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Hole probability for nodal sets of the cut-off Gaussian Free Field

Alejandro Rivera

September 1, 2016

Abstract

Let (Σ, g) be a closed connected surface equipped with a riemannian metric. Let $(\lambda_n)_{n \in \mathbb{N}}$ and $(\psi_n)_{n \in \mathbb{N}}$ be the increasing sequence of eigenvalues and the sequence of corresponding L^2 -normalized eigenfunctions of the laplacian on Σ . For each $L > 0$, we consider $\phi_L = \sum_{0 < \lambda_n \leq L} \frac{\xi_n}{\sqrt{\lambda_n}} \psi_n$ where the ξ_n are i.i.d centered gaussians with variance 1. As $L \rightarrow \infty$, ϕ_L converges a.s. to the Gaussian Free Field on Σ in the sense of distributions. We first compute the asymptotic behavior of the covariance function for this family of fields as $L \rightarrow \infty$. We then use this result to obtain the asymptotics of the probability that ϕ_L is positive on a given open proper subset with smooth boundary. In doing so, we also prove the concentration of the supremum of ϕ_L around $\frac{1}{\sqrt{2\pi}} \ln L$.

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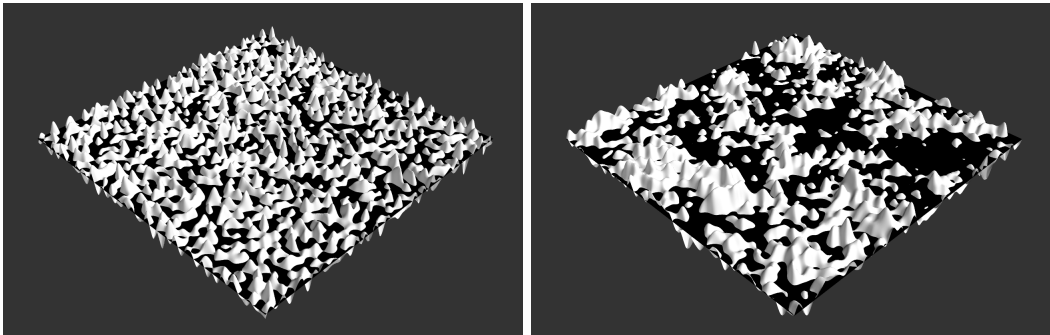
1 Introduction

1.1 Setting and main results

In recent years, there have been many developments in the study of random linear combinations of eigenfunctions of the laplacian on a closed manifold. In this paper we consider a different model with strong ties to statistical mechanics, mentioned both in [22] (Problem 2.4) and [28] (equation (97)). Let (Σ, g) be a smooth compact connected surface equipped with a riemannian metric. Let $\Delta = d^*d$ be the Laplace operator on Σ associated to g and $|dV_g|$ the volume density defined by g . Note that with our convention, if Σ is the flat torus with coordinates (x_1, x_2) , $\Delta = -\partial_{x_1}^2 - \partial_{x_2}^2$. For each $L > 0$, let U_L be the real vector space spanned by the eigenfunctions of Δ whose eigenvalues are positive and smaller than L . Then, U_L is finite dimensional and

$$(u, v) \mapsto \int_{\Sigma} g(\nabla u, \nabla v) |dV_g|$$

defines a scalar product on each U_L which induces a gaussian probability distribution on U_L . For each $L > 0$, let ϕ_L be random variable chosen with this distribution. Then, each instance of ϕ_L is a smooth function on Σ . In particular, for each $L > 0$, $(\phi_L(x))_{x \in \Sigma}$ defines a gaussian field on Σ . We will see that the field ϕ_L converges $L \rightarrow \infty$ almost surely in the sense of distributions to the Gaussian Free Field on Σ , a central object in contemporary statistical mechanics. Following [28], we choose to call ϕ_L **the cut-off Gaussian Free Field** on Σ or CGFF for short. While the definition of the CGFF is formally similar to that of the usual cut-off eigenfunction model (see for instance [21]), it is actually quite different. Indeed, while the cut-off model exhibits a local scale with polynomial correlations, the CGFF has global logarithmic correlations. We will prove that it is actually much closer to the discrete Gaussian Free Field. For this purpose we will combine methods from statistical mechanics and random geometry, thus creating a new interface between the two subjects.



An instance of the cut-off field (resp. the CGFF) on the left (resp. right) on the flat torus with $L = 1000$. The field is colored in white and the black surface is a horizontal square at height zero.

Let $D \subset \Sigma$ be a non-empty proper open subset of Σ with smooth boundary. We ask what the probability is that the field stays positive on D . The asymptotic behavior of this probability as $L \rightarrow \infty$ will be expressed in terms of **the capacity** of D , which we define as the infimum of the quantity $\frac{1}{2}\|\nabla h\|_2^2$ taken over all $h \in C^\infty(\Sigma)$ with zero mean, such that $\forall x \in D, h(x) \geq 1$ and which we denote by $cap_\Sigma(D)$ or $cap(D)$ when there is no ambiguity. More precisely, we prove the following theorem.

Theorem 1. *Let (Σ, g) be a smooth compact connected surface without boundary and let $(\phi_L)_{L>0}$ be the CGFF on (Σ, g) . Let D be a non-empty proper open subset of Σ with smooth boundary. Then,*

$$\lim_{L \rightarrow \infty} \frac{\ln(\mathbb{P}(\forall x \in D, \phi_L(x) > 0))}{\ln^2(\sqrt{L})} = -\frac{2}{\pi} cap(D).$$

Along the way, we will also prove that $cap(D) > 0$. The assumption that Σ has no boundary follows the tradition of the study of random sums of eigenfunctions. However, **all of the results presented in this paper stay valid in the case where Σ has a boundary** with minor modifications and provided we change the definitions of the CGFF and capacity accordingly (see section 5 for the corresponding statements). This is especially significant for two reasons. First, from Riemann's mapping theorem, two non-empty simply connected proper subsets of \mathbb{C} are conformally equivalent. Second, in this setting $cap(D)$ will be **conformally invariant** (see section 5 for more details). To the best of our knowledge, this is the first time a non-trivial conformal invariant emerges from the asymptotics of random sums of eigenfunctions. The event of staying positive on a given set has been studied before in the case of sections of complex line bundles on complex manifolds (see [26]) and in the case of the discrete Gaussian Free Field (or DGFF) on a box in the square lattice (see [6]). In [26], Shiffman, Zrebiec and Zelditch actually prove much stronger results relying on large deviation estimates that work because the field they consider has exponential decay in correlations. As will be apparent in the statement of Theorem 3, this is not the case in our model, which, like [6], has logarithmic correlations. In this article, Bolthausen, Deuschel and Giacomin prove the following result

Theorem 2. *Let $V_N = \{1, \dots, N\}^2$ be a square box in \mathbb{Z}^2 . Let ϕ_N be the discrete Gaussian Free Field on V_N with Dirichlet boundary conditions. Let $D \subset [0, 1]^2$ be an open subset with smooth boundary and at positive distance of $\partial[0, 1]^2$. Let D_N be the set of points $y \in V_N$ such that $\frac{1}{N}y \in D$. Then,*

$$\lim_{N \rightarrow \infty} \frac{1}{(\ln N)^2} \ln\left(\mathbb{P}(\forall x \in D_N, \phi_N(x) \geq 0)\right) = -\frac{8}{\pi} cap_V(D).$$

Here, $cap_V(D)$ is the infimum of $\frac{1}{2}\|\nabla h\|_2^2$ over all the $h \in C^\infty(V)$ with compact support in \mathring{V} such that $h \geq 1$ in D . In Theorem 1, the connectedness assumption simplifies the proofs and is not very restrictive since the CGFF is independent between different components. The assumption that D be an open set with smooth

boundary allows us to use classical results concerning the potential $\text{cap}(D)$ and is already present in the discrete setting. Finally, since the field we consider has zero mean, it cannot stay positive on $D = \Sigma$ so we assume $D \neq \Sigma$. We keep the square root inside the logarithm in the statement because it emerges from the proof as a more natural scale in this problem. Our approach follows the structure of [6]. However, we consider fields in a continuous setting, and more importantly, unlike the DGFF, the CGFF does not seem to have a Markov property. The relation with [6] as well as the strategies employed to deal with these issues will be explained below. Finally, we need to estimate the covariance function of the field. This step is central in our strategy since it is only through this object that we can manipulate the field. We prove the following theorem, which is significant in its own right.

Theorem 3. *Let (Σ, g) be a compact riemannian surface without boundary and let $(\phi_L)_{L>0}$ be the CGFF on (Σ, g) . For each $L > 0$ and $p, q \in \Sigma$, let*

$$G_L(p, q) = \mathbb{E}[\phi_L(p)\phi_L(q)].$$

Then, there exists $\varepsilon > 0$ such that for each $p, q \in \Sigma$ satisfying $d_g(p, q) \leq \varepsilon$ and for each $L > 0$,

$$G_L(p, q) = \frac{1}{2\pi} \left(\ln(\sqrt{L}) - \ln_+(\sqrt{L}d_g(p, q)) \right) + \rho_L(p, q)$$

where $\ln_+(a) = \max(\ln(a), 0)$, d_g is the riemannian distance and $\rho_L(p, q)$ is bounded uniformly with respect to p, q and L .

The proof of this theorem relies on Hörmander's estimates for the spectral kernel of an elliptic operator in [16]. The result is reminiscent of the well known analogue for the DGFF (see for instance Lemma 2.2 of [7]). Aside from Theorem 3, one important step in the proof of Theorem 1 is to control the supremum of the field on a given domain. We prove the following theorem.

Theorem 4. *Let (Σ, g) be a smooth compact riemannian surface without boundary and let $(\phi_L)_{L>0}$ be the CGFF on (Σ, g) . Let D be a non-empty open subset of Σ . Then for each $\eta > 0$,*

$$\limsup_{L \rightarrow \infty} \frac{\ln \left(\mathbb{P} \left(\sup_{\Sigma} \phi_L > \left(\sqrt{\frac{2}{\pi}} + \eta \right) \ln(\sqrt{L}) \right) \right)}{\ln(\sqrt{L})} \leq -2\sqrt{2\pi}\eta + O(\eta^2)$$

and there exists $a > 0$ such that for L large enough,

$$\mathbb{P} \left(\sup_D \phi_L \leq \left(\sqrt{\frac{2}{\pi}} - \eta \right) \ln(\sqrt{L}) \right) \leq \exp \left(-a \ln^2(\sqrt{L}) \right).$$

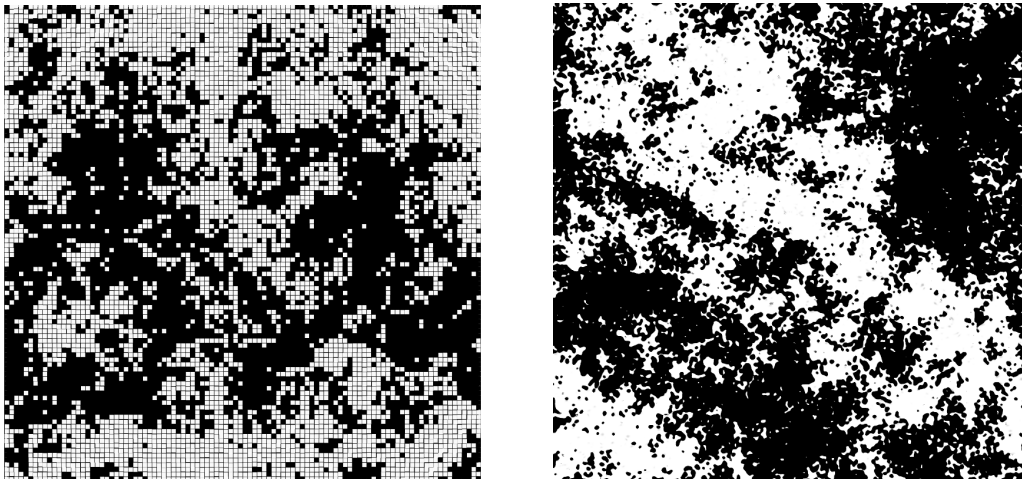
The maxima of random fields on smooth manifolds have been studied, for instance for holomorphic sections of line bundles on Kähler manifolds in [25] and for

another eigenfunction model in [9]. However, these fields are not log-correlated so the probabilistic arguments employed are quite different. This theorem is an analogue of Theorem 2 in [6] and the proof relies on it. The supremum of general discrete log-correlated fields has been studied in [10]. In the case of the DGFF, as well as a large class of continuous log-correlated fields, the law of the supremum has been studied with much higher precision, see for instance [7], [8], [4], [1], [10] and [18]. To obtain such results for the CGFF, one would need more precise estimates for the covariance function. Note that while [1] and [18] deal with continuous fields similar to the CGFF, the left tail estimate in Theorem 4 does not appear in these works.

The paper is organised as follows. In the rest of this section, we give an outline of the proof and introduce some basic notation in order to give a more concrete definition of the CGFF. In section 2 we prove Theorem 4. In section 3 we prove Theorem 1. Section 4 is dedicated to the proof of the analytical tools used before, most notably, Theorem 3. In section 5, we cover the case where Σ has a boundary. In the appendix 6, we recall some classical results regarding the laplacian and the capacity.

1.2 Comparison with the discrete setting

Part of this paper is written in the spirit of [6] which studies the hole probability of the discrete Gaussian Free Field. In this section we outline our proof strategy with [6] in mind and explain the new ideas introduced to deal with this model. We will use the notations introduced in Theorem 2.



On the left, an instance of the DGFF on the square of side $N = 100$ with periodic boundary conditions. On the right, an instance of the CGFF on the flat torus with $\sqrt{L} = 100$. In both cases, the square is colored white where the field is positive and black where it is negative.

To begin with, the field we consider is defined as a random linear combination of eigenfunctions of the laplacian on a compact surface Σ and some work is required to obtain a tractable expression for the covariance function. This is Theorem 3 and is proved in section 4. In particular we use Hörmander’s estimates of **the spectral function of the laplacian** from [16]. The analog in the discrete case is taken for granted in [6]. Since this part uses different techniques than the rest of the paper, we present it at the end. We first relate the kernel G_L to the Schwartz kernel of the orthogonal projector onto U_L in order to apply the aforementioned result by Hörmander. This yields an expression of G_L as an integral involving a kind of generalized Bessel function, which we control thanks to **the stationary phase method**. It emerges from Theorem 3 that the CGFF (ϕ_L) is a log-correlated field that varies at a scale of $L^{-1/2}$. Consequently, in the analogy with [6], Σ will play the role of the square box of size N where $N \simeq L^{1/2}$. This is why we choose to write $\ln(\sqrt{L})$ instead of $\frac{1}{2} \ln L$ in our main results above.

The proof of Theorem 1 goes as follows. The first step is to estimate the supremum of the field. This is the object of section 2 and the result is Theorem 4. In [6], the bound on the right tail comes from a simple union bound. In our case, since the space is continuous, the field could fluctuate at scales smaller than $L^{-1/2}$. To control these fluctuations, we use **a Sobolev inequality** on small disks of this scale (see Lemma 6). Note that this requires control of the successive derivatives of the field. The result then follows from a union bound applied to a covering of Σ by such disks (see Proposition 5). To control the left tail of the maximum, Bolthausen, Deuschel and Giacomin use **the Markov property** of the field to construct a tree-like structure and, inspired by branching random walks, use large deviation results to conclude. At this point we cannot follow the original proof because the CGFF does not seem to have a Markov property. Instead, we restrict the CGFF to a discrete box and use a method from the much more recent [10] in order to apply a gaussian comparison inequality between the restricted field and the DGFF. Thus we recover the bound from the original one (see Proposition 9).

Once we have Theorem 4, we can start studying the probability that the CGFF stays positive on a given domain D . This is section 3. For the lower bound, [6] uses an entropy inequality and the capacity appears by discrete approximation. In our case (see Proposition 16) it seemed more natural to apply **the barrier method**, already used in [19] and [14]. The idea is to decompose the field into a random multiple of a function h that is greater than one on D and an independent fluctuation. Then, we use the bound on the right tail from Theorem 4 to control the supremum of the fluctuation. We then vary h to minimize the cost of this procedure and end up with the capacity of D . The lower bound (see Proposition 17) is more subtle. Indeed, Bolthausen, Deuschel and Giacomin use the Markov property once again to decompose the DGFF (ϕ_N) into two independent gaussian fields. One is “tamer” while the other is “wilder”. We call this **the two-scale decomposition** of the

CGFF. Roughly speaking, to stop the wilder field from making ϕ_N negative, the tamer field will need to be close to the expected maximum on a large enough portion of D . This will come at a cost that will be related to the capacity. In our paper the Markov property is once again absent. To construct this decomposition, we split the eigenvalue interval in two and obtain a decomposition of ϕ_L as an independent sum. The tameness of the tamer field will be immediate. On the other hand the wilder field will require a finer analysis. We will first prove an approximate version of the independence afforded by the Markov property of the DGFF in this decomposition. Then, we will apply once again **the gaussian comparison method** from [10] to conclude.

For the convenience of the reader, we summarize the above discussion in the following table.

Proof step	Discrete case	Continuous case
Covariance function	\emptyset	spectral asymptotics
Right tail for the supremum	union bound	Sobolev inequality + union bound
Left tail of the maximum	Markov property + large deviations	gaussian comparison method
Lower bound for hole probability	entropy inequality	barrier method
Upper bound for hole probability	Markov property + two-scale decomposition	two-scale decomposition + decorrelation estimates + gaussian comparison method

1.3 The cut-off Gaussian Free Field

From now on, (Σ, g) will be a smooth, compact, connected surface without boundary, equipped with a riemannian metric. Let $|dV_g|$ be the density and $\Delta = d^*d$ the Laplace operator defined by g . We will denote by $L^2(\Sigma)$ and $H^m(\Sigma)$ for any integer $m \geq 1$ respectively the space of square integrable functions over Σ with respect to the measure $|dV_g|$ and the L^2 Sobolev space of order m with respect to this same measure (see for instance Definition B.1.1 of [17]). For any of these spaces, say $E(\Sigma)$, we will denote by $E_0(\Sigma)$ – or E_0 when no ambiguity is possible – the subspace of $E(\Sigma)$ consisting of functions of zero mean on Σ . We will denote by $\langle \cdot, \cdot \rangle_2$ the L^2 scalar product on Σ . We will use the same notation in the following case. If X, Y are two vector-fields on Σ , $\langle X, Y \rangle_2 = \int_{\Sigma} g_p(X_p, Y_p) |dV_g|(p)$. By the Poincaré-Wirtinger inequality (see Theorem 1 of section 5.8.1 of [11]) the bilinear form

$$\langle u, v \rangle_{\nabla} := \langle \nabla u, \nabla v \rangle_2.$$

defines a scalar product equivalent to the standard one on $H_0^1(\Sigma)$ called the Dirichlet inner product. It is well known (see for instance Theorem 4.43 of [13]) that there exist $(\psi_n)_{n \in \mathbb{N}} \in C^\infty(\Sigma)^{\mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots, \lambda_n \xrightarrow[n \rightarrow \infty]{} \infty$

$+\infty$, such that $(\psi_n)_{n \in \mathbb{N}}$ is a Hilbert basis for $L^2(\Sigma)$ and such that for each $n \in \mathbb{N}$, $\Delta \psi_n = \lambda_n \psi_n$. In addition, ψ_0 is constant and, consequently, $\forall n \geq 1$, $\psi_n \in C_0^\infty(\Sigma)$. Stokes' theorem shows that $\left(\frac{1}{\sqrt{\lambda_n}} \psi_n\right)_{n \geq 1}$ is a Hilbert basis of $(H_0^1, \langle \cdot, \cdot \rangle_\nabla)$. For each $L > 0$, let $(U_L, \langle \cdot, \cdot \rangle_\nabla)$ be the subspace of $(H_0^1, \langle \cdot, \cdot \rangle_\nabla)$ spanned by the functions $\frac{1}{\sqrt{\lambda_n}} \psi_n$ such that $0 < \lambda_n \leq L$. Let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d real centered gaussian random variables with variance 1. Then, for each $L > 0$ we define the **cut-off Gaussian Free Field**, or CGFF, as

$$\phi_L = \sum_{0 < \lambda_n \leq L} \frac{\xi_n}{\sqrt{\lambda_n}} \psi_n. \quad (1)$$

Hence, for each $L > 0$ and $p, q \in \Sigma$,

$$G_L(p, q) := \mathbb{E}[\phi_L(p)\phi_L(q)] = \sum_{0 < \lambda_n \leq L} \frac{1}{\lambda_n} \psi_n(p)\psi_n(q). \quad (2)$$

Note that defining the CGFF by equation (1) amounts to saying that it is a random function in U_L with probability density proportional to $e^{-\frac{1}{2}\|\phi\|_\nabla^2} d\phi$ where $d\phi$ is the Lebesgue measure on $(U_L, \langle \cdot, \cdot \rangle_\nabla)$. These definitions imply in particular that ϕ_L converges almost surely to the Gaussian Free Field in the sense of distributions as $L \rightarrow \infty$ (see for instance section 2.4 of [24]).

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2 The maximum of the CGFF

The aim of this section is to prove Theorem 4. The proof is split in two parts, one for the right tail of the maximum and one for the left. More precisely, Theorem 4 follows immediately from Proposition 5 and Proposition 9 below.

2.1 Binding the right tail the maximum

The aim of this section is to prove the following proposition.

Proposition 5. *Let (Σ, g) be a smooth compact surface and let $(\phi_L)_L$ be the CGFF on (Σ, g) . Then, for each $\eta > 0$,*

$$\limsup_{L \rightarrow \infty} \frac{\ln \left(\mathbb{P} \left(\sup_{\Sigma} \phi_L > \left(\sqrt{\frac{2}{\pi}} + \eta \right) \ln(\sqrt{L}) \right) \right)}{\ln(\sqrt{L})} \leq -2\sqrt{2\pi}\eta + O(\eta^2).$$

Let us begin by introducing some notation. For each $p \in \Sigma$, $t > 0$ and $L > 0$, let $D_L(p, t)$ be the riemannian disk of radius $\frac{t}{\sqrt{L}}$ around p . For the proof of Proposition 5, we will need the following two results.

Lemma 6. *Let (Σ, g) be a smooth compact surface and let $(\phi_L)_L$ be the CGFF on (Σ, g) . Then there is a constant $C > 0$ such that for each $p \in \Sigma$ and for $L > 0$ large enough,*

$$\begin{aligned} \mathbb{E} \left[\sup_{D_L(p,1)} |\phi_L| \right] &\leq C\sqrt{\ln(L)} \\ \mathbb{E} \left[\sup_{D_L(p,1)} |\nabla \phi_L| \right] &\leq C\sqrt{L}. \end{aligned}$$

This lemma is to be compared with Proposition 2.1 of [14]. We postpone its proof till the end of the section. The second result is Theorem 2.1.1 of [2] specialised to continuous fields.

Proposition 7 (Borell-Tsirelson-Ibragimov-Sudakov inequality). *Let T be a separable topological space and $(\phi_t)_{t \in T}$ be a centered gaussian field over T which is almost surely bounded and continuous. Then, $\mathbb{E}[\sup_{t \in T} \phi_t] < \infty$ and for all $u > 0$,*

$$\mathbb{P} \left(\sup_{t \in T} \phi_t - \mathbb{E}[\sup_{t \in T} \phi_t] > u \right) \leq \exp \left(-\frac{u^2}{2\sigma_T^2} \right)$$

where $\sigma_T^2 = \sup_{t \in T} \text{Var}(\phi_t)$.

Let us now prove Proposition 5 using Lemma 6 and Proposition 7.

Proof of Proposition 5. According to Theorem 3 for $x = y$, there is a constant C such that for each $p \in \Sigma$,

$$\sup_{q \in D_L(p,1)} \text{Var}(\phi_L(q)) = \sup_{q \in D_L(p,1)} G_L(q, q) \leq \frac{1}{2\pi} \ln(\sqrt{L}) + C.$$

Let $\eta > 0$. We apply Proposition 7 to $(\phi_L(q))_{q \in D_L(p,1)}$ using Lemma 6 to deduce that for each $0 < \eta' < \eta$, for L large enough and for all $p \in \Sigma$,

$$\mathbb{P} \left(\sup_{D_L(p,1)} \phi_L > \left(\sqrt{\frac{2}{\pi}} + \eta \right) \ln(\sqrt{L}) \right) \leq \exp \left(- \left(2 + 2\sqrt{2\pi}\eta' + O(\eta'^2) \right) \ln(\sqrt{L}) \right).$$

Choose some $0 < \eta' < \eta$. Since Σ is compact, there exists another constant which we also denote by C , such that for each L , Σ is covered by CL disks of radius $\frac{1}{\sqrt{L}}$. Thus,

$$\begin{aligned} \mathbb{P}\left(\sup_{\Sigma} \phi_L > \left(\sqrt{\frac{2}{\pi}} + \eta\right) \ln(\sqrt{L})\right) &\leq CL \exp\left(-\left(2 + 2\sqrt{2\pi}\eta' + O(\eta'^2)\right) \ln(\sqrt{L})\right) \\ &\leq C(\sqrt{L})^{-2\sqrt{2\pi}\eta' + O(\eta'^2)}. \end{aligned}$$

Since $-\phi_L$ has the same law as ϕ_L , we have the analogous result for the minimum. Therefore, for each $0 < \eta' < \eta$ there is an L_0 such that for each $L \geq L_0$,

$$\mathbb{P}\left(\sup_{\Sigma} |\phi_L| > \left(\sqrt{\frac{2}{\pi}} + \eta\right) \ln(\sqrt{L})\right) \leq (\sqrt{L})^{-2\sqrt{2\pi}\eta' + O(\eta'^2)}.$$

□

To prove Lemma 6, we use Theorem 3 and the following proposition. The proof of both of these results is presented in the last section.

Proposition 8. *Let Q_1 and Q_2 be differential operators on Σ of respective orders d_1 and d_2 and let $d = d_1 + d_2$. Suppose that $d \geq 1$. Then there exists $C > 0$ such that for each $p \in \Sigma$ and $L > 0$,*

$$\left| (Q_1 \otimes Q_2)G_L(p, p) \right| \leq C(1 + L^{d/2}).$$

Proof of Lemma 6. Let $p \in \Sigma$. We apply the Sobolev inequality from paragraph 5.2.4 of [12] with $m = 2$ and $N = 2$. The inequality implies there exist constants $C, L_0 > 0$ such that for all $L \geq L_0$,

$$\begin{aligned} \sup_{q \in D_L(p, 1)} |\phi_L(q)| &\leq C \left[\left(\frac{1}{\text{Vol}(D_L(p, 2))} \int_{D_L(p, 2)} \phi_L(q)^2 |dV_g|(q) \right)^{\frac{1}{2}} \right. \\ &\quad + L^{-\frac{1}{2}} \left(\frac{1}{\text{Vol}(D_L(p, 2))} \int_{D_L(p, 2)} |\nabla \phi_L(q)|^2 |dV_g|(q) \right)^{\frac{1}{2}} \\ &\quad \left. + L^{-1} \left(\frac{1}{\text{Vol}(D_L(p, 2))} \int_{D_L(p, 2)} |\nabla^2 \phi_L(q)|^2 |dV_g|(q) \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Here ∇^2 denotes the hessian defined by the metric g . Note that in order to bind the supremum of a function on a two-dimensional space by L^2 Sobolev norms, one must use derivatives up to order at least two. By compactness, C and L_0 may be chosen independent of p . The same inequality holds for expectations and applying Jensen's

inequality to the right hand side we obtain :

$$\begin{aligned} \mathbb{E}\left[\sup_{q \in D_L(p,1)} |\phi_L(q)|\right] &\leq C \left[\left(\frac{1}{\text{Vol}(D_L(p,2))} \int_{D_L(p,2)} \mathbb{E}[\phi_L(q)^2] |dV_g|(q) \right)^{\frac{1}{2}} \right. \\ &\quad + L^{-\frac{1}{2}} \left(\frac{1}{\text{Vol}(D_L(p,2))} \int_{D_L(p,2)} \mathbb{E}[|\nabla \phi_L(q)|^2] |dV_g|(q) \right)^{\frac{1}{2}} \\ &\quad \left. + L^{-1} \left(\frac{1}{\text{Vol}(D_L(p,2))} \int_{D_L(p,2)} \mathbb{E}[|\nabla^2 \phi_L(q)|^2] |dV_g|(q) \right)^{\frac{1}{2}} \right]. \end{aligned}$$

In the above inequality, for any tensor T , $|T|$ denotes the norm of T induced by g on the corresponding tensor bundle. Since for any differential operator P over Σ , $\mathbb{E}[P\phi_L(q)^2] = (P \otimes P)G_L(q, q)$, we get

$$\begin{aligned} \mathbb{E}\left[\sup_{q \in D_L(p,1)} |\phi_L(q)|\right] &\leq C \left[\left(\frac{1}{\text{Vol}(D_L(p,2))} \int_{D_L(p,2)} G_L(q, q) |dV_g|(q) \right)^{\frac{1}{2}} \right. \\ &\quad + L^{-\frac{1}{2}} \left(\frac{1}{\text{Vol}(D_L(p,2))} \int_{D_L(p,2)} |(\nabla \otimes \nabla)G_L(q, q)| |dV_g|(q) \right)^{\frac{1}{2}} \\ &\quad \left. + L^{-1} \left(\frac{1}{\text{Vol}(D_L(p,2))} \int_{D_L(p,2)} |(\nabla^2 \otimes \nabla^2)G_L(q, q)| |dV_g|(q) \right)^{\frac{1}{2}} \right]. \end{aligned} \tag{3}$$

Now, from Proposition 8, there is a constant $C > 0$ such that for all $p \in \Sigma$ and $L > 0$,

$$|G_L(p, p)| \leq C \ln(L); \quad |(\nabla \otimes \nabla)G_L(p, p)| \leq CL; \quad |(\nabla^2 \otimes \nabla^2)G_L(p, p)| \leq CL^2.$$

Applying these three inequalities to equation (3), we deduce that there is a constant $C > 0$ such that for all $p \in \Sigma$ and $L > 0$,

$$\mathbb{E}\left[\sup_{q \in D_L(p,1)} |\phi_L(q)|\right] \leq C\sqrt{\ln(L)}.$$

This proves the first statement. The proof carries over to the second statement almost verbatim, using in addition the following estimate from Proposition 8.

$$|(\nabla^3 \otimes \nabla^3)G_L(p, p)| \leq CL^3.$$

□

2.2 Binding the left tail of the maximum

In this section we prove the following proposition.

Proposition 9. *Let (Σ, g) be a smooth compact surface and let $(\phi_L)_L$ be the CGFF on (Σ, g) . Let $D \subset \Sigma$ be a non-empty open subset of Σ . Then, for each $\eta > 0$ there is a constant $a > 0$ such that for L large enough,*

$$\mathbb{P}\left(\sup_D \phi_L \leq \left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln(\sqrt{L})\right) \leq \exp\left(-a \ln^2(\sqrt{L})\right).$$

We now introduce some notation. For each $N \in \mathbb{N}_{\geq 1}$, let V_N be the set of points $(x_1, x_2) \in \mathbb{Z}^2$ such that $0 \leq x_1, x_2 \leq N - 1$, let V'_N the set of points in V_N at distance at least $N/4$ from the boundary and let x_N be one of the points nearest to its center. For each $x, y \in V_N$, $|x - y|$ will denote the euclidian distance between x and y . For each $t > 0$, $\ln_+ t$ will denote $\max(\ln t, 0)$.

Proposition 9 will follow from the two following results.

Proposition 10. *Let $(X_t)_{t>1}$ be a family of random fields such that for all $t > 1$, X_t is defined over the box $V_{\lfloor t \rfloor}$. Suppose there is a constant $C > 0$ such that for each $t > 1$ and $x, y \in V_{\lfloor t \rfloor}$,*

$$|\mathbb{E}[X_t(x)X_t(y)] - \ln t + \ln_+ |x - y|| \leq C.$$

Then for each $\eta > 0$ there is a constant $a > 0$ depending only on C and η such that for $t > 1$ large enough,

$$\mathbb{P}\left(\sup_{V_{\lfloor t \rfloor}} X_t \leq (2 - \eta) \ln t\right) \leq \exp(-a(\ln t)^2).$$

Lemma 11. *Fix $0 < \delta < \frac{1}{2\sqrt{2}}$ and $L > 0$. Let $\iota : V_{\lfloor \sqrt{L} \rfloor} \rightarrow \Sigma$ be an injection of the \mathbb{Z}^2 -box of side-length $\lfloor \sqrt{L} \rfloor$ into Σ such that for any distinct $x, y \in V_{\lfloor \sqrt{L} \rfloor}$,*

$$\frac{\delta}{2} \leq \frac{\sqrt{L}d_g(\iota(x), \iota(y))}{|x - y|} \leq 2\delta.$$

Then, there is a constant $C(\delta) > 0$ independent of L and ι such that for L large enough and for each $x, y \in V_{\lfloor \sqrt{L} \rfloor}$,

$$\left| \mathbb{E}[\phi_L(\iota(x))\phi_L(\iota(y))] - \frac{1}{2\pi} \ln(\sqrt{L}) + \ln_+ |x - y| \right| \leq C(\delta).$$

Lemma 11 is just the specialisation of Lemma 14 from section 3 to the case $\alpha = 0$. In the following proof we use Proposition 10 and Lemma 11.

Proof of Proposition 9. Choose some $\delta > 0$ and ι satisfying the properties required to apply Lemma 11 and such that the image of ι is contained in D for L large enough. Then, by Lemma 11, the family $(X_t)_t$ defined by, $\forall t > 0$, $X_t := \sqrt{2\pi}\phi_{t^2} \circ \iota$ satisfies the hypotheses of Proposition 10. In particular, for each $\eta > 0$, there is a constant $a > 0$ such that for all $L > 0$,

$$\begin{aligned} \mathbb{P}(\sup_D \phi_L \leq \left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln(\sqrt{L})) &\leq \mathbb{P}\left(\sup_{V_{\lfloor \sqrt{L} \rfloor}} X_{\sqrt{L}} \leq (2 - \sqrt{2\pi}\eta) \ln(\sqrt{L})\right) \\ &\leq \exp\left(-a \ln^2(\sqrt{L})\right). \end{aligned}$$

□

To prove Proposition 10, we use the two following results. The first is a special case of Theorem 2 (b) of [6].

Theorem 12. *Let $(\phi_N)_N$ be the DGFF on V_N with the standard normalization. For each $\eta > 0$ there is a constant $c > 0$ such that for each $N \in \mathbb{N}$ large enough,*

$$\mathbb{P}\left(\sup_{V'_N} \phi_N \leq \left(\sqrt{\frac{8}{\pi}} - \eta\right) \ln N\right) \leq \exp(-c(\ln N)^2).$$

Lemma 13 (Slepian's Lemma, see Theorem 2.2.1 of [2]). *Let T be a separable topological space and $(Z(p))_{p \in T}$, $(Y(p))_{p \in T}$ be two continuous centered gaussian fields on T satisfying the following two properties.*

1. *For each $p \in T$, $\mathbb{E}[Z(p)^2] = \mathbb{E}[Y(p)^2]$.*
2. *For each $p, q \in T$, $\mathbb{E}[Z(p)Z(q)] \leq \mathbb{E}[Y(p)Y(q)]$.*

Then, for each $u \in \mathbb{R}$,

$$\mathbb{P}(\sup_{p \in T} Z(p) > u) \geq \mathbb{P}(\sup_{p \in T} Y(p) > u).$$

We now deduce Proposition 10 from Proposition 12 and Lemma 13. The following proof is inspired by that of Lemma 2.8 of [10].

Proof of Proposition 10. Choose any $\eta > 0$ and some $j \in \mathbb{N}$ to be fixed later. For each $t > 2^j$, let $N = N(t) = \lfloor 2^{-j}t \rfloor$ and let Z_N be gaussian field defined on V_N by setting, for each $x \in V_N$, $Z_N(x) = X_t(2^j x)$. Then, for distinct $x, y \in V_N$,

$$\begin{aligned} |\mathbb{E}[Z_N(x)^2] - \ln N - j \ln 2| &\leq C \\ |\mathbb{E}[Z_N(x)Z_N(y)] - \ln N + \ln_+ |x - y|| &\leq C. \end{aligned}$$

Let Y_N be the DGFF on V_N multiplied by $\sqrt{\frac{\pi}{2}}$. From Lemma 2.2 of [7] there is a universal constant $C_0 > 0$ such that for any distinct $x, y \in V'_N$,

$$\begin{aligned} |\mathbb{E}[Y_N(x)^2] - \ln N| &\leq C_0 \\ |\mathbb{E}[Y_N(x)Y_N(y)] - \ln N + \ln_+ |x - y|| &\leq C_0. \end{aligned}$$

Since $1/4 < \ln 2 < 1$, there is j_0 depending only on C (and the universal constant C_0) such that for each $j \geq j_0$ and for each $x \in V'_N$,

$$j/4 \leq \mathbb{E}[Z_N(x)^2] - \mathbb{E}[Y_N(x)^2] \leq j.$$

Let

$$a_N(x) = \sqrt{j^{-1}(\mathbb{E}[Z_N(x)^2] - \mathbb{E}[Y_N(x)^2])} \in [1/2, 1]$$

and choose ξ a centered gaussian random variable with variance 1 independent from the fields previously introduced. Then, there is $j \geq j_0$ depending only on C such that for each $x, y \in V'_N$ distinct,

$$\begin{aligned} \mathbb{E}[Z_N(x)^2] &= \mathbb{E}[(Y_N(x) + \sqrt{j}\xi a_N(x))^2] \\ \mathbb{E}[Z_N(x)Z_N(y)] &\leq \mathbb{E}[(Y_N(x) + \sqrt{j}\xi a_N(x))(Y_N(y) + \sqrt{j}\xi a_N(y))]. \end{aligned}$$

Thus, by Lemma 13,

$$\begin{aligned}
\mathbb{P}\left(\sup_{V_{\lfloor t \rfloor}} X_t \leq (2 - \eta) \ln t\right) &\leq \mathbb{P}\left(\sup_{V'_N} Z_N \leq (2 - \eta) \ln t\right) \\
&\leq \mathbb{P}\left(\sup_{x \in V'_N} \left[Y_N(x) + \sqrt{j} \xi a_N(x)\right] \leq (2 - \eta) \ln t\right) \\
&\leq \mathbb{P}\left(\sup_{x \in V'_N} Y_N(x) \leq (2 - (\eta/2)) \ln t\right) + \\
&\quad \mathbb{P}\left(\xi \geq \frac{\eta}{2\sqrt{j} \sup_{V'_N} a_N} \ln t\right).
\end{aligned}$$

For t large enough, $(2 - (\eta/2)) \ln t \leq (2 - (\eta/3)) \ln(N)$. From standard tail estimates for gaussian variables applied to ξ and Theorem 12 applied to $\sqrt{\frac{2}{\pi}} Y_N$, there is a constant $a > 0$ such that for t large enough,

$$\mathbb{P}\left(\sup_{V'_{\lfloor t \rfloor}} X_t \leq (2 - \eta) \ln t\right) \leq \exp(-a(\ln t)^2).$$

Moreover, a depends only on C, η . \square

3 Hole probability for the CGFF

The aim of this section is to prove Theorem 1. From now on, we fix an open proper subset D of Σ with smooth boundary. This assumption implies that there exist functions $f \in C_0^\infty(\Sigma)$ greater or equal to 1 on D . We want to estimate the probability of the event

$$\Omega_L^+ = \left\{ \forall x \in D, \phi_L(x) > 0 \right\}.$$

We divide the proof into two parts, one for the lower bound and one for the upper bound. Theorem 1 will thus follow from immediately from Proposition 16 and Proposition 17 below. Note that, if D is non-empty, then Proposition 29 implies $\text{cap}(D) > 0$.

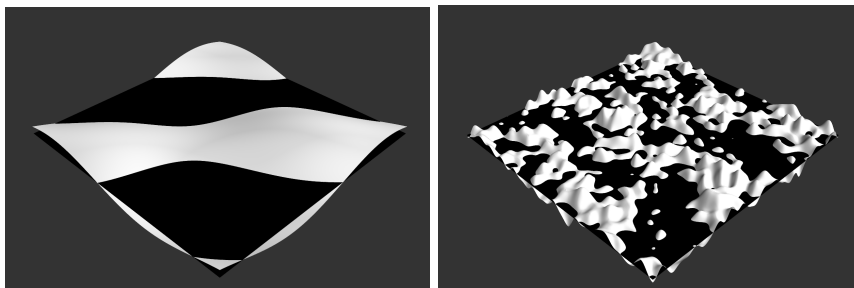
3.1 The two-scale decomposition

In this section, we introduce the two-scale decomposition of the CGFF used below. We will use notations from sections 1.3 and 2. For each $\alpha \in]0, 1]$ and each $L > 0$, we denote by $\psi_{\alpha, L}$ the field $\psi_{\alpha, L} = \sum_{L^\alpha < \lambda_n \leq L} \frac{\xi_n}{\sqrt{\lambda_n}} \psi_n$. Note that

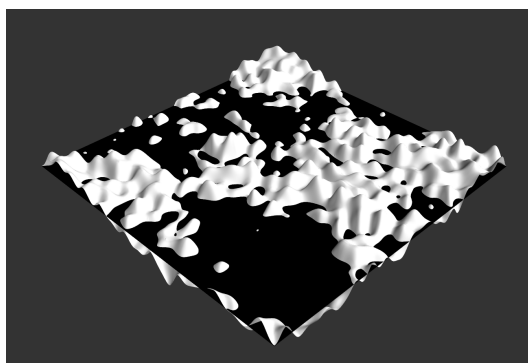
$$\phi_L = \psi_{\alpha, L} + \phi_{L^\alpha} \tag{4}$$

and that $\psi_{\alpha, L}$ and ϕ_{L^α} are independent. Moreover, observe that the two point correlation function of $\psi_{\alpha, L}$ is $G_{L^\alpha, L}(p, q) = \sum_{L^\alpha < \lambda_n \leq L} \frac{1}{\lambda_n} \psi_n(p) \psi_n(q) = G_L(p, q) - G_{L^\alpha}(p, q)$. The asymptotics of $G_{L^\alpha, L}$ will follow easily from those of G_L . The field

ϕ_{L^α} will vary at scale $L^{-\alpha/2}$ while $\psi_{\alpha,L}$ will vary at scale $L^{-1/2}$ and will decorrelate at large distances. While the first fact is immediate, the other two require some justification.



An instance of the field ϕ_{L^α} on the left and of the field $\psi_{\alpha,L}$ on the right for $\alpha = 0.25$ and $L = 400$ on the flat torus. The fields are colored in white and the black surface is a horizontal square at height zero.



The total field $\phi = \phi_{L^\alpha} + \psi_{\alpha,L}$ obtained from the instances above.

We will prove the following two results.

Lemma 14. Fix $0 < \delta < \frac{1}{2\sqrt{2}}$, $\alpha \in [0, 1[$ and $L > 0$. Let $\iota : V_{\lfloor L^{(1-\alpha)/2} \rfloor} \rightarrow \Sigma$ be an injection of the \mathbb{Z}^2 -box of side-length $\lfloor L^{(1-\alpha)/2} \rfloor$ into Σ such that for distinct $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$,

$$\frac{\delta}{2} \leq \frac{\sqrt{L}d_g(\iota(x), \iota(y))}{|x - y|} \leq 2\delta.$$

Then, there is a constant $C(\delta) > 0$ independent of α , L and ι such that for L large enough and for each $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$,

$$|\mathbb{E}[\psi_{\alpha,L}(\iota(x))\psi_{\alpha,L}(\iota(y))] - \frac{1-\alpha}{2\pi} \ln(\sqrt{L}) + \ln_+ |x - y|| \leq C(\delta).$$

This lemma shows that $\psi_{\alpha,L}$ does indeed vary at scale $L^{-1/2}$ and we will use it to prove that its maximum on a box of sidelength $L^{\alpha/2}$ will be close to $\frac{1-\alpha}{2\pi} \ln(\sqrt{L})$. Note that Lemma 11 is just Lemma 14 with $\alpha = 0$ as announced above. Before proving Lemma 14, let us state the second result we will need concerning $\psi_{\alpha,L}$.

Proposition 15. *Choose $0 < \alpha < 1$. For each $\delta > 0$ and $0 < \beta < \alpha$,*

$$G_{L^\alpha,L}(p, q) \xrightarrow{L \rightarrow \infty} 0.$$

This proposition shows that the field decorrelates at large distances. The proof is completely analytical so we leave it for section 4.

Proof of Lemma 14. For L large enough, then $V_L := \iota(V_{\lfloor L^{(1-\alpha)/2} \rfloor})$ has diameter smaller than the ε in the statement of Theorem 3. For each $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$,

$$\begin{aligned} \mathbb{E}[\psi_{\alpha,L}(\iota(x))\psi_{\alpha,L}(\iota(y))] &= G_{L^\alpha,L}(\iota(x), \iota(y)) \\ &= G_L(\iota(x), \iota(y)) - G_{L^\alpha}(\iota(x), \iota(y)). \end{aligned}$$

By Theorem 3 applied for $L' = L$ or $L' = L^\alpha$, there is a constant $C > 0$ such that for each $L > 0$ large enough, for each $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$,

$$G_L(\iota(x), \iota(y)) = \frac{1}{2\pi} \left(\ln(\sqrt{L}) - \ln_+(\sqrt{L}d_g(\iota(x), \iota(y))) \right) + \rho_L^1(x, y).$$

and

$$\begin{aligned} G_{L^\alpha}(\iota(x), \iota(y)) &= \frac{1}{2\pi} \left(\ln(\sqrt{L^\alpha}) - \ln_+(L^{\alpha/2}d_g(\iota(x), \iota(y))) \right) + \rho_L^2(x, y) \\ &= \frac{1}{2\pi} \left(\alpha \ln(\sqrt{L}) - \ln_+(L^{\alpha/2}d_g(\iota(x), \iota(y))) \right) + \rho_L^2(x, y). \end{aligned}$$

where $|\rho_L^j(x, y)| \leq C$ for $j = 1, 2$. Therefore,

$$\left| \mathbb{E}[\psi_L(\iota(x))\psi_L(\iota(y))] - \frac{1-\alpha}{2\pi} \ln(\sqrt{L}) + \ln_+(\sqrt{L}d_g(\iota(x), \iota(y))) - \ln_+(L^{\alpha/2}d_g(\iota(x), \iota(y))) \right| \leq 2C.$$

For each $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$, $|x - y| \leq \sqrt{2} \lfloor L^{(1-\alpha)/2} \rfloor$ so that, since $\delta < \frac{1}{2\sqrt{2}}$,

$$L^{\alpha/2}d_g(\iota(x), \iota(y)) \leq 2\delta L^{(\alpha-1)/2} |x - y| < 1.$$

Therefore, $\ln_+(L^{\alpha/2}d_g(\iota(x), \iota(y))) = 0$. Now, for each $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$,

$$\frac{1}{2}\delta |x - y| \leq \sqrt{L}d_g(\iota(x), \iota(y)) \leq 2\delta |x - y|$$

so that

$$\ln_+ |x - y| - \ln 2 + \ln(\delta) \leq \ln_+(\sqrt{L}d_g(\iota(x), \iota(y))) \leq \ln_+ |x - y| + \ln 2 + \ln(\delta).$$

This concludes the proof of the lemma. \square

3.2 The lower bound

In this section we use the upper bound in Theorem 4 to prove the lower bound in Theorem 1. In other words, we will prove the following.

Proposition 16. *Let (Σ, g) be a compact smooth surface equipped with a riemannian metric and (ϕ_L) be the CGFF on Σ . Let D be a proper open subset of Σ and Ω_L^+ be the event that $\phi_L(x) > 0$ for each $x \in D$. Then,*

$$\liminf_{L \rightarrow \infty} \frac{\ln \mathbb{P}(\Omega_L^+)}{\ln^2(\sqrt{L})} \geq -\frac{2}{\pi} \text{cap}(D).$$

Our approach in the following proof is inspired by that of Nazarov and Sodin in section 3 of [19] and that of Gayet and Welschinger in section 2.2 of [14].

Proof of Proposition 16. Let us choose $\varepsilon > 0$ and a function $h \in C_0^\infty(\Sigma)$ such that for each $x \in D$, $h(x) \geq 1$. Since $u \mapsto \|\nabla u\|_2$ is C^1 continuous, by Lemma 28, for L large than some L_0 , there is a function $f \in U_L$ such that $\forall x \in \Sigma$, $|f(x) - h(x)| \leq \varepsilon$ and such that $\|\nabla f - \nabla h\|_2 \leq \varepsilon$. Now, for L large enough, the random field ϕ_L can be decomposed as the independent sum $\xi \frac{f}{\|\nabla f\|_2} + \tilde{\phi}_L$ where ξ is a real centered gaussian random variable with variance 1 and $\tilde{\phi}_L$ is some gaussian field. Choose $\tilde{\xi}$ another real centered gaussian random variable with variance 1, independent from all the former random variables, and set

$$\phi_L^\pm = \pm \tilde{\xi} \frac{f}{\|\nabla f\|_2} + \tilde{\phi}_L.$$

Then, ϕ_L^\pm are random fields with the same law as ϕ_L but independent from ξ . Furthermore,

$$\tilde{\phi}_L = \frac{\phi_L^- + \phi_L^+}{2}.$$

We now introduce a constant $A > 0$ which we will fix later. The field ϕ_L will be positive on D if the following three equations are satisfied.

$$\xi > \|\nabla f\|_2 A \frac{1}{1 - \varepsilon}$$

$$\forall x \in D, \phi_L^-(x) \leq A$$

$$\forall x \in D, \phi_L^+(x) \leq A.$$

Therefore, by independence of ξ and ϕ_L^\pm ,

$$\mathbb{P}(\Omega_L^+) \geq \left(1 - 2\mathbb{P}\left(\sup_D \phi_L > A\right)\right) \mathbb{P}(\xi > \|\nabla f\|_2 A (1 - \varepsilon)^{-1}).$$

Choose $\delta > 0$ and $A = \left(\sqrt{\frac{2}{\pi}} + \delta\right) \ln(\sqrt{L})$ for $L \geq L_0$. From Theorem 4 we have

$$\mathbb{P}\left(\sup_D \phi_L \geq \left(\sqrt{\frac{2}{\pi}} + \delta\right) \ln(\sqrt{L})\right) \rightarrow 0.$$

Moreover, from gaussian tail estimates (see equation (1.2.2) of [2]), for L large enough,

$$\mathbb{P}\left(\xi > (1 - \varepsilon)^{-1} \|\nabla f\|_2 \left(\sqrt{\frac{2}{\pi}} + \delta\right) \ln(\sqrt{L})\right)$$

is greater than

$$\frac{(1 - \varepsilon) \exp\left(-\frac{1}{2}(1 - \varepsilon)^{-2} \|\nabla f\|_2^2 \left(\sqrt{\frac{2}{\pi}} + \delta\right)^2 \ln^2(\sqrt{L})\right)}{2 \|\nabla f\|_2 \left(\sqrt{\frac{2}{\pi}} + \delta\right) \ln(\sqrt{L})}.$$

Therefore,

$$\begin{aligned} \liminf_{L \rightarrow \infty} \frac{\ln \mathbb{P}(\Omega_L^+)}{\ln^2(\sqrt{L})} &\geq -\left(\sqrt{\frac{2}{\pi}} + \delta\right)^2 \frac{1}{2} (1 - \varepsilon)^{-2} \|\nabla f\|_2^2 \\ &\geq -\left(\sqrt{\frac{2}{\pi}} + \delta\right)^2 \frac{1}{2} (1 - \varepsilon)^{-2} (\|\nabla h\|_2 + \varepsilon)^2. \end{aligned}$$

Taking the infimum over $\delta > 0$ and $\varepsilon > 0$ we obtain

$$\liminf_{L \rightarrow \infty} \frac{\ln \mathbb{P}(\Omega_L^+)}{\ln^2(\sqrt{L})} \geq -\frac{2}{\pi} \frac{1}{2} \|\nabla h\|_2^2.$$

By taking the infimum of $\frac{1}{2} \|\nabla h\|_2^2$ over all h , we get

$$\liminf_{L \rightarrow \infty} \frac{\ln \mathbb{P}(\Omega_L^+)}{\ln^2(\sqrt{L})} \geq -\frac{2}{\pi} \text{cap}(D).$$

□

3.3 The upper bound

The aim of this section is to prove the following proposition.

Proposition 17.

$$\limsup_{L \rightarrow \infty} \frac{\ln(\Omega_L^+)}{\ln^2(\sqrt{L})} \leq -\frac{2}{\pi} \text{cap}(D).$$

Just as in section 3 of [6], we proceed by dichotomy with respect to the following event. Let $K \in \mathbb{N} \setminus \{0\}$, $\eta > 0$, $\alpha \in]0, 1]$ and

$$A_{K, \eta, \alpha} = \left\{ \text{Vol}\left[x \in D \mid \phi_{L^\alpha}(x) < \left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln(\sqrt{L})\right] \leq \frac{K}{L^{\alpha/2}} \right\}.$$

Proposition 17 is an immediate consequence of the two following results, to be compared with lemmas 9 and 10 of [6].

Lemma 18. *Let $\eta > 0$. For any integer $K > 0$ and $\alpha \in]0, 1]$,*

$$\limsup_{L \rightarrow \infty} \frac{\ln(\mathbb{P}(A_{K,\eta,\alpha} \cap \Omega_L^+))}{\ln^2(\sqrt{L})} \leq -\left(\sqrt{\frac{2}{\pi}} - \eta\right)^2 \text{cap}(D).$$

Lemma 19. *For any $\eta > 0$ and $\kappa > 0$ there exist $\alpha \in]0, 1[$ and an integer $K > 0$ such that*

$$\limsup_{L \rightarrow \infty} \frac{\ln(\mathbb{P}(A_{K,\eta,\alpha}^c \cap \Omega_L^+))}{\ln^2(\sqrt{L})} \leq -\kappa.$$

Proof of Lemma 18. Fix $\eta, K > 0$ and $\alpha \in]0, 1]$. Choose $f \in C^\infty(\Sigma)$ positive on D . For each $L > 0$ let \mathcal{E}_L be the random set and X_L the real random variable defined as follows.

$$\begin{aligned} \mathcal{E}_L &= \left\{ x \in D \mid \phi_{L^\alpha}(x) \geq \left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln(\sqrt{L}) \right\} \\ X_L &= \int_D f(p) \phi_{L^\alpha}(p) |dV_g|(p). \end{aligned}$$

X_L is a centered gaussian variable with variance

$$\text{Var}(X_L) = \int_\Sigma \int_\Sigma \mathbf{1}_D(p) f(p) G_{L^\alpha}(p, q) \mathbf{1}_D(q) f(q) |dV_g|(p) |dV_g|(q).$$

Therefore, according to equation (19),

$$\text{Var}(X_L) \xrightarrow{L \rightarrow \infty} \sigma(\mathbf{1}_D f).$$

Moreover, on $A_{K,\eta,\alpha} \cap \Omega_L^+$, $\text{Vol}(D \setminus \mathcal{E}_L) \leq \frac{K}{L^{\alpha/2}}$ and $X_L \geq \left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln(\sqrt{L}) \int_{\mathcal{E}_L} f$ so that

$$X_L \geq \left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln(\sqrt{L}) \left(\int_D f - \frac{K \|f\|_\infty}{L^{\alpha/2}} \right).$$

Therefore,

$$\mathbb{P}(A_{K,\eta,\alpha} \cap \Omega_L^+) \leq \mathbb{P}\left(X_L \geq \left(\int_D f - \frac{K \|f\|_\infty}{L^{\alpha/2}} \right) \left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln(\sqrt{L})\right).$$

By standard gaussian tail estimates (see again equation (1.2.2) of [2]), for L large enough, the right hand side of the above inequality is smaller than

$$\frac{\sqrt{\text{Var}(X_L)} \exp\left(-\frac{\left(\int_D f - \frac{K \|f\|_\infty}{L^{\alpha/2}}\right)^2}{2\text{Var}(X_L)}\right) \left(\sqrt{\frac{2}{\pi}} - \eta\right)^2 \ln^2(\sqrt{L})}{\left(\int_D f - \frac{K \|f\|_\infty}{L^{\alpha/2}}\right) \left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln(\sqrt{L})}.$$

Thus,

$$\limsup_{L \rightarrow \infty} \frac{\ln(\mathbb{P}(A_{K,\eta,\alpha} \cap \Omega_L^+))}{\ln^2(\sqrt{L})} \leq -\left(\sqrt{\frac{2}{\pi}} - \eta\right)^2 \frac{\left(\int_D f\right)^2}{2\sigma(\mathbf{1}_D f)}.$$

From Proposition 29, we complete the proof by taking the supremum over f in the last inequality. \square

To prove Lemma 19 we will need the following technical result, which we prove at the end of the section. This result is analogous to Lemma 12 of [6].

Lemma 20. *Let $\eta > 0$ and $\alpha \in]0, 1]$. For each $\delta > 0$, let F_δ be the event defined by*

$$F_\delta = \left\{ \sup_{d_g(p,q) \leq \delta L^{-\alpha/2}} |\phi_{L^\alpha}(p) - \phi_{L^\alpha}(q)| \geq (\eta/2) \ln(\sqrt{L}) \right\}.$$

Then, for each $\kappa > 0$ there is $\delta_0 > 0$ such that for each $\delta < \delta_0$,

$$\mathbb{P}(F_\delta) \leq \exp(-\kappa \ln^2(\sqrt{L})).$$

The proof itself goes roughly as follows. On the event $A_{K,\eta,\alpha}^c$ the field ϕ_{L^α} takes low values on an abnormally large set \mathcal{E}_L^c . On this set we will consider K small disks that will be so far apart from each other that, from Proposition 15, the values of the field $\psi_{\alpha,L}$ on different disks will decorrelate. Now Lemma 14 tells us that for α small enough, on each of these disks, $\psi_{\alpha,L}$ will be likely to spike downwards and make the sum negative. Taking K large enough will yield the desired inequality.

Proof of Lemma 19. Let us begin by fixing $\eta > 0$ and $\kappa > 0$. We introduce constants $K > 0$ and $\alpha \in]0, 1[$ which we will fix later. As in the previous lemma, for every $L > 0$ let

$$\mathcal{E}_L = \left\{ x \in D \mid \phi_{L^\alpha}(x) \geq \left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln(\sqrt{L}) \right\}$$

By Lemma 20 there is $\delta_0 > 0$ such that for any $\delta < \delta_0$,

$$\mathbb{P}(\Omega_L^+ \cap A_{K,\eta,\alpha}^c \cap F_\delta) \leq \mathbb{P}(F_\delta) \leq \exp(-\kappa \ln^2(\sqrt{L})).$$

We now provide an upper bound for $\mathbb{P}(\Omega_L^+ \cap A_{K,\eta,\alpha}^c \cap F_\delta^c)$. On the event $A_{K,\eta,\alpha}^c$, there is a constant $c > 0$ such that for L large enough, $D \setminus \mathcal{E}_L$ contains at least K points whose mutual distance and distance to \mathcal{E}_L is at least $2cL^{-\alpha/4}$. When both $A_{K,\eta,\alpha}^c$ and F_δ^c are satisfied, for L large enough, there is a collection $(D_j)_{j \in J}$ of K disks of radius $\delta L^{-\alpha/2}$ on which ϕ_{L^α} is smaller than $\left(\sqrt{\frac{2}{\pi}} - (\eta/2)\right) \ln(\sqrt{L})$ such that for each $i, j \in J$ distinct,

$$\inf_{p \in D_i, q \in D_j} d_g(p, q) \geq cL^{-\alpha/4}.$$

Recall that $\psi_{\alpha,L}$, defined in equation (4), is independent from ϕ_{L^α} . Let $\tilde{\mathbb{P}}$ be the conditional probability with respect to ϕ_{L^α} . On $A_{K,\eta,\alpha}^c \cap F_\delta^c$ consider the events

$$T_{K,\eta,\alpha}^\pm = \left\{ \phi_{L^\alpha} \in A_{K,\eta,\alpha}^c \cap F_\delta^c \text{ and } \forall j \in J, \sup_{D_j} \pm \psi_{\alpha,L} \leq \left(\sqrt{\frac{2}{\pi}} - (\eta/2) \right) \ln(\sqrt{L}) \right\}.$$

Since, $\psi_{\alpha,L}$ has the same law as $-\psi_{\alpha,L}$, the two events have the same probability. Moreover, since $\phi_L = \phi_{L^\alpha} + \psi_{\alpha,L}$, $\Omega_L^+ \cap A_{K,\eta,\alpha}^c \cap F_\delta^c$ clearly implies $T_{K,\eta,\alpha}^-$. We will prove that for an adequate choice of K and α , and for L large enough, on the event $A_{K,\eta,\alpha}^c \cap F_\delta^c$,

$$\tilde{\mathbb{P}}\left(T_{K,\eta,\alpha}^+\right) \leq \exp\left(-\kappa \ln^2(\sqrt{L})\right). \quad (5)$$

Passing to expectations with respect to ϕ_{L^α} , this will imply that

$$\mathbb{P}\left(\Omega_L^+ \cap A_{K,\eta,\alpha}^c \cap F_\delta^c\right) \leq \mathbb{P}\left(T_{K,\eta,\alpha}^+\right) \leq \exp\left(-\kappa \ln^2(\sqrt{L})\right).$$

For each $j \in J$, consider $\iota_{j,\alpha,L} : V_{\lfloor L^{(1-\alpha)/2} \rfloor} \rightarrow D_j$ such that for any distinct $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$

$$\frac{\delta}{8} \leq \frac{\sqrt{L} d_g(\iota_{\alpha,L}(x), \iota_{\alpha,L}(y))}{|x - y|} \leq \frac{\delta}{2}.$$

Here, as in Lemma 14, V_N is the box of sidelength N in \mathbb{Z}^2 . Now, for each $j \in J$, define $\psi_{j,\alpha,L}$ a random field on $V_{\lfloor L^{(1-\alpha)/2} \rfloor}$ with the same law as $\psi_{\alpha,L} \circ \iota_{j,\alpha,L}$ such that the collection $(\psi_{j,\alpha,L})_{j \in J}$ is independent overall and of the previously defined fields. According to Lemma 14 and Proposition 10, there is a constant $a > 0$ depending only on η and δ such that on the event $A_{K,\eta,\alpha}^c \cap F_\delta^c$,

$$\forall j \in J, \tilde{\mathbb{P}}\left(\sup \psi_{j,\alpha,L} \leq \left(\sqrt{\frac{2}{\pi}} - (\eta/16)\right)(1-\alpha) \ln(\sqrt{L})\right) \leq \exp(-a \ln^2(\sqrt{L})). \quad (6)$$

At this point, we fix $\alpha > 0$ such that $\left(\sqrt{\frac{2}{\pi}} - (\eta/16)\right)(1-\alpha) \geq \left(\sqrt{\frac{2}{\pi}} - (\eta/8)\right)$. For each $j \in J$, define $V_j = \iota_{j,\alpha,L}(V_{\lfloor L^{(1-\alpha)/2} \rfloor})$. Take $\varepsilon > 0$. By Proposition 15 with $\beta = \alpha/2$, there is L_0 such that for any $L \geq L_0$, any distinct $i, j \in J$ and any $p \in V_i, q \in V_j$,

$$|\mathbb{E}[\psi_{\alpha,L}(p)\psi_{\alpha,L}(q)]| \leq \varepsilon^2. \quad (7)$$

We now introduce, for each $j \in J$ and $p \in V_j$, a real random variable ξ_p , as well as an additional random variable ξ , all independent from previously introduced variables such that the family $(\xi, (\xi_p)_p)$, is independent and each variable is a centered gaussian with variance 1. We will consider the following events

$$\begin{aligned}
\Lambda_1 &= \left\{ \forall j \in J, \sup_{p \in V_j} \psi_{\alpha,L}(p) \leq \left(\sqrt{\frac{2}{\pi}} - (\eta/2) \right) \ln(\sqrt{L}) \right\} \\
\Lambda_2 &= \left\{ \exists j \in J, p \in V_j \mid \varepsilon \xi_p \geq (\eta/4) \ln(\sqrt{L}) \right\} \\
\Lambda_3 &= \left\{ \forall j \in J, \sup_{p \in V_j} \left[\psi_{\alpha,L}(p) + \varepsilon \xi_p \right] \leq \left(\sqrt{\frac{2}{\pi}} - (\eta/4) \right) \ln(\sqrt{L}) \right\} \\
\Lambda_4 &= \left\{ \forall j \in J, \sup_{V_{\lfloor L^{(1-\alpha)/2} \rfloor}} \left[\psi_{j,\alpha,L} + \varepsilon \xi \right] \leq \left(\sqrt{\frac{2}{\pi}} - (\eta/4) \right) \ln(\sqrt{L}) \right\} \\
\Lambda_5 &= \left\{ \forall j \in J, \sup_{V_{\lfloor L^{(1-\alpha)/2} \rfloor}} \psi_{j,\alpha,L} \leq \left(\sqrt{\frac{2}{\pi}} - (\eta/8) \right) \ln(\sqrt{L}) \right\} \\
\Lambda_6 &= \left\{ \varepsilon \xi > (\eta/8) \ln(\sqrt{L}) \right\}.
\end{aligned}$$

We have the following inclusions

$$\begin{aligned}
T_{K,\eta,\alpha}^+ &\subset \Lambda_1 \subset \Lambda_2 \cup \Lambda_3 \\
\Lambda_4 &\subset \Lambda_5 \cup \Lambda_6.
\end{aligned}$$

Consider the two following gaussian fields defined over $\sqcup_{j \in J} V_j$. First $(\psi_{\alpha,L}(p) + \varepsilon \xi_p)_p$, then $p \mapsto \psi_{j,\alpha,L} \circ \iota_{j,\alpha,L}^{-1}(p) + \varepsilon \xi$ where $j \in J$ is such that $p \in V_j$. From equation (7), these two fields satisfy the conditions for Lemma 13 so that

$$\tilde{\mathbb{P}}(\Lambda_3) \leq \tilde{\mathbb{P}}(\Lambda_4).$$

Therefore,

$$\tilde{\mathbb{P}}(T_{K,\eta,\alpha}^+) \leq \tilde{\mathbb{P}}(\Lambda_1) \leq \tilde{\mathbb{P}}(\Lambda_2) + \tilde{\mathbb{P}}(\Lambda_5) + \tilde{\mathbb{P}}(\Lambda_6). \quad (8)$$

Firstly, by standard estimates on tails of gaussian variables, for L large enough,

$$\begin{aligned}
\tilde{\mathbb{P}}(\Lambda_2) &\leq \frac{4K\varepsilon \lfloor L^{(1-\alpha)/2} \rfloor^2}{\eta \ln(\sqrt{L})} \exp\left(-\frac{\eta^2}{32\varepsilon^2} \ln^2(\sqrt{L})\right) \\
\tilde{\mathbb{P}}(\Lambda_6) &\leq \frac{8\varepsilon}{\eta \ln(\sqrt{L})} \exp\left(-\frac{\eta^2}{128\varepsilon^2} \ln^2(\sqrt{L})\right)
\end{aligned} \quad (9)$$

Secondly, by independence of the $\psi_{j,\alpha,L}$ and by equation (6)

$$\begin{aligned}
\tilde{\mathbb{P}}(\Lambda_5) &= \prod_{j \in J} \tilde{\mathbb{P}}\left(\sup \psi_{j,\alpha,L} \leq \left(\sqrt{\frac{2}{\pi}} - (\eta/8) \right) \ln(\sqrt{L})\right) \\
&\leq \exp(-aK \ln^2(\sqrt{L})).
\end{aligned}$$

From equations (8) and (9), taking K large enough and ε small enough, we obtain inequality 5 and the lemma is proved. \square

We now prove Lemma 20. The proof is very similar to that of Proposition 5.

Proof of Lemma 20. Firstly, note that if $0 < \delta < \delta'$ then $F_\delta \subset F_{\delta'}$. Therefore, for each $C > 0$ it is enough to find $\delta > 0$ such that $\mathbb{P}(F_\delta) \leq e^{-C \ln(\sqrt{L})^2}$. Choose $p, q \in \Sigma$ at distance smaller or equal to $\delta L^{-\alpha/2}$. There is a smooth path γ in Σ , parametrized by arclength, such that $\gamma(0) = p$, $\gamma(2\delta L^{-\alpha/2}) = q$ and for each $0 \leq t \leq 2\delta L^{-\alpha/2}$, $|\gamma'(t)| = 1$. Thus,

$$\begin{aligned} |\phi_{L^\alpha}(p) - \phi_{L^\alpha}(q)| &\leq \int_0^{2\delta L^{-\alpha/2}} |(\phi_{L^\alpha} \circ \gamma)'(t)| dt \\ &\leq 2\delta L^{-\alpha/2} \sup_{D_{L^\alpha}(p, 2\delta)} |\nabla \phi_{L^\alpha}|. \end{aligned}$$

We now choose a local trivialisation the tangent bundle of $D_{L^\alpha}(p, 2\delta)$ in which $\nabla \phi_{L^\alpha}$ has coordinates $(\nabla^1 \phi_{L^\alpha}, \nabla^2 \phi_{L^\alpha})$. From Proposition 8, there is a constant $C > 0$ independent of p such that for any such q and for $j = 1, 2$,

$$\text{Var}(\nabla_q^j \phi_{L^\alpha}) \leq CL^\alpha.$$

With this information as well as the second inequality in Lemma 6, we apply the BTIS inequality (Proposition 7) to the random fields $(\nabla_q^j \phi_{L^\alpha}(v_q))_{q \in D_{L^\alpha}(p, 2\delta)}$ for $j = 1, 2$ and deduce there is a constant $C > 0$ such that for L large enough,

$$\mathbb{P}\left(\sup_{D_{L^\alpha}(p, 2\delta)} |\nabla \phi_{L^\alpha}| \geq \frac{\eta L^{\alpha/2}}{4\delta} \ln(\sqrt{L})\right) \leq \exp\left(-\frac{C\eta^2}{\delta^2} \ln^2(\sqrt{L})\right).$$

By the triangle inequality,

$$\mathbb{P}\left(\sup_{q_1, q_2 \in D_{L^\alpha}(p, \delta)} |\phi_{L^\alpha}(q_1) - \phi_{L^\alpha}(q_2)| \geq (\eta/2) \ln(\sqrt{L})\right) \leq \exp\left(-\frac{C\eta^2}{\delta^2} \ln^2(\sqrt{L})\right). \quad (10)$$

There exists $C > 0$ such that for each $L > 0$, there is a covering of Σ by a collection $(D_j)_{j \in J}$ of at most $CL^\alpha \delta^{-2}$ disks of radius $\delta L^{-\alpha/2}$. For each j , inequality (10) applies on D_j . Therefore,

$$\begin{aligned} \mathbb{P}\left(\sup_{d_g(p, q) \leq \delta L^{-\alpha/2}} |\phi_{L^\alpha}(p) - \phi_{L^\alpha}(q)| \geq (\eta/2) \ln(\sqrt{L})\right) &\leq CL^\alpha \exp\left(-\frac{C\eta^2}{\delta^2} \ln^2(\sqrt{L})\right) \\ &\leq \exp\left(-\frac{2C\eta^2}{\delta^2} \ln^2(\sqrt{L})\right) \end{aligned}$$

for L large enough. This inequality ends the proof of the lemma. \square

4 The covariance function

The aim of this section is to prove Theorem 3, Proposition 8 and Proposition 15. For each $L > 0$, let E_L be the Schwartz kernel of the orthogonal projector in $L^2(\Sigma)$ onto U_L . Then,

$$E_L(p, q) = \sum_{0 < \lambda_n \leq L} \psi_n(p) \psi_n(q). \quad (11)$$

The asymptotic behavior of this kernel as $L \rightarrow \infty$ has been studied extensively by Hörmander in [16] (see also [3] and [15]). In order to prove Theorem 3 we will express G_L in terms of E_L and use the aforementioned results to extract an explicit formula for G_L .

4.1 Preliminary results

We begin with the following proposition, which establishes the link between G_L and E_L .

Proposition 21. *There is a function $R \in C^\infty(M \times M, \mathbb{R})$ such that for each $L > 0$,*

$$G_L = \frac{E_L}{L} + \int_1^L \frac{E_\lambda}{\lambda^2} d\lambda + R.$$

Proof. For any $p, q \in \Sigma$, the functions $L \mapsto E_L(p, q)$ and $L \mapsto G_L(p, q)$ define distributions over $]0, +\infty[$. In what follows, ∂_L will mean differentiation in the sense of distributions. First of all, for any $p, q \in \Sigma$

$$\begin{aligned} \partial_L E_L(p, q) &= \sum_{0 < \lambda_n} \psi_n(p) \psi_n(q) \delta_{\lambda_n}(L) \\ \partial_L G_L(p, q) &= \sum_{0 < \lambda_n} \frac{1}{\lambda_n} \psi_n(p) \psi_n(q) \delta_{\lambda_n}(L) = \sum_{0 < \lambda_n} \frac{1}{L} \psi_n(p) \psi_n(q) \delta_{\lambda_n}(L) = \frac{1}{L} \partial_L E_L(p, q). \end{aligned}$$

Consequently $\partial_L \left(G_L - \frac{E_L}{L} + \int_1^L \frac{E_\lambda}{\lambda^2} d\lambda \right) = 0$. Therefore

$$R = G_L - \frac{E_L}{L} + \int_1^L \frac{E_\lambda}{\lambda^2} d\lambda$$

is independent of L . The right hand side is a linear combination of functions $(p, q) \mapsto \psi_n(p) \psi_n(q)$ so it belongs to $C^\infty(M \times M)$. \square

Now, we use Theorem 5.1 of [16] to obtain an explicit description of the integral term in the equation of the previous proposition. Let us fix $p_0 \in \Sigma$. According to Theorem 5.1 of [16] there is an open neighborhood U of p in Σ with a chart $\phi : U \rightarrow \phi(U)$ such that $(d_{p_0} \phi^*)^{-1}$ is an isometry from $T_{p_0}^* \Sigma$ with the metric induced by g to \mathbb{R}^n with the euclidian metric, as well as a real valued function $\theta \in C^\infty(U \times T^*U)$

satisfying the phase condition (see Definition 2.3 of [16]) and a constant $C > 0$ such that for each $p, q \in U$ and each $L > 0$,

$$\left| E_L(p, q) - \frac{1}{(2\pi)^2} \int_{|\xi|^2 \leq L} e^{i\theta(p, q, \xi)} d_q \eta(\xi) \right| \leq C\sqrt{L} \quad (12)$$

where $d_q \eta$ is the measure associated to the metric induced by g on T^*U . Let $\phi_* \theta \in C^\infty(\phi(U) \times \phi(U) \times \mathbb{R}^2)$ be defined by

$$\forall x, y \in \phi(U), w \in \mathbb{R}^2, \phi_* \theta(x, y, w) = \theta(\phi^{-1}(x), \phi^{-1}(y), d_{\phi^{-1}(y)} \phi^* w).$$

Here $d_{\phi^{-1}(y)} \phi^*$ is the adjoint of the differential of ϕ at $\phi^{-1}(y)$.

Theorem 5.1 of [16] provides the following information concerning θ .

1. For each $x, y \in \phi(U)$ and $w \in \mathbb{R}^2$ such that $\langle x - y, w \rangle = 0$, $\phi_* \theta(x, y, w) = 0$.
2. For each $y \in \phi(U)$ and $w \in \mathbb{R}^2$, $\partial_x(\phi_* \theta)(x, y, w)|_{x=y} = w$.

In particular, these equations have the following consequence. Choose $y \in \phi(U)$, $t \geq 0$, $v \in S^1$, $\alpha \in \mathbb{N}^2$ a multiindex and $w \in \mathbb{R}^2$. Then, by the Taylor-Young estimate applied to $\partial_w^\alpha \phi_* \theta(y + \lambda v, y, w)$ with respect λ , for λ small enough,

$$\partial_w^\alpha \phi_* \theta(y + \lambda v, y, w) = \lambda \partial_w^\alpha \langle v, w \rangle + O(\lambda^2) \quad (13)$$

where the $O(\lambda^2)$ is uniform when y and w are restricted to any compact set.

Before we proceed any further, let us introduce some notation. For each $q \in \Sigma$, let S_q be the unit circle in $(T_q^* \Sigma, g_q)$ and $d_q \nu$ the measure induced by the restriction of g_q to S_q . Also, for each $y \in \phi(U)$, let \tilde{S}_y be

$$\tilde{S}_y = \{w \in \mathbb{R}^2 \mid d_{\phi^{-1}(y)} \phi^* w \in S_{\phi^{-1}(y)}\}. \quad (14)$$

Lemma 22. *For each $t > 0$ and $p, q \in U$, let*

$$J(p, q, t) = \int_{S_q} e^{i\theta(p, q, t\omega)} d_q \nu(\omega). \quad (15)$$

Then, there is a constant $C > 0$ such that for each $L > 0$ and $p, q \in U$,

$$\left| \int_1^L E_\lambda(p, q) \lambda^{-2} d\lambda - \frac{1}{4\pi^2} \left(\int_1^{\sqrt{L}} J(p, q, t) t^{-1} dt - L^{-1} \int_1^{\sqrt{L}} J(p, q, t) t dt \right) \right| \leq C.$$

Proof. First of all, from equation (12) there is $C > 0$ such that for all p, q and L ,

$$\left| \int_1^L E_\lambda(p, q) \lambda^{-2} d\lambda - \frac{1}{4\pi^2} \int_1^L \lambda^{-2} \int_{|\xi|^2 \leq \lambda} e^{i\theta(p, q, \xi)} d_p \eta(\xi) d\lambda \right| \leq C \int_1^L \lambda^{-3/2} d\lambda \leq 2C. \quad (16)$$

Now, applying the polar change of coordinates $(t, \omega) \mapsto t\omega = \xi$,

$$\int_{|\xi|^2 \leq \lambda} e^{i\theta(p, q, \xi)} d_p \eta(\xi) = \int_0^{\sqrt{\lambda}} J(p, q, t) t dt.$$

Next, we apply the change of variables $u = \sqrt{\lambda}$.

$$\int_1^L \lambda^{-2} \int_0^{\sqrt{\lambda}} J(p, q, t) t dt d\lambda = \int_1^{\sqrt{L}} \int_0^u J(p, q, t) t dt 2u^{-3} du.$$

We split the inner integral in two $\int_0^u = \int_0^1 + \int_1^u$. Note that the integral from 0 to 1 has lost any dependence on u or L . Since $\int_0^\infty u^{-3/2} du < \infty$ that term is bounded. Therefore applying Fubini's theorem,

$$\begin{aligned} \frac{1}{4\pi^2} \int_1^L \lambda^{-2} \int_0^{\sqrt{\lambda}} J(p, q, t) t dt d\lambda &= \frac{1}{4\pi^2} \int_1^{\sqrt{L}} \int_1^u J(p, q, t) t dt 2u^{-3} du \\ &= \frac{1}{4\pi^2} \int_1^{\sqrt{L}} J(p, q, t) t \int_t^{\sqrt{L}} 2u^{-3} du dt \\ &= \frac{1}{4\pi^2} \left(\int_1^{\sqrt{L}} J(p, q, t) t^{-1} dt - L^{-1} \int_1^{\sqrt{L}} J(p, q, t) t dt \right). \end{aligned}$$

Together with equation (16) this implies that

$$\int_1^L E_\lambda(p, q) \lambda^{-2} d\lambda - \frac{1}{4\pi^2} \left(\int_1^{\sqrt{L}} J(p, q, t) t^{-1} dt - L^{-1} \int_1^{\sqrt{L}} J(p, q, t) t dt \right)$$

is bounded as required. \square

To proceed any further, we need to control the behavior of $J(p, q, t)$ when $t \rightarrow \infty$. We will use the stationary phase method to prove the following proposition.

Proposition 23. *There exist $V \subset U$ an open neighborhood of p_0 and a constant $C > 0$, such that for all $p, q \in V$ and $t \in [0, +\infty[$,*

$$|J(p, q, t)| \leq \frac{C}{\sqrt{1 + d_g(p, q)t}}. \quad (17)$$

The function J is an oscillatory integral over the circle. To obtain the bound for large t uniformly with respect to p, q distinct, we should apply the stationary

phase method to the phase $w \mapsto \frac{\phi_*\theta(x,y,w)}{|x-y|}$ with parameter $|x-y|t$ where ϕ is the chart appearing just above equation (12) and $\phi_*\theta$ is defined just below it. In order to obtain these bounds we will first apply the adequate change of variables in order to compactify the space of definition near the diagonal.

Proof of Proposition 23. Let $K \subset \phi(U)$ be a compact neighborhood of $\phi(p)$. Since K is compact, there exists a constant $\alpha > 0$ such that for each $x \in K$, $D(x, \alpha) \subset \phi(U)$. Here $D(x, \alpha)$ is the open disk centered at x and of radius α . Let us define the following sets.

$$\begin{aligned} A &= \{(x, y, w) \mid x, y \in K, x \neq y, |x - y| < \alpha, w \in \tilde{S}_y\} \\ B &= S^1 \times]0, \alpha[\times \{(y, w) \in K \times \mathbb{R}^2 \mid y \in K, w \in \tilde{S}_y\}. \end{aligned}$$

where \tilde{S}_y is defined in equation (14). The following map is a diffeomorphism

$$\begin{aligned} f : A &\rightarrow B \\ (x, y, w) &\mapsto \left(\frac{x - y}{|x - y|}, |x - y|, y, w \right). \end{aligned}$$

Set

$$\begin{aligned} \psi : A &\rightarrow \mathbb{R} \\ (x, y, w) &\rightarrow \frac{\phi_*\theta(x, y, w)}{|x - y|}. \end{aligned}$$

Now, applying equation (13) with $\alpha = 0$ we deduce that for any $(v, \lambda, y, w) \in B$,

$$\psi \circ f^{-1}(v, \lambda, y, w) = \langle v, w \rangle + O(\lambda)$$

where $O(\lambda)$ is uniform with respect to v, y, w . Thus, $\Psi := \psi \circ f^{-1}$ extends by continuity to \bar{B} so that for all $y \in K$, $w \in \tilde{S}_y$ and $v \in S^1$, $\Psi(v, 0, y, w) = \langle v, w \rangle$. Equation (13) also shows that the map

$$\lambda \mapsto \left((v, y, w) \mapsto \Psi(v, \lambda, y, w) \right)$$

is continuous from $[0, \alpha]$ to the space of continuous functions on $S^1 \times \{(y, w) \in K \times \mathbb{R}^2 \mid w \in \tilde{S}_y\}$ which are C^∞ with respect to w . Let us momentarily fix v and set $y = \phi(p_0)$. The curve \tilde{S}_y is a circle since $d_{p_0}\phi^{-1}$ is an isometry. Hence, the map $w \mapsto \Psi(v, 0, y, w) = \langle v, w \rangle$ defined over \tilde{S}_y is a Morse function with two critical points. Thus, for λ small enough and y close enough to $\phi(p_0)$, $w \mapsto \Psi(v, \lambda, y, w)$ is also a Morse function with two critical points. Moreover, it depends C^4 -continuously on v, t , and y . Therefore there exists $\beta > 0$ and a constant $C > 0$ such that for any $x, y \in D(\phi(p_0), \beta)$ distinct we have the following.

- $\psi_{x,y} : w \mapsto \psi(x, y, w)$ is a Morse function with two critical points $z_1(x, y)$ and $z_2(x, y)$.

- For $j = 1, 2$, $|\det(d_{z_j(x,y)}^2 \psi_{x,y})| > C^{-1}$ (Here $d_{z_j(x,y)}^2 \psi_{x,y}$ is the hessian of $\psi_{x,y}$ which is well defined since $z_j(x,y)$ are critical points, see for instance the definition following Lemma 1.6 of [20]).
- $\|\psi_{x,y}\|_{C^4} \leq C$.

Consequently, $u = 1$ and $f = \psi_{x,y}$ satisfy the hypotheses of Theorem 7.7.5 of [17] with $k = 1$, from which we deduce that there is $C > 0$ such that for these same x, y and λ ,

$$J(\phi^{-1}(x), \phi^{-1}(y), \lambda) \leq \frac{C}{\sqrt{|x-y|\lambda}}.$$

Moreover, for all p, q and t , $|J(p, q, t)| \leq \int_{S_q} 1 d_q \nu = 2\pi$. Let $V = \phi^{-1}(D(\phi(p_0, \beta)))$. From the two last inequalities, there is a constant $C > 0$ such that for any $p, q \in V$ and $t \geq 0$,

$$J(p, q, t) \leq \frac{C}{\sqrt{1 + d_q(p, q)t}}.$$

□

4.2 Proof of Theorem 3

We now use the results and notations of the previous section to prove Theorem 3. We will estimate G_L when p and q are in a neighborhood of p_0 and use the compactness of Σ to make the result global.. Firstly, from Proposition 21, $G_L = \frac{E_L}{L} + \int_1^L \frac{E_\lambda}{\lambda^2} d\lambda + O(1)$. From equation (12), $L^{-1}E_L$ is bounded. Now, from Lemma 22, there is $C > 0$ such that for each $p, q \in U$

$$\left| \int_1^L E_\lambda(p, q) \lambda^{-2} d\lambda - \frac{1}{4\pi^2} \left(\int_1^{\sqrt{L}} J(p, q, t) t^{-1} dt - L^{-1} \int_1^{\sqrt{L}} J(p, q, t) t dt \right) \right| \leq C.$$

Let $V \subset U$ be as in Proposition 23. From equation (17), there is a constant $C > 0$ such that for all $p, q \in V$,

$$\left| L^{-1} \int_1^{\sqrt{L}} J(p, q, t) t dt \right| \leq C.$$

Note that, from equation (13), $\theta(p, q, 0) = 0$ so that $J(p, q, 0) = 2\pi$. Choose $p, q \in U$ distinct and set $r = d_g(p, q)$. Then, applying the change of variables $a = rt$,

$$\begin{aligned} \int_1^{\sqrt{L}} J(p, q, t) t^{-1} dt &= \int_r^{r\sqrt{L}} J(p, q, r^{-1}a) a^{-1} da \\ &= 2\pi \int_r^{r\sqrt{L}} \frac{\mathbf{1}_{a \leq 1}}{a} da + \int_r^{r\sqrt{L}} \frac{J(p, q, r^{-1}a) - 2\pi \mathbf{1}_{a \leq 1}}{a} da \end{aligned}$$

where $\mathbf{1}_{a \leq 1}$ equals 1 if $a \leq 1$ and 0 otherwise. Recall the notation introduced in equation (14). Now, from equation (13), for $0 < a \leq r$ and $x, y \in \phi(V)$, uniformly for $w \in \tilde{S}_y$,

$$\phi_*\theta(x, y, (a/r)w) = \frac{\langle x - y, w \rangle a}{r} + O(|x - y|^2 a/r)$$

so $f : a \mapsto \frac{J(p, q, r^{-1}a) - 2\pi \mathbf{1}_{a \leq 1}}{a}$ admits a continuous extension at $a = 0$ equal to

$$\int_{S_q} \frac{i \langle \phi(p) - \phi(q), (d_q \phi^*)^{-1} \omega \rangle}{r} d_q \nu(\omega).$$

Let $W \subset V$ be an open neighborhood of p_0 on which ϕ is bi-lipschitz and recall that $r = d_g(p, q)$. Then, there exists $C > 0$ independent of L such that for all $p, q \in W$, $|\phi(p) - \phi(q)| \leq Cr$ and such that for all $q \in W$ and $\omega \in S_q$, $\|(d_q \phi^*)^{-1} \omega\|_{eucl} \leq C$. (Here, $\|\cdot\|_{eucl}$ denotes the euclidean norm on \mathbb{R}^2 .) Thus, the aforementioned continuous extension of f is uniformly bounded for $p, q \in W$. Moreover, by equation (17), f is $O(a^{-3/2})$ uniformly with respect to r when $a \rightarrow \infty$. Therefore it is integrable with uniform bounds over p and q . Consequently, for any distinct $p, q \in W$ such that $d_g(p, q) < 1$,

$$\begin{aligned} \int_1^{\sqrt{L}} \lambda^{-2} E_\lambda d\lambda &= \frac{1}{2\pi} \int_{d_g(p, q)}^{d_g(p, q)\sqrt{L}} \frac{\mathbf{1}_{a \leq 1}}{a} da + O(1) \\ &= \frac{1}{2\pi} \left(\ln(\min(1, \sqrt{L}d_g(p, q))) - \ln(\min(1, d_g(p, q))) \right) + O(1) \\ &= \frac{1}{2\pi} \left(\ln(\sqrt{L}) - \ln_+(\sqrt{L}d_g(p, q)) \right) + O(1). \end{aligned}$$

Here, the bounds implied by the O 's are uniform with respect to p and q . For the last equality we use the fact that $\ln_+(d_g(p, q))$ is bounded for $p, q \in W$. The case $p = q$ follows by continuity. Moreover, by compactness, one can cover Σ with a finite number of such W 's so that there is a constant $\varepsilon > 0$ independent of L for which the constants are uniform with respect to any p, q such that $d_g(p, q) < \varepsilon$.

4.3 Proof of Propositions 8 and 15

Proof of Proposition 8. From Theorem 2.3 of [15] with $M = \Sigma$ and $P = \Delta$, there is a constant $C > 0$ such that for all $p \in \Sigma$ and $L \geq 0$,

$$|(Q_1 \otimes Q_2)E_L(p, p)| \leq C(1 + L^{1+d/2})$$

Therefore, from Proposition 21 for all $p \in \Sigma$ and $L > 0$,

$$\begin{aligned}
|(Q_1 \otimes Q_2)G_L(p, p)| &\leq L^{-1}|(Q_1 \otimes Q_2)E_L(p, p)| + \\
&+ \int_1^L \lambda^{-2}|(Q_1 \otimes Q_2)E_\lambda(p, p)|d\lambda + |(Q_1 \otimes Q_2)R(p, p)| \\
&\leq C\left(1 + L^{d/2} + \int_1^L \lambda^{-1+d/2}d\lambda\right) \\
&\leq C'(1 + L^{d/2}).
\end{aligned}$$

□

In order to prove Proposition 15, we will need the following technical result, which we deduce from Theorem 5.1 of [16].

Lemma 24. *For each $\beta > 0$ and $\delta > 0$, there is $C > 0$ such that for each $L, \lambda > 0$ and $p, q \in \Sigma$ such that $d_g(p, q) \geq \delta L^{-\beta/2}$,*

$$|E_\lambda(p, q)| \leq CL^{\beta/4}\lambda^{3/4}.$$

Proof. First of all, by Theorem 5.1 of [16], there is $\varepsilon > 0$ and C such that for each $p_0 \in \Sigma$ and $q \in \Sigma$,

1. If $d_g(p_0, q) \geq \varepsilon$ then, for all $\lambda > 0$, $|E_\lambda(p_0, q)| \leq C\sqrt{\lambda}$ (see equation (5.4) of [16]).
2. There is a chart (ϕ, U) such that $D(p_0, \varepsilon) \subset U$, around p_0 as well as a function θ such that equations (13) and (12) hold with the same constant $C > 0$.

From 1. we need only deal with the case where p_0 and q are ε -close. Now, by a polar coordinate change, and with the same notations as in Lemma 22, for all $\lambda > 0$,

$$\int_{|\xi|^2 \leq \lambda} e^{i\theta(p_0, q, \xi)} d_q \eta(\xi) = \int_0^{\sqrt{\lambda}} J(p_0, q, t) t dt.$$

By equation (17) there is $C > 0$ independent of p_0 and q such that for all $\lambda > 0$ and all $q \neq p_0$,

$$\left| \int_0^{\sqrt{\lambda}} J(p_0, q, t) t dt \right| \leq C \frac{\lambda^{3/4}}{\sqrt{d_g(p_0, q)}}.$$

This concludes the proof. □

Proof of Proposition 15. From Proposition 21,

$$G_{L^\alpha, L} = G_L - G_{L^\alpha} = \frac{E_L}{L} - \frac{E_{L^\alpha}}{L^\alpha} + \int_{L^\alpha}^L \frac{E_\lambda}{\lambda^2} d\lambda.$$

Now from Lemma 24 there is a constant $C > 0$ such that for any $L > 0$ and any $p, q \in \Sigma$ such that $d_g(p, q) \geq \delta L^{-\beta/2}$,

$$\begin{aligned} |L^{-1}E_L(p, q)| &\leq CL^{(\beta-1)/4} \\ |L^{-\alpha}E_{L^\alpha}(p, q)| &\leq CL^{(\beta-\alpha)/4} \\ \left| \int_{L^\alpha}^L \frac{E_\lambda}{\lambda^2} d\lambda \right| &\leq CL^{\beta/4} \int_{L^\alpha}^L \lambda^{-5/4} d\lambda \leq 2CL^{(\beta-\alpha)/4} \end{aligned}$$

and each term tends to 0 since $0 < \beta < \alpha < 1$. \square

5 The case of a surface with boundary

In this paper, we studied a random zero mean field over a compact surface Σ with $\partial\Sigma = \emptyset$. The bulk of our results remain valid in some sense if Σ has a smooth boundary $\partial\Sigma \neq \emptyset$. The zero mean condition is then replaced by a Dirichlet zero boundary condition. In this section we state definitions and results for this setting and explain the minor modifications needed in each proof. Be advised that **we use the same notations as before to denote slightly different objects**.

5.1 Definitions

Let (Σ, g) be a smooth surface with smooth boundary $\partial\Sigma$ equipped with a riemannian metric. We will denote the interior of Σ by $\overset{\circ}{\Sigma} = \Sigma \setminus \partial\Sigma$. If $E(\Sigma)$ is a topological vector-space of functions on Σ of which the $C^\infty(\Sigma)$ is a dense subspace, $E_0(\Sigma)$ will denote the closure in $E(\Sigma)$ of smooth functions with compact support in $\overset{\circ}{\Sigma}$. We begin by defining the CGFF on Σ . As in the introduction, the bilinear form $\langle u, v \rangle_\nabla := \int_\Sigma g(\nabla u, \nabla v) |dV_g|$ defines a scalar product on $H_0^1(\Sigma)$. From Theorem 4.43 of [13] there exist $(\psi_n)_{n \geq 1} \in C_0^\infty(\Sigma)^\mathbb{N}$ and $(\lambda_n)_{n \geq 1} \in \mathbb{R}^\mathbb{N}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \dots, \lambda_n \xrightarrow[n \rightarrow \infty]{} +\infty$, such that $(\psi_n)_{n \geq 1}$ is a Hilbert basis for $L_0^2(\Sigma)$ and such that for each $n \in \mathbb{N}$, $\Delta\psi_n = \lambda_n\psi_n$. Then $\left(\frac{1}{\sqrt{\lambda_n}}\psi_n\right)_{n \geq 1}$ is a Hilbert basis of $(H_0^1, \langle \cdot, \cdot \rangle_\nabla)$. For each $L > 0$, let $(U_L, \langle \cdot, \cdot \rangle_\nabla)$ be the subspace of $(H_0^1, \langle \cdot, \cdot \rangle_\nabla)$ spanned by the functions $\frac{1}{\sqrt{\lambda_n}}\psi_n$ such that $\lambda_n \leq L$. Let $(\xi_n)_{n \in \mathbb{N}}$ be an i.i.d sequence of centered gaussians with variance one. Then, for each $L > 0$ we define the CGFF on Σ as

$$\phi_L = \sum_{\lambda_n \leq L} \frac{\xi_n}{\sqrt{\lambda_n}} \psi_n. \quad (18)$$

For each $L > 0$ and $p, q \in \Sigma$, let $G_L(p, q) = \mathbb{E}[\phi_L(p)\phi_L(q)]$.

Let D be an open subset of $\overset{\circ}{\Sigma}$ at positive distance from $\partial\Sigma$. Then, we define the capacity of D relative to Σ as the infimum of $\frac{1}{2}\|\nabla h\|^2$ over all $h \in C_0^\infty(\overset{\circ}{\Sigma})$ and denote it by $\text{cap}_\Sigma(D)$ or $\text{cap}(D)$. If D is non-empty and has smooth boundary, we will see that $\text{cap}(D) > 0$. Note that in this setting, the capacity is conformally

invariant. More precisely, consider (Σ_1, g_1) and (Σ_2, g_2) are two riemannian surfaces with smooth boundary such that there exists a conformal isomorphism, $f : \Sigma_1 \rightarrow \Sigma_2$. Let $D_1 \subset \Sigma_1$ and $D_2 = f(D_1)$. We claim that $\text{cap}_{\Sigma_1}(D_1) = \text{cap}_{\Sigma_2}(D_2)$. Indeed, for each $h \in C_0^\infty(\Sigma_2)$ such that $h \geq 1$ on D_2 , we have $h \circ f \in C_0^\infty(\Sigma_1)$ and $h \circ f \geq 1$ on D_1 . The same is true for f^{-1} if we exchange 1 and 2 so f defines a bijection between the sets whose infimum define the capacities. Moreover, since f is a conformal map, for any $p \in \Sigma_1$ and $v \in T_p \Sigma_1$, $g_2(d_p f v, d_p f v) = |\det(d_p f)| g_1(v, v)$ where $\det(d_p f)$ is the determinant of the matrix of $d_p f$ written in orthonormal bases. It follows that from the change of variables formula that for any $h \in C_0^\infty(\Sigma_2)$, $\|h\|_\nabla = \|h \circ f\|_\nabla$. This proves the claim.

5.2 Main results

Theorem 25. *Let (Σ, g) be a smooth compact surface with smooth non-empty boundary and let $(\phi_L)_{L>0}$ be the CGFF on (Σ, g) . Let D be a non-empty open subset of $\overset{\circ}{\Sigma}$ with smooth boundary and positive distance to $\partial\Sigma$. Then,*

$$\lim_{L \rightarrow \infty} \frac{\ln(\mathbb{P}(\forall x \in D, \phi_L(x) > 0))}{\ln^2(\sqrt{L})} = -\frac{2}{\pi} \text{cap}(D).$$

The only change in section 3, which contains the heart of the argument, is in the proof of Proposition 29. Though the statement is the same, Σ has a boundary and the definition of $\text{cap}(D)$ has changed. For a proof in this case we refer the reader to Lemma 2.1 of [5] where one should simply replace V by Σ and $\langle f, \phi \rangle_V - \frac{1}{2} \langle f, K_V f \rangle_V$ by $\frac{1}{2\sigma(\mathbf{1}_D f)} \left(\int_D f \right)^2$.

Theorem 26. *Let (Σ, g) be a compact riemannian surface with smooth non-empty boundary and let $(\phi_L)_{L>0}$ be the CGFF on (Σ, g) . For each $L > 0$ and $p, q \in \Sigma$, let*

$$G_L(p, q) = \mathbb{E}[\phi_L(p)\phi_L(q)].$$

Then, there exists $\varepsilon > 0$ such that for each $p, q \in \overset{\circ}{\Sigma}$ satisfying $d_g(p, q) \leq \varepsilon$ and for each $L > 0$,

$$G_L(p, q) = \frac{1}{2\pi} \left(\ln(\sqrt{L}) - \ln_+(\sqrt{L}d_g(p, q)) \right) + \rho_L(p, q)$$

where, for any compact subset $K \subset \overset{\circ}{\Sigma}$, $\rho_L(p, q)$ is uniformly bounded for $L > 0$ and $p, q \in K$.

For the proof of Theorem 26 we fix a compact subset $K \subset \overset{\circ}{\Sigma}$ and proceed as in the original proof but replace the statements “uniform with respect to $p, q \in \Sigma$ ” by “uniform with respect to $p, q \in K$ ”. This works because Theorem 5.1 of [16] is valid on non-compact manifolds except the bound on the remainder term is only uniform on compact sets.

Theorem 27. *Let (Σ, g) be a smooth compact riemannian surface with non-empty boundary and let $(\phi_L)_{L>0}$ be the CGFF on (Σ, g) . Let D be a non-empty open subset of $\mathring{\Sigma}$ at positive distance of $\partial\Sigma$ and K be a compact subset of $\mathring{\Sigma}$. Then for each $\eta > 0$,*

$$\limsup_{L \rightarrow \infty} \frac{\ln \left(\mathbb{P} \left(\sup_K \phi_L > \left(\sqrt{\frac{2}{\pi}} + \eta \right) \ln(\sqrt{L}) \right) \right)}{\ln(\sqrt{L})} \leq -2\sqrt{2\pi}\eta + O(\eta^2)$$

and there exists $a > 0$ such that for L large enough,

$$\mathbb{P} \left(\sup_D \phi_L \leq \left(\sqrt{\frac{2}{\pi}} - \eta \right) \ln(\sqrt{L}) \right) \leq \exp \left(-a \ln^2(\sqrt{L}) \right).$$

The proof of this theorem is essentially the same as the original. In the statement of Lemma 6, one should introduce a compact $K \subset \mathring{\Sigma}$. The constant C is then uniform for $p \in K$. In Proposition 5 the supremum is taken over K and in the proof one should cover K by small disks instead of Σ .

6 Appendix

In section we recall some classical results from spectral theory used in the article and give an alternate characterization of the capacity which we use in Lemma 19.

6.1 Classical spectral theory results

In the following discussion we adopt the notations introduced in section 1.3. We begin by proving an approximation result for eigenfunctions of the laplacian.

Lemma 28. *For any integers $h, k \geq 1$, the vector space spanned by the sequence $(\psi_n)_{n \geq 1}$ is dense in $C_0^h(\Sigma)$ and in $H_0^k(\Sigma)$.*

Proof. By the Sobolev inequalities (see for instance, Theorem 5.6 (ii) of [11]), for any h there exists k_0 such that for $k \geq k_0$, $H_0^k \subset C_0^h$ and the inclusion is continuous. On the other hand, if $h \geq k$, $C_0^h \subset H_0^k$ and the inclusion is continuous. Therefore, it suffices to prove that the space spanned by the $(\psi_n)_{n>0}$ is dense in all the H_0^k . By Garding's inequality (see Theorem (1), section 8 of [27]), for any $k \geq 1$, there is $C > 0$ such that the scalar product

$$(u | v)_k := \langle u, (C + \Delta^k)v \rangle_2$$

is equivalent to the standard H_0^k scalar product. Let $f \in C_0^\infty$ and suppose that for any $n \geq 1$, $(f | \psi_n)_k = 0$. Then

$$0 = (f | \psi_n)_k = \langle f, (C + \Delta^k)\psi_n \rangle_2 = (C + \lambda_n^k) \langle f, \psi_n \rangle_2.$$

Since $f \in L_0^2$ and $(\psi_n)_{n \geq 1}$ is dense in L_0^2 , $f = 0$. Since C_0^∞ is dense in H_0^k , we conclude that so is $(\psi_n)_{n \geq 1}$. \square

Now we discuss the functional calculus of the laplacian. This will be useful in the alternate characterization of the capacity. The operator Δ is a symmetric differential operator of order 2 so it defines a bounded operator from $H^2(\Sigma)$ to $L^2(\Sigma)$ which, according to its spectral decomposition (see section 1.3) and by Lemma 28, defines an isomorphism $\Delta : H_0^2 \rightarrow L_0^2$. Let Δ^{-1} be its inverse, which we extend to $L^2(\Sigma)$ by setting $\Delta^{-1}f = 0$ for any constant function f . Since Δ is symmetric, Δ^{-1} is self-adjoint. Moreover, for any $n \geq 1$, $\Delta^{-1}\psi_n = \lambda_n^{-1}\psi_n$. For any $f \in L^2(\Sigma)$, we denote by $\sigma(f)$ the quadratic form $\sigma(f) := \langle f, \Delta^{-1}f \rangle_2$. By Parseval's formula,

$$\sigma(f) = \sum_{n \geq 1} \frac{1}{\lambda_n} \langle \psi_n, f \rangle_2^2. \quad (19)$$

Finally, since Δ and Δ^{-1} are positive and Δ is a differential operator, then $\Delta : C_0^\infty \rightarrow C_0^\infty$ and $\Delta^{-1} : C_0^\infty \rightarrow C_0^\infty$ both admit square roots $\Delta^{1/2}$ and $\Delta^{-1/2}$ respectively, which are symmetric pseudo-differential operators of orders 1 and -1 (see [23]). These define bounded operators

$$\begin{aligned} \Delta^{1/2} &: H^2 \rightarrow H^1 \\ \Delta^{1/2} &: H^1 \rightarrow L^2 \\ \Delta^{-1/2} &: L^2 \rightarrow L^2. \end{aligned}$$

Finally, by construction, these operators restrict to H_0^k and L_0^2 and for any $f \in H_0^2$, $\Delta^{-1/2}\Delta^{1/2}f = f$.

6.2 The capacity

For any subset $D \subset \Sigma$ that is not dense in Σ , there exist functions $h \in C_0^\infty(\Sigma)$ such that $h \geq 1$ on D . We define **the capacity** of D the infimum over all such h of $\frac{1}{2}\|\nabla h\|_2^2$ and denote it by $cap(D)$. This is a variation on the relative capacity used in [5] and [6]. Similarly to Lemma 2.1 of [5], in the case where D is a non-empty proper open set with smooth boundary we have the following alternative characterization of the capacity.

Proposition 29. *Let D be a non-empty proper open subset of Σ with smooth boundary. Then,*

$$cap(D) = \sup \left\{ \frac{1}{2\sigma(\mathbf{1}_D f)} \left(\int_D f \right)^2 \mid f \in C^\infty(\Sigma) \ f \geq 0 \text{ on } D. \right\}$$

In the proof of this proposition we will use the maximum principle and existence of solutions to certain elliptic PDEs. For this purpose, recall that we defined the laplacian to be $\Delta = d^*d$.

Proof of Proposition 29. Let $h \in C_0^\infty(\Sigma)$ such that for each $x \in D$, $h(x) \geq 1$. Let $f \in C^\infty(\Sigma)$, non-negative on D .

$$\begin{aligned} \left(\int_D f \right)^2 &\leq \left(\int_D hf \right)^2 = \langle \mathbf{1}_D f, h \rangle_2^2 = \langle \mathbf{1}_D f, \Delta^{-1/2} \Delta^{1/2} h \rangle_2^2 = \langle \Delta^{-1/2} \mathbf{1}_D f, \Delta^{1/2} h \rangle_2^2 \\ &\leq \langle \Delta^{-1/2} \mathbf{1}_D f, \Delta^{-1/2} \mathbf{1}_D f \rangle_2 \langle \Delta^{1/2} h, \Delta^{1/2} h \rangle_2 \text{ by Cauchy-Schwarz} \\ &= \langle \mathbf{1}_D f, \Delta^{-1} \mathbf{1}_D f \rangle_2 \langle h, \Delta h \rangle_2 = \sigma(\mathbf{1}_D f) \|\nabla h\|_2^2. \end{aligned}$$

Passing to the infimum and supremum, since D is non-empty, we deduce that

$$\sup \left\{ \frac{1}{2\sigma(\mathbf{1}_D f)} \left(\int_D f \right)^2 \mid f \in C^\infty(\Sigma) \text{ } f \geq 0 \text{ on } D. \right\} \leq \text{cap}(D).$$

We now show that one can find f and h so as to make the above functionals arbitrarily close to each other. The previous Cauchy-Schwarz inequality is close to equality when $\Delta^{-1} \mathbf{1}_D f$ is close to h . Recall that we defined Δ^{-1} to be zero when applied to constant functions. This means that we must find f so that $\mathbf{1}_D f - \Delta h$ is close to a constant. We now define a would-be minimizer h . Let $\tau > 0$ and h be a continuous function constant equal to 1 on D , smooth on $U = \Sigma \setminus \overline{D}$ such that Δh is equal to $-\tau$ on $\Sigma \setminus \overline{D}$ (see Theorems 5 of section 6.2 and 3 of section 6.3 of [11] for the existence of h). Then h is smooth up to the boundary of U (see Theorem 6 of section 6.3 of [11]). We denote by $\partial_\nu h$ the outward normal of h , defined on ∂U and by $d\sigma$ the volume density induced by g on $\partial U = \partial D$. Since $\Delta h \leq 0$ on U , by the maximum principle, $h < 1$ on U . Therefore, there exists τ such that h has zero mean. Let Δh be the laplacian of h in the sense of distributions, which is well defined since h is continuous. Before approximating h , we must find an explicit expression for Δh . For any $u \in C^\infty(\Sigma)$, by Stokes' theorem,

$$\begin{aligned} \langle \Delta h, u \rangle_2 &= \langle h, \Delta u \rangle_2 = \int_D h \Delta u |dV_g| + \int_U h \Delta u |dV_g| \\ &= \int_D \langle \nabla h, \nabla u \rangle_g |dV_g| + \int_{\partial D} h (-\partial_\nu u) d\sigma \\ &\quad + \int_U \langle \nabla h, \nabla u \rangle_g |dV_g| + \int_{\partial U} h \partial_\nu u d\sigma \\ &= \int_U \langle \nabla h, \nabla u \rangle_g |dV_g| = \int_U u \Delta h |dV_g| + \int_{\partial U} u \partial_\nu h d\sigma \\ &= \int_{\partial U} (\partial_\nu h) u d\sigma - \tau \int_U u |dV_g|. \end{aligned}$$

Therefore,

$$\langle \Delta h + \tau, u \rangle_2 = \tau \int_D u |dV_g| + \int_{\partial(\Sigma \setminus D)} (\partial_\nu h) u d\sigma.$$

Again, by the maximum principle, $\partial_\nu h$ is positive. Let $(\tilde{h}_\varepsilon)_{\varepsilon > 0}$ be a sequence of smooth functions with zero mean converging uniformly to h in Σ that coincide with

h on $\Sigma \setminus D$. Then, for each $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $h_\varepsilon = (1 + \delta(\varepsilon))\tilde{h}_\varepsilon \geq 1$ on D . Let $\tau_\varepsilon = (1 + \delta(\varepsilon))\tau$. Then, in the sense of distributions, $f_\varepsilon = \Delta h_\varepsilon + \tau_\varepsilon$ is non-zero and non-negative on Σ , and vanishes on $\Sigma \setminus D$. Since it is smooth, it satisfies these properties in the classical sense as well. Moreover

$$\begin{aligned} \left(\int_D f_\varepsilon\right)^2 &\geq (\sup_D h_\varepsilon)^{-2} \left(\int_D f_\varepsilon h_\varepsilon\right)^2 \\ &= (\sup_D h_\varepsilon)^{-2} \langle f_\varepsilon, h_\varepsilon \rangle_2^2 = (\sup_D h_\varepsilon)^{-2} \langle f_\varepsilon, \Delta^{-1} f_\varepsilon \rangle_{L^2} \langle h_\varepsilon, \Delta h_\varepsilon \rangle_2 \\ &= (\sup_D h_\varepsilon)^{-2} \sigma(f_\varepsilon) \|\nabla h_\varepsilon\|_2^2 = (\sup_D h_\varepsilon)^{-2} \sigma(f_\varepsilon \mathbf{1}_D) \|\nabla h_\varepsilon\|_2^2. \end{aligned}$$

Therefore,

$$\text{cap}(D) \leq \frac{1}{2} \|\nabla h_\varepsilon\|_2^2 \leq \frac{(\sup_D h_\varepsilon)^2}{2\sigma(f_\varepsilon \mathbf{1}_D)} \left(\int_D f_\varepsilon\right)^2.$$

Since $\sup_D h_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 1$, this concludes the proof of the lemma. \square

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