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LIPSCHITZ REGULARITY FOR INTEGRO-DIFFERENTIAL EQUATIONS WITH COERCIVE HAMILTONIANS AND APPLICATION TO LARGE TIME BEHAVIOR.

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Abstract. In this paper, we provide suitable adaptations of the "weak version of Bernstein method" introduced by the first author in 1991, in order to obtain Lipschitz regularity results and Lipschitz estimates for nonlinear integro-differential elliptic and parabolic equations set in the whole space. Our interest is to obtain such Lipschitz results to possibly degenerate equations, or to equations which are indeed "uniformly elliptic" (maybe in the nonlocal sense) but which do not satisfy the usual "growth condition" on the gradient term allowing to use (for example) the Ishii-Lions' method. We treat the case of a model equation with a superlinear coercivity on the gradient term which has a leading role in the equation. This regularity result together with comparison principle provided for the problem allow to obtain the ergodic large time behavior of the evolution problem in the periodic setting.

1. Introduction

The starting point of this article and its main motivation comes from the study of the large time behavior of solutions of nonlinear, nonlocal parabolic partial differential equations. This study requires, in general, two main arguments : Lipschitz estimates which are needed both to prove the compactness of solutions of the evolution equation and to solve the expected stationary limit *ergodic problem*; and a Strong Maximum Principle for either the stationary and/or evolution equation to actually prove the convergence. It is worth pointing out that such Strong Maximum Principle is often obtained through a linearization of the equation, which also uses the gradient bound and therefore Lipschitz estimates may be also used indirectly. In this short description of the method, we would like to stress on the fact that Lipschitz estimates play a central role in all the steps.

In order to be more specific, we turn to [12] where the large time behavior of solutions of (local) nonlinear parabolic PDEs is studied through two main cases : the *sub and super quadratic cases*, the point being that the Lipschitz estimates are obtained in different ways in these two cases.

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In the subquadratic case, this Lipschitz estimate comes from the uniform elliptic second-order operator (the Laplacian in [12]) and the subquadratic assumption on the nonlinear terms also related with the x-dependence of the Hamiltonian. Technically, this is done using the Ishii-Lions' method [29].

On the contrary, in the superquadratic case, the Lipschitz estimate comes from the nonlinear term through the *weak Bernstein's method* ([5]), which has the advantage of being able to handle degenerate cases and Hamiltonians with arbitrary growth (as the classical Bernstein's method).

For nonlocal equations, this program is carried out in the "subquadratic" case in a series of papers : the Lipschitz estimate is obtained in [7] and the large time behavior in [8] using the Strong Maximum Principle of [18]. In the superquadratic case, the contribution of [10] is to obtain $C^{0,\alpha}$-type estimates which are sufficient to obtain some large time behavior but for purely nonlocal operator (no mixing of second-order differential operator and nonlocal one). This is one of the rare cases where the Lipschitz estimates can be avoided.

The aim of this paper is to complete this study by providing for some model equations Lipschitz regularity results by using a weak Bernstein's method for bounded viscosity solutions of such nonlocal PDEs.

To the best of our knowledge, there is no general extension of the weak Bernstein's method to the case of nonlocal equations. The reason for such lack of extension may come from the fact that, for PDEs, Bernstein's method (weak or classical) uses a change of variable and such change is not easy to handle for nonlocal equations. But in [12], only one exponential change is used and it turns out that it fits well with a variety of nonlocal equations including models which are not covered by previous results. To show it, we have decided to treat the case of a rather simple equation, but involving the relevant difficulty, in order to emphasize the new point, namely the additional needed estimates to treat the nonlocal part of the equation.

Our model equations are

$$\lambda u - \mathrm{Tr}(A(x)D^2u) - \mathcal{I}^j(u,x) + H(x,Du) = 0 \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

in the stationary case and its time-dependent version

$$\partial_t u - \mathrm{Tr}(A(x)D^2u) - \mathcal{I}^j(u(\cdot,t),x) + H(x,Du) = 0 \quad \text{in } Q, \tag{1.2}$$

where $Q = \mathbb{R}^d \times (0,+\infty)$. In both cases, $u : \mathbb{R}^d \to \mathbb{R}$ is the unknown function, Du, D^2u denote respectively its gradient and Hessian matrix. The main assumptions are $\lambda \geq 0$, $H \in C(\mathbb{R}^d \times \mathbb{R}^d)$ is superlinear in the gradient term, A takes values in the set $\mathbb{S}^d_+$ of nonnegative symmetric matrices and $\mathcal{I}^j$ is a nonlocal operator in the *Lévy-Ito* form, defined as

$$\mathcal{I}^j(\phi,x) = \int_{\mathbb{R}^d} [\phi(x+j(x,z)) - \phi(x) - \mathbf{1}_B(z)\langle D\phi(x), j(x,z)\rangle]\nu(dz), \tag{1.3}$$

for $x \in \mathbb{R}^d$. Here ϕ is a bounded function which is C^2 in a neighborhood of x, the function $j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is the *jump function* and ν is a *Lévy* *type measure*, which is regular and nonnegative. Finally $\mathbf{1}_B$ is the indicator function of the unit ball B. Precise assumptions over the data will be given later on. Note that $-\mathcal{I}^j$ is, up to a normalizing constant, the fractional Laplacian of order $\sigma \in (0,2)$ when $j(x,z) = z$ and $\nu(dz) = |z|^{-d-\sigma}dz$, see [23].

We emphasize the fact that equations (1.1) and (1.2) may be degenerate in the second-order and/or in the nonlocal term. Hence, our results rely on the coercivity of H in the gradient term, but, in contrast to [10] we can get the result in some cases when the growth degree of coercivity of H in p is less than the order of the nonlocal operator.

We recall that, for local equations, the formal idea of the Bernstein's method is to show that $|Du|^2$ is a subsolution of a suitable elliptic equation. The estimate of Du is then obtained by applying the Maximum Principle, either in a bounded domain or in the whole space. The analysis provided in the introduction of [5] shows that this is possible if the equation satisfies some "structure condition". However, such a condition is not directly verifiable at first glance and it is necessary to perform a change of variables which leads to a new equation satisfying this property. The second key information in [5] is that the formal analysis, consisting in differentiating the equation and therefore requiring smooth solutions, can be justified by viscosity solutions' method and therefore for just continuous solutions.

As in [5], in our case the impossibility to differentiate the equation is carried out by the above mentioned viscosity argument that allows us to contrast the Lipschitz bounds of the solution to the problem with respect to the Lipschitz bounds of the data, and for this reason we must restrict ourselves to Lipschitz x-dependent problems.

Of course, the main difference of the application of the method in the current setting is the presence of the nonlocal term. Recalling the definition of $\mathcal{I}^j$ in (1.3), it is worth to mention that the application of Bernstein method when $|\nu| < \infty$ and/or j does not depend on x provide Lipschitz bounds with few extra efforts compared with the already known second-order case. When the measure is finite there is no differential effect coming from the nonlocal term in the equation and the corresponding term is easily controllable, while if j does not depend on x, then the operator is translation invariant which is a favorable situation in the Bernstein's method where the x-dependence of each differential term is important. For this reason we concentrate in details on the most difficult scenario of singular measures ν and x-dependence of j in the definition of $\mathcal{I}^j$, and whose treatment is summarized through Lemma 3.2 below. As in the local setting, Lipschitz conditions must be requested on j, ad-hoc to the integral configuration of the problem, and these assumptions are sufficient to control the influence of the nonlocal term with the stronger coercive effect of the gradient term, no matter the "order" of the singularity of the Lévy measure ν is.

In the last section of this paper, we apply these Lipschitz regularity results to the study of the large time behavior of the associated evolution problem in the periodic setting. As we already mentioned above, one of the main consequences of this regularity result is a "linearization" procedure of the Hamiltonian that allows us to prove a version of the Strong Maximum Principle for the evolution problem. Roughly speaking, after the mentioned linearization procedure, we can propagate the maximum value of a solution of the corresponding linearized problem in the directions of the uniform ellipticity of the second-order term as it is performed by Bardi and Da Lio in [2], meanwhile it is propagated in the directions of the degeneracy of the second-order term due to a covering property of the support of the measure defining the nonlocal term in the flavour of Coville [19, 20]. The novelty is to combine in a better way these two types of (very different) arguments: this leads to a simpler formulation and a slight improvement of the results of Ciomaga in [18].

Once comparison principle, Lipschitz regularity and strong maximum principles are available, we follow the lines presented in [13] for first-order equations, [13, 34] for second-order equations and [8, 10] for nonlocal problems to conclude the solution of the evolution problem behaves, up to a linear factor in time, as the solution of the so-called *ergodic problem*, which can be understood as an homogenization of (1.1) when we let $\lambda \to 0$.

We finish this introduction section mentioning that most of the results of this paper can be extended to equations which are nonlinear in the second-order and nonlocal term, such as Bellman-Isaacs-type nonlinearities arising in game theory. This can be explained by the homogeinity of such operators together with its sub/superaditivity related to Pucci-type associated extremal operators, which do not change the arguments consistently provided there is a weak coupling with the gradient term. However, we do not pursue in this direction for simplicity of the presentation.

Basic notation and organization of the paper. In this paper we consider the notion of viscosity solution, see [1, 9] for a definition of this concept in the integro-differential framework.

We use the notation USC, LSC, $\mathrm{C_b}$ and BUC, for upper and lower semicontinuous functions, continuous bounded functions and bounded uniformly continuous functions, respectively.

We recall that $Q = \mathbb{R}^d \times (0, +\infty)$, we write $Q_T = \mathbb{R}^d \times (0, T]$ for $T > 0$. For $a > 0$ and $x \in \mathbb{R}^d$ we denote $B_a(x)$ the ball of center x and radius a, B_a when x is the origin and simply B with in addition $a = 1$.

For a set $A \subset \mathbb{R}^d$, $x, p \in \mathbb{R}^d$ and ϕ a bounded function, we define

$$\mathcal{I}^j[A](\phi, x, p) = \int_A [\phi(x + j(x,z)) - \phi(x) - \mathbf{1}_B(z)\, \langle p, j(x,z)\rangle]\nu(dz).$$

If $A = \mathbb{R}^d$ we write $\mathcal{I}^j(\phi, x, p) := \mathcal{I}^j[\mathbb{R}^d](\phi, x, p)$. When $\phi \in C^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we also write $\mathcal{I}^j[A](\phi, x) := \mathcal{I}^j[A](\phi, x, D\phi(x))$.

Note that under mild assumptions on the jump function j and on the measure ν (see (M) below), by its definitions the operator (1.3) is well-defined for a bounded smooth function ϕ and can be written as

$$\mathcal{I}^j(\phi, x) = \mathcal{I}^j[\mathbb{R}^d](\phi, x, D\phi(x)).$$

The paper is organized as follows: in section 2 we provide the well-posedness to the stationary and evolution problem in a general framework. Section 3 is devoted to the main result of the paper, which is the Lipschitz regularity for solutions of the stationary and evolution problem using the weak Bernstein method. Finally, in section 4 we restrict ourselves to the periodic setting and provide a strong maximum principle from which we deduce the large time behavior result.

2. Comparison principle and its consequences.

In this section we study the well-posedness of the parabolic problem when (1.2) is associated with the initial data

$$u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d, \tag{2.1}$$

where u_0 is, at least, a bounded, continuous function in $\mathbb{R}^d$. The initial condition is satisfied in the sense of viscosity solutions which reduce here to the classical sense by Lemma 2.4 (see also Remark 1).

The well-posedness follows by comparison principle for bounded sub and supersolutions obtained under the following assumptions.

(A) *There exists a continuous function $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$, $k \leq d$, such that $A(x) = \sigma(x)\sigma^T(x)$ for each $x \in \mathbb{R}^d$ and there exists $L_\sigma \geq 0$ such that*

$$|\sigma(x)| \leq L_\sigma, \quad |\sigma(x) - \sigma(y)| \leq L_\sigma |x - y|, \quad \text{for all } x, y \in \mathbb{R}^d.$$

(H1) *There exists $m > 1$ and $K, b_m > 0$ such that for all $\mu \in (0,1)$, $x, p \in \mathbb{R}^d$*

$$\mu H(x, \mu^{-1}p) - H(x,p) \geq (1-\mu)\Big(b_m |p|^m - K\Big).$$

(H2) *Let m be as in (H1). There exist moduli of continuity ζ_1, ζ_2 such that, for all $x, y, p, q \in \mathbb{R}^d$, $|q| \leq 1$, we have*

$$H(y, p+q) - H(x,p) \leq \zeta_1(|x-y|)(1 + |p|^m) + \zeta_2(|q|)(1 + |p|^{m-1}).$$

(M) *There exists $C_{\nu,j} > 0$ such that*

$$\int_{B^c} \nu(dz) \leq C_{\nu,j} \quad \text{and} \quad \int_B |j(x,z)|^2 \nu(dz) \leq C_{\nu,j},$$

where $a \wedge b = \min(a,b)$.

(MJ) *For any $R \geq 1$, there exists constant $C^0(R), C^1(R), C^2(R) > 0$ such that for all $x, y \in \mathbb{R}^d$*

$$\int_{B_R \setminus B} |j(x,z)|\nu(dz) \leq C^0(R) ,$$

$$\int_{B_R \setminus B} |j(x,z) - j(y,z)|\nu(dz) \leq C^1(R)|x-y| ,$$

$$\int_{B_R} |j(x,z) - j(y,z)|^2\nu(dz) \leq C^2(R)|x-y|^2 .$$

We point out that (A) is a classical assumption to prove uniqueness for degenerate equations, see [29].

The hypothesis on the nonlocal term are classical for Lévy-Itô operators. Assumption (M) is the so-called *Lévy condition* over ν, and it allows to give a sense to the nonlocal operator for bounded C^2 functions. On the other hand, (MJ) is a continuity condition to treat the nonlocal terms in the comparison proof.

Concerning the conditions over the Hamiltonian, (H1) gives the structure allowing to apply the weak Bernstein method. When H is smooth, (H1) reduces to $H_p(x,p)p - H(x,p) \geq b_m|p|^m - K$ (see [13, 32]). In particular, H is coercive, see (4.4). Assumption (H2) states the continuity of the Hamiltonian, which is relative to its degree of coercivity m stated in (H1). Examples of such Hamiltonians can be found in [10] and references therein (see also [24] for some examples described in a detailed way).

Notice that (H2) implies the function $x \mapsto H(x,0)$ is uniformly continuous.

2.1. Comparison principle.

Proposition 2.1. *Let $u_0 \in C_b(\mathbb{R}^d)$, A satisfying (A), $\mathcal{I}^j$ defined as in* (1.3) *satisfying (M) and (MJ). Assume H satisfies (H1),(H2). Let u be an USC subsolution and v a LSC supersolution to the problem* (1.2)-(2.1) *such that u, v are bounded in $\mathbb{R}^d \times [0,T]$ for each $T > 0$. Then,*

$$u \leq v \quad \text{in } \bar{Q}.$$

Before proving the above proposition we introduce some technical results that are going to be used in different frameworks in the rest of the paper. The cell tool is the power function

$$\varphi_\alpha(x,y) = |x-y|^\alpha, \quad \text{for } x, y \in \mathbb{R}^d, \tag{2.2}$$

where $\alpha > 0$. This function will play different roles on the arguments to come, taking into account different values of α.

For $x, y \in \mathbb{R}^d$ with $x \neq y$, define the matrix

$$Z_\alpha = I_d + (\alpha - 2)\widehat{(x-y)} \otimes \widehat{(x-y)}, \tag{2.3}$$

where I_d is the $d \times d$ identity matrix and $\hat{x} = x/|x|$ for $x \neq 0$.

With this notation, note that for $x \neq y$, direct computations show that

$$\begin{aligned} D_x\varphi_\alpha(x,y) &= -D_y\varphi_\alpha(x,y) = \alpha|x-y|^{\alpha-1}(\widehat{x-y}),\\ D^2_{xx}\varphi_\alpha(x,y) &= D^2_{yy}\varphi_\alpha(x,y) = \alpha|x-y|^{\alpha-2}Z_\alpha, \\ D^2_{xy}\varphi_\alpha(x,y) &= D^2_{yx}\varphi_\alpha(x,y) = -\alpha|x-y|^{\alpha-2}Z_\alpha. \end{aligned} \tag{2.4}$$

Lemma 2.2. *Let $\alpha > 0$, φ_α defined in (2.2) and let $A : \mathbb{R}^d \to \mathbb{S}^d$ satisfying assumption (A). Let $\bar{x}, \bar{y} \in \mathbb{R}^d$ (with $\bar{x} \neq \bar{y}$ if $\alpha < 2$) and assume there exist two matrices $X, Y \in \mathbb{S}^d$ satisfying the inequality*

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq D^2_{(x,y)}\varphi_\alpha(\bar{x},\bar{y}). \tag{2.5}$$

Then, we have the estimate

$$\mathrm{Tr}(A(\bar{x})X - A(\bar{y})Y) \leq \alpha(1+|\alpha-2|)L_\sigma^2|\bar{x}-\bar{y}|^\alpha.$$

Proof: Multiplying the left inequality (2.5) by the nonnegative matrix

$$\begin{bmatrix} \sigma(\bar{x})\sigma^T(\bar{x}) & \sigma(\bar{x})\sigma^T(\bar{y}) \\ \sigma(\bar{y})\sigma^T(\bar{x}) & \sigma(\bar{y})\sigma^T(\bar{y}) \end{bmatrix} \tag{2.6}$$

and taking traces, the resulting inequality drives us to

$$\begin{aligned} &\mathrm{Tr}(A(\bar{x})X - A(\bar{y})Y) \\ &\leq \mathrm{Tr}\Big(\sigma(\bar{x})\sigma^T(\bar{x})D^2_{xx}\varphi_\alpha(\bar{x},\bar{y}) + \sigma(\bar{x})\sigma^T(\bar{y})D^2_{xy}\varphi_\alpha(\bar{x},\bar{y}) \\ &\qquad + \sigma(\bar{y})\sigma^T(\bar{x})D^2_{yx}\varphi_\alpha(\bar{x},\bar{y}) + \sigma(\bar{y})\sigma^T(\bar{y})D^2_{yy}\varphi_\alpha(\bar{x},\bar{y})\Big). \end{aligned}$$

Using the computations for the derivatives of φ_α and the definition of Z_α, we have

$$\begin{aligned} &\mathrm{Tr}(A(\bar{x})X - A(\bar{y})Y) \\ &\leq \alpha|\bar{x}-\bar{y}|^{\alpha-2}\mathrm{Tr}\Big((\sigma(\bar{x})\sigma^T(\bar{x}) - \sigma(\bar{x})\sigma^T(\bar{y}) - \sigma(\bar{y})\sigma^T(\bar{x}) + \sigma(\bar{y})\sigma^T(\bar{y}))Z_\alpha\Big) \\ &= \alpha|\bar{x}-\bar{y}|^{\alpha-2}\mathrm{Tr}\Big((\sigma(\bar{x})-\sigma(\bar{y}))(\sigma(\bar{x})-\sigma(\bar{y}))^TZ_\alpha\Big), \end{aligned}$$

but using Schwarz inequality, we obtain that

$$\mathrm{Tr}(A(\bar{x})X - A(\bar{y})Y) \leq \alpha(1+|\alpha-2|)|\bar{x}-\bar{y}|^{\alpha-2}|\sigma(\bar{x})-\sigma(\bar{y})|^2.$$

Finally, applying condition (A) we conclude the result. □

Due to the lack of compactness of $\mathbb{R}^d$, some localization argument in x is needed. In the sequel we use a nonnegative function $\psi \in C^2_b(\mathbb{R}^d)$ satisfying the following properties

$$\begin{cases} \psi = 0 \quad \text{in } B, \\ \psi = \Psi \quad \text{in } B_2^c \quad \text{for some constant } \Psi > 0, \\ 0 \leq \psi \leq \Psi \quad \text{in } B_2 \setminus B; \quad \text{and} \\ ||D\psi||_\infty, ||D^2\psi||_\infty \leq \Lambda \quad \text{for some } \Lambda > 0. \end{cases} \tag{2.7}$$

Next lemma states the estimates for a localization function based on ψ.

Lemma 2.3. *Assume (M) hold. Let ψ satisfying the properties listed in* (2.7) *and for $\beta > 0$, define the function*

$$\psi_\beta(x) = \psi(\beta x), \quad x \in \mathbb{R}^d. \tag{2.8}$$

Then, ψ_β satisfies

$$\begin{aligned}
&||D\psi_\beta||_\infty \leq \Lambda\beta, \quad ||D^2\psi_\beta||_\infty \leq \Lambda\beta^2,\\
&||\mathcal{I}^j[B_\delta \cap A](\psi_\beta, \cdot)||_\infty \leq \Lambda\beta^2 o_\delta(1),\\
&||\mathcal{I}^j[B_\delta^c \cap A](\psi_\beta, \cdot)||_\infty \leq \Lambda o_\beta(1),
\end{aligned}$$

where $o_\beta(1), o_\delta(1) \to 0$ as $\beta, \delta \to 0$ respectively and $o_\beta(1)$ depends only on Ψ, ν, j and $o_\delta(1)$ depends only on ν, j.

Proof of Lemma 2.3. The estimates for $D\psi_\beta$ and $D^2\psi_\beta$ are obvious.

Now we consider $\beta > 0$, $0 < \delta \leq 1$, $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$ measurable. Using the smoothness of ψ, we have

$$\mathcal{I}^j[B_\delta \cap A](\psi_\beta, x) = \frac{1}{2}\int_{B_\delta \cap A}\int_0^1 \langle D^2\psi_\beta(x + \theta j(x,z))j(x,z), j(x,z)\rangle d\theta\nu(dz),$$

from which, we easily deduce that

$$|\mathcal{I}^j[B_\delta \cap A](\psi_\beta, x)| \leq \frac{1}{2}||D^2\psi_\beta||_\infty \int_{B_\delta \cap A} |j(x,z)|^2\nu(dz).$$

Then, using (M) and the estimates for $D^2\psi_\beta$ we get

$$|\mathcal{I}^j[B_\delta \cap A](\psi_\beta, x)| \leq \frac{1}{2}\beta^2\Lambda \int_{B_\delta \cap A} |j(x,z)|^2\nu(dz) = \beta^2\Lambda o_\delta(1). \tag{2.9}$$

From the definition of ψ_β, we have, for all $x, y \in \mathbb{R}^d$, $|\psi_\beta(x) - \psi_\beta(y)| \leq \Psi$ and $\psi_\beta(x) \to 0$ as $\beta \to 0$. Therefore, since $\int_{B^c}\nu(dz) < \infty$ by (M), from the Dominated Convergence Theorem, we obtain

$$\mathcal{I}^j[B^c \cap A](\psi_\beta, x) = o_\beta(1), \tag{2.10}$$

where $o_\beta(1)$ depends only on Ψ, ν, j.

From (2.9) (with $\delta = 1$) and (2.10) it follows

$$\mathcal{I}^j[A](\psi_\beta, x) = o_\beta(1) + \beta^2\Lambda o_1(1) = o_\beta(1),$$

and from here we finally get that

$$\mathcal{I}^j[B_\delta^c \cap A](\psi_\beta, \cdot) = \mathcal{I}^j[A](\psi_\beta, \cdot) - \mathcal{I}^j[B_\delta \cap A](\psi_\beta, \cdot) = o_\beta(1),$$

from which the result follows. □

Next, we have the following

Lemma 2.4. *Let A satisfying (A), $\mathcal{I}^j$ defined in* (1.3) *such that its components satisfy (M), (MJ) and let H satisfying (H1). Let u be an USC subsolution and v a LSC supersolution to problem* (1.2)-(2.1)*, bounded in $\mathbb{R}^d \times [0,T]$ for each $T > 0$. Then, $u(x,0) \leq u_0(x) \leq v(x,0)$ for all $x \in \mathbb{R}^d$.*

Remark 1. The initial condition has to be understood in the viscosity sense, see [14, Definition 2.1]. Lemma 2.4 states that actually it holds in the classical sense. We refer to [14] (see also [22]) for a proof of this result in the case of Dirichlet problem in bounded domains. In the current setting, the proof must be slightly modified by a standard localization procedure that allows to deal with the lack of compactness of $\mathbb{R}^d$. Hence, we remark that uniform continuity of the initial data is not necessary in the proof of the comparison principle.

Proof of Proposition 2.1. We will argue over the finite horizon problem

$$\begin{cases} \partial_t u - \mathrm{Tr}(AD^2u) - \mathcal{I}^j(u,x) + H(x,Du) &= 0 \qquad \text{in } Q_T \\ u(x,0) &= u_0(x) \qquad x \in \mathbb{R}^d, \end{cases}$$

We are going to prove that $M := \sup_{Q_T}(u - v) \leq 0$; the general result on Q follows since T is arbitrary.

We argue by contradiction, assuming that $M > 0$ and for $\mu, \eta \in (0,1)$ to be fixed later, we define $(x,t) \mapsto \bar{u}(x,t) := \mu u(x,t) - \eta t$. Since u is bounded, we have $\sup_{Q_T}(\bar{u} - v) \geq M/2 > 0$ if η is small enough and μ sufficiently close to 1. We notice that since u is a subsolution to (1.2) then $\bar{u}$ satisfies

$$\partial_t \bar{u} - \mathrm{Tr}(AD^2\bar{u}) - \mathcal{I}^j(\bar{u},x) + \mu H(x, \mu^{-1}D\bar{u}) \leq -\eta \quad \text{in } Q_T,$$

in the viscosity sense.

Next we consider for $\epsilon > 0$ the function

$$(x,y,s,t) \mapsto \Phi(x,y,t) := \bar{u}(x,t) - v(y,t) - \phi(x,y),$$

where $\phi(x,y) := \epsilon^{-2}|x-y|^2 + \psi_\beta(y)$ and ψ_β is defined as in Lemma 2.3 with a localization function ψ defined as in (2.7) with $\Psi = 2(||u||_{L^\infty(Q_T)} + ||v||_{L^\infty(Q_T)})$. With this we see that for all β small enough

$$\overline{M} := \sup_{\bar{Q}_T \times \bar{Q}_T} \Phi > 0,$$

and this supremum is achieved at some point $(\bar{x}, \bar{y}, \bar{t}) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T]$. However, in view of Lemma 2.4 and the positiveness of $\overline{M}$ we necessarily have that $\bar{t} > 0$, and standard arguments in the viscosity theory imply this sequence of points (depending on ϵ and β) satisfies

$$\begin{aligned} &\epsilon^{-2}|\bar{x} - \bar{y}|^2 \to 0 \quad \text{if } \epsilon \to 0,\ \beta \text{ fixed}, \\ &\overline{M} \to \sup_{Q_T}(\bar{u} - v - \psi_\beta) \quad \text{if } \epsilon \to 0,\ \beta \text{ fixed}, \\ &\sup_{Q_T}(\bar{u} - v - \psi_\beta) \to \sup_{Q_T}(\bar{u} - v) > 0 \quad \text{when } \beta \to 0. \end{aligned} \tag{2.11}$$

Then we can apply the nonlocal parabolic version of Ishii-Jensen Lemma provided in [9] (see also [30, 28, 21] for second-order parabolic equations) which avoids the doubling in the time variable in the definition of Φ above.

Thus, for all $\delta > 0$ and all $\rho > 0$ small enough we have

$$
\begin{aligned}
&\varpi - \mathrm{Tr}(A(\bar{x})X_\rho) - \mathcal{I}^j[B_\delta](\phi(\cdot,\bar{s},\bar{y},\bar{t}),\bar{x}) \\
&\qquad -\mathcal{I}^j[B_\delta^c](\bar{u},\bar{x},\bar{p}) + \mu H(\bar{x},\mu^{-1}\bar{p}) \le o_\rho(1) - \eta, \\
&\varpi - \mathrm{Tr}(A(\bar{y})Y_\rho) - \mathcal{I}^j[B_\delta](-\phi(\bar{x},\bar{s},\cdot,\bar{t}),\bar{y}) \\
&\qquad -\mathcal{I}^j[B_\delta^c](v,\bar{y},\bar{p}+\bar{q}) + H(\bar{y},\bar{p}+\bar{q}) \ge o_\rho(1)
\end{aligned} \tag{2.12}
$$

where $\varpi \in \mathbb{R}$, $\bar{p} = 2\epsilon^{-2}(\bar{x}-\bar{y})$, $\bar{q} = -D\psi_\beta(\bar{y})$ and the matrices $X_\rho, Y_\rho \in \mathbb{S}^d$ satisfy the inequality

$$
-\rho^{-1} I_{2d} \le \begin{bmatrix} X_\rho & 0 \\ 0 & -Y_\rho \end{bmatrix} \le D^2_{(x,y)}\phi(\bar{x},\bar{s},\bar{y},\bar{t}) + o_\rho(1). \tag{2.13}
$$

We remark that all the terms $o_\rho(1)$ arising in (2.12) and (2.13) satisfy $o_\rho(1) \to 0$ for $\epsilon, \beta > 0$ fixed. Subtracting both inequalities in (2.12) we get

$$
\mathcal{H} \le \mathcal{A} + \mathcal{B}_\delta + \mathcal{B}^\delta - \eta + o_\rho(1), \tag{2.14}
$$

where

$$
\begin{aligned}
\mathcal{A} &= \mathrm{Tr}(A(\bar{x})X_\rho) - \mathrm{Tr}(A(\bar{y})Y_\rho) \\
\mathcal{H} &= \mu H(\bar{x},\mu^{-1}\bar{p}) - H(\bar{y},\bar{p}+\bar{q}) \\
\mathcal{B}_\delta &= \mathcal{I}^j[B_\delta](\phi(\cdot,\bar{s},\bar{y},\bar{t}),\bar{x}) - \mathcal{I}^j[B_\delta](\phi(\bar{x},\bar{s},\cdot,\bar{t}),\bar{y}), \\
\mathcal{B}^\delta &= \mathcal{I}^j[B_\delta^c](\bar{u}(\cdot,\bar{s}),\bar{x},\bar{p}) - \mathcal{I}^j[B_\delta^c](v(\cdot,\bar{t}),\bar{y},\bar{p}+\bar{q}).
\end{aligned}
$$

In what follows we estimate each term arising in (2.14).

1.- *Estimate of $\mathcal{A}$:* In view of the definition of φ in (2.2) we can write

$$
\phi(x,y,t) = \epsilon^{-2}\varphi_2(x,y) + \psi_\beta(y).
$$

Hence, using the estimate given by Lemma 2.2 and applying the estimates for the second derivatives of ψ_β in Lemma 2.3, we conclude that

$$
\mathrm{Tr}(A(\bar{x})X_\rho - A(\bar{y})Y_\rho) \le 2L_\sigma^2\epsilon^{-2}|\bar{x}-\bar{y}|^2 + \beta^2 L_\sigma^2 \Lambda + o_\rho(1).
$$

From this, using (2.11), we conclude

$$
\mathcal{A} \le m_\beta(\epsilon) + o_\beta(1) + o_\rho(1)\,, \tag{2.15}
$$

where, here and below, $m_\beta(\epsilon)$ denotes various quantities which tend to 0 when $\epsilon \to 0$, β remaining fixed and $o_\beta(1) \to 0$ as $\beta \to 0$, μ and η being fixed.

2.- *Estimate for $\mathcal{H}$:* Recalling that $|\bar{q}| \to 0$ as $\beta \to 0$, we have $|\bar{q}| \le 1$ for β small enough and we can apply (H1) and (H2) to get

$$
\begin{aligned}
\mathcal{H} \;\ge\; & [(1-\mu)b_m - \zeta_1(|\bar{x}-\bar{y}|)]\,|\bar{p}|^m - \zeta_2(|\bar{q}|)(1+|\bar{p}|^{m-1}) \\
& -K(1-\mu) - \zeta_1(|\bar{x}-\bar{y}|),
\end{aligned}
$$

and applying Young's inequality, we can write

$$\begin{aligned}\mathcal{H} \geq & \Big[(1-\mu)b_m - \zeta_1(|\bar{x}-\bar{y}|) - (m-1)m^{-1}\zeta_2(|\bar{q}|)^{m(m-1)^{-1}/2}\Big]|\bar{p}|^m \\ & - m^{-1}\zeta_2(|\bar{q}|)^{m/2} - \zeta_2(|\bar{q}|) - K(1-\mu) - \zeta_1(|\bar{x}-\bar{y}|).\end{aligned}$$

At this point, we fix $\mu = \mu_\eta < 1$ close to 1 in order to have $K(1-\mu_\eta) \leq \eta/4$. Considering ϵ and β small enough depending on μ and b_m, we can suppose that $|\bar{x}-\bar{y}|$ and $|\bar{q}|$ are small enough to make positive the term in the squared brackets in the last inequality. From this, we conclude that

$$\mathcal{H} \geq -o_\beta(1) - m_\beta(\epsilon) - \eta/4. \tag{2.16}$$

3.- *Estimate for* $\mathcal{B}_\delta$*:* Using the definition of ϕ and Lemma 2.3, we obtain

$$\begin{aligned}\mathcal{B}_\delta \leq & \frac{1}{2}\frac{||D^2\varphi_2(\cdot,\bar{y})||_\infty}{\epsilon^2}\int_{B_\delta}|j(\bar{x},z)|^2\nu(dz) \\ & +\frac{1}{2}\left(\frac{||D^2\varphi_2(\bar{x},\cdot)||_\infty}{\epsilon^2} + ||D^2\psi_\beta||_\infty\right)\int_{B_\delta}|j(\bar{y},z)|^2\nu(dz),\end{aligned}$$

and therefore, using (M) we conclude that

$$\mathcal{B}_\delta \leq (\epsilon^{-2} + \Lambda\beta^2)o_\delta(1), \tag{2.17}$$

where $o_\delta(1) \to 0$ as $\delta \to 0$.

4.- *Estimate for* $\mathcal{B}^\delta$*:* We start writing $\mathcal{B}^\delta = \mathcal{B}^{\delta,1} + \mathcal{B}^{\delta,2}$ with

$$\begin{aligned}\mathcal{B}^{\delta,1} &= \mathcal{I}^j[B\cap B_\delta^c](\bar{u}(\cdot,\bar{t}),\bar{x},\bar{p}) - \mathcal{I}^j[B\cap B_\delta^c](v(\cdot,\bar{t}),\bar{y},\bar{p}+\bar{q}), \\ \mathcal{B}^{\delta,2} &= \mathcal{I}^j[B^c](\bar{u}(\cdot,\bar{t}),\bar{x},\bar{p}) - \mathcal{I}^j[B^c](v(\cdot,\bar{t}),\bar{y},\bar{p}+\bar{q}).\end{aligned}$$

Since $(\bar{x},\bar{y},\bar{t})$ is a maximum for Φ, for each $\xi,\xi' \in \mathbb{R}^d$ we have

$$\begin{aligned}& \bar{u}(\bar{x}+\xi,\bar{t}) - v(\bar{y}+\xi',\bar{t}) - \epsilon^{-2}|\bar{x}+\xi-\bar{y}-\xi'|^2 - \psi_\beta(\bar{y}+\xi') \\ & \leq \bar{u}(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - \epsilon^{-2}|\bar{x}-\bar{y}|^2 - \psi_\beta(\bar{y}).\end{aligned}$$

Using this inequality with $\xi = j(\bar{x},z), \xi' = j(\bar{y},z)$ we obtain

$$\begin{aligned}& \bar{u}(\bar{x}+j(\bar{x},z),\bar{t}) - \bar{u}(\bar{x},\bar{t}) - (v(\bar{y}+j(\bar{y},z),\bar{t}) - v(\bar{y},\bar{t})) \\ & \leq \epsilon^{-2}|j(\bar{x},z)-j(\bar{y},z)|^2 + 2\epsilon^{-2}\langle\bar{x}-\bar{y}, j(\bar{x},z)-j(\bar{y},z)\rangle \\ & \quad + \psi_\beta(\bar{y}+j(\bar{y},z)) - \psi_\beta(\bar{y},z),\end{aligned} \tag{2.18}$$

and replacing this inequality into the definition of $\mathcal{B}^{\delta,1}$ we get

$$\begin{aligned}\mathcal{B}^{\delta,1} & \leq \epsilon^{-2}\int\limits_{B\setminus B_\delta}|j(\bar{x},z)-j(\bar{y},z)|^2\nu(dz) + \mathcal{I}^j[B\cap B_\delta^c](\psi_\beta,\bar{y}) \\ & \leq C^2(1)\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} + \Lambda o_\beta(1)\end{aligned}$$

where we have used (MJ) with $R=1$ and Lemma 2.3 to control the nonlocal term applied to ψ_β.

Thus, by (2.11), we conclude that

$$\mathcal{B}^{\delta,1} \leq m_\beta(\epsilon) + o_\beta(1). \tag{2.19}$$

Now we address the estimate of $\mathcal{B}^{\delta,2}$. Note that

$$\mathcal{B}^{\delta,2} = \int_{B^c} [\bar{u}(\bar{x} + j(\bar{x},z),\bar{t}) - v(\bar{y} + j(\bar{y},z),\bar{t}) - (\bar{u}(\bar{x},\bar{t}) - v(\bar{y},\bar{t}))]\nu(dz).$$

For any $R > 1$, we divide this integral in two parts, $\mathcal{B}^{\delta,2} = \mathcal{B}^{\delta,2}[B_R^c] + \mathcal{B}^{\delta,2}[B_R \setminus B]$, where the first one is integrated over B_R^c and the second one over $B_R \setminus B$.

For the first one, we have

$$\mathcal{B}^{\delta,2}[B_R^c] \leq \Psi \int_{B_R^c} \nu(dz),$$

where we recall that $\Psi = 2(||u||_{L^\infty(Q_T)} + ||v||_{L^\infty(Q_T)})$.

From (M), we can choose $R = R_\eta$ large enough in order that the above term is less than $\eta/4$.

For the second term, using (2.18) we can get

$$\begin{aligned}
\mathcal{B}^{\delta,2}[B_{R_\eta} \setminus B] \ &\leq \int\limits_{B_{R_\eta}\setminus B} [\epsilon^{-2}|j(\bar{x},z) - j(\bar{y},z)|^2 + 2\epsilon^{-2}\langle \bar{x} - \bar{y}, j(\bar{x},z) - j(\bar{y},z)\rangle]\nu(dz) \\
&\qquad + \int\limits_{B_{R_\eta}\setminus B} [\psi_\beta(\bar{y} + j(\bar{y},z)) - \psi_\beta(\bar{y})]\nu(dz) \\
&\leq 2\epsilon^{-2} \int\limits_{B_{R_\eta\setminus B}} \Big(|j(\bar{x},z) - j(\bar{y},z)|^2 + |\bar{x} - \bar{y}||j(\bar{x},z) - j(\bar{y},z)|\Big)\nu(dz) \\
&\qquad + ||D\psi_\beta||_\infty \int\limits_{B_{R_\eta}\setminus B} |j(\bar{y},z)|\nu(dz) \\
&\leq (C^2(R_\eta) + C^1(R_\eta))\epsilon^{-2}|\bar{x} - \bar{y}|^2 + \Lambda\beta C^0(R_\eta),
\end{aligned}$$

by using (MJ) with $R = R_\eta$. Finally, we obtain

$$\mathcal{B}^{\delta,2} \leq \eta/4 + m_\beta(\epsilon) + o_\beta(1).$$

Using this and estimate (2.19), we get

$$\mathcal{B}^{\delta} \leq \eta/4 + m_\beta(\epsilon) + o_\beta(1), \tag{2.20}$$

where $m_\beta(\epsilon), o_\beta(1)$ depend on μ, η but are independent of δ.

5.- *Conclusion:* Joining (2.15), (2.16), (2.17) and (2.20), and using them into (2.14), we conclude that

$$-\epsilon^{-2}o_\delta(1) - m_\beta(\epsilon) - o_\beta(1) - \eta/2 \leq -\eta + o_\rho(1).$$

Finally, letting $\rho, \delta \to 0$ first, then $\epsilon \to 0$ for a fixed, small enough β, we obtain a contradiction since $\eta > 0$. It follows that $M \leq 0$ and the proof is complete. $\square$

In what follows we discuss some important consequences of the comparison principle.

2.2. **Well-posedness.** The consideration of the following boundedness condition

(H0) *There exists a constant* $H_0 > 0$ *such that* $||H(\cdot, 0)||_\infty \leq H_0$

allows us to provide existence to (1.2)-(2.1) via Perron's method.

Corollary 2.5. *Let* $u_0 \in \mathrm{C}_b(\mathbb{R}^d)$*, A satisfying (A),* $\mathcal{I}^j$ *defined as in* (1.3) *in such a way its components satisfy assumptions (M), (MJ), and H satisfying assumptions (H0), (H1), (H2). Then, there exists a unique viscosity solution* $u \in C(\bar{Q})$ *to problem* (1.2)-(2.1)*, which is also in* $L^\infty(\bar{Q}_T)$ *for all* $T > 0$.

The proof of this result follows classical arguments, see [28, 21]. It is posible to argument in Q_T first and then extend it to the infinite time horizon. The role of the global sub and supersolution present in Perron's method is played by functions with the form $(x, t) \mapsto C_1 t + C_2$, for suitable constants C_1, C_2 depending on the data and T. Uniqueness comes from Proposition 2.1.

For the stationary case we can follow closely the previous arguments and obtain the analogous well-posedness result provided the equations is strictly proper.

Proposition 2.6. *Let* $\lambda > 0$*, A satisfying (A),* $\mathcal{I}^j$ *defined as in* (1.3) *in such a way its components satisfy assumptions (M), (MJ), and H satisfying assumptions (H0), (H1), (H2). Let u be an USC bounded viscosity subsolution and v be a LSC bounded viscosity supersolution to equation* (1.1)*. Then,* $u \leq v$ *in* $\mathbb{R}^d$.

Moreover, if in addition we assume (H0), then there exists a unique viscosity solution $u \in C_b(\mathbb{R}^d)$ *to equation* (1.1)*, for which we have the following bound*

$$||u||_\infty \leq \lambda^{-1} H_0. \tag{2.21}$$

2.3. **Continuity results coming from comparison.** Comparison principle given by Proposition 2.1 allows us to obtain important continuity results for the solution of the addressed problems. The first result gives Lipschitz regularity in time if the initial data is smooth.

Proposition 2.7. *Consider the hypotheses of Proposition 2.1 and assume further that* $u_0 \in C_b^2(\mathbb{R}^d)$ *with* $||u_0||_{C^2(\mathbb{R}^d)} < +\infty$*. Then, there exists a constant* Λ_0 *depending on the datas and* $||u_0||_{C^2(\mathbb{R}^d)}$ *such that each viscosity solution u to problem* (1.2)-(2.1) *in* $C(\bar{Q}) \cap L^\infty(\bar{Q}_T)$ *for each* $T > 0$ *satisfies*

$$|u(x,t) - u(x,s)| \leq \Lambda_0 |t - s| \quad \textit{for all } 0 \leq s, t \leq T, \ x \in \mathbb{R}^d.$$

The proof of this result relies in comparison with functions with the form $(x,t) \mapsto \Lambda_0 t + u_0(x)$ for $\Lambda_0 \in \mathbb{R}$ adequate, together with translation invariance in time, see [34].

A less direct consequence is stated in the following

Proposition 2.8. *Let $u_0 \in BUC(\mathbb{R}^d)$ and assume that $\mathcal{I}^j$ and H satisfy the assumptions of Corollary 2.5. Denote by $u \in C(\bar{Q})$ the unique viscosity solution to* (1.2)-(2.1) *given in Corollary 2.5. Assume further that*

(i) There exists a modulus of continuity m_j such that for each $R > 0$ there exists $C_R > 0$ satisfying

$$|j(x,z) - j(y,z)| \leq C_R \; m_j(|x-y|), \quad \textit{for all } x, y \in \mathbb{R}^d, \ |z| \leq R.$$

(ii) For each $R > 0$ there exists a constant $C_R > 0$ such that

$$|H(x,p)| \leq C_R \quad \textit{for all } \ x \in \mathbb{R}^d, \ |p| \leq R.$$

Then, there exists a modulus of continuity m_T depending on the data, T and $||u||_{L^\infty(Q_T)}$ such that

$$|u(x,t) - u(y,t)| \leq m_T(|x-y|) \quad \textit{for all } x, y \in \mathbb{R}^d.$$

Proof: We only sketch the proof since most of the arguments are tedious but easily checkable.

By contradiction we assume that u is not uniformly continuous in x. Then, there exist sequences $x_k, y_k \in \mathbb{R}^d$, $t_k \in [0,T]$ such that $x_k - y_k$ tends to 0 and $u(x_k,t_k) - u(y_k,t_k) > \eta > 0$.

We consider the function $v_k(x,t) = u(x+x_k,t)$. Denoting $u_{0,k}(x) = u_0(x+x_k)$, $A_k(x) = A(x+x_k)$, $j_k(x,z) = j(x+x_k,z)$ and $H_k(x,p) = H(x+x_k,p)$ we see that v_k is a solution to the problem

$$\partial_t v_k - \mathcal{I}^{j_k}(v_k(\cdot,t),x) - \mathrm{Tr}(A_k D^2 v_k) + H_k(x, Dv_k) = 0 \quad \text{in } Q_T,$$

with initial data $v_k(\cdot,0) = u_{0,k}$ in $\mathbb{R}^d$.

The assumptions of the problem considered in the existence and uniqueness result together with the extra assumptions (i) and (ii) imply that the sequences of functions $\{u_{k,0}\}_k, \{A_k\}_k, \{H_k(\cdot,p)\}_k$, and $\{j_k(\cdot,z)\}_k$ are locally uniformly continuous in $\mathbb{R}^d$ and bounded, for each $z, p \in \mathbb{R}^d$. Then, up to subsequences there exist functions $\tilde{u}_0, \tilde{A}, \tilde{H}$ and $\tilde{j}$ which are respective limit functions to the previous sequences, and the convergence is locally uniform in $\mathbb{R}^d$. Then, defining the functions

$$\bar{v}(x,t) = \limsup_{y \to x, s \to t, k \to \infty} v_k(y,s), \quad \underline{v}(x,t) = \liminf_{y \to x, s \to t, k \to \infty} v_k(y,s)$$

we apply the half-relaxed limits method (see [11]) to conclude that $\bar{v}, \underline{v}$ are, respectively, bounded viscosity sub and supersolutions to the problem

$$\partial_t v - \mathcal{I}^{\tilde{j}}(v(\cdot,t),x) - \mathrm{Tr}(\tilde{A} D^2 v) + \tilde{H}(x, Dv) = 0 \quad \text{in } Q_T,$$

with the initial condition $v(\cdot,0) = \tilde{u}_0$ in $\mathbb{R}^d$. Noticing that the limit problem satisfies the hypotheses of Proposition 2.1, we conclude that $\bar{v} \leq \underline{v}$ and therefore they are equal. But $u(x_k,t_k)-u(y_k,t_k) > \eta > 0$ can be interpreted as $v_k(0,t_k) - v_k(y_k - x_k, t_k) > \eta > 0$ and this would lead to a contradiction after taking limit as $k \to \infty$. This concludes the result. $\square$

Similar arguments can be given for the stationary problem (1.1) to conclude the following

Proposition 2.9. *Let $\lambda > 0$ and assume $\mathcal{I}^j, H$ satisfy the assumptions of Proposition 2.6, and denote $u \in C(\mathbb{R}^d)$ the unique viscosity solution to* (1.1). *Assume further conditions $(i), (ii)$ in Proposition 2.8. Then, there exists a modulus of continuity m depending on the data and λ such that*

$$|u(x) - u(y)| \leq m(|x-y|) \quad \textit{for all } x, y \in \mathbb{R}^d.$$

3. Lipschitz regularity - Bernstein method.

In this section we provide the Lipschitz regularity for solutions of problem (1.1). This is accomplished by the introduction of a change of variables in the direction of Bernstein method, see [5]. As it can be seen in the literature, it is natural to assume Lipschitz regularity on the data to apply the mentioned method. Keeping assumptions (H1) and (A) as in the previous section, we require to strengthen assumptions (H2), (M) and (MJ) in the following sense.

(H2') *Let m as in (H1). There exists $L_H > 0$ and a modulus of continuity ζ such that, for all $x, y, p, q \in \mathbb{R}^d$, $|q| \leq 1$, we have*

$$H(y, p+q) - H(x,p) \leq L_H|x-y|(1+|p|^m) + \zeta(|q|)(1+|p|^{m-1}).$$

(M') *There exists $C_\nu > 0$ such that*

$$\int_{\mathbb{R}^d} 1 \wedge |z|^2 \nu(dz) \leq C_\nu,$$

(J1) *There exists a constant $C_j > 0$ such that for all $x, y, z \in \mathbb{R}^d$*

$$|j(x,z)| \leq C_j|z| \quad \text{and} \quad |j(x,z) - j(y,z)| \leq C_j|z||x-y|.$$

(J2) *For each $a > 0$, there exists a constant $C_a > 0$ such that for all $x, y \in \mathbb{R}^d$*

$$\int_{B_a^c} |j(x,z) - j(y,z)|\nu(dz) \leq C_a|x-y|$$

Remark 2. Once we consider assumptions (A), (H1), together with (H2'), (M'), (J1) and (J2), we can apply all the results of the previous section since (M'), (J1) and (J2) imply (M),(MJ).

The above assumptions are natural since they reflect the Lipschitz regularity of the data. This is evident in (J1) and (J2), meanwhile because of (H2') we have the function $x \mapsto H(x, 0)$ is Lipschitz continuous.

Notice that (J2) reflects a compatibility condition among the measure ν and the jumps j. For example, (J2) is fullfilled by measures ν satisfying (M') and jumps satisfying

$$|j(x,z) - j(y,z)| \leq C|x-y|\eta(z), \quad \text{for all } x, y \in \mathbb{R}^d, z \in B^c,$$

with $\eta(z)\nu(dz)$ finite in B^c.

The main result of this paper is the following

Theorem 3.1. (Lipschitz regularity) *Let $\lambda \geq 0$. Assume A satisfies (A), H satisfies (H1), (H2'), and $\mathcal{I}^j$ defined in* (1.3) *such that ν, j satisfy assumptions (M'), (J1) and (J2). Let u be a bounded uniformly continuous viscosity solution to problem* (1.1)*. Then, there exists $L > 0$ large enough such that*

$$|u(x) - u(y)| \leq L|x-y| \quad \text{for all } x, y \in \mathbb{R}^d.$$

The constant L depends only on the data and $\mathrm{osc}(u)$.

As we will see next, Theorem 3.1 is accomplished by an exponential change of variables inspired by the Bernstein method for viscosity solutions. In fact, we note that if $u \in C_b(\mathbb{R}^d)$ is a solution of (1.1), replacing u by $u - \inf\{u\} + 1$ we can assume $u \geq 1$ and then, under the change of variables $u = e^v$, we can prove that $v \geq 0$ satisfies the equation [1]

$$\lambda - \mathrm{Tr}(AD^2 v) - \mathcal{J}^j(v, x) + e^{-v} H(x, e^v Dv) - |\sigma^T Dv|^2 = 0 \quad \text{in } \mathbb{R}^d, \tag{3.1}$$

where $\mathcal{J}^j(v, x)$ is defined as

$$\mathcal{J}^j(v,x) = \int_{\mathbb{R}^d} [e^{v(x+j(x,z))-v(x)} - 1 - \mathbf{1}_B(z)\langle Dv(x), j(x,z)\rangle]\nu(dz). \tag{3.2}$$

For simplicity, for $x, p \in \mathbb{R}^d$, $r \in \mathbb{R}$ we introduce the notation

$$\tilde{H}(x, r, p) = \lambda + e^{-r} H(x, e^r p) - |\sigma^T(x) p|^2. \tag{3.3}$$

With this, using equation (3.1) and the above notation, u is a viscosity solution to (1.1) if and only if v defined as $u = e^v$ is a viscosity solution to the equation

$$\tilde{H}(x, v(x), Dv(x)) - \mathrm{Tr}(A(x)D^2 v(x)) - \mathcal{J}^j(v, x) = 0, \quad x \in \mathbb{R}^d. \tag{3.4}$$

By the above discussion, it is going to be convenient to argue over the equivalent equation (3.4). As we mentioned in the introduction, of main interest is the treatment of the nonlocal term, and the nonlinearity arising in $\mathcal{J}^j$ after the exponential change of variables is an extra difficulty.

[1]In fact, H should be changed into $H + \lambda(\inf\{u\} - 1)$. This new Hamiltonian has the same properties of H since $\lambda \inf\{u\}$ is bounded independently of λ by (2.21). Hence we keep the notation H for the sake of simplicity.

Now we present the proof of Theorem 3.1. In its proof we use several technical estimates which are precisely stated and proved in the Appendix.

Proof of Theorem 3.1: Note that under the change $u = e^v$ with $u \geq 1$, we have that $\operatorname{osc}(v) \leq \operatorname{osc}(u)$, and by the boundedness of v we see that v is still uniformly continuous. Then, if we get v is Lipschitz continuous then we get the result for u.

We will argue over the equation (3.4). Assume by contradiction that for each $L > 0$ large enough (we can assume $L > 1$), there exists $\epsilon_L > 0$ such that

$$\sup_{\mathbb{R}^d \times \mathbb{R}^d} \{v(x) - v(y) - L|x-y|\} \geq \ 2\epsilon_L.$$

and therefore, there exist $x_L, y_L \in \mathbb{R}^d$ such that

$$v(x_L) - v(y_L) - L|x_L - y_L| \geq \epsilon_L.$$

At this point we consider a nonnegative function $\psi \in C_b^2(\mathbb{R}^d)$ satisfying assumptions (2.7) with $\Psi = \operatorname{osc}(v)$ and $\Lambda > 0$ depending only on $\operatorname{osc}(v)$, and for $\beta > 0$ we consider ψ_β as in (2.8). Then, setting $\phi(x,y) := L|x-y| + \psi_\beta(y)$ we have

$$\sup_{\mathbb{R}^d \times \mathbb{R}^d} \{v(x) - v(y) - \phi(x,y)\} \geq v(x_L) - v(y_L) - L|x_L - y_L| - \psi_\beta(y_L) \geq \epsilon_L > 0 \tag{3.5}$$

for all β small enough to have $1/\beta > |y_L|$. Moreover, since $\psi_\beta = \operatorname{osc}(v)$ in $B^c_{2/\beta}$, the supremum in (3.5) is achieved at some point $(\bar{x}, \bar{y}) \in \mathbb{R}^d \times \mathbb{R}^d$, with $\bar{x} \neq \bar{y}$ for each $L, \beta > 0$, and we get

$$L|\bar{x} - \bar{y}| \leq \operatorname{osc}(v). \tag{3.6}$$

Using Proposition 2.8, let $\omega : [0, +\infty) \to [0, +\infty)$, $\omega(0) = 0$, be an increasing modulus of continuity for v. Then we can write

$$|v(x) - v(y)| \leq \omega(|x-y|) \quad \text{for all } x, y \in \mathbb{R}^d.$$

By (3.5) we have $\epsilon_L \leq v(\bar{x}) - v(\bar{y})$ and therefore

$$0 < \omega^{-1}(\epsilon_L) \leq |\bar{x} - \bar{y}|, \tag{3.7}$$

which is a lower bound for $|\bar{x} - \bar{y}|$, uniform in terms of β.

Now we would like to use the viscosity inequality for v at $\bar{x}$ and $\bar{y}$. Using Ishii-Jensen lemma for nonlocal equations, for all $\delta > 0$ and all $\rho > 0$ small enough we have

$$\begin{gathered}
\tilde{H}(\bar{x}, v(\bar{x}), L\bar{p}) - \operatorname{Tr}(A(\bar{x})X_\rho) \\
-\mathcal{J}^j[B_\delta](\phi(\cdot, \bar{y}), \bar{x}) - \mathcal{J}^j[B^c_\delta](v, \bar{x}, L\bar{p}) \leq o_\rho(1), \\
\tilde{H}(\bar{y}, v(\bar{y}), L\bar{p} - \bar{q}) - \operatorname{Tr}(A(\bar{y})Y_\rho) \\
-\mathcal{J}^j[B_\delta](-\phi(\bar{x}, \cdot), \bar{y}) - \mathcal{J}^j[B^c_\delta](v, \bar{y}, L\bar{p} - \bar{q}) \geq -o_\rho(1),
\end{gathered} \tag{3.8}$$

where $\bar{p} = (\bar{x}-\bar{y})/|\bar{x}-\bar{y}|$ and $\bar{q} = D\psi_\beta(\bar{y})$. The matrices $X_\rho, Y_\rho \in \mathbb{S}^d$ satisfy the inequality

$$-\rho^{-1} I_{2d} \leq \begin{bmatrix} X_\rho & 0 \\ 0 & -Y_\rho \end{bmatrix} \leq D^2\phi(x,y) + o_\rho(1). \tag{3.9}$$

We remark that all the terms $o_\rho(1)$ arising in (3.8) and (3.9) satisfy $o_\rho(1) \to 0$ if $L, \beta > 0$ are fixed.

Subtracting both inequalities in (3.8), we get

$$\mathcal{H} \leq \mathcal{A} + \mathcal{B}_\delta + \mathcal{B}^\delta, \tag{3.10}$$

where

$$\begin{aligned} \mathcal{H} &= \tilde{H}(\bar{x}, v(\bar{x}), L\bar{p}) - \tilde{H}(\bar{y}, v(\bar{y}), L\bar{p} - \bar{q}), \\ \mathcal{B}_\delta &= \mathcal{J}^j[B_\delta](\phi(\cdot, \bar{y}), \bar{x}) - \mathcal{J}^j[B_\delta](-\phi(\bar{x}, \cdot), \bar{y}), \\ \mathcal{B}^\delta &= \mathcal{J}^j[B_\delta^c](v, \bar{x}, L\bar{p}) - \mathcal{J}^j[B_\delta^c](v, \bar{y}, L\bar{p} - \bar{q}), \\ \mathcal{A} &= \mathrm{Tr}(A(\bar{x})X_\rho) - \mathrm{Tr}(A(\bar{y})Y_\rho) + o_\rho(1). \end{aligned} \tag{3.11}$$

In what follows, we estimate each term present in (3.10) in order to get the desired contradiction by taking L large enough in terms of the data.

1.- *Estimates for $\mathcal{A}$.* Note that in this case we have

$$\phi(x,y) = L\varphi_1(x,y) + \psi_\beta(y),$$

where φ_1 is defined in (2.2).

Hence, using the estimate given by Lemma 2.2 and applying the estimates for the second derivatives of ψ_β in Lemma 2.3, we conclude that

$$\mathrm{Tr}(A(\bar{x})X_\rho - A(\bar{y})Y_\rho) \leq 2LL_\sigma^2|\bar{x}-\bar{y}| + \beta^2 L_\sigma^2 \Lambda + o_\rho(1)$$

and therefore

$$\mathcal{A} \leq 2L_\sigma^2 L|\bar{x}-\bar{y}| + o_\beta(1) + o_\rho(1). \tag{3.12}$$

2.- *Estimates for $\mathcal{H}$.* By definition of $\tilde{H}$ in $\mathcal{H}$ we have

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2,$$

with

$$\begin{aligned} \mathcal{H}_1 &= e^{-v(\bar{x})} H(\bar{x}, e^{v(\bar{x})} L\bar{p}) - e^{-v(\bar{y})} H(\bar{y}, e^{v(\bar{y})}(L\bar{p} - \bar{q})) \\ \mathcal{H}_2 &= -|\sigma^T(\bar{x})L\bar{p}|^2 + |\sigma^T(\bar{y})(L\bar{p} - \bar{q})|^2. \end{aligned}$$

For $\mathcal{H}_2$, using the estimates for the gradient of ψ_β and the fact that $|\bar{p}| = 1$, we can write

$$\begin{aligned} \mathcal{H}_2 &= L^2(|\sigma^T(\bar{y})(\bar{p} - L^{-1}\bar{q})|^2 - |\sigma^T(\bar{x})\bar{p}|^2) \\ &\geq L^2(|\sigma^T(\bar{y})\bar{p}|^2 - |\sigma^T(\bar{x})\bar{p}|^2 - 2L^{-1}\beta^2 L_\sigma^2 \Lambda), \end{aligned}$$

and applying the boundedness and the Lipschitz estimates of σ given in (A), we conclude

$$\mathcal{H}_2 \geq -2L_\sigma^2 L^2 |\bar{x}-\bar{y}| - 2L_\sigma^2 \Lambda^2 \tilde{C} \beta^2 L. \tag{3.13}$$

The estimate from below of $\mathcal{H}_1$ is a crucial step in the weak Bernstein method. It is based on the superlinearity of the Hamiltonian which is encoded in Assumption (H1) (see (4.4)). We define $\mu = e^{v(\bar{y})-v(\bar{x})}$ and we notice that $0<\mu<1$ since $v(\bar{y})<v(\bar{x})$ for all L,β. Using this we can write

$$\mathcal{H}_1 = e^{-v(\bar{y})}\Big(\mu H(\bar{x},\mu^{-1}e^{v(\bar{y})}L\bar{p}) - H(\bar{y}, e^{v(\bar{y})}(L\bar{p}-\bar{q}))\Big).$$

Applying (H1), (H2') with β small enough in order that $|e^{v(\bar{y})}\bar{q}|\le 1$ and recalling that $m>1$ and $v\ge 0$, we can write

$$\begin{aligned}\mathcal{H}_1 \ge\ & e^{-v(\bar{y})}\Big((1-\mu)(b_m L^m e^{mv(\bar{y})} - K) - L_H|\bar{x}-\bar{y}|(1+L^m e^{mv(\bar{y})})\\ &\quad - \zeta(|e^{v(\bar{y})}\bar{q}|)L^{m-1}e^{(m-1)v(\bar{y})} - \zeta(|e^{v(\bar{y})}\bar{q}|\Big)\\ \ge\ & L^m e^{(m-1)v(\bar{y})}\Big((b_m - KL^{-m})(1-\mu) - L_H|\bar{x}-\bar{y}|(1+L^{-m})\Big) - o_\beta(1),\end{aligned}$$

where $o_\beta(1)\to 0$ as $\beta\to 0$, but depending on L and $||v||_\infty$. Taking L satisfying

$$L^m \ge \max\{1, 2^{-1}b_m^{-1}K\}, \tag{3.14}$$

we conclude that

$$\mathcal{H}_1 \ge L^m e^{(m-1)v(\bar{y})}\Big(\frac{b_m}{2}(1-\mu) - 2L_H|\bar{x}-\bar{y}|\Big) - o_\beta(1).$$

Now, we have

$$1-\mu = e^{v(\bar{y})-v(\bar{x})}(e^{v(\bar{x})-v(\bar{y})}-1) \ge e^{v(\bar{y})-v(\bar{x})}(v(\bar{x})-v(\bar{y})) \ge e^{-\mathrm{osc}(v)}L|\bar{x}-\bar{y}|$$

since $\mathrm{osc}(v)\ge v(\bar{x})-v(\bar{y})\ge L|\bar{x}-\bar{y}|$ by maximality of $(\bar{x},\bar{y})$ in (3.5). From this, taking

$$L \ge 8L_H e^{\mathrm{osc}(v)} b_m^{-1} \tag{3.15}$$

and using that $v\ge 0$ and $m>1$, we conclude that

$$\mathcal{H}_1 \ge \frac{b_m}{4}e^{-\mathrm{osc}(v)}L^{m+1}|\bar{x}-\bar{y}| - o_\beta(1).$$

Recalling that $\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2$, we join the last estimate and (3.13) to obtain

$$\mathcal{H} \ge (\frac{b_m}{4}e^{-\mathrm{osc}(v)}L^{m+1} - 2L_\sigma^2 L^2)|\bar{x}-\bar{y}| - o_\beta(1),$$

and therefore, since $m>1$, taking

$$L^{m-1} \ge 16 L_\sigma^2 b_m^{-1} e^{\mathrm{osc}(v)}, \tag{3.16}$$

we finally conclude that

$$\mathcal{H} \ge cL^{m+1}|\bar{x}-\bar{y}| - o_\beta(1), \tag{3.17}$$

where $c = b_m e^{-\mathrm{osc}(v)}/8 > 0$ and $o_\beta(1)\to 0$ as $\beta\to 0$ for fixed L.

3.-*Estimates for $\mathcal{B}_\delta$.* At this point, recalling that $|\bar{x}-\bar{y}|>0$ by (3.7), we are going to consider $\delta>0$ small enough depending on $|\bar{x}-\bar{y}|$ such that $|j(\bar{x},z)|, |j(\bar{y},z)| \leq |\bar{x}-\bar{y}|/2$ for each $|z| \leq \delta$. This is possible by assumption (J1).

Thus, we have $x \mapsto |\bar{x}-\bar{y}+x|$, $y \mapsto |\bar{x}-\bar{y}-y|$ are smooth in B_δ. Then, applying Lemma A.1 we can write

$$\mathcal{B}_\delta \leq \mathcal{I}^j[B_\delta](\phi(\cdot,\bar{y}),\bar{x}) + \mathcal{I}^j[B_\delta](\phi(\bar{x},\cdot),\bar{y}) + C(L^2+\beta^2\Lambda^2)o_\delta(1),$$

and by definition of ϕ together with Lemma 2.3 and assumptions (M'), (J1) we arrive at

$$\mathcal{B}_\delta \leq \Big(L|\bar{x}-\bar{y}|^{-1} + L^2 + o_\beta(1)\Big)o_\delta(1). \tag{3.18}$$

Note that the estimate above is appropriate since we will send $\delta \to 0$ first in the global estimate in Step 5.

4.- *Estimates for $\mathcal{B}^\delta$.* This is the key new estimate to perform the weak Bernstein's method in the nonlocal case and therefore we state it as a lemma.

Lemma 3.2. *There exists $C > 0$ depending only on the datas $C_j, C_{\nu,j}, C_\nu$ and on $\mathrm{osc}(v)$ such that*

$$\mathcal{B}^\delta \leq CL^2|\bar{x}-\bar{y}| + o_\beta(1), \tag{3.19}$$

where $o_\beta(1) \to 0$ as $\beta \to 0$ for $L > 0$ fixed.

Then, the rest of Step 4 is devoted to the proof of this lemma. We start with the following notation: for a function $f : \mathbb{R}^d \to \mathbb{R}$ and $x, z \in \mathbb{R}^d$, we denote

$$\Delta_x f = \Delta_x f(z) = f(x + j(x,z)) - f(x).$$

We also consider $\Theta_i(z)$ for $i = 1,2,3$, $z \in \mathbb{R}^d$, defined as

$$\begin{aligned}
\Theta_1(z) &= L(|\bar{x}-\bar{y}+j(\bar{x},z)| - |\bar{x}-\bar{y}|),\\
\Theta_2(z) &= -L(|\bar{x}-\bar{y}-j(\bar{y},z)| - |\bar{x}-\bar{y}|) - \Delta_{\bar{y}}\psi_\beta,\\
\Theta_3(z) &= L(|\bar{x}-\bar{y}+j(\bar{x},z)-j(\bar{y},z)| - |\bar{x}-\bar{y}|) + \Delta_{\bar{y}}\psi_\beta,
\end{aligned}$$

and notice that the maximality of $(\bar{x},\bar{y})$ in (3.5) implies, for each $z \in \mathbb{R}^d$, the following inequalities

$$\Delta_{\bar{x}}v \leq \Theta_1, \quad \Delta_{\bar{y}}v \geq \Theta_2, \quad \Delta_{\bar{x}}v - \Delta_{\bar{y}}v \leq \Theta_3. \tag{3.20}$$

We write $\mathcal{B}^\delta = \mathcal{B}_1 + \mathcal{B}_2^\delta$ with

$$\begin{aligned}
\mathcal{B}_1 &= \int_{B^c}[e^{\Delta_{\bar{x}}v} - e^{\Delta_{\bar{y}}v}]\nu(dz)\\
\mathcal{B}_2^\delta &= \int_{B\setminus B_\delta}[e^{\Delta_{\bar{x}}v} - e^{\Delta_{\bar{y}}v} - L\langle\bar{p}, j(\bar{x},z)-j(\bar{y},z)\rangle - \langle\bar{q}, j(\bar{y},z)\rangle]\nu(dz).
\end{aligned}$$

Our aim is to estimate from above the integrals. Of main importance is to get estimates for the integral over $\mathcal{B}_2^\delta$ independent of δ since we are going to take $\delta \to 0$ first at the end of this proof.

We start with the estimate of $\mathcal{B}_1$. We first remark that we can integrate only on the set $\mathcal{P}_1$ where $e^{\Delta_{\bar{x}}v} - e^{\Delta_{\bar{y}}v} \geq 0$, i.e., where $\Delta_{\bar{x}}v - \Delta_{\bar{y}}v \geq 0$.

On this set, we have

$$e^{\Delta_{\bar{x}} v} - e^{\Delta_{\bar{y}} v} = e^{\Delta_{\bar{x}} v}(1 - e^{\Delta_{\bar{y}} v - \Delta_{\bar{x}} v}) \leq e^{\operatorname{osc}(v)}(\Delta_{\bar{x}} v - \Delta_{\bar{y}} v) \leq e^{\operatorname{osc}(v)}\Theta_3$$

since $\Delta_{\bar{x}} v \leq \operatorname{osc}(v)$ and $0 \leq 1 - e^{-r} \leq r$ for $r \geq 0$.

Noticing that $\Theta_3 \leq L|j(\bar{x}, z) - j(\bar{y}, z)| + \Delta_{\bar{y}} \psi_\beta$ and using (J2) and Lemma 2.3, we get

$$\mathcal{B}_1 \leq e^{\operatorname{osc}(v)} \left(L \int_{B^c \cap \mathcal{P}_1} |j(\bar{x}, z) - j(\bar{y}, z)| \nu(dz) + \mathcal{I}^j[B^c \cap \mathcal{P}_1](\psi_\beta, \bar{y}) \right),$$

and using (J2) for the first integral and Lemma 2.3 for the second term in the righ-hand side, we arrive at

$$\mathcal{B}_1 \leq e^{\operatorname{osc}(v)}(LC_1|\bar{x} - \bar{y}| + o_\beta(1)), \tag{3.21}$$

where C_1 is given by (J2) for $a = 1$ and we point out that $o_\beta(1) \to 0$ as $\beta \to 0$ uniformly in all the other variables.

Now we deal with the estimate of $\mathcal{B}_2^\delta$. A key fact is that it is enough to integrate

$$\Psi(z) := e^{\Delta_{\bar{x}} v} - e^{\Delta_{\bar{y}} v} - L\langle \bar{p}, j(\bar{x}, z) - j(\bar{y}, z) \rangle - \langle \bar{q}, j(\bar{y}, z) \rangle$$

only on the set $\mathcal{P}_2$ where $\Psi(z) \geq 0$. This consideration allows us to get the relevant estimates to apply the "linearization" procedure provided by Lemma A.2, which is proven in the Appendix. In fact, notice that by the third inequality in (3.20), applying (J1) and the properties of ψ_β, for each z we can write

$$\Delta_{\bar{x}} v - \Delta_{\bar{y}} v \leq C(L|\bar{x} - \bar{y}| + o_\beta(1))|z|,$$

where $C > 0$ depends only on the data.

On the other hand, for each $z \in \mathcal{P}_2$ we can write

$$-(LC_j|\bar{x} - \bar{y}| + o_\beta(1))|z| \leq L\langle \bar{p}, j(\bar{x}, z) - j(\bar{y}, z) \rangle + \langle \bar{q}, j(\bar{y}, z) \rangle \leq e^{\Delta_{\bar{x}} v} - e^{\Delta_{\bar{y}} v},$$

where the first inequality comes from (J1) and the properties of ψ_β, and the second one from the definition of $\mathcal{P}_2$. Now, since

$$e^{\Delta_{\bar{x}} v} - e^{\Delta_{\bar{y}} v} = e^{\Delta_{\bar{x}} v}(1 - e^{\Delta_{\bar{y}} v - \Delta_{\bar{x}} v}) \leq e^{\Delta_{\bar{x}} v}(\Delta_{\bar{x}} v - \Delta_{\bar{y}} v),$$

we join this inequality and the previous one to conclude that

$$-Ce^{\operatorname{osc}(v)}(L|\bar{x} - \bar{y}| + o_\beta(1))|z| \leq \Delta_{\bar{x}} v - \Delta_{\bar{y}} v$$

where $C > 0$ depends only on the data. Thus, the upper and lower bound for $\Delta_{\bar{x}} v - \Delta_{\bar{y}} v$ can be summarized as

$$|\Delta_{\bar{x}} v - \Delta_{\bar{y}} v| \leq C(L|\bar{x} - \bar{y}| + o_\beta(1))|z| \quad \text{for } z \in \mathcal{P}_2 \tag{3.22}$$

where C denotes a constant which may vary line to line but depends only the data and $\operatorname{osc}(v)$.

Now, using the first and second inequalities in (3.20) together with (J1) and the properties of ψ_β we can write

$$\Delta_{\bar{x}} v \leq LC_j|z|, \quad \Delta_{\bar{y}} v \geq -\left(LC_j + o_\beta(1)\right)|z|.$$

These inequalities and (3.22) allows us to obtain, for $z \in \mathcal{P}_2$

$$|\Delta_{\bar{x}} v|, |\Delta_{\bar{y}} v| \leq C\left(L + L|\bar{x} - \bar{y}| + o_\beta(1)\right)|z|.$$

Finally, since we can assume $L > 1$ and by (3.6) we conclude that

$$|\Delta_{\bar{x}} v|, |\Delta_{\bar{y}} v| \leq C\left(L + o_\beta(1)\right)|z|. \tag{3.23}$$

Recalling that we also have that $|\Delta_{\bar{x}} v|, |\Delta_{\bar{y}} v| \leq \mathrm{osc}(v)$, in view of (3.22) and (3.23) we can apply Lemma A.2 with $g(x,z) = \Delta_x v(z)$, C_1 just depending on the data, $C_2 = \mathrm{osc}(v), b = o_\beta(1)$ and $\mathcal{P} = \mathcal{P}_2$ to conclude that

$$e^{\Delta_{\bar{x}} v} - e^{\Delta_{\bar{y}} v} \leq \Delta_{\bar{x}} v - \Delta_{\bar{y}} v + C(L^2|\bar{x} - \bar{y}| + o_\beta(1))|z|^2, \quad \text{for } z \in \mathcal{P}_2.$$

Then, by using this estimate and the last inequality in (3.20), for each $z \in \mathcal{P}_2$ we get

$$\Psi(z) \leq \Psi_1(z) + \Psi_2(z),$$

where

$$\Psi_1(z) = L\Big(|\bar{x} - \bar{y} + j(\bar{x}, z) - j(\bar{y}, z)| - |\bar{x} - \bar{y}| - \langle \bar{p}, j(\bar{x}, z) - j(\bar{y}, z)\rangle\Big),$$
$$\Psi_2(z) = \Delta_{\bar{y}} \psi_\beta - \langle \bar{q}, j(\bar{y}, z)\rangle + C(L^2|\bar{x} - \bar{y}| + o_\beta(1))|z|^2.$$

For the integral term concerning Ψ_2, we note that the integral of the first two terms is exactly $\mathcal{I}^j[(B \setminus B_\delta) \cap \mathcal{P}_2](\psi_\beta, \bar{y})$ which is $o_\beta(1)$ by Lemma 2.3. Using (M') to estimate the last term, we infer the existence of a constant $C > 0$ not depending on δ, β or L such that

$$\int_{\mathcal{P}_2 \cap (B \setminus B_\delta)} \Psi_2(z)\nu(dz) \leq C(L^2|\bar{x} - \bar{y}| + o_\beta(1)),$$

where $o_\beta(1) \to 0$ when $L > 0$ is fixed.

For the estimate of the integral term related to Ψ_1 we use the estimate given by Lemma A.3 proven in the Appendix to conclude that

$$\int_{\mathcal{P}_2 \cap (B \setminus B_\delta)} \Psi_1(z)\nu(dz) \leq CL|\bar{x} - \bar{y}|,$$

for some $C > 0$ depending on the data.

In view of the above estimates and since we assume $L > 1$, we finally arrive at

$$\mathcal{B}_2^\delta \leq CL^2|\bar{x} - \bar{y}| + o_\beta(1).$$

Putting together this last estimate and the estimate for $\mathcal{B}_1$ in (3.21) we finally obtain (3.19) as stated in Lemma 3.2.

5.- *Conclusion of the proof of the Theorem.* Replacing (3.12), (3.17), (3.18) and (3.19) into (3.10), we obtain

$$\begin{aligned} cL^{m+1}|\bar{x} - \bar{y}| \leq\ & CL^2|\bar{x} - \bar{y}| \\ & + (L|\bar{x} - \bar{y}|^{-1} + L^2 + o_\beta(1))o_\delta(1) + o_\beta(1) + o_\rho(1), \end{aligned} \tag{3.24}$$

where $o_\rho(1) \to 0$ as $\rho \to 0$, $o_\delta(1) \to 0$ as $\delta \to 0$ uniformly in the remaining variables, $o_\beta(1) \to 0$ as $\beta \to 0$ when $L > 0$ is fixed and $L, C, c > 0$ depend only on the datas $C_\nu, C_{\nu,j}, C_j, L_\sigma, L_H, b_m, K$ and $\mathrm{osc}(v)$.

More precisely, we can fix L at the beginning such that (3.14), (3.15), (3.16) hold and in addition

$$L \geq (c^{-1}C)^{1/(m-1)} + 1.$$

With this choice and since $|\bar{x} - \bar{y}|$ is uniformly positive in terms of β by (3.7), making $\rho \to 0, \delta \to 0,\ \beta \to 0$ we arrive at a contradiction with (3.24), which ends the proof of the theorem. $\square$

We can extend the Lipschitz regularity in the x variable for the solution to the parabolic problem (1.2)-(2.1).

Proposition 3.3. *Let u_0 be bounded and Lipschitz function in $\mathbb{R}^d$ with Lipschitz constant $L_0 > 0$. Assume A, H and $\mathcal{I}^j$ defined in* (1.3) *satisfy the assumptions of Theorem 3.1. Let $u \in C(\bar{Q})$ be the unique viscosity solutions to to problem* (1.2)-(2.1) *given by Corollary 2.5.*

Then, there exists $L > 0$ depending on the data, L_0 and

$$\mathrm{osc}_T(u) := \sup_{t\in[0,T]} \mathrm{osc}(u(\cdot,t)) = \sup_{t\in[0,T]} \{\sup_{\mathbb{R}^d} u(\cdot,t) - \inf_{\mathbb{R}^d} u(\cdot,t)\}$$

such that

$$|u(x,t) - u(y,t)| \leq L|x-y|, \quad \textit{for all } x, y \in \mathbb{R}^d,\ t \in [0,T].$$

Sketch of the Proof: As in the proof of Theorem 3.1, we may assume without loss of generality that $u \geq 1$ and we argue over the function v defined though the change of variables $u(x,t) = e^{v(x,t)}$ for all $(x,t) \in \bar{Q}$. Hence, proving the Lipschitz continuity in x for v, we conclude the desired property for u.

We start by proving the result for $u_0 \in C^2(\mathbb{R}^d)$ with $||u_0||_{C^2(\mathbb{R}^d)} < +\infty$ in order to be able to use Proposition 2.7.

The new function v solves the problem

$$\begin{cases} \partial_t v - \mathrm{Tr}(AD^2 v) - \mathcal{J}^j(v(\cdot,t),x) + \tilde{H}(x,v,Dv) & = 0 \qquad \text{in } Q \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v(\cdot,0) & = e^{u_0} \qquad \text{in } \mathbb{R}^d, \end{cases}$$

where $\mathcal{J}^j_x$ is defined in (3.2) and $\tilde{H}$ is defined by (3.3) with $\lambda = 0$.

As in Theorem 3.1, we argue by contradiction, assuming that for all $L \geq 1$ large enough, there exists $x_L, y_L \in \mathbb{R}^d$, $t_L \in [0,T]$ and $\epsilon_L > 0$ such that

$$\begin{aligned} &\sup_{(x,y,t)\in\mathbb{R}^d\times\mathbb{R}^d\times[0,T]} \{v(x,t) - v(y,t) - L|x-y|\} \\ &\qquad \geq\ v(x_L,t_L) - v(y_L,t_L) - L|x_L - y_L| \geq 2\epsilon_L. \end{aligned}$$

Hence, introducing the localization function ψ_β defined in (2.8) as in Theorem 3.1 but replacing $\mathrm{osc}(v)$ by $\mathrm{osc}_T(v)$, for all $\beta > 0$ small enough with respect to L and all $\eta > 0$ we have

(3.25)
$$\sup_{(x,y,s,t)\in(\mathbb{R}^d)^2\times[0,T]^2} \{v(x,s) - v(y,t) - L|x-y| - \psi_\beta(y) - \eta^{-1}(s-t)^2\} \geq 2\epsilon_L.$$

Applying Proposition 2.7 with a constant $\Lambda_0 > 0$ depending on $||u_0||_{C^2(\mathbb{R}^d)}$ and using the definition of osc_T we can write

$$\begin{aligned} & v(x,s) - v(y,t) - L|x-y| - \psi_\beta(y) - \eta^{-1}(s-t)^2 \\ & \le \mathrm{osc}_T(v) - \psi_\beta(y) - L|x-y| + \Lambda_0|s-t| - \eta^{-1}(s-t)^2. \end{aligned}$$

Notice that $\Lambda_0 a - \eta^{-1}a^2 \le \Lambda_0^2\eta/4$ for all $a > 0$. Using this and the properties of ψ_β we get the inequality

$$v(x,s) - v(y,t) - L|x-y| - \psi_\beta(y) - \eta^{-1}(s-t)^2 \le -L|x-y| + \Lambda_0^2\eta/4$$

for all $|y| \ge 2/\beta$. It follows that for all η small enough in terms of ϵ_L and Λ_0, the supremum in (3.25) is achieved at some point $(\bar{x}, \bar{y}, \bar{s}, \bar{t}) \in (\mathbb{R}^d)^2 \times [0,T]^2$ and $\bar{x}, \bar{y}$ are in a bounded set depending only on $\mathrm{osc}_T(v)$ and β. Moreover, by classical results, $\eta^{-1}(\bar{s}-\bar{t})^2 \to 0$ as $\eta \to 0$ uniformly in L, β.

In particular, up to take a subsequence $\eta \to 0$, we get $\bar{x} \to \bar{x}^*$, $\bar{y} \to \bar{y}^*$ and $\bar{s}, \bar{t} \to t^*$. Choosing $L > L_0$ implies $t^* > 0$ since otherwise passing to the limit as $\eta \to 0$ in (3.25) we arrive to a contradiction with u_0 L_0-Lipschitz continuous. It follows that $\bar{s}, \bar{t} > 0$ for sufficiently small η. Likewise, we obtain that $\bar{x} \neq \bar{y}$ for η sufficiently small.

It follows that, for η small enough, we can write the viscosity inequalities for the subsolution v at $(\bar{x}, \bar{s})$ and for the supersolution v at $(\bar{y}, \bar{t})$ to obtain exactly (3.10), that is

$$\mathcal{H} \le \mathcal{A} + \mathcal{B}_\delta + \mathcal{B}^\delta, \tag{3.26}$$

where $v(\bar{x})$ is replaced by $v(\bar{x}, \bar{s})$ and $v(\bar{y})$ by $v(\bar{y}, \bar{t})$ in (3.11).

Sending $\eta \to 0$, we still have (3.26) where $\bar{x}$ is replaced by $\bar{x}^*$, $\bar{y}$ by $\bar{y}^*$, $\bar{s}, \bar{t}$ by t^* and $(\bar{x}^*, \bar{y}^*, t^*)$ is a maximum point of

$$\sup_{(x,y,t) \in (\mathbb{R}^d)^2 \times [0,T]} \{v(x,s) - v(y,t) - L|x-y| - \psi_\beta(y)\} \ge 2\epsilon_L. \tag{3.27}$$

By compactness of $[0,T]$, we can assume that $t^* \to \hat{t}$ as $\beta \to 0$. Using Proposition 2.7 and (3.27), it follows

$$\begin{aligned} 2\epsilon_L &\le v(\bar{x}^*, t^*) - v(\bar{y}^*, t^*) \\ &\le v(\bar{x}^*, t^*) - v(\bar{x}^*, \hat{t}) + v(\bar{x}^*, \hat{t}) - v(\bar{y}^*, \hat{t}) + v(\bar{y}^*, \hat{t}) - v(\bar{y}^*, t^*) \\ &\le \omega_{\hat{t}}(|\bar{x}^* - \bar{y}^*|) + 2\Lambda_0|t^* - \hat{t}|, \end{aligned}$$

where $\omega_{\hat{t}}(\cdot)$ is the modulus of continuity associated with $v(\cdot, \hat{t})$. Therefore, for all β small enough, we get

$$\epsilon_L \le \omega_{\hat{t}}(|\bar{x}^* - \bar{y}^*|),$$

which gives (3.7) with the modulus $\omega_{\hat{t}}$ independent of β yielding a lower bound for $|\bar{x}^* - \bar{y}^*|$, uniform in terms of β.

From this point, we continue the proof with the arguments given in the proof of Theorem 3.1. We point out that the constant L, which gives the Lipschitz bound, depends on the constants appearing in the assumptions, on the Lipschitz constant L_0 of u_0 and on $\mathrm{osc}_T(u)$ but not directly neither on $||u(\cdot,t)||_\infty$ nor on T.

In particular, L is independent on $||u_0||_{C^2(\mathbb{R}^d)}$ so we get the result for any Lipschitz continuous function u_0 by approximation. □

4. Application: large time behavior in the periodic setting.

In this section we provide the large time behavior result for the problem (1.2)-(2.1) in the case when the datas are $\mathbb{Z}^d-$periodic. Hence, we argue on the problem

$$(4.1) \qquad \partial_t u - \mathrm{Tr}(A(x)D^2u) - \mathcal{I}^j(u(\cdot,t),x) + H(x,Du) = 0 \quad \text{in } \mathcal{Q} := \mathbb{T}^d \times (0,+\infty),$$

$$(4.2) \qquad u(\cdot,0) = u_0 \quad \text{in } \mathbb{T}^d,$$

which is (1.2)-(2.1) in the periodic setting with

$$\mathcal{I}^j(u(\cdot,t),x) = \int_{\mathbb{T}^d} [u(x+j(x,z),t) - u(x,t) - \mathbf{1}_B(z)\langle Du(x,t), j(x,z)\rangle]\nu(dz).$$

In order to have comparison principle/well-posedness results of Section 2 and the regularity results given in Section 3, from now on, we assume that (A), (M'), (H0)-(H1)-(H2'), (J1)-(J2) hold with periodic datas with respect to x, z.

4.1. Solvability of the ergodic problem.

Proposition 4.1. *There exists a unique constant $c \in \mathbb{R}$ for which the stationary ergodic problem*

$$(4.3) \qquad -\mathrm{Tr}(A(x)D^2u) - \mathcal{I}^j(u,x) + H(x,Du) = -c, \quad \text{in } \mathbb{T}^d$$

has a solution $w \in W^{1,\infty}(\mathbb{T}^d)$.

In the proof of the above proposition we require an appropriate compactness property over the family of solutions $\{u_\lambda\}$ of problem (1.1) (in the torus) as $\lambda \to 0$. This is the purpose of the following lemma, whose proof follows closely the arguments of [32].

Lemma 4.2. *Let $\lambda > 0$ and let u be a continuous solution to* (1.1) *in $\mathbb{T}^d$. Then, there exists $C > 0$ not depending on λ such that* $\mathrm{osc}(u) \le C$.

Proof: We start claiming that, under Assumption (H1), there exists a constant $\eta > 0$ just depending on the data, and a sequence $L \to +\infty$ such that H satisfies

$$(4.4) \qquad H(x,Lp) - LH(x,p) \ge \eta L^m |p|^m - \eta^{-1}$$

for $x, p \in \mathbb{R}^d$. Note that, in particular, this proves that H is superlinear. Now, we consider $L \ge 1$ to be fixed and

$$M := \max_{x,y\in\mathbb{T}^d} \{u(x) - Lu(y) + (L-1)\min\{u\} - L|x-y|\}.$$

Note that if there exists L such that $M \leq 0$, then for all $x, y \in \mathbb{T}^d$ we can write

$$u(x) - Lu(y) \leq (1-L)\min\{u\} + L\sqrt{d},$$

and hence, taking $x, y \in \mathbb{T}^d$ such that $u(x) = \max\{u\}$ and $u(y) = \min\{u\}$, we get the result with $C = L\sqrt{d}$.

Then, we assume that $M > 0$ for all $L \geq 1$. In particular, the maximum in M is attained at $(\bar{x}, \bar{y})$ with $\bar{x} \neq \bar{y}$. Thus, denoting $\varphi(x,y) = -(L-1)\min\{u\} + L|x-y|$, we can use $x \mapsto \varphi(x, \bar{y})$ as a test function for u at $\bar{x}$ and $y \mapsto -\varphi(\bar{x}, y)$ as a test function for $v := Lu$ at $\bar{y}$. Then, for each $\delta > 0$ we see that

$$\lambda u(\bar{x}) - \mathrm{Tr}(A(\bar{x})X) - \mathcal{I}^j[B_\delta](\varphi(\cdot, \bar{y}), \bar{x}) - \mathcal{I}^j[B_\delta^c](u, \bar{p}, \bar{x}) + H(\bar{x}, L\bar{p}) \leq 0$$
$$\lambda v(\bar{x}) - \mathrm{Tr}(A(\bar{y})Y) - \mathcal{I}^j[B_\delta](\varphi(\bar{x}, \cdot), \bar{y}) - \mathcal{I}^j[B_\delta^c](v, \bar{p}, \bar{y}) + LH(\bar{x}, \bar{p}) \geq 0,$$

where $\bar{p} = (\bar{x} - \bar{y})/|\bar{x} - \bar{y}|$ and X, Y satisfy (2.5) with $\varphi = \varphi_1$.

We subtract both inequalities and estimate each term arising in this operation. Note that by (2.21) we have

$$\lambda(u(\bar{x}) - v(\bar{y})) \leq (1+L)H_0,$$

by Lemma 2.2 we have

$$\mathrm{Tr}(A(\bar{x})X) - \mathrm{Tr}(A(\bar{y})Y) \leq CL|\bar{x} - \bar{y}|,$$

for some constant $C > 0$ just depending on the data. Considering δ small in terms of $|\bar{x} - \bar{y}|$, we use (2.4) (M') and (J1) similarly as in (3.18) to get

$$\mathcal{I}^j[B_\delta](\varphi(\cdot, \bar{y}), \bar{x}) + \mathcal{I}^j[B_\delta](\varphi(\bar{x}, \cdot), \bar{y}) \leq L|\bar{x} - \bar{y}|^{-1} o_\delta(1).$$

By using that $(\bar{x}, \bar{y})$ is the maximum point in M we obtain

$$\begin{aligned} &\mathcal{I}^j[B_\delta^c](u, \bar{p}, \bar{x}) - \mathcal{I}^j[B_\delta^c](v, \bar{p}, \bar{y}) \\ &\leq L \int_{B_\delta^c} (|\bar{x} - \bar{y} + j(\bar{x}, z) - j(\bar{y}, z)| - |\bar{x} - \bar{y}| - \mathbf{1}_B(z)\langle \bar{p}, j(\bar{x}, z) - j(\bar{y}, z)\rangle)\nu(dz), \end{aligned}$$

and performing a similar analysis as the one done in the proof of Theorem 3.1 (see (A.3)-(A.4)), we conclude that

$$\mathcal{I}^j[B_\delta^c](u, \bar{p}, \bar{x}) - \mathcal{I}^j[B_\delta^c](v, \bar{p}, \bar{y}) \leq CL|\bar{x} - \bar{y}|,$$

for $C > 0$ and all L large in terms of the data, and not depending on δ.

Finally, using (4.4) we see that

$$H(\bar{x}, L\bar{p}) - LH(\bar{y}, \bar{p}) \geq \eta L^m - \eta^{-1},$$

for L large enough.

Then, joining the above estimates we conclude that there exists $C > 0$ such that, for all L large just in terms of the data, we get

$$\eta L^m \leq C(1+L) + L|\bar{x} - \bar{y}|^{-1} o_\delta(1),$$

and therefore, taking $\delta \to 0$ and enlarging L if this is necessary, we arrive to a contradiction. $\square$

Proof of Proposition 4.1: For $\lambda > 0$, we denote u_λ the unique bounded uniformly continuous solution to (1.1) given by Proposition 2.6. By Lemma 4.2, $\mathrm{osc}(u_\lambda)$ is bounded independently of λ. Therefore, by Theorem 3.1, u_λ is Lipschitz continuous with a constant independent of λ.

Defining

$$v_\lambda(x) = u_\lambda(x) - u_\lambda(0), \quad x \in \mathbb{T}^d,$$

it follows that the family $\{v_\lambda\}_{\lambda>0}$ is bounded and equi-Lipschitz continuous in $C(\mathbb{T}^d)$. By standard viscosity arguments, v_λ satisfies the equation

$$\lambda v - \mathrm{Tr}(A(x)D^2 v) - \mathcal{I}^j(v,x) + H(x, Dv) = -\lambda u_\lambda(0) \quad \text{in } \mathbb{T}^d,$$

where $\lambda u_\lambda(0)$ is bounded as $\lambda \to 0$ by (2.21). Finally, by Ascoli Theorem and stability, up to subsequences, there exists $w \in W^{1,\infty}(\mathbb{T}^d)$ and $c \in \mathbb{R}$ such that $v_\lambda \to w$, $\lambda u_\lambda(0) \to c$ as $\lambda \to 0$ and the pair (w, c) satisfies (4.3).

For the uniqueness of c, we note that if there exist two pairs $(w_i, c_i), i = 1, 2$, both solutions to (4.3), then the functions $(x,t) \mapsto w_i(x) + c_i t, i = 1, 2$ are bounded viscosity solutions to equation (1.2). Hence, comparing them by the use of Proposition 2.1, we obtain

$$(c_1 - c_2)t \leq 2||w_1 - w_2||_\infty, \quad \text{for all } t > 0.$$

Dividing by t and letting $t \to \infty$ we conclude $c_1 \leq c_2$. Since we can exchange the roles of c_1 and c_2, we conclude the uniqueness of c. □

4.2. **Strong maximum principle.** Before presenting our strong maximum principle result, we need to introduce some notation. For a Lévy measure ν and a jump function j, for each $x \in \mathbb{T}^d$ we define the push-forward measure ν_x^j associated to ν through the function $z \mapsto j(x,z)$, that is, for each borel set $A \subset \mathbb{R}^d$ we have $\nu_x^j(A) = \nu(j^{-1}(x, A))$. Thus, for each $x \in \mathbb{T}^d$ we define

$$X_0(x) = \{x\}, \quad X_{n+1} = \bigcup_{\xi \in X_n} \{\xi + \mathrm{supp}\{\nu_\xi^j\}\}, \ n \in \mathbb{N},$$

where supp denotes the support of the measure, and the set

$$\mathcal{X}(x) = \overline{\bigcup_{n \in \mathbb{N}} X_n(x)}.$$

Finally, for $x \in \mathbb{R}^d$ we denote $E_0(x)$ the eigenspace associated to the null eigenvalue of $A(x)$.

Proposition 4.3. (Strong maximum principle) *Assume the hypotheses of Theorem 3.1 hold, with in addition, $H(x,p)$ locally Lipschitz in p. Assume the existence of a constant $r_0 > 0$ such that, for each $x \in \mathbb{T}^d$,*

$$B_{r_0}(x) \cap \{x + E_0(x)\} \subset \mathcal{X}(x). \tag{4.5}$$

Let $u, v \in C(\mathcal{Q})$ be two solutions to (4.1), associated to Lipschitz initial datum u_0, v_0, respectively. Assume that $u - v$ achieves a maximum in $\mathcal{Q} = \mathbb{T}^d \times (0, +\infty)$ at (x_0, t_0), that is,

$$(u - v)(x_0, t_0) = \sup_{\mathcal{Q}}\{u - v\}.$$

Then, the function $u - v$ is constant in $\mathbb{T}^d \times [0, t_0]$. Moreover, we have

$$(u - v)(x, t) = \sup_{x \in \mathbb{T}^d} \{u_0(x) - v_0(x)\}, \quad \text{for all } (x, t) \in \bar{\mathcal{Q}}.$$

We remark that by the available Lipschitz regularity results, it is possible to reduce Proposition 4.3 to a linear framework and this is the aim of the following lemma. We would like to stress on the role of the assumption (4.5) in its proof: in the directions of the second-order uniform ellipticity of the matrix A the propagation of maxima follows classical arguments, and therefore the mixed operator extend this propagation in the directions of degeneracy of A through the sequential covering property of the nonlocal operator.

Lemma 4.4. *Let A be a matrix satisfying (A), $\mathcal{I}^j$ as in* (1.3) *with ν satisfying (M') and j satisfying (J1), and both ν, j satisfying assumption* (4.5). *Let $\beta \in L^\infty(\mathcal{Q}; \mathbb{R}^d)$ and w be a bounded USC viscosity subsolution to the problem*

$$\partial_t w - \mathrm{Tr}(A(x)D^2 w) - \mathcal{I}^j(w, x) + \langle \beta(x, t), Dw \rangle = 0 \quad \text{in } \mathcal{Q}. \tag{4.6}$$

If there exists $(x_0, t_0) \in \mathcal{Q}$ such that $M := w(x_0, t_0) = \sup_{\mathcal{Q}}\{w\}$, then $w(x, t_0) = M$ for all $x \in \mathbb{T}^d$.

Proof: It is sufficient to prove that under the assumptions of the problem, if (x_0, t_0) is a global maximum point for w subsolution to (4.6), then w is constant equal to $w(x_0, t_0)$ in $B_{r_0/4}(x_0) \times \{t_0\}$ where r_0 appears in (4.5).

In fact, iterating the argument presented below a finite number of times, we conclude the main result. Denote

$$K = \{x \in \mathbb{T}^d : w(x, t_0) = M\}$$

which is a nonempty closed set containing x_0.

Let $x^* \in K$. Since (x^*, t_0) is a global maximum point for w we can use a constant function as a test function for w at (x^*, t_0). Thus, for each $\delta > 0$ we have

$$-\int_{B_\delta^c} [w(x^* + j(x^*, z), t_0) - w(x^*, t_0)]\nu(dz) \leq 0,$$

and since $w(x^* + j(x^*, z), t_0) \leq w(x^*, t_0)$, we obtain

$$\int_{B_\delta^c} [w(x^* + j(x^*, z), t_0) - w(x^*, t_0)]\nu(dz) = 0,$$

Therefore, since $z \mapsto w(x^* + j(x^*, z), t_0) - w(x^*, t_0)$ is upper semicontinuous and $\delta > 0$ is arbitrary, we get $w(x, t_0) = M$ for each $x \in x^* + \mathrm{supp}\{\nu^j_{x^*}\}$. We apply the same argument inductively to conclude that $\mathcal{X}(x^*) \subseteq K$.

Noting that $\mathcal{X}(x) \subseteq \mathcal{X}(x^*)$ for all $x \in \mathcal{X}(x^*)$, by the use of (4.5) we have

$$B_{r_0}(x) \cap \{x + E_0(x)\} \subset \mathcal{X}(x^*), \quad \text{for each } x \in \mathcal{X}(x^*) \text{ and } x^* \in K. \tag{4.7}$$

Consider the open set $\Gamma := B_{r_0/4}(x_0)\backslash K$. If $\Gamma = \emptyset$, then the result follows. From now on, we argue by contradiction assuming that $\Gamma \neq \emptyset$. It follows that there exists $\bar{x} \in \Gamma$, $0 < R < r_0/4$ and $x^* \in \partial K$ such that

$$B_R(\bar{x}) \subset \Gamma \quad \text{and} \quad x^* \in \partial B_R(\bar{x}) \cap K.$$

Up to replace $\bar{x}$ by $(\bar{x} + x^*)/2$ if needed, we may assume $\partial B_R(\bar{x}) \cap K = \{x^*\}$.

At this point, for $\gamma, h > 0$ to be fixed, we introduce the function

$$\phi(x,t) = e^{-\gamma R^2} - e^{-\gamma d(x,t)} \quad \text{with } d(x,t) = |x - \bar{x}|^2 + h(t - t_0)^2.$$

Direct computations say that for each $(x,t) \in \mathbb{T}^d$

$$\begin{aligned} \partial_t \phi(x,t) &= 2\gamma h e^{-\gamma d(x,t)}(t - t_0) \\ D\phi(x,t) &= 2\gamma e^{-\gamma d(x,t)}(x - \bar{x}) \\ D^2\phi(x,t) &= 2\gamma e^{-\gamma d(x,t)}[I_d - 2\gamma(x - \bar{x}) \otimes (x - \bar{x})], \end{aligned}$$

meanwhile, following [18], there exists $C_{\nu,j} > 0$ depending only on the data, such that

$$\mathcal{I}^j(\phi(\cdot,t),x) \leq \gamma e^{-\gamma d(x,t)} C_{\nu,j}.$$

With these estimates and applying (A), for each $x \in \mathbb{T}^d$ we have

$$\begin{aligned} & \mathcal{E}(\phi,x,t_0) \\ := \;& \partial_t\phi(x,t_0) - \mathrm{Tr}(A(x)D^2\phi(x,t_0)) - \mathcal{I}^j(\phi(\cdot,t_0),x) + \langle \beta(x,t), D\phi(x,t_0)\rangle \\ \geq \;& 2\gamma e^{-\gamma d(x,t)}\Big(h(t-t_0) - L_\sigma + \gamma|\sigma^T(x)(x-\bar{x})|^2 - C_{\nu,j} - ||\beta||_\infty |x - \bar{x}|\Big) \end{aligned}$$

Note that $\bar{x} \in B_{r_0/4}(x^*)$. If $\bar{x} - x^* \in E_0(x^*)$, by (4.7) we would have $\bar{x} \in \mathcal{X}(x^*) \subseteq K$, which is a contradiction with the choice of $\bar{x}$. Thus, $\sigma^T(x^*)(x^* - \bar{x}) \neq 0$. By continuity, there exists $\eta(x^*), R^* > 0$ such that

$$|\sigma^T(x)(x - \bar{x})| \geq \eta(x^*) \quad \text{for all } x \in B_{R^*}(x^*).$$

This allows us to get the inequality

$$\mathcal{E}(\phi,x,t_0) \geq 2\gamma e^{-\gamma d(x,t)}\Big(h|t-t_0| + \gamma\eta(x^*)^2 - L_\sigma - C_{\nu,j} - 2R||\beta||_\infty\Big),$$

for each $x \in B_{R^*}(x^*)$. Thus, taking γ large in terms of R, h, t_0 and the data, we conclude v that is a strict supersolution to (4.6) in $B_{R^*}(x^*) \times (0, t_0 + 1)$.

On the other hand, since $\bar{B}_R(\bar{x}) \cap K = \{x^*\}$, there exists $\rho^* > 0$ such that

$$w(x,t_0) \leq M - \rho^* \quad \text{for all } x \in \bar{B}_R(\bar{x}) \setminus B_{R^*}(x^*),$$

and therefore, by upper semicontinuity of w, there exists $\tau^* \in (0,1)$ small enough such that

$$w \leq M - \rho^*/2 \quad \text{in } (\bar{B}_R(\bar{x}) \setminus B_{R^*}(x^*)) \times (t_0 - \tau^*, t_0 + \tau^*). \tag{4.8}$$

At this point, we fix $h > (R/\tau^*)^2$. Under this choice, the ellipsoid

$$\Sigma = \{(x,t) : |x - \bar{x}|^2 + h(t - t_0)^2 \leq R^2\}$$

satisfies $\Sigma \subset \bar{B}_R(\bar{x}) \times (t_0 - \tau^*, t_0 + \tau^*)$. Notice that $(x^*, t_0) \in \partial\Sigma$ since $|x^* - \bar{x}| = R$ and $w(x^*, t_0) = M$. Since $\phi > 0$ in Σ^c, for all $\epsilon > 0$ we have $w - \epsilon\phi < M$ in Σ^c, and by (4.8), taking $\epsilon > 0$ small in terms of ρ^*, we obtain

$$w - \epsilon\phi \leq M - \rho^*/2 + \epsilon||\phi||_{L^\infty(\Sigma)} < M \text{ in } \Sigma \setminus (B_{R^*}(x^*) \times (t_0 - \tau^*, t_0 + \tau^*)).$$

Hence, we conclude from this that $w - \epsilon\phi$ attains its global maximum at a point $(x', t') \in \Sigma$ with $x' \in B_{R^*}(x^*)$. Since w is a viscosity subsolution to (4.6), we get $\mathcal{E}(\epsilon\phi, x', t') \leq 0$.

By the linearity of (4.6) this drives us to the inequality

$$\mathcal{E}(\phi, x', t') \leq 0,$$

which contradicts the fact that v is a strict supersolution to (4.6) in $B_{R^*}(x^*) \times (0, t_0 + 1)$. □

The following lemma is a consequence of the comparison principle, see [13].

Lemma 4.5. *Assume assumptions of Proposition 2.1 hold. Let u, v be respectively a bounded USC subsolution and a bounded LSC supersolution to equation* (4.1) *and for $t \in [0, +\infty)$, define*

$$\kappa(t) = \sup_{x \in \mathbb{T}^d} \{u(x,t) - v(x,t)\}.$$

Then, for all $0 \leq s \leq t$, we have $\kappa(t) \leq \kappa(s)$.

The previous lemmas allows to provide the

Proof of Propostion 4.3: By Lemma 4.5, the continuity of $u - v$ and the fact that (x_0, t_0) is a global maximum point for $u - v$, we have $(u - v)(x_0, t_0) = \kappa(0) = \kappa(\tau)$ for all $\tau \in [0, t_0]$. Then, it is sufficient to prove that for each $\tau \in (0, t_0]$, $(u - v)(x, \tau) = \kappa(0)$ for all $x \in \mathbb{T}^d$, concluding the result up to $\tau = 0$ by continuity.

By Proposition 3.3, u and v are Lipschitz in space in $[0, t_0]$, with Lipschitz constant depending only on the data and t_0. Then, by classical arguments in the viscosity theory, the function $w := u - v$ is a viscosity subsolution to the problem

$$\partial w - \mathrm{Tr}(A(x)D^2 w) - \mathcal{I}^j(w(\cdot, t), x) + \langle \beta(x,t), Dw \rangle \leq 0 \quad \text{in } \mathbb{T}^d \times (0, t_0],$$

with $\beta \in L^\infty(\mathbb{T}^d \times [0, t_0]; \mathbb{R}^d)$ defined as

$$\beta(x,t) = \int_0^1 D_p H(x, sDu(x,t) + (1-s)Dv(x,t))ds.$$

Therefore, for all $\tau \in (0, t_0]$, there exists $x_\tau \in \mathbb{T}^d$ such that $w(x_\tau, \tau) = \kappa(\tau)$. By Lemma 4.4, we obtain $w(\cdot, \tau) = \kappa(\tau) = \kappa(0)$ and the results follows. □

4.3. **Large time behavior.** The above results are sufficient to get the large time behavior for (4.1)-(4.2).

Proposition 4.6. *Under the assumptions of this section and of Proposition 4.3, the continuous solution of* (4.3) *is unique up to a constant.*

Theorem 4.7. (Ergodic large time behavior) *Under the assumptions of this section, for any $u_0 \in W^{1,\infty}$, there exists a unique solution u to problem* (4.1)-(4.2)*. Under the additional assumptions of Proposition 4.3, there exists a pair (u_∞, c) solution to* (4.3) *such that, as $t \to \infty$*

$$u(\cdot,t) + ct \to u_\infty \quad \textit{in } W^{1,\infty}(\mathbb{T}^d).$$

The proof of Proposition 4.6 is an easy consequence of Propositions 2.1 and 4.3 and follows the same lines as in [8, 10]. To prove Theorem 4.7, we first notice that, by comparison, for every solution $(v, c) \in W^{1,\infty}(\mathbb{T}^d) \times \mathbb{R}$ of (4.3), there exists $M > 0$ such that $v(x) - M \le u(x,t) + ct \le v(x) + M$. It follows that $\mathrm{osc}(u(\cdot,t))$ is bounded independently of t.

Therefore, by Proposition 2.7, $u(\cdot,t)$ is Lipschitz continuous with a constant independent of t. Hence, $\{u(\cdot,t) + ct, t \ge 0\}$ is relatively compact in $W^{1,\infty}(\mathbb{T}^d)$. The proof of the convergence of the whole sequence then follows as in [8].

Appendix A

We provide the technical estimates used in Theorem 3.1.

We start with the following relationship between $\mathcal{I}^j$ and $\mathcal{J}^j$, see [18] for a proof. For $\mathcal{J}^j$ defined in (3.2) we adopt the analogous notations as those introduced for $\mathcal{I}^j$ at the end of the introduction.

Lemma A.1. *Let $g \in C^2(\mathbb{R}^d)$. Then, for each $\delta \in (0,1)$ we have*

$$\mathcal{J}^j[B_\delta](g,x) = \mathcal{I}^j[B_\delta](g,x) + ||Dg||^2_{L^\infty(B_\delta(x))} O(\delta^{2-\sigma}),$$

where the O-term is independent of g.

Next result is useful in the linearization of the exponential terms arising in $\mathcal{J}^j$.

Lemma A.2. *Let $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ a bounded measurable function. Assume there exist $L > 1$, $b \in (0,1)$, $C_1, C_2 > 0$, $\mathcal{P} \subset \mathbb{R}^d$ measurable and $\bar{x}, \bar{y} \in \mathbb{R}^d$ with $|\bar{x} - \bar{y}| \le 1$, such that for all $z \in \mathcal{P}$ we have*

$$|g(\bar{x},z) - g(\bar{y},z)| \le C_1(L|\bar{x} - \bar{y}| + b)|z|, \ \textit{and} \tag{A.1}$$

$$|g(\bar{x},z)|, |g(\bar{y},z)| \le \min\{C_2, C_1(L+b)|z|\}. \tag{A.2}$$

Then, there exists C_3 just depending on C_1, C_2 such that

$$e^{g(\bar{x},z)} - e^{g(\bar{y},z)} \le g(\bar{x},z) - g(\bar{y},z) + C_3(L^2|\bar{x} - \bar{y}| + Lb)|z|^2, \quad \textit{for } z \in \mathcal{P}.$$

Proof: By the Mean Value Theorem, for each $z \in \mathcal{P}$ we have

$$\begin{aligned} e^{g(\bar{x},z)} - e^{g(\bar{y},z)} &= e^{g(\bar{y},z)}(e^{g(\bar{x},z)-g(\bar{y},z)} - 1) \\ &\leq e^{g(\bar{y},z)}(g(\bar{x},z) - g(\bar{y},z) + e^{\xi_1(z)}(g(\bar{x},z) - g(\bar{y},z))^2/2) \end{aligned}$$

for some $\xi_1(z) \in [-2C_2, 2C_2]$ in view of (A.2). Using this and again the Mean Value Theorem (this time on the term $e^{g(\bar{y},z)}$) we obtain

$$\begin{aligned} e^{g(\bar{x},z)} - e^{g(\bar{y},z)} &\leq g(\bar{x},z) - g(\bar{y},z) + (e^{g(\bar{y},z)} - 1)(g(\bar{x},z) - g(\bar{y},z)) \\ &\quad + \frac{e^{3C_2}}{2}|g(\bar{x},z) - g(\bar{y},z)|^2 \\ &\leq g(\bar{x},z) - g(\bar{y},z) + e^{\xi_2(z)}|g(\bar{y},z)||g(\bar{x},z) - g(\bar{y},z)| \\ &\quad + \frac{e^{3C_2}}{2}|g(\bar{x},z) - g(\bar{y},z)|^2 \end{aligned}$$

with $\xi_2(z) \in [-C_2, C_2]$. Then, using this last fact together with (A.1) and (A.2) we infer

$$\begin{aligned} e^{g(\bar{x},z)} - e^{g(\bar{y},z)} &\leq g(\bar{x},z) - g(\bar{y},z) + C_1^2 e^{C_2}(L + b)(L|\bar{x} - \bar{y}| + b)|z|^2 \\ &\quad + \frac{e^{3C_3}}{2}C_1^2(L|\bar{x} - \bar{y}| + b)^2|z|^2 \\ &\leq g(\bar{x},z) - g(\bar{y},z) + C|z|^2(L|\bar{x} - \bar{y}| + b)(L + L|\bar{x} - \bar{y}| + b) \end{aligned}$$

for some $C > 0$ just depending on C_1 and C_2. Since $b, |\bar{x} - \bar{y}| \leq 1 \leq L$ we conclude the proof. $\square$

Lemma A.3. *Let $\bar{x}, \bar{y} \in \mathbb{R}^d$ with $|\bar{x} - \bar{y}| > 0$ and denote $\bar{p} = (\bar{x} - \bar{y})/|\bar{x} - \bar{y}|$. Define, for $z \in \mathbb{R}^d$ the function*

$$\Psi(z) = |\bar{x} - \bar{y} + j(\bar{x},z) - j(\bar{y},z)| - |\bar{x} - \bar{y}| - \langle \bar{p}, j(\bar{x},z) - j(\bar{y},z)\rangle.$$

Then, there exists $C > 0$ just depending on the data such that, for all $\delta > 0$ and $\mathcal{P} \subset \mathbb{R}^d$ measurable, we have the estimate

$$\int_{\mathcal{P} \cap B \setminus B_\delta} \Psi(z)\nu(z)dz \leq C|\bar{x} - \bar{y}|.$$

Proof: Getting nonnegative upper bounds for Ψ in the domain of integration, we can get rid of the intersection with $\mathcal{P}$ and therefore we omit it for simplicity.

Notice that for z such that $C_j|z| \leq 1/2$, by (J1) we have

$$|j(\bar{x},z) - j(\bar{y},z)| \leq |\bar{x} - \bar{y}|/2.$$

Then, for $|z| \leq (2C_j)^{-1}$ we can perform a Taylor expansion on $\Psi(z)$ (around the point $\bar{x} - \bar{y}$) to write

$$\Psi(z) \leq \frac{2}{|\bar{x} - \bar{y}|}|j(\bar{x},z) - j(\bar{y},z)|^2 \leq 2C_j^2|\bar{x} - \bar{y}||z|^2 \tag{A.3}$$

and introducing $\delta_0 = (2C_j)^{-1}$, by the above inequality and (M') we get

$$\int_{B_{\delta_0}\setminus B_\delta} \Psi(z)\nu(dz) \leq CL|\bar{x} - \bar{y}|,$$

where $C > 0$ just depend on the data.

On the other hand, when $|z| > (2C_j)^{-1}$ we apply triangular inequality, and the fact that $|\bar{p}| = 1$ together with Cauchy-Schwartz inequality to get, by using (J1), the inequality

$$\Psi(z) \leq 2|j(\bar{x}, z) - j(\bar{y}, z)| \leq 2C_j|\bar{x} - \bar{y}||z|. \tag{A.4}$$

Using this and (M') we can write

$$\int_{B\setminus B_{\delta_0}} \Psi(z)\nu(dz) \leq C|\bar{x} - \bar{y}|,$$

for some $C > 0$ depending on the data. This concludes the proof. $\square$

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