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From exact observability to identification of singular sources

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Abstract. We give general results about the identifiability of source terms for infinite dimensional linear systems that are exactly observable. We allow the source term to be unbounded, i.e., not contained in the state space, but in one of a sequence of extended spaces. We show that the operator from the source term to the output function is bounded from below, in suitable norms. We apply the main result to a system described by the wave equation in a bounded n-dimensional domain. We derive results of independent interest concerning the range of the input map of an exactly controllable system, when restricted to various spaces of smooth input functions.

1. Introduction

In this work we give some general results about the identifiability of source terms for infinite dimensional linear systems that are exactly observable. We allow the source term to be unbounded, i.e., not contained in the state space. The abstract main result (presented in Section 4) is a generalization of the main result of Alves *et al* [1], which in turn is inspired by Puel and Yamamoto [17]. The class of source terms that we consider is larger than in [1], which makes it possible to tackle a large range of applications to systems governed by partial differential equations. In the applications, the general aim is to show that the location of point sources can be determined from the output function in a continuous manner (in other words, we have a stability estimate for the considered inverse problem). For the physical background of such problems see our references.

Our motivating example is a wave equation with a point source and Neumann boundary observation on a part of the boundary:

$$\left\{\begin{array}{rcll} \dfrac{\partial^2 w}{\partial t^2} - \Delta w &=& \lambda(t)\delta_\xi & \text{on } \Omega\times(0,\tau), \\ w &=& 0 & \text{on } \partial\Omega\times(0,\tau), \\ w(x,0) &=& w_0(x), \quad \dfrac{\partial w}{\partial t}(x,0) = w_1(x) & \text{on } x\in\Omega, \\ y &=& \dfrac{\partial w}{\partial \nu} & \text{on } \Gamma\times(0,\tau), \end{array}\right. \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with C^2 boundary, $\tau>0$, λ is a given absolutely continuous function with $\lambda'\in L^1[0,\tau]$ and $\lambda(0)\neq 0$. The distribution δ_ξ is the Dirac mass located at $\xi \in \Omega$ and Γ is a nonempty open subset of $\partial\Omega$. The initial data satisfy $w_0 \in \mathcal{H}_0^1(\Omega)$ and $w_1 \in L^2(\Omega)$ and they are known. The well-posedness of the system (1.1) has been investigated for $n \in \{1, 2, 3\}$ in Triggiani [18, Theorem 2.1] (see also Meyer [16] for the case $n = 3$ and without the boundary observation).

The inverse problem is to determine ξ from y in a continuous way. In fact, we may replace in this problem the distribution δ_ξ with a finite sum of the type $\sum_{k=1}^N \delta_{\xi_k}$, with $\xi_k \in \Omega$, without essentially changing the difficulty of the problem, and then the problem is to determine all the points ξ_k. To keep the exposition simple, in this Introduction we stick to a single Dirac mass at ξ. This problem (with a finite sum of Dirac masses) has been considered by Komornik and Yamamoto [12, Section 4] under the assumptions that Ω is a ball in $\mathbb{R}^n$ and Γ is its whole boundary. There is a large engineering literature on related localization problems with point sources and point measurements, see for instance A.J. Weiss [21] and the references therein.

We say that the wave equation on Ω with the Neumann trace operator on Γ is *exactly observable in time* $\tau_0 > 0$ if the following estimate holds for smooth solutions of (1.1) with $\lambda = 0$:

$$\|y\|_{L^2([0,\tau_0];L^2(\Gamma))} \geqslant k \left(\|\nabla w_0\|^2_{L^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} ,$$

where $k > 0$. We refer to [20, Chapter 7] for more details on this concept. If $\partial\Omega$ is sufficiently smooth, then the above observability property is equivalent to the geometric optics condition due to Bardos, Lebeau and Rauch [2] (see also Burq and Gérard [3]).

For any Hilbert space U, any $m \in \mathbb{N}$ and any $\tau > 0$ we set

$$\mathcal{H}_L^m(0,\tau;U) = \left\{ u \in \mathcal{H}^m(0,\tau;U) \mid u(0) = \cdots = u^{m-1}(0) = 0 \right\} ,$$

$$\mathcal{H}_R^m(0,\tau;U) = \left\{ u \in \mathcal{H}^m(0,\tau;U) \mid u(\tau) = \cdots = u^{m-1}(\tau) = 0 \right\} .$$

For $m = 0$ the above spaces are defined to be $L^2([0,\tau];U)$. We denote, as is the standard practice, $\mathcal{H}_0^m(0,\tau;U) = \mathcal{H}_L^m(0,\tau;U) \cap \mathcal{H}_R^m(0,\tau;U)$. We denote by $W^{1,1}(0,\tau)$ the space of absolutely continuous functions on the interval $(0,\tau)$ whose derivative is in $L^1[0,\tau]$.

Our main result, Theorem 4.4 requires some preliminaries, so we state here only its consequences when applied to the above wave equation with a point source term.

Theorem 1.1. *Assume that the wave equation on Ω with the Neumann trace operator on Γ is exactly observable in time $\tau_0 > 0$. Let $\varepsilon > 0$ be such that the open set*

$$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

is not empty. Let $m \in \mathbb{N}$ be such that $m > n/2$ and let $\tau > \tau_0$. Let $\lambda \in W^{1,1}(0,\tau)$.

Then there exists a constant $K_{\varepsilon,m,\tau} > 0$ (which depends also on λ) with the following property: if $y^{(1)}$ and $y^{(2)}$ are the outputs of the system (1.1) corresponding to $\xi = \xi^{(1)} \in \Omega_\varepsilon$ and $\xi = \xi^{(2)} \in \Omega_\varepsilon$, respectively, with the same initial data, then

$$|\xi^{(1)} - \xi^{(2)}| \leqslant K_{\varepsilon,m,\tau} \left\| y^{(1)} - y^{(2)} \right\|_{[\mathcal{H}_R^{m-1}(0,\tau;L^2(\Gamma))]'} . \tag{1.2}$$

Here $[\mathcal{H}_R^{m-1}(0,\tau;L^2(\Gamma))]'$ is the dual of $\mathcal{H}_R^{m-1}(0,\tau;L^2(\Gamma))$ with respect to the pivot space $L^2([0,\tau];L^2(\Gamma))$, and $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. For the proof of this theorem we refer to Section 5.

The theorem tell us that we can recover ξ from y in a continuous way, with respect to the specified norm on the output functions. Of course, the $[\mathcal{H}_R^m]'$ norm decreases as m increases, so that formally we have stronger statements for larger m, but in applications it is probably enough to take one integer m that satisfies $m > n/2$. For $m \leqslant n/2$ the right-hand side of (1.2) may become meaningless, because $y^{(1)}, y^{(2)}$ may not be in the required space (whose norm we are using).

This theorem is a generalization of the result in [12, Section 4], in the sense that in [12] Ω is restricted to be a ball. On the other hand, for a ball, the result in [12, Section 4] is more general than ours in the sense that they consider related estimates for all $m \geqslant \frac{n+1}{2}$ (not necessarily integer). Moreover, in [12], the left side of (1.2) appears at a positive power, so that the estimates are not equivalent. While the analysis in [12] is based on series expansions into spherical functions, we use an abstract result given in Section 4. This abstract result can be applied also to other examples, for example, systems described by the heat, plate or Schrödinger equation.

An interesting related problem is to determine both the localization ξ and the intensity λ from boundary measurements. This is, in the general case, an open problem. Some results in this direction can be found in El Badia and Ha-Duong [6] and [1, Section 6].

To prove our abstract result from Section 4, we need some new results about exactly controllable systems: the range of the input map when restricted to inputs in certain smooth Sobolev spaces. These new results are derived in Section 2 (and the dual results in Section 3). We believe that these sections are of independent interest.

2. Controllability with smooth inputs for smooth final states

For linear infinite-dimensional systems, exact controllability in some time τ means that, starting from the initial state zero, with a suitably chosen input function u of class L^2, we can reach any final state at time τ. This property and its dual, exact observability, have been discussed in a very large number of papers and books, many inspired by the book Lions [15]. Relatively few authors have considered the following question: if the desired final state is in some smoother subspace, for example, the domain of the generator to some power, can the corresponding input function also be chosen in a smoother subspace, for example, a Sobolev space with positive index. A systematic study of this question has been undertaken in a recent paper by Ervedoza and Zuazua [7]. For the specific case of the wave equation with bounded (distributed) control, it was considered in Dehman and Lebeau [5]. There are several related results in Section 11.3 of [20]. Here we derive more results on this topic, that generalize certain results from [7] and [20].

Throughout this section, X and U are Hilbert spaces.

Our **first standing assumption** for this section is that $\mathbb{T}$ is a strongly continuous semigroup of operators on X, with generator $A : \mathcal{D}(A) \to X$. For $m \in \mathbb{N}$, we denote $X_m = \mathcal{D}(A^m)$ and $X_m^d = \mathcal{D}(A^{*m})$, each with the graph norm, so that they are Hilbert spaces too. We set $X_0 = X_0^d = X$. We denote

$$X_{-m} = (X_m^d)', \qquad X_{-m}^d = (X_m)',$$

all these dual spaces being with respect to the pivot space X. Note that, for each $k \in \mathbb{Z}$, the original semigroup $\mathbb{T}$ has a restriction (or an extension) to X_k that is the image of $\mathbb{T}$

through the unitary operator $(\beta I - A)^{-k} \in \mathcal{L}(X, X_k)$, where $\beta \in \rho(A)$ (the resolvent set of A). We refer to [20, Remark 2.10.5] for a proof of the last statement.

The **second standing assumption** for this section is that $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$. The admissibility assumption means that for some $\tau > 0$, the operator Φ_τ defined by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) \mathrm{d}\sigma \qquad \forall\, u \in L^2([0,\tau]; U),$$

a priori with range Ran Φ_τ contained in X_{-1}, has its range contained in X. We refer again to [20] for more material on this concept. Here we only mention that it follows from the admissibility assumption that $\Phi_\tau \in \mathcal{L}(L^2([0,\tau]; U), X)$ holds for all $\tau \geqslant 0$, and (with u as above) $\Phi_t u$ depends continuously on $t \in [0,\tau]$. The operators Φ_τ are called input maps corresponding to the pair (A, B).

Lemma 2.1. *If $u \in \mathcal{H}^m_L(0,\tau; U)$ (with $m \in \mathbb{N}$) and $z(t) = \Phi_t u$ for every $t \in [0,\tau]$, then*

$$z \in C^m([0,\tau]; X), \qquad z^{(j)}(0) = 0 \qquad \forall\, j \in \{0, 1, \dots m\} \tag{2.1}$$

and, for all $t \in [0,\tau]$,

$$z^{(m)}(t) = A z^{(m-1)}(t) + B u^{(m-1)}(t). \tag{2.2}$$

Proof. The case $m = 1$ is contained in Proposition 4.2.5 and Lemma 4.2.8 of [20]. Now suppose that the statement is true for $m - 1$, i.e., for $u \in \mathcal{H}^{m-1}_L(0,\tau; U)$ we have

$$z \in C^{m-1}([0,\tau]; X), \qquad z^{(j)}(0) = 0 \qquad \forall\, j \in \{0, 1, \dots m-1\} \tag{2.3}$$

and, for all $t \in [0,\tau]$,

$$z^{(m-1)}(t) = A z^{(m-2)}(t) + B u^{(m-2)}(t). \tag{2.4}$$

Suppose that $u \in \mathcal{H}^m_L(0,\tau; U)$, so that $Bu^{(m-2)} \in \mathcal{H}^2_L(0,\tau; X_{-1}) \subset C^1([0,\tau]; X_{-1})$. It is clear from (2.3) that also $Az^{(m-2)} \in C^1([0,\tau]; X_{-1})$. Thus, (2.4) implies that $z^{(m-1)} \in C^1([0,\tau]; X_{-1})$. Differentiating both sides of (2.4) we obtain that (2.2) holds as an equality in X_{-1}, for all $t \in [0,\tau]$. In particular, for $t = 0$ we obtain that $z^{(m)}(0) = 0$.

Denote $x = z^{(m-1)}$ and $v = u^{(m-1)}$, so that $x \in C^1([0,\tau]; X_{-1})$, $v \in \mathcal{H}^1_L(0,\tau; U)$ and (according to (2.2)) $\dot{x} = Ax + Bv$ holds on $[0,\tau]$ and $x(0) = 0$. According to Proposition 4.2.5 in [20] we have $x(t) = \Phi_t v$ for every $t \in [0,\tau]$ and according to Lemma 4.2.8 in [20] we have $x \in C^1([0,\tau]; X)$. This implies that $z \in C^m([0,\tau]; X)$ (so that and (2.2) holds as an equality in X). Thus, by induction, we have proved the lemma. ■

Lemma 2.2. *For every $m \in \mathbb{N}$ define the space $Z_m \subset X$ by*

$$Z_m = X_m + (\beta I - A)^{-1} BU + (\beta I - A)^{-2} BU \dots + (\beta I - A)^{-m} BU, \tag{2.5}$$

where $\beta \in \rho(A)$. We set $Z_0 = X$. Then Z_m is independent of the choice of $\beta \in \rho(A)$.

Proof. For brevity, in this proof we denote $R_\mu = (\mu I - A)^{-1}$. Let β be the number used in (2.5). We break the proof into a sequence of claims.

Claim 0. We have $Z_m \subset Z_{m-1}$, for all $m \in \mathbb{N}$. This is easy to see.

Claim 1. We have $R_\beta Z_{m-1} \subset Z_m$, for all $m \in \mathbb{N}$. This is obvious.

Claim 2. For any $\mu \in \rho(A)$ we have $R_\mu BU \subset Z_m$, for all $m \in \mathbb{N}$. This is shown by induction. It is obviously true for $m = 0$. Suppose it is true for $m-1$, for some $m \in \mathbb{N}$. From the resolvent identity we have

$$R_\mu BU \subset R_\beta BU + R_\beta R_\mu BU \subset Z_m + R_\beta Z_{m-1}.$$

Using Claim 1, we obtain that $R_\mu BU \subset Z_m$.

Claim 3. We have the recurrence relation

$$Z_m = R_\beta \left[Z_{m-1} + BU \right],$$

for all $m \in \mathbb{N}$. This is easy to see.

Claim 4. We have $R_\mu Z_{m-1} \subset Z_m$, for all $m \in \mathbb{N}$ and for all $\mu \in \rho(A)$. This is proved by induction. The case $m = 1$ is obvious. Suppose that the claim holds for some $m \in \mathbb{N}$. Applying R_μ to the recurrence relation in Claim 3 and then using Claim 2, we get

$$R_\mu Z_m = R_\beta R_\mu Z_{m-1} + R_\beta R_\mu BU \subset R_\beta Z_m + R_\beta Z_m.$$

Finally, using Claim 1 we get $R_\mu Z_m \subset Z_{m+1}$, i.e., the claim holds for $m+1$.

Claim 5. For some $\mu \in \rho(A)$ and all $m \in \mathbb{N}$ define

$$\tilde{Z}_m = X_m + R_\mu BU + R_\mu^2 BU \ldots + R_\mu^m BU.$$

Then $\tilde{Z}_m \subset Z_m$. Indeed, the first term X_m in the above sum is obviously a subset of Z_m. For the other terms we have, using Claim 2 and then repeatedly Claims 0 and 4, that for every $k \in \mathbb{N}$,

$$R_\mu^k BU \subset R_\mu^{k-1} R_\mu BU \subset R_\mu^{k-1} Z_m \subset Z_m.$$

Claim 6. We have $\tilde{Z}_m = Z_m$, for all $m \in \mathbb{N}$. Indeed, by redoing the above steps with the roles of β and μ reversed, we obtain that $Z_m \subset \tilde{Z}_m$. ∎

Proposition 2.3. *For every $m \in \{0, 1, 2, \ldots\}$ and for all $\tau > 0$,*

$$\Phi_\tau \mathcal{H}_L^m(0, \tau; U) \subset Z_m, \qquad \Phi_\tau \mathcal{H}_0^m(0, \tau; U) \subset X_m,$$

where Z_m is the space introduced in Lemma 2.2.

Proof. It is well known that if $\beta \in \mathbb{C}$ then B is admissible also for the semigroup generated by $A - \beta I$. It is easy to see that the range of the operator Φ_τ, when applied to one of the spaces $\mathcal{H}_L^m(0, \tau; U)$ or $\mathcal{H}_0^m(0, \tau; U)$, does not change if we replace A with $A - \beta I$. For these reasons, without loss of generality we may assume in this proof that $\mathbb{T}$ is exponentially stable. Therefore we may take $\beta = 0$ in (2.5).

Let $u \in \mathcal{H}_L^m(0, \tau; U)$ and let $z : [0, \tau] \to X$ be defined by $z(t) = \Phi_t u$. We know from Lemma 2.1 that (2.1) and (2.2) hold, for any $k \in \mathbb{N}$ with $k \leqslant m$ in place of m. Applying A^{-1} to both sides of (2.2) we obtain that (for $1 \leqslant k \leqslant m$)

$$z^{(k-1)}(\tau) = A^{-1} z^{(k)}(\tau) - A^{-1} B u^{(k-1)}(\tau).$$

Substituting the case $k = 2$ into the case $k = 1$, then substituting the case $k = 3$ into the resulting formula and so on, we obtain that

$$z(\tau) = A^{-m} z^{(m)}(\tau) - A^{-1} B u(\tau) - A^{-2} B u^{(1)}(\tau) \ldots - A^{-m} B u^{(m-1)}(\tau). \tag{2.6}$$

According to (2.5) we get $z(\tau) \in Z_m$. If we assume that $u \in \mathcal{H}_0^m(0,\tau;U)$ then all the terms containing u on the right-hand side of (2.6) vanish, so that we get $z(\tau) \in X_m$. ■

The **third standing assuption** that will be used in the remaining part of this section is that the pair (A,B) is exactly controllable in some time $\tau_0 > 0$. This ssumption means that Ran $\Phi_{\tau_0} = X$ (when using inputs of class L^2). In this case, Ran $\Phi_\tau = X$ holds for all $\tau > \tau_0$, as it is easy to see. The reader may look up, for instance, [20, Chapter 11] for some consequences of this concept, equivalent conditions and examples.

Lemma 2.4. *Assume that $\mathbb{T}$ is exponentially stable. Then there exists a constant $c > 0$ with the following property: For any $\lambda > 0$ consider the two systems with states $w(t) \in X$ and $u(t) \in U$ and the common input function v, described by*

$$\dot{w} = Aw + Bv\,, \qquad \dot{u} = \lambda u + v\,. \tag{2.7}$$

These systems are simultaneously exactly controllable in any time

$$\tau > \tau_0 + \frac{c}{\lambda}. \tag{2.8}$$

This means that for any $w_0 \in X$ and any $u_0 \in U$, there exists $v \in L^2([0,\tau];U)$ such that the solutions w, u of (2.7) corresponding to $w(0) = 0$ and $u(0) = 0$ satisfy

$$w(\tau) = w_0\,, \qquad u(\tau) = u_0\,. \tag{2.9}$$

The dual (and hence equivalent) version of this lemma (with some additional information on the constant c) has appeared as Lemma 11.3.5 in [20], so that we omit the proof. If U is finite-dimensional, then a slightly stronger conclusion (corresponding to $c = 0$ in (2.8)) can be obtained from [19, Theorem 3.3] (see also [20, Corollary 11.3.3]).

Theorem 2.5. *For every $m \in \{0,1,2,\ldots\}$ and for all $\tau > \tau_0$,*

$$\Phi_\tau \mathcal{H}_L^m(0,\tau;U) = Z_m\,, \qquad \Phi_\tau \mathcal{H}_0^m(0,\tau;U) = X_m\,.$$

Proof. We prove the theorem using induction. The case $m = 0$ is true by the definition of exact controllability (since $Z_0 = X_0 = X$). Now suppose that the theorem is true for $m-1$, where $m \in \mathbb{N}$. We have to prove that the theorem is true for m. While proving this, we may assume, without loss of generality, that $\mathbb{T}$ is exponentially stable (this was explained at the beginning of the proof of Proposition 2.3). In view of Proposition 2.3, we only have to prove the converse inclusions.

The proof of $\Phi_\tau \mathcal{H}_L^m(0,\tau;U) \supset Z_m$:

Choose $\tau > \tau_0$ and then choose $\lambda > 0$ such that (2.8) holds. Consider the two systems with states $w(t) \in X$ and $u(t) \in U$ and the common input function v, described by (2.7). According to Lemma 2.4, these systems are simultaneously exactly controllable in time τ. This means that the combined system described by $\dot{q} = \mathcal{A}q + \mathcal{B}v$, with

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & \lambda I \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} B \\ I \end{bmatrix},$$

with state space $\mathcal{X} = X \times U$, is exactly controllable in time τ. Let us denote by $\tilde{\Phi}_\tau$ the input to state operator for the combined system, given by

$$\tilde{\Phi}_\tau = \begin{bmatrix} \Phi_\tau \\ \varphi_\tau \end{bmatrix}, \quad \text{where} \quad \varphi_\tau v = \int_0^\tau e^{\lambda(\tau-\sigma)} v(\sigma) \mathrm{d}\sigma\,.$$

According to the theorem applied for $m-1$, we have

$$\tilde{\Phi}_\tau \mathcal{H}_L^{m-1}(0,\tau;U) \,=\, \mathcal{Z}_{m-1}\,, \tag{2.10}$$

where $\mathcal{Z}_{m-1}$ is defined similarly as in (2.5), so that

$$\mathcal{Z}_{m-1} \,=\, Z_{m-1} \times U\,.$$

For an arbitrary $z^0 \in Z_m$ choose $w^0 \in Z_{m-1},\ u^0 \in U$ such that

$$z^0 \,=\, A^{-1}[w^0 - Bu^0]\,. \tag{2.11}$$

Indeed, this is possible according to the recurrence relation in Claim 3 of the proof of Lemma 2.2. From (2.10) there exists an input signal $v \in \mathcal{H}_L^{m-1}(0,\tau;U)$ such that the solutions $w,\ u$ of (2.7) satisfy

$$w(0) \,=\, 0\,,\quad w(\tau) \,=\, w^0 - \lambda z^0\,,\quad u(0) \,=\, 0\,,\quad u(\tau) \,=\, u^0\,. \tag{2.12}$$

Moreover, it is easy to see that $u \in \mathcal{H}_L^m(0,\tau;U)$. We define $z \in C([0,\tau];X)$ by

$$z(t) \,=\, (A-\lambda I)^{-1}[w(t) - Bu(t)]\,.$$

It is clear that $z(0) = 0$. It is easy to see that

$$z(\tau) \,=\, (A-\lambda I)^{-1}[w^0 - Bu^0 - \lambda z^0] \,=\, (A-\lambda I)^{-1}[Az^0 - \lambda z^0] \,=\, z^0\,.$$

This part of the proof will be complete if we show that z is a solution of

$$\dot{z}(t) \,=\, Az(t) + Bu(t)\,,$$

so that $z(\tau) = \Phi_\tau u$. First we verify that z satisfies the differential equation

$$\dot{z}(t) \,=\, \lambda z(t) + w(t) \qquad \forall\, t \in [0,\tau]\,. \tag{2.13}$$

Indeed, we have (using the definition of z)

$$\begin{aligned}\dot{z}(t) \,&=\, (A-\lambda I)^{-1}[\dot{w}(t) - B\dot{u}(t)] \,=\, (A-\lambda I)^{-1}[Aw(t) - \lambda Bu(t)]\\ &=\, (A-\lambda I)^{-1}[(A-\lambda I)w(t) + \lambda(w(t) - Bu(t))] \,=\, w(t) + \lambda z(t)\,.\end{aligned}$$

Note that (2.13) implies that $z \in C^1([0,\tau];X)$. Now from (2.13) we get, using again the definition of z,

$$\begin{aligned}\dot{z}(t) \,&=\, (\lambda I - A + A)(A-\lambda I)^{-1}[w(t) - Bu(t)] + w(t)\\ &=\, -\,[w(t) - Bu(t)] + A(A-\lambda I)^{-1}[w(t) - Bu(t)] + w(t)\\ &=\, Az(t) + Bu(t)\,.\end{aligned}$$

The proof of $\Phi_\tau \mathcal{H}_0^m(0,\tau;U) \supset X_m$:

This part of the proof is very similar to the previous part. The difference is that now instead of (2.10) we use

$$\tilde{\Phi}_\tau \mathcal{H}_0^{m-1}(0,\tau;U) \,=\, \mathcal{X}_{m-1}\,,$$

where $\mathcal{X}_{m-1} = \mathcal{D}(\mathcal{A}^{m-1}) = \mathcal{D}(A^{m-1}) \times U$. This time we choose $u^0 = 0$, $v \in \mathcal{H}_0^{m-1}(0, \tau; U)$ and we obtain that $u \in \mathcal{H}_0^m(0, \tau; U)$. ■

We remark that the equality $\Phi_\tau \mathcal{H}_L^1(0, \tau; U) = Z_1$ (which follows from the above theorem) constitutes Theorem 11.3.6 in [20]. The equality $\Phi_\tau \mathcal{H}_L^2(0, \tau; U) = Z_2$, for finite-dimensional U, constitutes Proposition 11.3.8 in [20]. Our proof above uses ideas from the proofs of the results cited. The inclusion $\Phi_\tau \mathcal{H}_0^m(0, \tau; U) \supset X_m$ is contained in Theorem 1.4 of [7], but under the additional assumption that the semigroup $\mathbb{T}$ is invertible.

We define a norm on the space Z_m from (2.5) as follows:

$$\|z\|_{Z_m}^2 = \inf \left\{ \|x\|_m^2 + \|u_1\|^2 + \|u_2\|^2 \ldots + \|u_m\|^2 \right\}, \tag{2.14}$$

the infimum being computed over all the collections $(x, u_1, u_2, \ldots u_m) \in X_m \times U^m$ such that $z = x + (\beta I - A)^{-1} B u_1 + (\beta I - A)^{-2} B u_2 \ldots + (\beta I - A)^{-m} B u_m$. Then Z_m can be regarded as a subspace of $X_m \times U^m$, namely, the orthogonal complement of the space

$$\mathcal{N} = \left\{ ((x, u_1, u_2, \ldots u_m) \in X_m \times U^m \;\middle|\; x + \sum_{k=1}^{m} (\beta I - A)^{-k} B u_k = 0 \right\}.$$

This shows that Z_m is complete (hence, it is a Hilbert space). It is easy to see that for all $m \in \mathbb{N}$ the embeddings

$$X_m \subset Z_m \subset Z_{m-1}$$

are continuous. (For $m = 1$ see [20, p. 119].) Usually, X_m is not dense in Z_m.

3. Extensions of output maps

In this section we formulate and prove the dual versions of the results from Section 2 and we give a very simple example. Throughout this section, X and Y are Hilbert spaces.

Our **first standing assumption** for this section is that $\mathbb{T}$ is a strongly continuous semigroup of operators on X, with generator $A : \mathcal{D}(A) \to X$. For $m \in \mathbb{Z}$, the spaces X_m, X_m^d, X_{-m} and X_{-m}^d are defined as in Section 2.

The **second standing assumption** for this section is that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. The admissibility assumption means that for some $\tau > 0$, the operator Ψ_τ defined by

$$(\Psi_\tau z_0)(t) = C \mathbb{T}_t z_0 \qquad \forall\, z_0 \in X_1,$$

has an extension to an operator $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau], Y)$. Equivalently, there is a positive number k such that $\int_0^\tau \|C \mathbb{T}_t z_0\|^2 \mathrm{d}t \leqslant k^2 \|z_0\|^2$ for all $z_0 \in \mathcal{D}(A)$. We refer to [20, 23] for more material on this concept. Here we only mention that it follows from the admissibility assumption that $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau]; Y)$ holds for all $\tau \geqslant 0$. The operators Ψ_τ are called output maps corresponding to the pair (A, C).

The dual result of Proposition 2.3 is the following:

Proposition 3.1. *For every $m \in \{0, 1, 2, \ldots\}$ and for all $\tau > 0$, Ψ_τ has unique extensions*

$$\Psi_\tau \in \mathcal{L}((Z_m^d)', [\mathcal{H}_R^m(0, \tau; Y)]'), \qquad \Psi_\tau \in \mathcal{L}((X_{-m}, \mathcal{H}^{-m}(0, \tau; Y)), \tag{3.1}$$

where

$$Z_m^d = X_m^d + (\beta I - A^*)^{-1}C^*U + (\beta I - A^*)^{-2}C^*U \ldots + (\beta I - A^*)^{-m}C^*U, \tag{3.2}$$

for some $\beta \in \rho(A^*)$. The norm on Z_m^d is defined similarly as in (2.14)*, with A^* and C^* in place of A and B. Here and below, dualities are computed with respect to the pivot spaces X and $L^2([0,\tau];Y)$, respectively.*

Proof. Denote by Φ_τ^d the input maps corresponding to the pair (A^*, C^*). We know from [20, Section 4.4] that

$$\Phi_\tau^d = \Psi_\tau^* \boldsymbol{Я}_\tau, \tag{3.3}$$

where $\boldsymbol{Я}_\tau$ is the reflection operator on $L^2([0,\tau];Y)$ defined by $\boldsymbol{Я}_\tau u(t) = u(\tau - t)$. Notice that $\boldsymbol{Я}_\tau$ is self-adjoint and also unitary. Since $\boldsymbol{Я}_\tau$ is an isomorphism from $\mathcal{H}_L^m(0,\tau;Y)$ to $\mathcal{H}_R^m(0,\tau;Y)$, it follows from Proposition 2.3 (with A^* and C^* in place of A and B) that

$$\Psi_\tau^* \mathcal{H}_R^m(0,\tau;Y) \subset Z_m^d, \qquad \Psi_\tau^* \mathcal{H}_0^m(0,\tau;Y) \subset X_m^d.$$

where Z_m^d is the counterpart of Z_m for the pair (A^*, C^*), so that it is given by (3.2). By the closed graph theorem we have that Ψ_τ^* is bounded between the spaces indicated above. Now taking adjoints we obtain (3.1). ■

The **third standing assuption** for the remaining part of this section is that the pair (A, C) is exactly observable in some time $\tau_0 > 0$. This assumption means that Ψ_{τ_0} is bounded from below. In this case, Ψ_τ is bounded from below for all $\tau > \tau_0$, as it is easy to see. We refer to [20, Chapter 6] for more material on this concept.

The dual result of Theorem 2.5 is the following:

Theorem 3.2. *With the notation of Proposition* 3.1*, for every $m \in \{0,1,2,\ldots\}$ and each $\tau > \tau_0$, there exists a constant $c_{m,\tau} > 0$ such that, for every $f \in (Z_m^d)'$, we have*

$$\|\Psi_\tau f\|_{[\mathcal{H}_R^m(0,\tau;Y)]'} \geqslant c_{m,\tau} \|f\|_{(Z_m^d)'}. \tag{3.4}$$

Similarly, for every $m \in \{0,1,2,\ldots\}$ and each $\tau > \tau_0$, there exists a constant $k_{m,\tau} > 0$ such that, for every $f \in X_{-m}$, we have

$$\|\Psi_\tau f\|_{\mathcal{H}^{-m}(0,\tau;Y)} \geqslant k_{m,\tau} \|f\|_{X_{-m}}. \tag{3.5}$$

Proof. We denote again by Φ_τ^d the input maps corresponding to the pair (A^*, C^*). We know from [20, Theorem 11.2.1] that (A^*, C^*) is exactly controllable in time τ_0. According to Theorem 2.5 (applied to the pair (A^*, C^*)), for every $m \in \{0,1,2,\ldots\}$ and for all $\tau > \tau_0$, Φ_τ^d maps $\mathcal{H}_L^m(0,\tau;Y)$ onto Z_m^d, and it also maps $\mathcal{H}_0^m(0,\tau;Y)$ onto X_m^d. Since $\Psi_\tau^* = \Phi_\tau^d \boldsymbol{Я}_\tau$ (see (3.3)), it follows that

$$\Psi_\tau^* \mathcal{H}_R^m(0,\tau;Y) = Z_m^d, \qquad \Psi_\tau^* \mathcal{H}_0^m(0,\tau;Y) = X_m^d.$$

By the closed graph theorem, Ψ_τ^* is bounded between these spaces. By a well known result about surjective operators (see, for instance, [20, Proposition 12.1.3]), Ψ_τ is bounded from below between the corresponding dual spaces. This fact is expressed in (3.4) and (3.5). ■

Example 3.3. Consider the boundary observed wave equation on the interval $(0,\pi)$:

$$w_{tt} = w_{xx}, \quad w(0,t) = w(\pi,t) = 0, \tag{3.6}$$

with the initial conditions

$$w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x), \tag{3.7}$$

and with the measurements

$$y_1(t) = -\frac{\partial w}{\partial x}(0,t), \qquad y_2(t) = \frac{\partial w}{\partial x}(\pi,t).$$

We refer to Section 5 for an n-dimensional version of this system.

To associate to these equations a system in the sense considered earlier, first we introduce $H = L^2[0,\pi]$ and $A_0 : \mathcal{D}(A_0) \to H$ by

$$\mathcal{D}(A_0) = \mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi), \qquad A_0 w = -w_{xx}.$$

Then A_0 is a strictly positive operator, so that for every $\alpha > 0$ we can introduce the space $H_\alpha = \mathcal{D}(A_0^\alpha)$, with $\|z\|_\alpha = \|A_0^\alpha z\|_H$. We set $H_0 = H$ and $H_{-\alpha} = H'_\alpha$ (duality with respect to the pivot space H). We have

$$H_{\frac{1}{2}} = \mathcal{H}_0^1(0,\pi),$$

see for instance [20, Section 3.5]. The semigroup $\mathbb{T}$ associated to our PDE is defined on the state space $X = H_{\frac{1}{2}} \times H$, with $X_1 = \mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$ and

$$A \begin{bmatrix} g \\ f \end{bmatrix} = \begin{bmatrix} f \\ -A_0 g \end{bmatrix}.$$

This A is skew-adjoint and hence $\mathbb{T}$ is unitary. We have $Y = \mathbb{C}^2$ and the observation operator $C \in \mathcal{L}(X_1, Y)$ is given by

$$C \begin{bmatrix} g \\ f \end{bmatrix} = \begin{bmatrix} -g_x(0) \\ g_x(\pi) \end{bmatrix} \qquad \forall \begin{bmatrix} g \\ f \end{bmatrix} \in H_1 \times H_{\frac{1}{2}}.$$

The PDE can be solved by elementary methods and the corresponding semigroup is isomorphic to a periodic left shift semigroup on $[0,2\pi]$, see for instance Weiss [22, Section 5]. However, in order to express Ψ_τ for small τ, we do not need the solution formulas for the system (3.6), (3.7). It will be enough to recall the elementary fact that if w is a solution of (3.6),(3.7) then for $x \in [0,\pi]$ and $\delta \in \mathbb{R}$ such that $x + \delta \in [0,\pi]$ we have

$$\dot{w}(x,t) - w_x(x,t) = \dot{w}(x+\delta, t+\delta) - w_x(x+\delta, t+\delta). \tag{3.8}$$

Similarly, if $x \in [0,\pi]$ and $\delta \in \mathbb{R}$ are such that $x - \delta \in [0,\pi]$, we have

$$\dot{w}(x,t) + w_x(x,t) = \dot{w}(x-\delta, t+\delta) + w_x(x-\delta, t+\delta). \tag{3.9}$$

The expression in the last two formulas are called Riemann invariants of the wave equation and these formulas say that they are constant along characteristics.

If we take $x = \pi$, $t \in [0,\pi]$ and $\delta = -t$ in (3.8), we get

$$w_x(\pi,t) = w_x(\pi - t, 0) - \dot{w}(\pi - t, 0) = w_{0,x}(\pi - t) - w_1(\pi - t).$$

Similarly, if we take $x = 0$, $t \in [0,\pi]$ and $\delta = -t$ in (3.9), we get

$$w_x(0,t) = w_x(t,0) + \dot{w}(t,0) = w_{0,x}(t) + w_1(t).$$

As usual, we denote by Ψ_τ the output maps corresponding to the pair (A, C). From the last two formulas, denoting $g = w_0$ and $f = w_1$, it follows that

$$\Psi_\pi \begin{bmatrix} g \\ f \end{bmatrix} = \begin{bmatrix} -g_x - f \\ \mathbf{Я}_\pi(g_x - f) \end{bmatrix} \qquad \forall \begin{bmatrix} g \\ f \end{bmatrix} \in X. \tag{3.10}$$

This shows, in particular, that (A, C) is exactly observable in time π, because both g_x and f can be continuously recovered from $g_x + f$ and from $g_x - f$.

Sometimes (such as in the example discussed in Section 5) we are interested to know if it is possible to recover an initial state of the special structure $\left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right]$, and then the equivalent question is if Ψ_τ is bounded from below on the subspace of such initial states. We can see from (3.10) that this is indeed the case for $\tau \geqslant \pi/2$, i.e., if we know that the initial state has the special structure $\left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right]$, then we need only half the time to recover it than we would otherwise need, see also Komornik and Yamamoto [10].

A simple computation shows that

$$C^* = -\begin{bmatrix} D \\ 0 \end{bmatrix},$$

where $D : \mathbb{C}^2 \to H$ is the Dirichlet map:

$$D \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} (x) = \left(1 - \frac{x}{\pi}\right) u_1 + \frac{x}{\pi} u_2,$$

and this easily implies that

$$Z_1^d = X_1^d + A^{-1} C^* Y = (\mathcal{H}^2(0, \pi) \cap \mathcal{H}_0^1(0, \pi)) \times \mathcal{H}^1(0, \pi)$$

(we have used that $X_1^d = \mathcal{D}(A^*) = \mathcal{D}(A)$, because A is skew-adjoint). According to Proposition 3.1 with $m = 1$, for every $\tau \geqslant 0$, Ψ_τ is bounded from $(Z_1^d)'$ to $\left[\mathcal{H}_R^1(0, \tau; \mathbb{C}^2)\right]'$, and for $\tau > \pi$, according to Theorem 3.2, this operator is bounded from below.

Remark 3.4. In this remark we signal what we think to be a small mistake in two papers of Komornik and Yamamoto [10, 12], appearing also in Cipolatti and Yamamoto [4], where a certain inequality seems to contradict our Proposition 3.1.

In Section 4 of [12] the authors study the wave equation on an n-dimensional ball Ω, with Neumann boundary observation over the whole boundary. Let us denote $A_0 = -\Delta$, $\mathcal{D}(A_0) = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, the spaces H_α (with $\alpha \in \mathbb{R}$) are defined as for the one dimensional case above and $Y = L^2(\partial\Omega)$. In [12, Proposition 4.2] it is claimed that the output maps Ψ_τ of this system can be extended so that they boundedly map initial states of the form $\left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right]$ with $f \in H_{-\frac{s}{2}}$ $(s \geqslant 0)$ into $[\mathcal{H}_R^s(0, \tau; Y)]'$ (which is defined by interpolation). This boundedness is expressed as the existence of $c > 0$ such that

$$\left\| \Psi_\tau \begin{bmatrix} 0 \\ f \end{bmatrix} \right\|_{\left[\mathcal{H}_R^s(0, \tau; Y)\right]'} \leqslant c \|f\|_{H_{-\frac{s}{2}}} \tag{3.11}$$

(this is the second half of [12, estimate (4.3)]). (Actually, this is claimed only for τ larger than the radius of the ball, but if it were true, then this would easily imply that it is true for any $\tau \geqslant 0$.) Moreover, it is claimed in the same place that if τ is larger than the radius of the ball, then this map is bounded from below. This result (the boundedness part) contradicts our Proposition 3.1, because it would mean that (at least in the given context) Ψ_τ boundedly maps X_{-s} into $[\mathcal{H}^s_R(0,\tau;Y)]'$. It is claimed in [4, Proposition 2.2] that the same result generalizes to any bounded open set Ω with smooth boundary.

To clarify if there is indeed a mistake, we consider the one dimensional domain $\Omega=(0,\pi)$ and we take $s=1$ and $\tau=\pi$, so that we can use our computations from Example 3.3. Then (3.11) with the help of (3.10) reduces to

$$\left\| \begin{bmatrix} f \\ \boldsymbol{\mathcal{A}}_\pi f \end{bmatrix} \right\|_{\left[\mathcal{H}^1_R(0,\pi;Y)\right]'} \leqslant c\|f\|_{\mathcal{H}^{-1}(0,\pi)},$$

which would imply in particular that

$$\|f\|_{\left[\mathcal{H}^1_R(0,\pi)\right]'} \leqslant c\|f\|_{\mathcal{H}^{-1}(0,\pi)} \qquad \forall\, f\in L^2[0,\pi]. \tag{3.12}$$

To disprove (3.12) we construct a sequence (f_n) of functions in $L^2[0,\pi]$ such that

$$\|f_n\|_{\mathcal{H}^{-1}(0,\pi)}\to 0, \qquad \|f_n\|_{\left[\mathcal{H}^1_R(0,\pi)\right]'} \not\to 0.$$

Such a sequence is given by

$$f_n(x) = ne^{-nx}.$$

We have (using the duality pairing between $\mathcal{H}^{-1}(0,\pi)$ and $\mathcal{H}^1_0(0,\pi)$)

$$\|f_n\|_{\mathcal{H}^{-1}(0,\pi)} = \sup_{\varphi\in\mathcal{H}^1_0(0,\pi),\ \|\varphi'\|_{L^2}=1} |\langle f_n,\varphi\rangle| = \sup_{\varphi\in\mathcal{H}^1_0(0,\pi),\ \|\varphi'\|_{L^2}=1} \left|\int_0^\pi e^{-nx}\varphi'(x)\mathrm{d}x\right|.$$

Now we see from the Cauchy-Schwarz inequality that indeed $\|f_n\|_{\mathcal{H}^{-1}(0,\pi)}\to 0$. Finally, to show that (f_n) does not converge to zero in $\left[\mathcal{H}^1_R(0,\pi)\right]'$, we take $\varphi\in\mathcal{H}^1_R(0,\pi)$ defined by $\varphi(x)=\pi-x$. Then we have the duality pairing

$$\langle f_n,\varphi\rangle = -\left[e^{-nx}\varphi(x)\right]_0^\pi + \int_0^\pi e^{-nx}(-1)\mathrm{d}x = \pi - \frac{1-e^{-n\pi}}{n} \to \pi.$$

The same problem appears in the discussion of the one dimensional case in [10]. Indeed, the first part of the estimate (12) from [10] is (3.12), with the interval $(0,\pi)$ replaced with $(0,1)$, both in space and in time. The proof of this estimate given in [10] (with an erratum given in [11]) is valid for $f\in F$, where F is a certain dense subspace of $\mathcal{H}^{-1}(0,\pi)$. The mistake is to conclude from here that the estimate holds for $f\in\left[\mathcal{H}^1_R(0,\pi)\right]'$, because F is not dense in $\left[\mathcal{H}^1_R(0,\pi)\right]'$.

4. The main result

In this section we give our abstract result regarding the identification of source terms. Let X and Y be Hilbert spaces. We consider systems of the type

$$\begin{aligned} \dot z(t) &= Az(t)+\lambda(t)f, \quad z(0)=0, \\ y(t) &= Cz(t), \end{aligned} \tag{4.1}$$

where A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X, $C\in\mathcal{L}(X_1,Y)$ is an admissible observation operator for $\mathbb{T}$ and (A,C) is exactly observable in time τ_0.

If $f \in X$ and λ is a continuous (scalar) function, then it is easy to see that the solution of (4.1) on any interval $[0,\tau]$ gives the following output function:

$$y(t) = (\mathcal{J}_\tau f)(t) = \int_0^t \lambda(t-s)(\Psi_\tau f)(s)\mathrm{d}s \qquad \forall\, t \in [0,\tau], \tag{4.2}$$

where Ψ_τ is defined as at the beginning of Section 3.

Here we are concerned with extensions of this operator $\mathcal{J}_\tau$ (from f to y) to some larger spaces. For $m \in \mathbb{N}$, the spaces X_m, X_{-m}, X_m^d and X_{-m}^d are defined as at the beginning of Section 2. In our main result we consider $f \in (Z_m^d)'$ (these extended spaces were introduced in the previous section) and $\lambda \in W^{1,1}(0,\tau)$ (defined before Theorem 1.1). We may regard f as a control operator and λ as an input function, in the sense of [20], but we do not assume that f is an admissible control operator for $\mathbb{T}$. We do not even assume that $f \in X_{-1}$, we only have that $f \in X_{-m}$ (because $(Z_m^d)' \subset X_{-m}$). Systems with a control operator mapping into X_{-m} have been considered, for instance, in Latushkin *et al* [14].

Lemma 4.1. *Let $\tau > 0$, $k \in L^1[0,\tau]$ and define the operator $C_k \in \mathcal{L}(L^2([0,\tau];Y))$ by*

$$(C_k v)(t) = \int_0^t k(t-s)v(s)\mathrm{d}s.$$

Then $\sigma(C_k) = \{0\}$.

Proof. It is well known that $\|C_k\| \leqslant \|k\|_{L^1}$. For every $\gamma \in \mathbb{R}$ we define the operator $e_\gamma \in \mathcal{L}(L^1[0,\tau])$ by $(e_\gamma v)(t) = e^{\gamma t}v(t)$. It easy to check that

$$C_k = e_{-\gamma} C_{e_\gamma k}\, e_\gamma,$$

which shows that for any nonzero $\alpha \in \mathbb{C}$, $\alpha I - C_k$ is invertible iff $\alpha I - C_{e_\gamma k}$ is invertible. According to the dominated convergence theorem we have $\lim_{\gamma\to-\infty} \|e_\gamma k\|_{L^1} = 0$, so that $\|C_{e_\gamma k}\| \to 0$ as $\gamma \to -\infty$. This implies that indeed $\alpha I - C_{e_\gamma k}$ is invertible. ■

For results related to the above lemma we refer to Gohberg and Feldman [9, p. 41, p. 100] or Kress [13, p. 33, p. 34].

Remark 4.2. It is easy to see that C_k from the last lemma commutes with the operator of integration from 0 to t. This fact together with the lemma implies that for any $\alpha \in \mathbb{C}\setminus\{0\}$ and $m \in \mathbb{N}$, $\alpha I - C_k$ is an isomorphism (i.e., it is boundedly invertible) on $\mathcal{H}_L^m(0,\tau;Y)$.

Lemma 4.3. *Let $\tau > 0$ and $\lambda \in W^{1,1}(0,\tau)$ with $\lambda(0) \neq 0$. Define $S_\tau : L^2([0,\tau];Y) \to \mathcal{H}_L^1(0,\tau;Y)$ by*

$$(S_\tau g)(t) = \int_0^t \lambda(t-s)g(s)\mathrm{d}s. \tag{4.3}$$

Then S_τ is an isomorphism from $L^2([0,\tau];Y)$ onto $\mathcal{H}_L^1(0,\tau;Y)$. Moreover, for every $m \in \mathbb{N}$, the operator S_τ admits a unique extension to an isomorphism

$$S_\tau : [\mathcal{H}_R^m(0,\tau;Y)]' \to \left[\mathcal{H}_R^{m-1}(0,\tau;Y)\right]'.$$

Proof. Denote $y = S_\tau g$. Differentiating the sides of (4.3) we obtain that

$$\dot{y}(t) = \lambda(0)g(t) + \int_0^t \dot{\lambda}(t-s)g(s)\mathrm{d}s.$$

Thus, the operator from g to $\dot{y}$ is $\lambda(0)I + C_k$, with $k = \dot{\lambda}$. According to Lemma 4.1 this operator is an isomorphism on $L^2([0,\tau];Y)$. Since $y(t) = \int_0^t \dot{y}(\sigma)\mathrm{d}\sigma$, it follows that S_τ is an isomorphism from $L^2([0,\tau];Y)$ onto $\mathcal{H}^1_L(0,\tau;Y)$.

Denote $\mathcal{X} = L^2([0,\tau];Y)$, $\mathcal{D}(\mathcal{A}) = \mathcal{H}^1_L(0,\tau;Y)$ and let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ be the inverse of S_τ. For $m \in \mathbb{N}$ we denote by $\mathcal{X}_m$ the space $\mathcal{D}(\mathcal{A}^m)$ endowed with the graph norm. It is not difficult to check (using Remark 4.2) that

$$\mathcal{X}_m = \mathcal{H}^m_L(0,\tau;Y) \qquad \forall\, m \in \mathbb{N}.$$

The spaces $\mathcal{X}_{-m}$, with $m \in \mathbb{N}$ are defined as the completion of $\mathcal{X}$ with respect to the norm $\|z\|_{-m} = \|(\beta I - \mathcal{A})^{-m} z\|$, where $\beta \in \rho(\mathcal{A})$. We refer to [20, Section 2.10] for a detailed discussion of such spaces. In particular, it is well known that $\mathcal{X}_{-m}$ is the dual of $\mathcal{D}((\mathcal{A}^*)^m)$ with respect to the pivot space $\mathcal{X}$. It is easy to check that

$$(S^*_\tau h)(t) = \int_t^\tau \overline{\lambda}(s-t)h(s)\mathrm{d}s,$$

so that S^*_τ maps $L^2([0,\tau];Y)$ onto $\mathcal{H}^1_R(0,\tau;Y)$, whence $\mathcal{D}(\mathcal{A}^*) = \mathcal{H}^1_R(0,\tau;Y)$. More generally, by the dual version of Remark 4.2, $\mathcal{D}((\mathcal{A}^*)^m) = \mathcal{H}^m_R(0,\tau;Y)$, so that we have

$$\mathcal{X}_{-m} = [\mathcal{H}^m_R(0,\tau;Y)]' \qquad \forall\, m \in \mathbb{N}.$$

By denoting $\mathcal{X}_0 = \mathcal{X}$, $\mathcal{A}$ maps $\mathcal{X}_m$ onto $\mathcal{X}_{m-1}$ for every $m \in \mathbb{Z}$. Consequently, $S_\tau = \mathcal{A}^{-1}$ extends to an isomorphism from $[\mathcal{H}^m_R(0,\tau;Y)]'$ onto $\left[\mathcal{H}^{m-1}_R(0,\tau;Y)\right]'$. ■

Theorem 4.4. *Assume that the pair (A,C) is exactly observable in some time $\tau_0 > 0$ and that $\lambda \in W^{1,1}(0,\tau)$ with $\lambda(0) \neq 0$. Then for every $m \in \{0,1,2,\ldots\}$ and each $\tau > \tau_0$, $\mathcal{J}_\tau = S_\tau \Psi_\tau$ is one-to-one from $(Z^d_m)'$ to $[\mathcal{H}^{m-1}_R(0,\tau;Y)]'$ and there exists a constant $\kappa_{m,\tau} > 0$ such that*

$$\|f\|_{(Z^d_m)'} \leqslant \kappa_{m,\tau} \|\mathcal{J}_\tau f\|_{\left[\mathcal{H}^{m-1}_R(0,\tau;Y)\right]'}, \qquad \forall\, f \in (Z^d_m)'. \tag{4.4}$$

Proof. By using Lemma 4.3 and Theorem 3.2, there exists $M_\tau > 0$ such that

$$\|\mathcal{J}_\tau f\|_{\left[\mathcal{H}^{m-1}_R(0,\tau;Y)\right]'} \geqslant M_\tau \|\Psi_\tau f\|_{\left[\mathcal{H}^m_R(0,\tau;Y)\right]'} \geqslant M_\tau c_{m,\tau} \|f\|_{[Z^d_m]'}.$$ ■

Remark 4.5. The above theorem considers $f \in (Z^d_m)'$. This implies, according to the definition of Z^d_m, that the duality products $\langle f, (sI - A^*)^{-k} C^* y_0 \rangle$ are well defined for all $y_0 \in Y$ and all $k \in \{1,2,\ldots m\}$, so that the products $f^*(sI - A^*)^{-k} C^*$ make sense.

5. An application to the wave equation

Consider the system described by (1.1), with the stated assumptions on Ω, Γ and λ. Our aim here is to prove Theorem 1.1, which shows that the inverse problem of determining the location of the source ξ from the output y, is well-posed in appropriate spaces. Note that Theorem 1.1 implies the existence of $K > 0$ such that for any $\xi^{(1)}, \xi^{(2)} \in \Omega_\varepsilon$,

$$\|y^{(1)} - y^{(2)}\|_{L^2(0,\tau;L^2(\Gamma))} \geqslant K|\xi^{(1)} - \xi^{(2)}|,$$

where the L^2 norm on the left might be ∞.

For the proof we need to introduce some notation. We denote by $-A_0$ the Dirichlet Laplacian on Ω, so that $A_0 : \mathcal{D}(A_0) \to L^2(\Omega)$, where

$$\mathcal{D}(A_0) = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega),$$

and $A_0 > 0$. We denote $H = L^2(\Omega)$ and for every $\alpha > 0$ we define $H_\alpha = \mathcal{D}(A_0^\alpha)$, with the norm $\|z\|_\alpha = \|A_0^\alpha z\|_H$. The spaces $H_{-\alpha}$ are defined as the duals of H_α with respect to the pivot space H. We have (see, for instance, Section 3.6 in [20])

$$H_1 = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega), \quad H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega), \quad H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega).$$

We recall the following well-known result defining the Dirichlet map D for Ω (see, for instance, Propositions 10.6.1 and 10.6.2 in [20]).

Proposition 5.1. *For every $\mathrm{v} \in L^2(\partial\Omega)$, there exists a unique function $D\mathrm{v} \in L^2(\Omega)$ such that*

$$\int_\Omega (D\mathrm{v})(x) g(x)\,\mathrm{d}x = -\int_{\partial\Omega} \mathrm{v} \frac{\partial(A_0^{-1}g)}{\partial\nu}\,\mathrm{d}\sigma \qquad \forall\, g \in L^2(\Omega). \tag{5.1}$$

The operator D defined above (called the Dirichlet map) is linear and bounded from $L^2(\partial\Omega)$ into $L^2(\Omega)$ and its adjoint $D^* \in \mathcal{L}(L^2(\Omega), L^2(\partial\Omega))$ given by

$$D^* g = -\frac{\partial(A_0^{-1}g)}{\partial\nu} \qquad \forall\, g \in L^2(\Omega). \tag{5.2}$$

Moreover, for any $\mathrm{v} \in L^2(\partial\Omega)$ we have $D\mathrm{v} \in C^\infty(\Omega)$ and $\Delta D\mathrm{v} = 0$.

We mention that $\gamma_0 D\mathrm{v} = \mathrm{v}$ for all $\mathrm{v} \in L^2(\partial\Omega)$, where γ_0 is a suitable extension of the Dirichlet trace operator, see for instance [20, Proposition 10.6.4].

The system (1.1) can be rewritten in the abstract form (4.1), with the following spaces and operators:

$$X = H_{\frac{1}{2}} \times H, \qquad Y = L^2(\Gamma),$$

$$\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}, \qquad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \gamma_1 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 \\ \delta_\xi \end{bmatrix},$$

where γ_1 is the Neumann trace operator restricted to Γ. Note that $A^{-1} = \begin{bmatrix} 0 & -A_0^{-1} \\ I & 0 \end{bmatrix}$.

Lemma 5.2. *For every $\xi \in \Omega$ and $m > n/2$, we have $f \in (Z_m^d)'$.*

Proof. A simple calculation based on (5.2) shows that $C^* = -\begin{bmatrix} D \\ 0 \end{bmatrix}$, where D is the Dirichlet map defined in Proposition 5.1. Since A is skew-adjoint, we have

$$X_m^d = H_{\frac{m+1}{2}} \times H_{\frac{m}{2}}.$$

Hence, by a short computation, the space Z_m^d from (3.2) (using $\beta = 0$) is given by

$$Z_m^d = X_m^d + A^{-1}C^*Y + A^{-2}C^*Y \ldots + A^{-m}C^*Y = W_m \times V_m,$$

where

$$W_m = H_{\frac{m+1}{2}} + A_0^{-1}DY \ldots + A_0^{-[(m+1)/2]}DY\,,$$

$$V_m = H_{\frac{m}{2}} + A_0^{-1}DY \ldots + A_0^{-[m/2]}DY\,, \tag{5.3}$$

and $[\alpha]$ denotes the integer part of the real number α.

We claim that for $m > n/2$, $V_m \subset C(\Omega)$. Indeed, if $\varphi \in V_m$ then its first component according to the above decomposition is in $H_{\frac{m}{2}} \subset \mathcal{H}^m(\Omega) \subset C(\Omega)$ (we have used a well known Sobolev embedding theorem). The other components of φ are in $A_0^{-k}DY$, for various $k \in \mathbb{N}$. We know from Proposition 5.1 that $DY \subset C^\infty(\Omega)$. According to a regularity result for elliptic equations (see Evans [8, p. 309]), $A_0^{-k}DY \subset C^\infty(\Omega)$.

We claim that for every $\varepsilon > 0$ and for $m > n/2$, the operator $T_\varepsilon : V_m \to C(\overline{\Omega}_\varepsilon)$ defined by

$$T_\varepsilon \varphi = \varphi|_{\overline{\Omega}_\varepsilon}$$

is bounded. (On $C(\overline{\Omega}_\varepsilon)$ we use the supremum norm.) Indeed, since V_m is continuously embedded in H, it follows that T_ε is bounded from V_m to the larger space $L^2(\Omega_\varepsilon)$. Using the closed graph theorem, the boundedness of T_ε (to $C(\overline{\Omega}_\varepsilon)$) follows.

From the above claim it follows that for every $\xi \in \Omega$ and $m > n/2$, the Dirac mass δ_ξ is a bounded functional on V_m. Indeed, if we choose $\varepsilon > 0$ such that $\xi \in \Omega_\varepsilon$, then

$$|\langle \delta_\xi, \varphi \rangle| = |\varphi(\xi)| \leqslant \|T_\varepsilon\| \cdot \|\varphi\|_{V_m} \qquad \forall\, \varphi \in V_m\,.$$

From here, the lemma follows. ■

Proof of Theorem 1.1. Let $\varepsilon > 0$, $m > n/2$, $\xi^{(1)}, \xi^{(2)} \in \Omega_\varepsilon$, $\tau > \tau_0$. For $j \in \{1,2\}$ let $y^{(j)}$ be the output of (1.1) corresponding to $\xi = \xi^{(j)}$, with the initial data $\left[\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right] \in X$. These outputs are defined by continuous extension: we approximate $\delta_{\xi^{(j)}}$ by a sequence of functions $\delta_{k,j} \in L^2(\Omega)$, with convergence in V_m' (V_m was defined in (5.3)). Then from the decomposition $(Z_m^d)' = W_m' \times V_m'$ (see the proof of Lemma 5.2) we see that $\left[\begin{smallmatrix} 0 \\ \delta_{k,j} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} 0 \\ \delta_{\xi^{(j)}} \end{smallmatrix}\right]$ in $(Z_m^d)'$. For $\delta_{k,j}$ in place of $\delta_{\xi^{(j)}}$, the output of (1.1) on $[0,\tau]$ is

$$y_{k,j} = \Psi_\tau \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \mathcal{J}_\tau \begin{bmatrix} 0 \\ \delta_{k,j} \end{bmatrix},$$

where $\mathcal{J}_\tau$ is the operator introduced in (4.2), with our specific A and C. According to Theorem 4.4 the operator $\mathcal{J}_\tau$ has a continuous extension from $(Z_m^d)'$ to $\left[\mathcal{H}_R^{m-1}(0,\tau;Y)\right]'$. Thus, $y^{(j)}$ can be defined in the latter space as the limit of $y_{k,j}$.

Denoting $f^{(j)} = \left[\begin{smallmatrix} 0 \\ \delta_{\xi_j} \end{smallmatrix}\right]$, we obtain that

$$y^{(j)} = \Psi_\tau \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \mathcal{J}_\tau f^{(j)} \in \left[\mathcal{H}_R^{m-1}(0,\tau;Y)\right]'.$$

Subtracting these equations and using Theorem 4.4, we obtain that

$$\left\| f^{(1)} - f^{(2)} \right\|_{(Z_m^d)'} \leqslant \kappa_{m,\tau} \left\| y^{(1)} - y^{(2)} \right\|_{\left[\mathcal{H}_R^{m-1}(0,\tau;Y)\right]'}. \tag{5.4}$$

To end the proof we note that for every $j \in \{1, \dots, n\}$ there exists a function $\varphi_j \in \mathcal{D}(\Omega) \subset V_m$ such that

$$\varphi_j(x_1, x_2, \dots x_n) = x_j \qquad \forall\, (x_1, x_2, \dots, x_n) \in \Omega_\varepsilon$$

(this follows, for instance, from [20, Proposition 13.1.5]). Since $\xi^{(1)}, \xi^{(2)} \in \Omega_\varepsilon$, we have

$$\left\langle \delta_{\xi^{(1)}} - \delta_{\xi^{(2)}}, \varphi_j \right\rangle = \xi_j^{(1)} - \xi_j^{(2)}.$$

Thus, denoting $\eta_{m,\varepsilon} = \max_{j\in\{1,2,\dots\, n\}} \|\varphi_j\|_{V_m}$, it follows that

$$|\xi^{(1)} - \xi^{(2)}| \leqslant \sqrt{n}\, \eta_{m,\varepsilon} \left\| \delta_{\xi^{(1)}} - \delta_{\xi^{(2)}} \right\|_{V_m'} . \tag{5.5}$$

Since $\|f^{(j)}\|_{(Z_m^d)'} = \|\delta_{\xi_j}\|_{V_m'}$ and using (5.4), we get the desired estimate (1.2). ■

We mention that many interesting estimates related to (5.5) (also in the converse direction) can be found in Komornik and Yamamoto [12, Section 2].

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