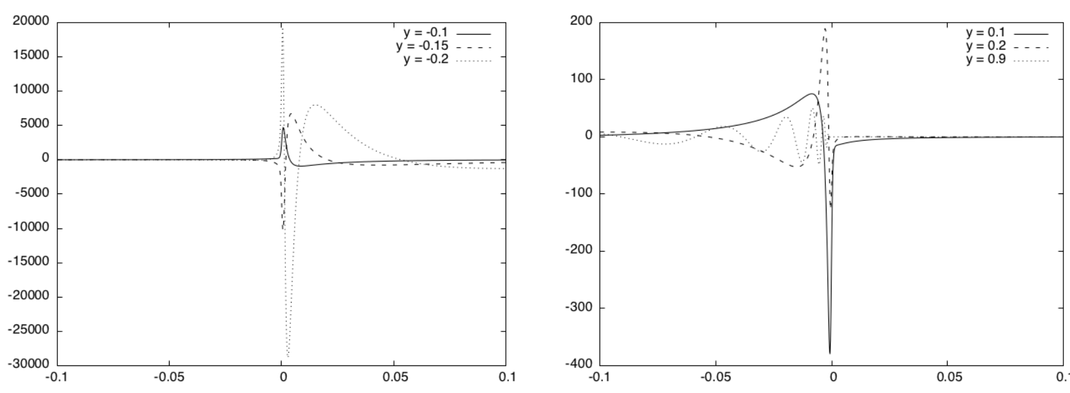

Figure 3: Real part of a singular manufactured solution $F_1^{w,\nu}$ defined as in Figure 1 but with $\delta(x, y) = 4y^2 + 1$ to enhance the phase oscillations in the x direction close to the resonance $x = 0$, a phenomenon that is reminiscent of some propagative singularities studied in [4]. Note that there are oscillations on both sides of the resonance, but because of the damping/growth phenomenon that occurs along y for $x > 0$ as discussed in Remark 17, they are visible on one side only.

Proof. Combining the third equations of (80) and (88), one has

$$g_2^{w,\nu} = -\partial_x C_3^{w,\nu} - i\delta F_1^{w,\nu} - (\alpha + i\nu)F_2^{w,\nu} = i(\sigma - \delta)F_1^{w,\nu} - (\alpha + i\nu)F_2^{w,\nu}. \tag{104}$$

Since $\frac{1}{x}(\sigma(y) - \delta(x,y)) = \frac{1}{x}(\delta(0,y) - \delta(x,y))$ is uniformly bounded in the support of w, one gains a factor x which makes $g_2^{w,\nu}$ bounded in $L^2_{\text{loc}}(\Omega)$ uniformly with respect to $0 < \nu < 1$, thanks to the bound (97) on $F_1^{w,\nu}$ and $F_2^{w,\nu}$. Similarly, combining the second equations of (80) and (88) gives

$$g_1^{w,\nu} = \partial_y C_3^{w,\nu} - (\alpha + i\nu)F_1^{w,\nu} + i\delta F_2^{w,\nu} = (rx - \alpha)F_1^{w,\nu} + i(\delta - \sigma)F_2^{w,\nu}. \tag{105}$$

For the first coefficient between parentheses, using that $r = \partial_x\alpha(0,y)$ and $\alpha(0,y) = 0$ we have

$$|r(y)x - \alpha(x,y)| = |x\partial_x\alpha(0,y) - \alpha(x,y)| = \left|\int_0^x \Big(\partial_x\alpha(0,y) - \partial_x\alpha(z,y)\Big)dz\right| \le x^2\|\partial_x^2\alpha\|_{L^\infty_{\text{loc}}}.$$

Since F_1^ν is uniformly bounded in $L^2_{x,\text{loc}}(\Omega)$, the first term $(rx-\alpha)F_1^{w,\nu}$ is then uniformly bounded in $L^2_{1/x,\text{loc}}(\Omega)$. This is evidently the case also for the second term $(\delta - \sigma)F_2^{w,\nu}$. Hence $g_1^{w,\nu}$ is uniformly bounded in $L^2_{1/x,\text{loc}}(\Omega)$. Finally the last term is

$$q_3^{w,\nu} = C_3^{w,\nu} + \partial_y F_1^{w,\nu} - \partial_x F_2^{w,\nu}.$$

Substituting $F_1^\nu = -\frac{1}{i\sigma}\partial_x C_3^\nu$ and $F_2^\nu = -\frac{1}{i\sigma}\partial_y C_3^\nu + \frac{rx+i\nu}{i\sigma}F_1^\nu$ from the second and third lines of (88), we obtain

$$q_3^{w,\nu} = C_3^{w,\nu} - \partial_y\left(\frac{1}{i\sigma}\partial_x C_3^\nu\right) + \partial_x\left(\frac{1}{i\sigma}\partial_y C_3^\nu - \frac{rx+i\nu}{i\sigma}F_1^\nu\right)$$

The crux is then an identity essentially identical to (87), which yields

$$\begin{aligned} q_3^{w,\nu} &= C_3^{w,\nu} + \partial_y\left(C_3^\nu \partial_x\left(\frac{1}{i\sigma}\right)\right) - \partial_x\left(C_3^\nu\partial_y\left(\frac{1}{i\sigma}\right) + \frac{rx+i\nu}{i\sigma}F_1^\nu\right) \\ &= C_3^{w,\nu} - \partial_x\left(\frac{1}{i\sigma}\left(-C_3^\nu\frac{\sigma'}{\sigma} + (rx+i\nu)F_1^\nu\right)\right) \end{aligned} \tag{106}$$

where we have used the fact that σ is a function of y only. We next observe that the first line of (88) gives that $C_3^\nu\frac{\sigma'}{\sigma} - \frac{rx+i\nu}{i\sigma}F_1^\nu = w(y)$ is independent of the variable x. Therefore

$$q_3^{w,\nu} = C_3^{w,\nu} \tag{107}$$

which incidentally shows that the singular manufactured solution satisfies $\operatorname{curl}\mathbf{F}_1^{w,\nu} = 0$. The estimate then follows from Proposition 17. Finally the limits (103) are readily derived from the relations (105), (104) and (107). □

Remark 18. *The essential ingredient in the proof is the control of the last term in (106) which actually vanishes.*

One can then pass to the limit in the integral identity (101) and obtain

$$\int_\Omega \left(B_3^+ F_2^w - E_2^+ C_3^w\right)(x,y)\varphi'(x)dxdy = \int_\Omega \left(g_1^{w,+}E_1^+ + g_2^{w,+}E_2^+ - q_3^{w,+}B_3^+\right)\varphi(x)dxdy. \tag{108}$$

6.3 A 2D weak formulation

Now that the limits are characterized by means of the manufactured solutions, it is sufficient to multiply (78) by convenient test functions. One obtains for example the following definition of a limit solution with the spaces

$$H^p_{0,\text{loc}}(\Omega) = \{u \in H^p_{\text{loc}}(\Omega),\ u = 0 \text{ on the line } x = 0\}, \quad p = 1,2.$$

We consider one (and only one) function $\varphi \in H^1_0(\mathbb{R})$ such that $\varphi(0) \neq 0$.

Definition 3. *The triplet* $(e_1, e_2, b_3) \in L^2_{x,\text{loc}}(\Omega) \times L^2_{\text{loc}}(\Omega) \times L^2_{\text{loc}}(\Omega)$ *is an admissible weak solution if*

i) *the functions satisfy the weak formulations*

$$\begin{cases} \displaystyle\int_\Omega (b_3 u_1 - e_1 \partial_y u_1 + e_2 \partial_x u_1)\, dx = 0, & \forall u_1 \in H^2_{0,\mathrm{loc}}(\Omega), \\ \displaystyle\int_\Omega (-b_3 \partial_y u_2 - \alpha e_1 u_2 - i\delta e_2 u_2)\, dx = 0, & \forall u_2 \in H^1_{\mathrm{loc}}(\Omega), \\ \displaystyle\int_\Omega (b_3 \partial_x u_3 + i\delta e_1 u_3 - \alpha e_2 u_3)\, dx = 0, & \forall u_3 \in H^1_{0,\mathrm{loc}}(\Omega). \end{cases} \tag{109}$$

ii) *the functions satisfy the integral relation (108) associated with either the regular manufactured solution* (84) *or the singular one* (98)*, and for one smooth cutoff function $\varphi \in C^1_0(\mathbb{R})$ such that $\varphi(0) \neq 0$ and for all cutoff functions $w \in H^1(\mathbb{R})$ with compact support.*

Note we have reinforce the regularity of φ for the sake of integrability. For $u_1 \in H^2_{0,\mathrm{loc}}(\Omega)$, one has that $\partial_y u_1 \in H^1_{0,\mathrm{loc}}(\Omega)$. So the product $e_1 \partial_y u_1$ is integrable thanks to the Hardy inequality. Similarly the product $\delta e_1 u_3$ is integrable thanks to the Hardy inequality. Many other weak formulations can be derived, in particular with dissipative inequalities.

6.4 An alternative approach with smooth auxiliary fields

One gains some perspective on the previous integral formulations by interpreting them in terms of smooth changes of variables. In the 1D case for instance, it is possible to view the integral relation (37), namely

$$\int_\Omega \left(F_2^+ B_3^+ - E_2^+ C_3^+\right) \varphi' dx = \int_\Omega \left(q_3^+ B_3^+ - g_2^+ E_2^+\right) \varphi dx, \quad \forall \varphi \in C^1_0(\Omega), \tag{110}$$

as a weak control on the derivative of $F_2^+ B_3^+ - E_2^+ C_3^+$. Specifically, one may see the construction of the manufactured solutions as a way to design an *auxiliary field*, here it is $F_2^+ B_3^+ - E_2^+ C_3^+$, with better control on the smoothness. In what follows we elaborate on this fundamental remark.

6.4.1 Smooth auxiliary fields in 1D

We try in dimension 1 to construct auxiliary fields $\widetilde{E}_2$ and $\widetilde{B}_3$ with control on the derivatives.
As E_2^ν is known to satisfy a uniform H^1 bound (see propositions 4 and 5), the simplest option consists in taking

$$\widetilde{E}_2^\nu = E_2^\nu. \tag{111}$$

Next for the auxiliary magnetic field, a natural construction is to correct B_2^ν with E_2^ν, since the former is L^2 but not H^1 (uniformly in ν). Hence set

$$\widetilde{B}_3^\nu = B_3^\nu - \beta^\nu E_2^\nu \tag{112}$$

and look for a correction weight β^ν such that $\widetilde{B}_3^\nu$ is uniformly in H^1. Since

$$(\widetilde{B}_3^\nu)' = (B_3^\nu)' - (\beta^\nu)' E_2^\nu - \beta^\nu (E_2^\nu)' = \left(\frac{\delta^2}{\alpha + i\nu} - (\alpha + i\nu) - (\beta^\nu)'\right) E_2^\nu - \beta^\nu B_3^\nu \tag{113}$$

one sees that $(\beta^\nu)'$ should behave like $\frac{\delta^2}{\alpha+i\nu}$ close to the singularity. This is easily realized with

$$\beta^\nu := \frac{\delta(0)^2}{r} \log(rx + i\nu) \tag{114}$$

where $\log(rx + i\nu) = \frac{1}{2}\log(r^2x^2 + \nu^2) + i \arg(rx + i\nu)$ is the principal value of the complex logarithm, see Remark 2. The function

$$\eta^\nu := \frac{\delta^2}{\alpha + i\nu} - (\beta^\nu)' = \frac{\delta^2}{\alpha + i\nu} - \frac{\delta(0)^2}{rx + i\nu} \tag{115}$$

then satisfies an L^∞ bound uniformly in ν, provided $\delta \in C^1$ and $\alpha \in C^1$ and vanishes only at $x = 0$ with $r = \alpha'(0) \neq 0$. The integral relations that allow us to control the smoothness of the auxiliary fields, namely

$-\int \widetilde{E}_2^\nu \varphi' = \int (\widetilde{E}_2^\nu)' \varphi$ and $-\int \widetilde{B}_3^\nu \varphi' = \int (\widetilde{B}_3^\nu)' \varphi$, take then the form

$$\begin{cases} -\displaystyle\int E_2^\nu \varphi' = \int B_3^\nu \varphi \\ -\displaystyle\int (B_3^\nu - \beta^\nu E_2^\nu)\varphi' = \int \Big((\eta^\nu - (\alpha + i\nu))E_2^\nu - \beta^\nu B_3^\nu \Big) \varphi \end{cases}$$

which correspond to the relation (37) with the two manufactured solutions proposed in (73) and (74), respectively. In view of the identity (29) that defines the second manufactured solution, one realizes that $\beta^\nu = \lambda C_3^\nu$ is essentially a convenient rescaling of the singular manufactured solution.

Remark 19. *Using* (113) *one finds that the 1D problem* (17) *rewrites in the auxiliary variables* (111)-(112) *as*

$$\frac{d}{dx}\widetilde{\mathbf{G}}^\nu(x) = \widetilde{M}^\nu(x)\widetilde{\mathbf{G}}^\nu(x), \qquad \widetilde{\mathbf{G}}^\nu(x) = \begin{pmatrix} \widetilde{E}_2^\nu(x) \\ \widetilde{B}_3^\nu(x) \end{pmatrix}, \quad \widetilde{M}^\nu(x) = \begin{pmatrix} 0 & 1 \\ \eta^\nu - (\alpha + i\nu) & -\beta^\nu \end{pmatrix}. \tag{116}$$

Here β^ν and η^ν given by (114), (115) *satisfy uniform $L^2(\Omega)$ bounds, hence also does the matrix $\widetilde{M}^\nu$. Since $\phi(x) := \|\widetilde{\mathbf{G}}^\nu(x)\|_2$ satisfies $|\phi(x)'| \le \|\frac{d}{dx}\widetilde{\mathbf{G}}^\nu(x)\|_2 \le \|\widetilde{M}^\nu(x)\|_2\phi(x)$ this gives a direct $L^\infty(\Omega)$ bound on the auxiliary field $\widetilde{\mathbf{G}}^\nu$, uniformly in ν, and also an a priori $H^1(\Omega)$ bound.*

6.4.2 Smooth auxiliary fields in 2D

The same principle can be applied in the 2D case. First, motivated by the observations from Remark 14 we see that the auxiliary electric field

$$\widetilde{\mathbf{E}}^\nu := \mathbf{E}^\nu - \nabla \left(\frac{B_3^\nu}{i\delta} \right) = \frac{\alpha + i\nu}{i\delta} \begin{pmatrix} E_2^\nu \\ -E_1^\nu \end{pmatrix} - B_3^\nu \nabla \left(\frac{1}{i\delta} \right) \tag{117}$$

should be in $H(\mathrm{curl})$ with a bound uniform in ν. Observing that this field generalizes the construction in (111) for the case of a plasma parameter δ varying in y, we next consider an auxiliary magnetic field obtained again by correcting the physical B field by the smooth (auxiliary) E field, i.e.,

$$\widetilde{B}_3^\nu = \gamma^\nu B_3^\nu - \beta^\nu \widetilde{E}_2^\nu = \gamma^\nu B_3^\nu - \beta^\nu \left(E_2^\nu - \partial_y \left(\frac{B}{i\delta} \right) \right), \tag{118}$$

where the additional weight function γ^ν has been added for a reason that will soon become clear. To identify the constraints on the weights we compute, using (78),

$$\begin{aligned} \partial_x \widetilde{B}_3^\nu &= \partial_x \gamma^\nu B_3^\nu + \gamma^\nu \partial_x B_3^\nu - \partial_x \beta^\nu \left(E_2^\nu - \partial_y \left(\frac{B_3^\nu}{i\delta} \right) \right) - \beta^\nu \partial_x \widetilde{E}_2^\nu \\ &= B_3^\nu \left(\partial_x \gamma^\nu - i\partial_x \beta^\nu \partial_y \left(\frac{1}{\delta} \right) \right) + \left(E_2^\nu - \frac{\partial_y B_3^\nu}{i\delta} \right) \left(\frac{\gamma^\nu \delta^2}{\alpha + i\nu} - \partial_x \beta^\nu \right) - \gamma^\nu (\alpha + i\nu) E_2^\nu - \beta^\nu \partial_x \widetilde{E}_2^\nu. \end{aligned} \tag{119}$$

Hence we see that the weights γ^ν and β^ν should be such that the functions

$$\zeta^\nu := \partial_x \gamma^\nu - i\partial_x \beta^\nu \partial_y \left(\frac{1}{\delta} \right) \qquad \text{and} \qquad \eta^\nu := \frac{\gamma^\nu \delta^2}{\alpha + i\nu} - \partial_x \beta^\nu \tag{120}$$

satisfy uniform L^2 bounds for $\nu \to 0$ (in passing we note that it is the dependence of δ on y that prevents using a constant weight $\gamma^\nu \equiv 1$ on B_3^ν). One may for instance look for weights satisfying

$$\gamma^\nu = 1 - \frac{i\sigma'(y)}{\sigma^2(y)}\beta^\nu \qquad \text{and} \qquad \frac{\sigma^2(y)}{rx + i\nu} - \frac{i\sigma'(y)}{rx + i\nu}\beta^\nu = \partial_x \beta^\nu$$

where $\sigma(y) := \delta(0, y)$ as above. Here the constant 1 in γ^ν allows us to recover the 1D case when $\sigma' = 0$, i.e. when δ is a function of x only. The solutions of the second equation (which has been seen already when constructing manufactured solutions, see the proof of Proposition 16) take the form

$$\beta^\nu = \frac{\sigma^2(y)}{i\sigma'(y)} \left(k - (rx + i\nu)^{-\frac{i\sigma'(y)}{r}} \right)$$

(with complex powers defined according to the principal value of the complex logarithm, as above) and again we take $k = 1$ in order to have a limit at $\sigma'(y) = 0$. Setting $\xi(\sigma') = (rx + i\nu)^{-\frac{i\sigma'}{r}}$, this yields indeed

$$\lim_{\sigma'(y)\to 0} \beta^\nu = \lim_{\sigma'(y)\to 0} \left(i\sigma^2(y) \frac{\xi(\sigma'(y)) - 1}{\sigma'(y)} \right) = i\sigma^2(y)\xi'(0) = \frac{\sigma^2(y)}{r} \log(rx + i\nu).$$

The resulting weights for the auxiliary B field (118) are then

$$\beta^\nu(x,y) := \begin{cases} \frac{\sigma^2(y)}{i\sigma'(y)} \left(1 - (r(y)x + i\nu)^{-\frac{i\sigma'(y)}{r(y)}}\right) & \text{if } \sigma'(y) \neq 0 \\ \frac{\sigma^2(y)}{r(y)} \log(r(y)x + i\nu) & \text{if } \sigma'(y) = 0 \end{cases} \qquad \text{and} \qquad \gamma^\nu(x,y) := (r(y)x + i\nu)^{-\frac{i\sigma'(y)}{r(y)}}. \tag{121}$$

To verify that this construction allows us to recover integral formulations similar to the ones derived above, e.g. (101), we rewrite the equalities

$$\int \operatorname{curl} \widetilde{\mathbf{E}}^\nu \psi = \int \widetilde{\mathbf{E}}^\nu \cdot \mathbf{curl}\, \psi \qquad \text{and} \qquad \int (\partial_x \widetilde{B}_3^\nu)\phi = -\int \widetilde{B}_3^\nu \partial_x \phi \tag{122}$$

in terms of the original fields. Here the second relation results from testing $\mathbf{curl}\, \widetilde{B}_3^\nu$ against a test function of the form $(0, \phi)$: we have chosen to discard test functions with components in the x direction because their contribution to (122) corresponds to an integration by parts along y, and does not give any control on the singular limits. For the first relation we have

$$\int \operatorname{curl} \widetilde{\mathbf{E}}^\nu \psi = \int \operatorname{curl} \mathbf{E}^\nu \psi = \int B_3^\nu \psi$$

and if the test function is a tensor product $\psi(x,y) = \varphi(x)w(y) \in C_0^1(\Omega)$, the RHS reads

$$\int \widetilde{\mathbf{E}}^\nu \cdot \mathbf{curl}\, \psi = \int \left(\widetilde{E}_1^\nu \varphi w' - \widetilde{E}_2^\nu \varphi' w \right) = \int \left(\left(\frac{\alpha + i\nu}{i\delta} E_2^\nu - B_3^\nu \partial_x \left(\frac{1}{i\delta} \right) \right) \varphi w' - E_2^\nu \varphi' w - \frac{B_3^\nu}{i\delta} \varphi' w' \right)$$

so that the first equality in (122) reads

$$\int \left(B_3^\nu \left(w + w' \partial_x \left(\frac{1}{i\delta} \right) \right) - \frac{\alpha + i\nu}{i\delta} E_2^\nu w' \right) \varphi = -\int \left(E_2^\nu w + \frac{B_3^\nu}{i\delta} w' \right) \varphi'. \tag{123}$$

In particular, we recover exactly the integral relation (82) with the first manufactured solution.

Remark 20. *We could also set $\widetilde{E}_1^\nu := 0$ in the auxiliary electric field, and still have $\widetilde{\mathbf{E}}^\nu$ in $H(\mathrm{curl};\Omega)$: indeed*

$$\partial_x \widetilde{E}_2^\nu = \partial_x E_2^\nu - \partial_x \partial_y \left(\frac{B_3^\nu}{i\delta} \right) = B_3^\nu + \partial_y E_1^\nu - \partial_y \left(\frac{\partial_x B_3^\nu}{i\delta} + B_3^\nu \partial_x \left(\frac{1}{i\delta} \right) \right) = B_3^\nu + \partial_y \left(\frac{\alpha + i\nu}{i\delta} E_2^\nu - B_3^\nu \partial_x \left(\frac{1}{i\delta} \right) \right) \tag{124}$$

which is uniformly L^2 under Assumption 2. Interestingly, this would not change the resulting relation (123).

Turning to the second equality of (122) and considering again a test function of the form $\phi(x,y) = \varphi(x)w(y)$ in $C_0^1(\Omega)$ we rewrite the RHS using (117) and (118), as

$$-\int \widetilde{B}_3^\nu \partial_x \phi = -\int \left(\gamma^\nu B_3^\nu - \beta^\nu \left(E_2^\nu - \partial_y \left(\frac{B_3^\nu}{i\delta} \right) \right) \right) w\varphi' = \int \left(B_3^\nu \left(\frac{1}{i\delta} \partial_y(\beta^\nu w) - \gamma^\nu w \right) + E_2^\nu (\beta^\nu w) \right) \varphi'.$$

In other terms, we have

$$-\int \widetilde{B}_3^\nu \partial_x \phi = -\int (B_3^\nu F_2^\nu - E_2^\nu C_3^\nu) \varphi'$$

with

$$\begin{cases} F_2^\nu = -\frac{1}{i\delta} \partial_y(\beta^\nu w) + \gamma^\nu w = -\frac{1}{i\delta} \partial_y \left(w \frac{\sigma^2}{i\sigma'} \left(1 - (rx + i\nu)^{-\frac{i\sigma'}{r}} \right) \right) + w(rx + i\nu)^{-\frac{i\sigma'}{r}} \\ C_3^\nu = \beta^\nu w = w \frac{\sigma^2}{i\sigma'} \left(1 - (rx + i\nu)^{-\frac{i\sigma'}{r}} \right). \end{cases}$$

We next combine (119)-(120) and (124) into

$$\partial_x \widetilde{B}_3^\nu = B_3^\nu\left(\zeta^\nu - \beta^\nu\right) + E_2^\nu\left(\eta^\nu - \gamma^\nu(\alpha+i\nu)\right) - (\partial_y B_3^\nu)\frac{\eta^\nu}{i\delta} - \beta^\nu \partial_y\left(\frac{\alpha+i\nu}{i\delta}E_2^\nu - B_3^\nu \partial_x\left(\frac{1}{i\delta}\right)\right)$$

which we plug in the LHS of (122). Integrating by parts the appropriate terms along y gives

$$\int (\partial_x \widetilde{B}_3^\nu)\phi = -\int \left(q_3^\nu B_3^\nu - g_2^\nu E_2^\nu\right)\varphi$$

with

$$\begin{cases} g_2^\nu = \Big(\eta^\nu - \gamma^\nu(\alpha+i\nu)\Big)w + \left(\dfrac{\alpha+i\nu}{i\delta}\right)\partial_y\left(w\beta^\nu\right) \\ q_3^\nu = (\beta^\nu - \zeta^\nu)w - \partial_y\left(\dfrac{w\eta^\nu}{i\delta}\right) + \partial_x\left(\dfrac{1}{i\delta}\right)\partial_y\left(w\beta^\nu\right). \end{cases}$$

Thus we find an integral relation of the same form than the one derived above with the second manufactured solution. In fact we can verify that both would be exactly the same if we had used $i\sigma w$ instead of w in the first line of the constraints (88).

6.5 Short conclusion in 2D

It remains to prove that, with convenient boundary conditions, formulations like (109) are well posed. If the coefficients are constant (90) the analysis can be done using a Fourier decomposition in the vertical direction. In this case it can be reasonably anticipated that the one dimensional problem is well posed Fourier mode per Fourier mode, which turns into the well posedness of the 2D problem. The general case is fully open.

A natural conjecture is that resonant multiD solutions have the same singularities than the singular manufactured solutions described in formulas (99-100) and in Figures 1, 2 and 3. This conjecture is natural since it is already the case in dimension one. The use of manufactured solutions, weak formulations and auxiliary variables for numerical discretization is a natural extension of the present work.

7 Conclusion

We have proposed several formulations to characterize the vanishing absorption solutions to the time-harmonic Maxwell equations in the presence of a resonant dielectric tensor. These formulations consist of complementing the original Maxwell system with additional constraints that take the form of either standard weak relations or dissipative inequalities reminiscent of entropy techniques. They are derived following a constructive approach where explicit functions called manufactured solutions are designed to reproduce the singular behavior of the limit resonant fields. In the 1D case we prove that these additional relations allow to complement the resonant Maxwell equations into a series of well-posed problems having the same solution, and in the 2D case we propose an explicit design of manufactured solutions that seem to exhibit original features for the resonant solutions. The proposed formulations provide a convenient framework for numerical approximations, which will be addressed in a further work.

A Basic coercive inequalities for the regularized problem (2)

The standard identity

$$\nabla\times\mathbf{K}\cdot\mathbf{G} - \mathbf{K}\cdot\nabla\times\mathbf{G} = \nabla\cdot(\mathbf{K}\times\mathbf{G}), \quad \mathbf{K},\mathbf{G}\in H(\mathrm{curl},\Omega) \tag{125}$$

yields that $\mathbf{E}^\nu$ solution of (2) admits after integration by parts against $\overline{\mathbf{E}^\nu}$ the integral identity

$$\|\nabla\times\mathbf{E}^\nu\|^2_{L^2(\Omega)} + \int_\Gamma \left((\nabla\times\mathbf{E}^\nu)\times\overline{\mathbf{E}^\nu}\right)\cdot\mathbf{n}\,d\sigma - \left(\underline{\underline{\varepsilon}}\mathbf{E}^\nu,\mathbf{E}^\nu\right)_{L^2(\Omega)} - i\nu\,\|\mathbf{E}^\nu\|^2_{L^2(\Omega)} = 0.$$

Using the fact that $\mathbf{V}\times\mathbf{n} = -((\mathbf{V}\times\mathbf{n})\times\mathbf{n})\times\mathbf{n}$ for all $\mathbf{V}\in\mathbb{C}^3$, and the boundary condition, one has

$$(\nabla\times\mathbf{E}^\nu)\times\overline{\mathbf{E}^\nu}\cdot\mathbf{n} = (\overline{\mathbf{E}^\nu}\times\mathbf{n})\cdot(\nabla\times\mathbf{E}^\nu) = -(\overline{\mathbf{E}^\nu}\times\mathbf{n})\cdot(((\nabla\times\mathbf{E}^\nu)\times\mathbf{n})\times\mathbf{n}) = -i\lambda\left|\overline{\mathbf{E}^\nu}\times\mathbf{n}\right|^2 + (\overline{\mathbf{E}^\nu}\times\mathbf{n})\cdot(\mathbf{f}\times\mathbf{n})$$

so that

$$\|\nabla \times \mathbf{E}^\nu\|^2_{L^2(\Omega)} - \left(\underline{\underline{\varepsilon}}\mathbf{E}^\nu, \mathbf{E}^\nu\right)_{L^2(\Omega)} - i\nu \|\mathbf{E}^\nu\|^2_{L^2(\Omega)} - i\lambda \|\mathbf{E}^\nu \times \mathbf{n}\|^2_{L^2(\Gamma)} = -\left(\mathbf{E}^\nu \times \mathbf{n}, \mathbf{f} \times \mathbf{n}\right)_{L^2(\Gamma)}. \qquad (126)$$

Notice that $\left(\underline{\underline{\varepsilon}}\mathbf{E}^\nu, \mathbf{E}^\nu\right)_{L^2(\Omega)} \in \mathbb{R}$ due to the hermitianity of the dielectric tensor. Since ν and λ are both positive, it immediatly yields a control of the imaginary terms under the form

$$\|\mathbf{E}^\nu \times \mathbf{n}\|_{L^2(\Gamma)} \le \frac{1}{\lambda} \|\mathbf{f} \times \mathbf{n}\|_{L^2(\Gamma)}, \qquad \|\mathbf{E}^\nu\|_{L^2(\Omega)} \le \frac{1}{\sqrt{\lambda\nu}} \|\mathbf{f} \times \mathbf{n}\|_{L^2(\Gamma)}.$$

By taking the real part one has $\|\nabla \times \mathbf{E}^\nu\|^2_{L^2(\Omega)} \le \|\underline{\underline{\varepsilon}}\|_{L^\infty(\Omega)} \|\mathbf{E}^\nu\|^2_{L^2(\Omega)} + \|\mathbf{E}^\nu \times \mathbf{n}\|_{L^2(\Gamma)} \|\mathbf{f} \times \mathbf{n}\|_{L^2(\Gamma)}$, hence

$$\|\nabla \times \mathbf{E}^\nu\|_{L^2(\Omega)} \le \left(\sqrt{\frac{\|\underline{\underline{\varepsilon}}\|_{L^\infty(\Omega)}}{\nu}} + 1\right) \frac{1}{\sqrt{\lambda}} \|\mathbf{f} \times \mathbf{n}\|_{L^2(\Gamma)}.$$

These a priori inequalities are enough to invoke the Lax-Milgram theorem, so the problem (2) with regularization parameter $\nu > 0$ is well posed in $H(\mathrm{curl}, \Omega, \Gamma)$ (existence and uniqueness of the solution).

References

[1] F. Assous, P. Ciarlet Jr, E. Garcia, J. Segré, Time-dependent Maxwell's equations with charges in singular geometries. Computer Methods in Applied Mechanics and Engineering, Elsevier, 2006, 196 (1-3), pp.665-681.

[2] A. Back, T. Hattori, S. Labrunie, J.-R. Roche and P. Bertrand, Electromagnetic wave propagation and absorption in magnetised plasmas: variational formulations and domain decomposition, M2AN Volume 49, Number 5, 2015.

[3] A. Bensalah and P. Joly, private communication, 2015.

[4] A.S. Bonnet-Ben Dhia, L. Chesnel and X. Claeys, Radiation condition for a non-smooth interface between a dielectric and a metamaterial. Math. Models Methods Appl. Sci. 23 (2013), no. 9, 1629-1662.

[5] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2011.

[6] A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr, T-coercivity for scalar interface problems between dielectrics and metamaterials. ESAIM: Mathematical Modeling and Numerical Analysis, EDP Sciences, 2012, 46, pp.1363-1387.

[7] K.G. Budden, Radio Waves in the Ionosphere, Cambridge University Press, 1961.

[8] P. Ciarlet, The finite element method for elliptic problems, North-Holland, 1978.

[9] P. Ciarlet Jr, T-coercivity: Application to the discretization of Helmholtz-like problems. Computers and Mathematics with Applications, Elsevier, 2012, 64 (1), pp.22-34.

[10] L. Chesnel, X. Claeys and S. Nazarov, A curious instability phenomenon for a rounded corner in presence of a negative material. Asymptot. Anal. 88 (2014), no. 1-2, 43-74.

[11] M. Costabel, M. Dauge and S. Nicaise, Corner singularities of Maxwell interface and eddy current problems. Operator theoretical methods and applications to mathematical physics, 241-256, Oper. Theory Adv. Appl., 147, Birkhuser, Basel, 2004.

[12] B. Després, L.M. Imbert-Gérard and O. Lafitte, Singular solutions for the plasma at the resonance, Hal preprint https://hal.archives-ouvertes.fr/hal-01097364

[13] B. Després, L.M. Imbert-Gérard and R. Weder, Hybrid resonance of Maxwell's equations in slab geometry. J. Math. Pures Appl. (9) 101 (2014), no. 5, 623-659.

[14] B. Després, F. Lagoutière and N. Seguin, Weak solutions to Friedrichs systems with convex constraints. Nonlinearity 24 (2011), no. 11, 3055-3081.

[15] P. Grisvard, Singularities in boundary value problems. Recherches en Mathématiques Appliquées, 22. Masson, Paris, Springer-Verlag, Berlin, 1992.

[16] E. Godlewski and P.A. Raviart, Numerical approximation of hyperbolic systems of conservation laws. Applied Mathematical Sciences, 118. Springer-Verlag, New York, 1996.

[17] R. Hiptmair, A. Moiola and I. Perugia, Stability results for the time-harmonic Maxwell equations with impedance boundary conditions. Math. Models Methods Appl. Sci., 21 (2011), no. 11, 2263-2287.

[18] P. Monk, Finite element methods for Maxwell's equations. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

[19] J.C. Nédélec, Mixed finite elements in $\mathbb{R}^3$, Numerische Mathematik (1980).

[20] H.-M. Nguyen, Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients, Arxiv preprint http://arxiv.org/abs/1507.01730.

[21] D.G. Swanson, Plasma Waves, Bristol Philadelphia, Institute of Physics Pub., 2003.

[22] R.B. White and F.F. Chen, Amplification and absorption of electromagnetic waves in overdense plasmas, Plasma Physics, 16 (1974), pp. 565–587.